Analytic Methods for Partial Differential Equations (Fall 2024)

Michael Weinstein
notes by Mark Ma

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This is a one-semester course on Introduction to Partial Differential Equations taught by Prof. Michael Weinstein in Fall 2024. The set of notes, taken by Kunyi Ma, serves as gratitude for such wonderful presentations. Any potential typos are due to the note taker.

Contents

1 Introduction

A brief summary of objectives for the semester.

- Laplace Equation. $-\Delta u = f$. Later: Elliptic Equation.
- Heat/Diffusion Equation. $\partial_t u = \Delta u$. Later: Parabolic.
- Wave Equation. $\frac{1}{c^2}\partial_{tt} u = \Delta u$. Later: Hyperbolic Equation.
- Schrödinger Equation. $i\partial_t \psi = -\Delta \psi$ for $\psi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$. Later: Dispersive Equation.

Question: Where do PDEs come from?

Example 1.1. *Take a wire frame, dip into soapy water. What is the shape of the soap film that spans the space which is interior to the wire frame? Imagine a surface of a wire frame Σ_u whose boundary is $\partial \Sigma_u = \{(x_{1,0}, x_{2,0}, f(x_{1,0}, x_{2,0}))\}$ in $\mathbb{R}^3$, with its projection to xy-plane $U := \{(x_1, x_2, 0)\}$ open with boundary $\partial U = \{(x_{1,0}, x_{2,0}, 0)\}$. In particular, one has $u : U \to \mathbb{R}$ s.t. $\Sigma_u = \{(x_1, x_2, u(x_1, x_2)) \mid (x_1, x_2) \in U\}$ with $u|_{\partial U} = f$. The set of all possible surfaces we consider is the admissible set*

$$\mathcal{A}_f := \{u(x_1, x_2) : U \to \mathbb{R} \mid u|_{\partial U} = f, \text{ and smoothness assumptions}\}$$

Physics principle: The shape taken by the soap film is given by that surface Σ_{u_*} where $x_3 = u_*(x_1, x_2)$, $u_*|_{\partial U} = f$, i.e., $u_* \in \mathcal{A}_f$, for which the surface area is minimized. Given $u \in \mathcal{A}_f$, let

$$\mathcal{E}[u] := \text{the surface area of } \Sigma_u$$

Purpose: Minimize $\mathcal{E}[u]$ among $u \in \mathcal{A}_f$. This is a classical problem in the calculus of variations. We claim that

$$\mathcal{E}[u] = \iint_U \sqrt{1 + |\nabla u(x_1, x_2)|^2}\, dx_1 dx_2$$

Intuitively, the area element is the measure of the cross product of tangent vectors at the point $(x_1, x_2, u(x_1, x_2))$. A priori, suppose that $u_* \in \mathcal{A}_f$ is a minimizer, one wish to derive the equation that u_* satisfies. Claim: $u_*(x)$ satisfies the following boundary value problem

$$\begin{cases} 2H_{\Sigma_{u_*}} := \nabla \cdot \left(\frac{1}{\sqrt{1+|\nabla u_*(x_1,x_2)|^2}} \nabla u_*(x) \right) = 0 & \forall\, x \in U \\ u_*(x) = f(x) & \forall\, x \in \partial U \end{cases} \tag{1}$$

This is mean curvature equation. If $|\nabla u_*(x_1, x_2)| \ll 1$, then $\Delta u_* = \nabla \cdot \nabla u_* = 0$ with $u_*|_{\partial U} = f$ gives the Laplace Equation. In fact, one impose the smoothness criterion $u \in C^2(\overline{U})$ in $\mathcal{A}_f$.

Proof of (1). Let $\varphi \in C^2(\overline{U})$ test function s.t. $\varphi|_{\partial U} = 0$, arbitrary. Recall we're assuming $u_* \in \mathcal{A}_f$ and by definition of the minimizing problem

$$\mathcal{E}[u_*] = \min_{u \in \mathcal{A}_f} \mathcal{E}[u]$$

so $u_* + \varepsilon\varphi \in \mathcal{A}_f$ for all $\varepsilon > 0$. Let's study the object $e : \mathbb{R} \to \mathbb{R}$ s.t.

$$e(\varepsilon) := \mathcal{E}[u_* + \varepsilon\varphi]$$

clearly e is smooth w.r.t. ε and it is minimized at the point $\varepsilon = 0$. From calculus we know $e'(0) = 0$. Let's write it out explicitly

$$\begin{aligned} e(\varepsilon) = \mathcal{E}[u_* + \varepsilon\varphi] &= \int_U \left(1 + |\nabla u_*(x) + \varepsilon \nabla \varphi(x)|^2\right)^{\frac{1}{2}} dx \\ &= \int_U \left(1 + |\nabla u_*|^2 + 2\varepsilon \nabla u_* \cdot \nabla\varphi + \varepsilon^2 |\nabla \varphi|^2\right)^{\frac{1}{2}} dx \\ e'(\varepsilon) &= \int_U \frac{1}{2} \left(1 + |\nabla u_*|^2 + 2\varepsilon \nabla u_* \cdot \nabla\varphi + \varepsilon^2 |\nabla \varphi|^2\right)^{-\frac{1}{2}} \left(2\nabla u_* \cdot \nabla \varphi + 2\varepsilon |\nabla\varphi|^2\right) dx \\ &= \int_U \left(1 + |\nabla u_*|^2 + 2\varepsilon \nabla u_* \cdot \nabla\varphi + \varepsilon^2 |\nabla \varphi|^2\right)^{-\frac{1}{2}} \left(\nabla u_* \cdot \nabla \varphi + \varepsilon |\nabla\varphi|^2\right) dx \\ 0 = e'(0) &= \int_U \left(1 + |\nabla u_*|^2\right)^{-\frac{1}{2}} \left(\nabla u_* \cdot \nabla \varphi\right) dx \end{aligned}$$

Hence the minimizer $u_* \in \mathcal{A}_f$ satisfies

$$\int_U \left(1 + |\nabla u_*|^2\right)^{-\frac{1}{2}} \left(\nabla u_* \cdot \nabla \varphi\right) dx = 0 \qquad \text{for all } \varphi \in C^2(\overline{U}) \text{ s.t. } \varphi|_{\partial U} = 0 \tag{2}$$

Recall an IBP lemma

Lemma 1.1. *For f, $g\,h \in C^2(\overline{U})$ smooth*

$$
\begin{aligned}
\nabla \cdot (h(\nabla f)g) &= (h\nabla f)\cdot \nabla g + \nabla \cdot (h\nabla f)g \\
\implies h\nabla f \cdot \nabla g &= \nabla \cdot (h\nabla f g) - \nabla \cdot (h\nabla f)g \\
\implies \int_U h\nabla f \cdot \nabla g &= \int_U \left(\nabla \cdot (h\nabla f g) - \nabla \cdot (h\nabla f)g\right)\, dx \\
&= \int_{\partial U} h\frac{\partial f}{\partial \nu} g\, dS - \int_U \nabla \cdot (h\nabla f) g\, dx \qquad (3)
\end{aligned}
$$

Apply (3) to (2) so that

$$
0 = \int_{\partial U} \left(1+|\nabla u_*|^2\right)^{-\frac{1}{2}} \varphi \frac{\partial u_*}{\partial \nu}\, dS - \int_U \nabla \cdot \left(\left(1+|\nabla u_*|^2\right)^{-\frac{1}{2}} \nabla u_*\right) \varphi\, dx \qquad for\ all\ \varphi \in C^2(\overline{U})\ s.t.\ \varphi|_{\partial U} = 0 \quad (4)
$$

But the first term vanishes due to choice of φ, so one conclude from (4) that

$$
0 = \int_U \nabla \cdot \left(\left(1+|\nabla u_*|^2\right)^{-\frac{1}{2}} \nabla u_*\right) \varphi\, dx \qquad for\ all\ \varphi \in C^2(\overline{U})\ s.t.\ \varphi|_{\partial U} = 0 \qquad (5)
$$

Lemma 1.2 (Fundamental Lemma of Calculus of Variations)**.** *Suppose $f \in C^0(U)$ for $U \subset \mathbb{R}^n$ open and $\varphi \in C^0(\overline{U})$ s.t. $\varphi|_{\partial U} = 0$ and that*

$$
\int_U f(x)\varphi(x)\, dx = 0 \qquad for\ all\ such\ \varphi
$$

Then $f \equiv 0$ on U

Proof of Lemma 1.2. Suppose there exists $x_0 \in U$ s.t. $f(x_0) \neq 0$. WLOG, let $f(x_0) > 0$. Then due to continuity of f, there exists r_0 s.t. $0 < f(x)$ for all $|x - x_0| < r_0$ small enough. Let $\varphi_0(x)$ be a function which is positive for $|x - x_0| < \frac{r_0}{2}$ and $\varphi_0(x) = 0$ for $|x - x_0| \geq r_0$. Then $0 = \int_U f(x)\varphi_0(x)\, dx$ by assumption, and moreover, $\int_U f(x)\varphi_0(x)\, dx = \int_{|x-x_0|<r_0} f(x)\varphi_0(x)\, dx > 0$ which is a contradiction. An example for such φ_0 is

$$
\varphi_0(x) := \begin{cases} \exp\left(-\frac{1}{r_0^2 - |x-x_0|^2}\right) & for\ |x - x_0| < r_0 \\ 0 & for\ |x - x_0| \geq r_0 \end{cases}
$$

Indeed such $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$. ☐

Hence apply Lemma 1.2 to (5) to obtain (1). ☐

2 Laplace and Poisson Equation

Lemma 2.1 (Green's Identity)**.** *For $u, v \in C^2(\overline{\Omega})$, for $\Omega \subset \mathbb{R}^n$ open. Assume $\partial\Omega$ is smooth*

$$u\Delta v = \nabla \cdot (u\nabla v) - \nabla u \cdot \nabla v \tag{6}$$
$$v\Delta u = \nabla \cdot (v\nabla u) - \nabla v \cdot \nabla u \tag{7}$$

Integrate (6) over Ω gives Green's first identity

$$\int_\Omega u\Delta v\, dx = \int_{\partial\Omega} u\frac{\partial v}{\partial n}\, dS - \int_\Omega \nabla u \cdot \nabla v\, dx \tag{8}$$

(6) minus (7) gives Green's second identity

$$u\Delta v - v\Delta u = \nabla \cdot (u\nabla v - v\nabla u)$$
$$\int_\Omega u\Delta v - v\Delta u\, dx = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS \tag{9}$$

2.1 Fundamental Solutions

Definition 2.1. *Laplacian operator $\Delta u = \Delta_r u + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}} u$ where radial Laplacian $\Delta_r v = v''(r) + \frac{n-1}{r}v'(r)$ if $v = v(|x|)$ for $r = |x|$ radial.*

Good about the radial equation is that its ODE, with

$$\begin{aligned}
v''(r) + \frac{n-1}{r}v'(r) &= 0\\
iff \quad (v')' + \frac{n-1}{r}v' &= 0\\
\implies letting\ w = v', \quad w' + \frac{n-1}{r}w &= 0\\
\implies w' &= -\frac{n-1}{r}w\\
\implies \frac{w'}{w} &= -\frac{n-1}{r}\\
\implies \log(|w|) &= -(n-1)\log(r) + C\\
\implies w(r) &= C'r^{1-n}
\end{aligned}$$

Hence $v'(r) = C'r^{1-n}$ gives

$$v(r) = \begin{cases} c'\frac{1}{2-n}r^{2-n} & n \neq 2\\ c'\log(r) & n = 2\end{cases}$$

Hence for $x \neq 0$, one writes

$$v(x) = \begin{cases} b|x|^{2-n} & n \neq 2\\ b\log(|x|) & n = 2\end{cases} \in C^\infty(\mathbb{R}) \tag{10}$$

and indeed satisfies $\Delta v = 0$. Note for $x \neq x_0 \in \mathbb{R}^n$, one has $v(x - x_0) \in C^\infty(\mathbb{R}^n \setminus \{x_0\})$ and $\Delta v(x - x_0) = 0$.

Definition 2.2. *Newtonian Potential Φ is*

$$\Phi(x) := \begin{cases} -\frac{1}{(2-n)\omega_n}|x|^{2-n} & n \neq 2\\ -\frac{1}{2\pi}\log(|x|) & n = 2\end{cases} \quad for\ x \neq 0 \tag{11}$$

for ω_n surface area of $\mathbb{S}^{n-1}$, and $\Phi(x) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ where $\Delta\Phi = 0$ for $x \neq 0$.

We seek for solution of Poisson's Equation $-\Delta u(x) = f(x)$ for any $x \in \mathbb{R}^n$ with any f defined on $\mathbb{R}^n$ with some conditions. For now say $f \in C^2(\mathbb{R}^n)$ with compact support, i.e., $\operatorname{supp}(f) := \overline{\{x \in \mathbb{R}^n \mid f(x) \neq 0\}}$ is compact. Our aim is to construct the inverse of Laplacian $(-\Delta)^{-1}f$.

Theorem 2.1 (Fundamental Theorem in Potential Theory)**.** *Define the fundamental solution as the convolution of Newtonian potential with force $f \in C_0^2(\mathbb{R}^n)$*

$$u(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\, dy = \int_{\mathbb{R}^n} \Phi(y)f(x-y)\, dy \tag{12}$$

- $u \in C^2(\mathbb{R}^n)$

- $-\Delta u(x) = f(x)$ *for any* $x \in \mathbb{R}^n$

Remark 2.1. *Look at* $\Phi(z) = \frac{1}{|z|^{n-2}}$ *for* $n \geq 3$, *this is actually singular. But this is weakly singular.*

$$
\begin{aligned}
|\int_{\mathbb{R}^n} \Phi(x-y)f(y)\,dy| &\leq C\int_{\mathbb{R}^n} |\Phi(x-y)||f(y)|\,dy \\
&\leq C\int_{|y|\leq R} \frac{1}{|x-y|^{n-2}}|f(y)|\,dy \leq \|f\|_\infty \int_{|y|\leq R} \frac{1}{|x-y|^{n-2}}\,dy \\
&\leq C\int_{|z|\leq R} \frac{1}{|z|^{n-2}}\,dz = \int_{\mathbb{S}^{n-1}}\int_0^R \frac{1}{r^{n-2}}r^{n-1}\,drd\theta = |\mathbb{S}^{n-1}|R^2 < \infty
\end{aligned}
$$

Remark 2.2. *Formally* $-\Delta\Phi(x) = \delta(x)$.

Proof of Theorem 2.1. For any $\varepsilon > 0$

$$
\begin{aligned}
-\Delta u(x) &= \int_{\mathbb{R}^n} \Phi(y)(-\Delta f(x-y))\,dy \\
&= \int_{|y|\leq R} \Phi(y)(-\Delta f(x-y))\,dy
\end{aligned}
$$

where $\tilde{R} < \infty$ depends on x due to compact support. So

$$
\begin{aligned}
-\Delta u(x) &= \int_{|y|<\epsilon} \Phi(y)(-\Delta f(x-y))\,dy + \int_{\varepsilon\leq|y|\leq R} \Phi(y)(-\Delta f(x-y))\,dy \\
&= I_\varepsilon(x) + J_\varepsilon(x)
\end{aligned}
$$

So

$$
\begin{aligned}
|I_\varepsilon(x)| &\leq \|\Delta f\|_\infty \int_{|y|<\epsilon} |\Phi(y)|\,dy \\
&\leq C\,\|\Delta f\|_\infty \int_0^\varepsilon \frac{1}{|y|^{n-2}}|y|^{n-1}\,dy = \frac{C}{2}\varepsilon^2\,\|\Delta f\|_\infty
\end{aligned}
$$

now

$$-\Delta u(x) = -\Delta(\Phi * f)(x) = I_\varepsilon(x) + J_\varepsilon(x) = J_\varepsilon(x) + O(\varepsilon^2)$$

Now we compute $J_\varepsilon(x)$ using (9) with $u = \Phi$ and $v = f$, and that f compactly supported

$$
\begin{aligned}
J_\varepsilon(x) &= -\int_{\varepsilon\leq|y|\leq R} \Phi(y)\Delta_y f(x-y)\,dy \\
&= -\left\{\int_{\varepsilon\leq|y|\leq R} f(x-y)\Delta_y\Phi(y)\,dy + \int_{\{|y|=\varepsilon\}\cup\{|y|=R\}} \Phi(y)\frac{\partial f(x-y)}{\partial n} - f(x-y)\frac{\partial\Phi(y)}{\partial n}\,dS(y)\right\} \\
&= \int_{\{|y|=\varepsilon\}\cup\{|y|=R\}} -\Phi(y)\frac{\partial f(x-y)}{\partial n} + f(x-y)\frac{\partial\Phi(y)}{\partial n}\,dS(y) \\
&= \int_{\{|y|=\varepsilon\}} -\Phi(y)\frac{\partial f(x-y)}{\partial n} + f(x-y)\frac{\partial\Phi(y)}{\partial n}\,dS(y) \\
&= J_{\varepsilon,1}(x) + J_{\varepsilon,2}(x)
\end{aligned}
$$

Look at $J_{\varepsilon,1}(x)$

$$|J_{\varepsilon,1}(x)| \leq \int \frac{1}{(n-2)\omega_n}|y|^{2-n}|\nabla_y f(x-y)\cdot n|\,dS(y) \leq C\,\|\nabla f\|_\infty \int_{|y|=\varepsilon} |y|^{2-n}\,dS \leq C\varepsilon$$

Hence

$$-\Delta u(x) = J_{\varepsilon,2}(x) + O(\varepsilon^2) + O(\varepsilon)$$

Now just compute $J_{\varepsilon,2}(x)$

$$
\begin{aligned}
J_{\varepsilon,2}(x) &= \int_{|y|=\varepsilon} f(x-y)\frac{\partial\Phi(y)}{\partial n}\,dS(y) \\
&= \int_{|y|=\varepsilon} f(x-y)\left[\frac{|y|^{1-n}}{\omega_n}\right]\,dS(y) \\
&= \int_{|w|=1} f(x-\varepsilon w)\frac{\varepsilon^{1-n}}{\omega_n}\varepsilon^{n-1}\,dS(w) = \frac{1}{\omega_n}\int_{|w|=1} f(x-\varepsilon w)\,dS(w) \to f(x)
\end{aligned}
$$

$\square$

Definition 2.3 (Hölder Space)**.** $0 < \alpha < 1$, $C^\alpha(\Omega) := \{f \in C^0(\Omega) \mid |f(x) - f(y)| \le C|x-y|^\alpha \text{ for any } x, y \in \Omega\}$

Remark 2.3. *We've done for $f \in C_0^2$. If $f \in C^1$, it still holds. But what if $f \in C^0$? Is $\Phi * f \in C^2$? No. But if $f \in C^\alpha$ for some $0 < \alpha \le 1$ Hölder, then still true via Schauder theory.*
If $f \in C^\alpha$, then $(\Phi * f) \in C^{2,\alpha}$ and $-\Delta \cdot (\phi * f) = f$. What about f on C^0? There are counter-examples.

2.2 Harmonic Functions

Definition 2.4. *$u \in C^2(\Omega)$, for Ω open, is harmonic if $\Delta u = 0$ for all $x \in \Omega$.*

Remark 2.4. *Intuition: $n = 2$. $u_{xx} + u_{yy} = 0$. If $u(i,j) = u_{ij}$, then $(u_{xx})(i,j) \sim u_{i+1,j} - 2u_{i,j} + u_{i-1,j}$ and $(u_{yy})(i,j) \sim u_{i,j-1} - 2u_{i,j} + u_{i,j+1}$ leading to*

$$0 = (\Delta u)_{ij} = u_{i+1,j} - 4u_{i,j} + u_{i-1,j} + u_{i,j-1} + u_{i,j+1} \implies u_{i,j} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j-1} + u_{i,j+1})$$

One wish to prove the following

- Mean Value Property: u is harmonic iff it satisfies a mean value property.
- Maximum Principle: If u harmonic on Ω but non-constant, then its maximum and minimum are attained only on the boundary $\partial\Omega$.

Notations: $B_r(x) = \{y \in \mathbb{R}^n \mid |y - x| < r\}$. $\partial B_r(x) = \{y \in \mathbb{R}^n \mid |y - x| = r\}$. Moreover, for $\omega_n = |\partial B_1(0)|$

$$|B_r(0)| = \frac{\omega_n}{n} r^n \qquad |\partial B_r(0)| = \omega_n r^{n-1}$$

One also introduce averages

$$\fint_A f \equiv \frac{1}{|A|} \int_A f$$
$$\fint_{B_r(x)} f(y)\,dy = \frac{1}{\frac{\omega_n r^n}{n}} \int_{|y-x|<r} f(y)\,dy$$
$$\fint_{\partial B_r(x)} f(y)\,dy = \frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} f(y)\,dS(y)$$

Theorem 2.2 (Mean Value Property)**.** *Assume u harmonic on Ω. Let $x \in \Omega$, assume $r > 0$ s.t. $B_r(x) \subset \Omega$. Then*

$$u(x) = \fint_{\partial B_r(x)} u = \frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} u(y)\,dS(y) \tag{13}$$
$$u(x) = \fint_{B_r(x)} u\,dy = \frac{1}{\frac{\omega_n r^n}{n}} \int_{|y-x|<r} u(y)\,dy \tag{14}$$

Proof by method of spherical averages. Introduce

$$\phi(r) = \frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} u(y)\,dS(y)$$

This is well-defined if r sufficiently small. Indeed, by Lebesgue Differentiation one has $\lim_{r\to 0} \phi(r) = u(x)$. On the other hand, if we're able to prove $\phi'(r) = 0$, then this immediately implies ϕ is constant independent of r. Combining with $\lim_{r\to 0} \phi(r) = u(x)$, one may conclude $u(x) = \phi(r)$ for r sufficiently small. It suffices to see $\phi'(r) = 0$. By a change of variables $y = x + rz$ for $|z| = 1$ so $dS(y) = r^{n-1} dS(z)$, one may use Gauss Theorem

$$\begin{aligned}
\phi(r) &= \frac{1}{\omega_n r^{n-1}} \int_{|z|=1} u(x + rz) r^{n-1}\,dS(z) \\
\phi'(r) &= \frac{1}{\omega_n} \int_{|z|=1} \nabla u(x + rz) \cdot z\,dS(z) \\
&= \frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} \nabla_y u(y) \cdot \frac{y - x}{r} dS(y) \\
&= \frac{1}{\omega_n r^{n-1}} \int_{|y-x|<r} \Delta_y u(y)\,dy = 0 \implies (13)
\end{aligned}$$

Now to prove (14),

$$\int_{|y-x|<r} u(y)\,dy = \int_0^r \int_{|y-x|=s} u(y)\,dS(y)\,ds$$
$$= \int_0^r \omega_n s^{n-1} u(x)\,ds = \omega_n u(x)\frac{r^n}{n} \implies (14)$$

□

Theorem 2.3 (Maximum/Minimum Principle for harmonic functions)**.** *Let $\Omega \subset \mathbb{R}^n$ bounded open, u harmonic in Ω, i.e., $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $\Delta u = 0$ in $x \in \Omega$. Then*

- *maximum over $\overline{\Omega}$ is attained on the boundary $\partial\Omega$, i.e., $\max\limits_{x\in\overline{\Omega}} u(x) = \max\limits_{x\in\partial\Omega} u(x)$*
- *If Ω is connected. If u attains its maximum over $\overline{\Omega}$ in Ω its interior, then u is constant throughout Ω.*
- *Since $-u$ is harmonic, the same holds true with* max *replaced by* min*. Note $\min\limits_{x\in\overline{\Omega}} u = -\max\limits_{x\in\overline{\Omega}}(-u)$*

Recall a connected set is not a disjoint union of 2 nonempty, closed and open sets.

Proof. We start with the second item. Let Ω open connected. Assume there exists $x_0 \in \Omega$ s.t. $u(x_0) = M = \max\limits_{x\in\overline{\Omega}} u(x)$. Consider the partition $\Omega = \{x \in \Omega \mid u(x) = M\} \cup \{x \in \Omega \mid u(x) < M\}$. Call the first A_1 and the second A_2. A_1 is nonempty as assumed. Clearly A_1 is closed by continuity of u. We wish to show that A_1 is open and since A_1 is nonempty, by connectedness, A_2 is empty. Suppose $0 < r < \operatorname{dist}(x_0, \partial\Omega)$. We know

$$M = u(x_0) = ⨍_{B_r(x_0)} u$$
$$M = ⨍_{B_r(x_0)} M = ⨍_{B_r(x_0)} u$$
$$\implies ⨍_{B_r(x_0)} (M - u(x))\,dx = 0$$

But $M - u(x) \geq 0$ on $B_r(x_0)$. Hence A_1 is open. □

Corollary 2.1 (Instantaneous Propagation of Boundary Information throughout Ω)**.** *$\Omega \subset \mathbb{R}^n$ connected. And $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ s.t. $\Delta u = 0$ for $x \in \Omega$ and $u|_{\partial\Omega} = g \in C^0(\overline{\Omega})$. Assume $g \geq 0$ all along $\partial\Omega$, and $g > 0$ somewhere on $\partial\Omega$. Then $u > 0$ everywhere on Ω.*

Remark 2.5. *Contrast with simple wave phenomena.*

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}u(x,t) = 0 \qquad u(x,0) = \varphi(x)$$

here $u(x,t) = \varphi(x-t)$ solves the equation. This has finite propagation speed.

Corollary 2.2 (Uniqueness to Dirichlet Problem)**.** *Ω bounded connected open. Dirichlet Problem for Poisson's Equation $-\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$. Then there exists at most one solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to the Dirichlet Problem.*

Proof. Suppose u_1 and u_2 are 2 solutions to Dirichlet Problem. Define $w = u_1 - u_2$. Then $\Delta w = 0$ in Ω and $w = 0$ on $\partial\Omega$. Hence w takes both maximum and minimum on $\partial\Omega$. Hence $w \equiv 0$. □

Definition 2.5 (Standard Mollifier)**.** *For $\varepsilon > 0$, and $\Omega \subset \mathbb{R}^n$ open,*

- *Let $\Omega_\varepsilon := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$.*
- *Define*

$$\eta(x) := \begin{cases} C\exp(\frac{1}{|x|^2-1}) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

  $C > 0$ is chosen so that $\int_{\mathbb{R}^n} \eta(x)\,dx = \int_{|x|\leq 1} \eta(x) = 1$. Notice $\eta \in C^\infty(\mathbb{R}^n)$ and $\operatorname{supp}(\eta) = B_1(0)$.

- *For any $\varepsilon > 0$, $\eta_\varepsilon(x) := \frac{1}{\varepsilon^n}\eta(\frac{x}{\varepsilon})$ so $\operatorname{supp}(\eta_\varepsilon) \subset B_\varepsilon(0)$. Notice*

$$\int_{\mathbb{R}^n} \eta_\varepsilon dx = \int_{|x|\leq\varepsilon} \frac{1}{\varepsilon^n}\eta\left(\frac{|x|}{\varepsilon}\right)\,dx = \int_{\mathbb{R}^n} \eta = 1$$

  If fix $x \in \Omega_\varepsilon$, then $y \mapsto \eta_\varepsilon(x-y)$ has support in Ω

- *If $f \in L^1_{loc}$, define $f^\varepsilon := (\eta_\varepsilon * f)(x)$ the mollification.*

Lemma 2.2. *One has tools from mollification*

- *For $f \in L^1_{loc}(\Omega)$, $f^\varepsilon \in C^\infty(\Omega_\varepsilon)$ for $\varepsilon > 0$.*
- *$f^\varepsilon \to f$ a.e. as $\varepsilon \to 0$ in Ω.*
- *If $f \in C^0(\Omega)$, then $f^\varepsilon \to f$ uniformly on compact subsets of Ω.*
- *$1 \le p < \infty$ and $f \in L^p_{loc}(\Omega)$, then $f^\varepsilon \to f$ in $L^p_{loc}(\Omega)$.*

One then wish to do the following

- Converse to mean value property of harmonic functions
- size estimates on harmonic functions and Liouville Theorem
- Green's functions and BVP.

Theorem 2.4 (Converse to Mean Value Property)**.** *Suppose $u \in C^0(\Omega)$ and u satisfies the mean value property on Ω. Then $u \in C^\infty(\Omega)$ and $\Delta u = 0$.*

Proof. Define $u^\varepsilon := (\eta_\varepsilon * u)(x)$. Strategy: To show $u \in C^\infty(\Omega)$, it suffices to show $u^\varepsilon(x) = u(x)$ for $x \in \Omega_\varepsilon$ and that $u^\varepsilon \in C^\infty(\Omega_\varepsilon)$, where the latter follows directly from DCT. To see the former,

$$\begin{aligned}
u^\varepsilon(x) &= \int_\Omega \eta_\varepsilon(x-y)u(y)\,dy = \int_\Omega \frac{1}{\varepsilon^n}\eta\left(\frac{|x-y|}{\varepsilon}\right)u(y)\,dy \\
&= \frac{1}{\varepsilon^n}\int_{|x-y|\le\varepsilon} \eta\left(\frac{|x-y|}{\varepsilon}\right)u(y)\,dy = \frac{1}{\varepsilon^n}\int_0^\varepsilon\int_{|x-y|=r}\eta\left(\frac{|x-y|}{\varepsilon}\right)u(y)\,dS(y)\,dr \\
&= \frac{1}{\varepsilon^n}\int_0^\varepsilon\int_{|x-y|=r}\eta\left(\frac{r}{\varepsilon}\right)u(y)\,dS(y)\,dr = \frac{1}{\varepsilon^n}\int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right)\int_{|x-y|=r}u(y)\,dS(y)\,dr \\
&= \frac{1}{\varepsilon^n}\int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right)\omega_n r^{n-1}\,u(x)\,dr = u(x)\int_{|x|\le\varepsilon}\eta_\varepsilon(x)\,dx = u(x)
\end{aligned}$$

One needs to show $\Delta u = 0$. By Gauss Green theorem and change of variables $y = x + rz$ for $|z| = 1$

$$\begin{aligned}
\int_{|x-y|<r}\Delta u(y)\,dy &= \int_{|x-y|<r}\nabla\cdot(\nabla u(y))\,dy = \int_{|x-y|=r}\frac{\partial u}{\partial n}(y)\,dS(y) \\
&= \int_{|x-y|=r}\nabla_y u(y)\cdot\frac{y-x}{r}\,dS(y) = \int_{|z|=1}\frac{\partial}{\partial r}u(x+rz)\,dS(z) \\
&= \frac{\partial}{\partial r}\int_{|z|=1}u(x+rz)\,dS(z) = \frac{\partial}{\partial r}(\omega_n u(x)) = 0
\end{aligned}$$

Divide the LHS by the volumne of the ball $\frac{\omega_n}{n}r^n$. But take $r \to 0$ on LHS to conclude $\Delta u(x) = 0$. $\square$

Remark 2.6.

$$u(x) = \int_{|y-x|=r}u(y)\frac{dS(y)}{\omega_n r^{n-1}} = \int_{|y-x|=r}u(y)\,d\mu(y)$$

where $d\mu(y) := \frac{dS(y)}{\omega_n r^{n-1}}$ is probability measure. In general this defines the harmonic measure

$$u(x) = \int_{\partial\Omega}u(y)\,d\mu_{x,\partial\Omega}(y)$$

Theorem 2.5 (Estimates on the Size of Harmonic Functions)**.** *Let u be harmonic in domain $\Omega \subset \mathbb{R}^n$ open. Let $x_0 \in \Omega$ and $B_r(x_0) \subset\subset \Omega$. Then*

$$|u(x_0)| \le \frac{n}{\omega_n}\frac{1}{r^n}\|u\|_{L^1(B_r(x_0))} \tag{15}$$

$$\left|\frac{\partial}{\partial x_i}u(x_0)\right| \le \frac{2^{n+1}n}{r^{n+1}}\left(\frac{n}{\omega_n}\right)\|u\|_{L^1(B_r(x_0))} \qquad \forall\ i \in \{1,\cdots,n\} \tag{16}$$

Proof. $u(x_0) = \frac{1}{\frac{\omega_n}{n}r^n}\int_{B_r(x_0)}u(y)\,dy$.

$$|u(x_0)| \le \frac{1}{\frac{\omega_n}{n}r^n}\int_{B_r(x_0)}|u(y)|\,dy \implies (15)$$

Note that u harmonic implies that u satisfies the MVP and further implies that $u \in C^\infty$. Hence we may take derivatives

$$0 = \frac{\partial}{\partial x_i}(\Delta u) = \Delta(\frac{\partial}{\partial x_i} u)$$

Hence $\frac{\partial}{\partial x_i} u(x) \equiv u_{x_i}(x)$ is harmonic.

$$u_{x_i}(x_0) = \frac{1}{\frac{\omega_n}{n}(\frac{r}{2})^n} \int_{B_{\frac{r}{2}}(x_0)} u_{y_i}(y)\, dy$$

Note that

$$u_{y_i}(y) = \nabla_y \cdot (0, \cdots, 0, u(y), 0, \cdots, 0)$$

so

$$\int_{B_{\frac{r}{2}}(x_0)} u_{y_i}(y)\, dy = \int_{\partial B_{\frac{r}{2}}(x_0)} u(y)\nu_i \, dS(y) \qquad \textit{where } \nu = (\nu_1, \cdots, \nu_n) \textit{ is unit outer normal}$$

now

$$\begin{aligned} u_{x_i}(x_0) = \frac{1}{\frac{\omega_n}{n}(\frac{r}{2})^n} \int_{\partial B_{\frac{r}{2}}(x_0)} u(y)\nu_i \, dS(y) &\le \frac{n2^n}{\omega_n r^n}\omega_n(\frac{r}{2})^{n-1} \max_{|y-x_0| \le \frac{r}{2}} |u(y)| \\ &= \frac{2n}{r} \max_{|y-x_0| \le \frac{r}{2}} |u(y)| \end{aligned}$$

now for any y s.t. $|y - x_0| = \frac{r}{2}$, $B_{\frac{r}{2}}(y) \subset B_r(x_0)$, so

$$|u(y)| \le \frac{n}{\omega_n} \frac{1}{(\frac{r}{2})^n} \|u\|_{L^1(B_{\frac{r}{2}}(y))} \le \frac{n}{\omega_n} \frac{1}{(\frac{r}{2})^n} \|u\|_{L^1(B_r(x_0))}$$

Hence

$$|u_{x_i}(x_0)| \le \frac{2^{n+1}n}{r^{n+1}} \left(\frac{n}{\omega_n}\right) \|u\|_{L^1(B_r(x_0))} \implies (16)$$

□

Theorem 2.6 (Liouville Theorem)**.** *Suppose $u : \mathbb{R}^n \to \mathbb{R}$, $u \in C^2$ with $\Delta u = 0$. Suppose there exists $M > 0$ s.t. $|u(x)| \le M$ for all $x \in \mathbb{R}^n$. Then $u(x) \equiv C$ constant.*

Proof. For $i = 1, \cdots, n$,

$$|\frac{\partial}{\partial x_i} u(x)| \le \frac{C_n}{r^{n+1}} \int_{|y-x|<r} |u(y)|\, dy \le \frac{C_n}{r^{n+1}} M \frac{\omega_n}{n} r^n = C' \frac{1}{r} \to 0 \qquad \textit{as } r \to \infty$$

□

2.3 Dirichlet Boundary Value Problems

Let $u, v \in C^2(\overline{\Omega})$.

$$u\Delta v - v\Delta u = \nabla \cdot (u\nabla v - v\nabla u) \implies \int_\Omega u\Delta v - v\Delta u \, dx = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS \ (9)$$

Now fix $u \in C^2(\overline{\Omega})$, fix $x \in \Omega$ and let $\Omega_\varepsilon := \Omega \setminus \overline{B_\varepsilon(x)}$

$$v(y) = \Phi(x - y) = \begin{cases} c_n|x-y|^{2-n} & n \ge 3 \\ \frac{1}{2\pi} \log|x-y| & n = 2 \end{cases} \in C^2(\overline{\Omega_\varepsilon})$$

Hence applying (9)

$$\int_{\Omega_\varepsilon} u(y)\Delta_y \Phi(x-y)\, dy = \int_{\Omega_\varepsilon} \Phi(x-y)\Delta u(y)\, dy + \int_{\partial\Omega_\varepsilon} u(y)\frac{\partial}{\partial n_y}\Phi(x-y) - \Phi(x-y)\frac{\partial}{\partial n_y}u(y)\, dS(y) \qquad (17)$$

For $\partial\Omega_\varepsilon = \partial\Omega \cup \{|x-y| = \varepsilon\}$

$$\begin{aligned} 0 = &\int_{\Omega_\varepsilon} \Phi(x-y)\Delta u(y)\, dy + \int_{\partial\Omega} u(y)\frac{\partial}{\partial n_y}\Phi(x-y) - \Phi(x-y)\frac{\partial}{\partial n_y}u(y)\, dS(y) \\ &+ \int_{\{|x-y|=\varepsilon\}} u(y)\frac{\partial}{\partial n_y}\Phi(x-y) - \Phi(x-y)\frac{\partial}{\partial n_y}u(y)\, dS(y) \end{aligned}$$

Thus sending $\varepsilon \to 0$ one get the Layer Potential Representation of function $u \in C^2(\overline{\Omega})$

$$u(x) = \int_\Omega \Phi(x-y)(-\Delta u(y))\,dy + \int_{\partial\Omega} \left(-u(y)\frac{\partial}{\partial n_y}\Phi(x-y) + \Phi(x-y)\frac{\partial}{\partial n_y}u(y)\right) dS(y) \tag{18}$$

where the first term is volume potential, the middle term is the double layer potential, and the last term is single layer potential. This might suggest that we can solve the boundary value problem

$$\begin{cases} -\Delta u = f & in\ \Omega \\ u|_{\partial\Omega} = g & on\ \partial\Omega \\ \frac{\partial}{\partial n}u\big|_{\partial\Omega} = h & on\ \partial\Omega \end{cases}$$

by taking $u(x) = VP[f](x) + DLP[g](x) + SLP[h](x)$. Recall Uniqueness Theorem to Dirichlet Problem

$$\begin{cases} -\Delta u = f & in\ \Omega \\ u|_{\partial\Omega} = g & on\ \partial\Omega \end{cases} \tag{19}$$

has at most 1 solution. Hence in fact, the uniqueness theorem says the solution is uniquely determined by f and g alone. In other words, h is something we can compute afterwards. Here is Green's observation. The layer potential representation formula (18) holds if one replace $\Phi(x-y)$ by $\Phi(x-y) - \phi(y)$ where

$$\begin{cases} -\Delta_y \phi(y) = 0\ in\ \Omega \\ \phi \in C^0(\overline{\Omega}) \cap C^2(\Omega) \end{cases} \tag{20}$$

one hence obtain the formula

$$u(x) = \int_\Omega [\Phi(x-y)-\phi(y)](-\Delta_y u(y))\,dy - \int_{\partial\Omega} u(y)\frac{\partial}{\partial n_y}(\Phi(x-y)-\phi(y))\,dS(y) + \int_{\partial\Omega} (\Phi(x-y)-\phi(y))\frac{\partial u}{\partial n_y}(y)\,dS(y) \tag{21}$$

for any ϕ satisfies (20).

Definition 2.6 (Corrector Function). *Fix any $x \in \Omega$, define the corrector function $\phi(y;x) \in C_y^2(\Omega)$*

$$\phi(y;x) \qquad s.t. \begin{cases} \Delta_y \phi(y;x) = 0 & y \in \Omega \\ \phi(y,x) = \Phi(x-y) & y \in \partial\Omega \end{cases} \tag{22}$$

where this is a family of Dirichlet Problems with specific family of Dirichlet Boundary conditions. This is extremely domain dependent.

If we can solve this specific family of BVPs for $-\Delta$ on Ω, then we define

Definition 2.7 (Green's Function).

$$G_{Dir,\Omega}(x,y) = \Phi(x-y) - \phi(y;x,\Omega) \qquad Green's\ Function \tag{23}$$

Define

$$u(x) = \int_\Omega G_{Dir,\Omega}(x,y)(-\Delta u(y))\,dy - \int_{\partial\Omega} u(y)\frac{\partial G_{Dir,\Omega}(x,y)}{\partial n_y}\,dS(y) \tag{24}$$

$$= \int_\Omega G_{Dir,\Omega}(x,y)f(y)\,dy - \int_{\partial\Omega} g(y)\frac{\partial G_{Dir,\Omega}(x,y)}{\partial n_y}\,dS(y)\ as\ solution\ to\ (19) \tag{25}$$

Claim: For nice domains, $G_{Dir,\Omega}(x,y)$ can be constructed and (24) solves the Dirichlet Problem (19).

Theorem 2.7 (Representation Formula for Dirichlet BVP). *Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ s.t. u solves (19). Suppose for any $x \in \Omega$, there exists $\phi(y;x)$ s.t. (22) holds, then $u(x)$ has the representation (25).*

Example 2.1. *Apply (24) to $u \equiv 1$ so*

$$1 = -\int_{\partial\Omega} \frac{\partial G_{Dir,\Omega}(x,y)}{\partial n_y}\,dS(y)$$

Notice that $\frac{\partial G_{Dir,\Omega}(x,y)}{\partial n_y}\,dS(y)$ is weighted surface measure. For harmonic function on balls, this collapses to Mean Value Property.

Lemma 2.3. *For $x, y \in \Omega$ s.t. $x \neq y$, $G_{Dir,\Omega}(x,y) = G_{Dir,\Omega}(y,x)$ is symmetric. This is essentially due to Δ is self-adjoint operator.*

Proof. Introduce $v(z) := G(x,z)$ and $w(z) := G(y,z)$ with x, y fixed. v singular at x while w singular at y. Remove small discs around the singularities of size ε small enough. Let $\Omega_\varepsilon := \Omega \setminus (B_\varepsilon(x) \cup B_\varepsilon(y))$. In Ω_ε

$$\begin{aligned} 0 &= v(z)\Delta_z w(z) - w(z)\Delta_z v(z) = \nabla \cdot (v(z)\nabla_z w(z) - w(z)\nabla_z v(z)) \\ 0 &= \int_{\Omega_\varepsilon} \nabla \cdot (v(z)\nabla_z w(z) - w(z)\nabla_z v(z))\, dz \\ &= \int_{\partial\Omega_\varepsilon} v(z)\frac{\partial w}{\partial n_z} - w(z)\frac{\partial v}{\partial n_z}\, dS(z) \end{aligned}$$

Look at $v(z)$ and $w(z)$ on $\partial\Omega$. $v(z) = G(x,z) \to 0$ for $x \in \Omega$ as $z \to \partial\Omega$ and $w(z) = G(y,z) \to 0$ for $y \in \Omega$ as $z \to \partial\Omega$, so

$$0 = \int_{|z-x|=\varepsilon} v(z)\frac{\partial w}{\partial n_z}(z) - w(z)\frac{\partial v}{\partial n_z}(z)\, dS(z) + \int_{|z-y|=\varepsilon} v(z)\frac{\partial w}{\partial n_z}(z) - w(z)\frac{\partial v}{\partial n_z}(z)\, dS(z)$$

now as $z \to x$, $v(z)$ looks like $\Phi(x-z) \sim \frac{1}{|x-z|^{n-2}}$, which is a weak singularity, while $w(z) = G(y,z) \to G(y,x)$ is bounded as $z \to x$. Also, as $z \to x$, $\frac{\partial w}{\partial n_z}(z) \sim \frac{1}{|y-x|^{n-1}}$ bounded, and apply similar argument to the second term

$$0 = o(1) - G(y,x) + G(x,y) - o(1) \implies G(y,x) = G(x,y)$$

□

Example 2.2. *Green's Function for $B_a(0) = \{y \mid |y| < a\}$. Need to seek $\phi(y; x, B_a)$ s.t. for all $x \in B_a(0)$, $\Delta_y \phi(y; x) = 0$, and that for any $y_0 \in \partial B_a$*

$$\lim_{y \to y_0\ |y|<a} \phi(y; x) = \frac{1}{\omega_n(n-2)} \frac{1}{|y_0 - x|^{n-2}} = \Phi(x - y_0)$$

Key Geometric Property of $B_a(0)$: Given any $|x| < a$, there exists x^* s.t. $|x^*| > a$ and for any $y_0 \in \partial B_a(0)$

$$\frac{|x^* - y_0|}{|x - y_0|} = C(a, |x|)$$

and

$$x^* = (\frac{a}{|x|})^2 x \qquad C(a,|x|) = \frac{a}{|x|}$$

x^* is reflection point to x w.r.t. ∂B_a. To apply this,

$$\frac{1}{|x-y_0|} = \frac{C(a,|x|)}{|x^*-y_0|} \implies \frac{1}{|x-y_0|^{n-2}} - \frac{(C(a,|x|))^{n-2}}{|x^*-y_0|^{n-2}} = 0 \implies \Phi(x-y_0) - C(a,|x|)^{n-2}\Phi(x^*-y_0) = 0$$

hence

$$\Phi(x-y) - C(a,|x|)^{n-2}\Phi(x^*-y)\big|_{y=y_0,\ |y_0|=a} = 0$$

where the left term is Newtonian Potential with singularity at $x \in B_a(0)$ and the right term is Newtonian Potential with singularity outside $B_a(x)$, and $\Delta\phi(y; x, B_a) = 0$ for $y \in B_a(0)$. Conclusion:

$$G_{Dir,B_a}(x,y) = \Phi(x-y) - C(a,|x|)^{n-2}\Phi(x^*-y) \qquad \textit{for all } x, y \in B_a(x) \textit{ s.t. } x \neq y$$

Indeed

$$-\Delta G_{Dir,B_a}(x,y) = \delta(x-y), \qquad |x| < a \implies \lim_{y\to y_0\ |y|<a} G_{Dir,B_a}(x,y) = 0$$

Proposed Representation of the Solution of

$$\begin{cases} -\Delta u = f & \textit{in } |x| < a \\ u\big|_{|x|=a} = g & \textit{on } |x| = a \end{cases}$$

for a given f defined on $B_a(0)$ and g defined on $\partial B_a(0)$ is

$$u(x) = \int_{|y|<a} G_{Dir,B_a}(x,y) f(y)\, dy - \int_{|y|=a} g(y)\frac{\partial G_{Dir,B_a}(x,y)}{\partial n_y}\, dS(y) \tag{26}$$

$$= \int_{|y|<a} G_{Dir,B_a}(x,y) f(y)\, dy + \int_{|y|=a} g(y) H(x,y)\, dS(y) \tag{27}$$

where for $n \geq 3$

$$G_{Dir,B_a}(x,y) = \Phi(x-y) - (\frac{a}{|x|})^{n-2}\Phi(x^*-y) \qquad \textit{for } x^* = (\frac{a}{|x|})^2 x \quad 0 < |x| < a,\ |y| < a \textit{ and } x \neq y$$

One may actually calculate Poisson Kernel for $B_a(0)$

$$H(x,y) = -\left.\frac{\partial G_{Dir,B_a}(x,y)}{\partial n_y}\right|_{|x|<a,|y|=a} = \frac{1}{a\omega_n}\frac{a^2 - |x|^2}{|x-y|^n} \tag{28}$$

Theorem 2.8. *$u \in C^2(|y|<a) \cap C^0(|y| \le a)$ solution to (19) with $\Omega = B_a(0)$, then u is given by (27).*

Theorem 2.9. *Let g denote any continuous function on $\partial B_a(0) = \{|x| = a\}$. Let $H(x,y)$ be the Poisson Kernel (28), define*

$$V(x) = \begin{cases} g(x) & |x| = a \\ \int_{|y|=a} H(x,y) g(y)\, dS(y) & |x| < a \end{cases}$$

Then V satisfies the Dirichlet Problem (19) with $f = 0$ and

- *$V \in C^2(|x| < a)$ with $\Delta V = 0$ for $|x| < a$*
- *$V \in C^0(|x| \le a)$, i.e., for any x_0 with $|x_0| = a$, one has*

$$\lim_{x \to x_0 \ |x|<a} V(x) = g(x_0) \tag{29}$$

Remark 2.7. *Set $x = 0$, so $H(0,y) = \frac{1}{a^{n-1}\omega_n}$, so $V(0) = \int_{|y|=a} g(y) H(0,y)\, dS(y) = \frac{1}{\omega_n a^{n-1}} \int_{|y|=a} g(y)\, dS(y)$. This may also be interpreted as probability of Brownian Motion starting at x escaping through $\Gamma \subset \partial B_a$ using $\int_\Gamma H(x,y)\, dS(y)$.*

Proposition 2.1. *Properties on $H(x,y)$*

(a) $H(x,y) \in C^\infty$ for $|y| \le a$, $|x| < a$ with $y \ne x$

(b) $\Delta_x H(x,y) = 0$ for any $|x| < a$ and $|y| = a$

(c) $\int_{|y|=a} H(x,y)\, dS(y) = 1$ for $|x| < a$

Proof. Look at $u \equiv 1$ and apply (27). □

(d) $H(x,y) > 0$ for $|x| < a$ and $|y| = a$

(e) Pick ζ with $|\zeta| = a$ with $\delta > 0$

$$\lim_{x \to \zeta} H(x,y) = 0 \ uniformly \ on \ \{|y| = a \mid |y - \zeta| \ge \delta > 0\}$$

Proof. $H(x,y) \sim \frac{a^2 - |x|^2}{|x-y|^n} \sim \frac{a^2 - |x|^2}{|\zeta - y|^n} \le \frac{a^2 - |x|^2}{\delta^n} \to 0$ uniformly in y for $|y| = a$ and $|y - \zeta| \ge \delta > 0$. □

Proof of Theorem 2.9. For any x_0 with $|x_0| = a$. For $|x| < a$

$$\begin{aligned} V(x) - g(x_0) &= \int_{|y|=a} g(y) H(x,y)\, dS(y) - g(x_0) \int_{|y|=a} H(x,y)\, dS(y) \\ &= \int_{|y|=a} (g(y) - g(x_0)) H(x,y)\, dS(y) \end{aligned}$$

Let $\varepsilon > 0$, it suffices to show that as $|x - x_0| \to 0$, $|V(x) - g(x_0)| \le \varepsilon$. For any $\delta > 0$

$$V(x) - g(x_0) = \int_{\substack{|y|=a \\ |y-x_0|<\delta}} H(x,y)(g(y) - g(x_0))\, dS(y) + \int_{\substack{|y|=a \\ |y-x_0|\ge\delta}} H(x,y)(g(y) - g(x_0))\, dS(y) := I_1(\delta) + I_2(\delta)$$

By continuity, there exists $\delta(\varepsilon) > 0$ s.t.

$$|y - x_0| < \delta(\varepsilon) \implies |g(y) - g(x_0)| < \varepsilon$$

Hence

$$|I_1(\delta)| \le \int_{\substack{|y|=a \\ |y-x_0|<\delta}} H(x,y) |g(y) - g(x_0)|\, dS(y) \le \varepsilon \int_{\substack{|y|=a \\ |y-x_0|<\delta}} H(x,y)\, dS(y) \le \varepsilon$$

Now send x to x_0

$$\begin{aligned} |I_2(\delta)| &\le |\int_{\substack{|y|=a \\ |y-x_0|\ge\delta}} H(x,y)(g(y) - g(x_0))\, dS(y)| \le 2\, \|g\|_\infty \int_{\substack{|y|=a \\ |y-x_0|\ge\delta}} H(x,y)\, dS(y) \\ &\le 2\, \|g\|_\infty \max_{|y-x_0|\ge\delta, |y|=a} H(x,y) \int_{|z|=a} 1\, dS(z) = 2\, \|g\|_\infty\, \omega_n a^{n-1} \max_{|y-x_0|\ge\delta, |y|=a} H(x,y) \\ &\to 0 \qquad as\ x \to x_0 \end{aligned}$$

This proves (29). □

2.4 Potential Theory Approach to solving general Dirichlet Problem (integral equations)

Suppose that u solves $\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = g \end{cases}$ The layer potential representation formula says

$$u(x) = \int_{\partial\Omega} \left(g(y) \frac{\partial}{\partial n_y} \Phi(x-y) - \Phi(x-y) \frac{\partial}{\partial n_y} u(y) \right) dS(y)$$

One seek solution

$$u(x) = \int_{\partial\Omega} \frac{\partial}{\partial n_y} \Phi(x-y) \mu(y) \, dS(y)$$

for μ to be determined. Note $x \in \Omega$ one has $\Delta_x u(x) = 0$. Choose $\mu = \mu_g$. One wish to Tune μ so that

$$\lim_{x \to x_0 \in \partial\Omega} u(x) = g(x_0)$$

One has the key observation that

$$\lim_{x \in \Omega \to x_0 \in \partial\Omega} u(x) = -\frac{1}{2}\mu(x_0) + \int_{\partial\Omega} K(x_0, y)\mu(y) \, dS(y)$$

where $K(x,y) = -\left. \frac{\partial \Phi(x-y)}{\partial n_y} \right|_{x \neq y \in \partial\Omega}$ Therefore, μ must satisfy (this is very practical)

$$g(x) = -\frac{1}{2}\mu(x) + \int_{\partial\Omega} K(x,y)\mu(y) \, dS(y)$$

which is a linear integral equation on the boundary. It is in fact Fredholm alternative operator. T_k is a compact operator on $L^2(\partial\Omega)$.

$$(-\frac{1}{2} I + T_K)\mu = g$$

Recall linear algebra $Ax = b$ solvable iff $b \perp ker(A^*)$. Claim:

$$(-\frac{1}{2} I + T_K)^* = -\frac{1}{2} I + T_{K^*}$$

for $K^*(x,y) = K(y,x)$ and the nullspace $Ker((-\frac{1}{2}I + T_K)^*) = \{0\}$. Note $-\frac{1}{2}I + T_{K^*}$ is associated to uniqueness of the Neumann boundary value problem $\begin{cases} \Delta u = 0 \\ \frac{\partial}{\partial n} u \big|_{\partial\Omega} = 0 \\ u \to 0 \ as \ |x| \to \infty \end{cases}$ This converts the existence of Dirichlet Boundary Value Problem to the Uniqueness of the Neumann Problem. For a sketch, look at $n = 3$ so $\Phi(x-y) \sim \frac{1}{|x-y|}$.

$$u(x) = \int_{\partial\Omega} \nabla_y \left(\frac{1}{|x-y|}\right) \cdot n_y \mu(y) \, dS(y) \sim \int_{\partial\Omega} \frac{1}{|x-y|^2} \left(\frac{x-y}{|x-y|} \cdot n_y\right) \mu(y) \, dS(y)$$

We try computing

$$|u(x)| \leq \int_{\partial\Omega} \frac{1}{|x-y|^2} \, dS(y) \, \|\mu\|_{L^\infty} \sim \int \frac{1}{|y|^2} |y| \, d|y|$$

But this fails. On the other hand, observe

$$\frac{x-y}{|x-y|} \cdot n_y \to 0 \ as \ x \to x_0 \ and \ y \to x_0$$

3 Heat/Diffusion Equation

Motivation: Think about a random walk on $\delta\mathbb{Z}$, i.e., $(-\infty, \cdots, -2\delta, -\delta, 0, \delta, 2\delta, \cdots, \infty)$ for $0 < \delta < 1$. For each time increment τ, $2\tau, \cdots$, the walker which starts at $x = 0$ jumps right or left with probability $\frac{1}{2}$. Question: What is the probability that the particle/walker which starts at $x = 0$, $t = 0$ is at position $x \in \delta\mathbb{Z}$ at time $t = n\tau$.

$$\delta = microscopic\ spatial\ scale\ and\ \tau = microscopic\ time\ scale$$

Write $v(x,t) = \mathbb{P}(X_n = x \mid t = n\tau)$ and look at

$$\begin{aligned}
v(x, t+\tau) &= \mathbb{P}(X_{n+1} = x \mid t = (n+1)\tau) \\
&= \mathbb{P}(X_n = x - \delta \mid t = n\tau \ and\ the\ walker\ jumps\ right) + \mathbb{P}(X_n = x + \delta \mid t = n\tau\ and\ the\ walker\ jumps\ left) \\
&= \mathbb{P}(X_n = x - \delta)\mathbb{P}(walker\ jumps\ right) + \mathbb{P}(X_n = x + \delta)\mathbb{P}(walker\ jumps\ left) \\
&= \frac{1}{2}v(x - \delta, t) + \frac{1}{2}v(x + \delta, t)\ for\ x \in \delta\mathbb{Z}\ and\ t = \tau\mathbb{N}
\end{aligned}$$

One may taylor expand so

$$\begin{aligned}
v(x,t) + v_t(x,t)\tau + O(\tau^2) &= \frac{1}{2}\left[v(x,t) + v_x(x,t)\delta + v_{xx}(x,t)\frac{\delta^2}{2!} + O(\delta^3) + v(x,t) - v_x(x,t)\delta + v_{xx}(x,t)\frac{(-\delta)^2}{2!} + O((-\delta)^3)\right] \\
&= v(x,t) + \frac{1}{2}\delta^2 v_{xx}(x,t) + O(\delta^3) \\
\implies v_t(x,t) + O(\tau) &= \frac{1}{2}(\frac{1}{\tau}\delta^2)v_{xx}(x,t) + O(\frac{1}{\tau}\delta^2\delta)
\end{aligned}$$

let $D = \frac{1}{\tau}\delta^2$ constant. Send $\delta, \tau \to 0$. This leads to $v_t = \frac{1}{2}Dv_{xx}$ with initial condition $v_0(x) = v(x,0) = \delta(x)$ and $v_0(x) \geq 0$ and $\int v_0(x)\,dx = 1$.

Second Motivation: $f(x,t)$ = concentration or density of a quantity at position x at time t. e.g. the density $\rho = \frac{mass}{volume}$. For $V \subset \mathbb{R}^3$

$$\int_V \rho(x,t)\,dx = Mass\ inside\ V\ at\ time\ t$$

with our physcial model

$$\frac{d}{dt}\int_V \rho(x,t)\,dx = -\int_{\partial V} F \cdot n\,dS$$

One compute using Gauss-Green

$$\frac{d}{dt}\int_V \rho(x,t)\,dx = \int_V \frac{d}{dt}\rho(x,t)\,dx = -\int_V \nabla \cdot F\,dx$$

hence

$$\partial_t\rho(x,t) = -\nabla_x \cdot F(x,t) \qquad \forall\ (x,t)$$

Assume $F = -a(x)\nabla\rho$. So

$$\partial_t\rho = \nabla \cdot (a(x)\nabla\rho)$$

for $a(x) \equiv a_0 > 0$ constant we simly have $\partial_t\rho = a_0\Delta\rho$

3.1 Initial Value Problem

Look for solution to initial value problem

$$\begin{cases} u_t = \Delta u & x \in \mathbb{R}^n\ t > 0 \\ u(x,0) = f(x) & t = 0 \end{cases} \tag{30}$$

3.1.1 Fourier Transform

Definition 3.1 (Schwartz Class $\mathcal{S}(\mathbb{R}^1)$). *$\phi \in \mathcal{S}(\mathbb{R}^1)$ if $\phi \in C^\infty$ and for any α, $\beta \geq 0$ integers, we have*

$$|x^\alpha\partial_x^\beta\phi(x)| \leq C_{\alpha,\beta,\phi} < \infty\ for\ all\ x$$

Definition 3.2 (Schwartz Class $\mathcal{S}(\mathbb{R}^n)$). *$\phi \in \mathcal{S}(\mathbb{R}^n)$ if $\phi \in C^\infty$ and for any α, $\beta \in \mathbb{N}^n$, let $x^\alpha = {x_1}^{\alpha_1}\cdots{x_n}^{\alpha_n}$ and $\partial_x^\beta = \partial_{x_1}^{\beta_1}\cdots\partial_{x_n}^{\beta_n}$.*

$$\sup_{x\in\mathbb{R}^n}|x^\alpha\partial_x^\beta\phi(x)| \leq C_{\alpha,\beta,\phi} < \infty \iff |x^\alpha\partial_x^\beta\phi(x)| \leq C_{\alpha,\beta,\phi} < \infty\ for\ all\ x$$

Some further notation, $|\alpha| = \sum_{i=1}^n \alpha_i$.

For example, $e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^n)$. But $e^{-|x|}$ is not.

Theorem 3.1. *$\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for any $1 \le p < \infty$. Let $f \in L^p(\mathbb{R}^n)$, there exists $f_j \subset \mathcal{S}(\mathbb{R}^n)$ s.t.*

$$\|f_j - f\|_{L^p} \to 0 \text{ as } j \to \infty$$

Definition 3.3 (Fourier Transform)**.** *For any $f \in L^1(\mathbb{R}^n)$, define*

$$\hat{f}(\xi) = (\mathcal{F}f)(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-iy\cdot\xi} f(y)\, dy$$

Lemma 3.1 (Riemann-Lebesgue)**.** *For $f \in L^1(\mathbb{R}^n)$, the simplest bound is*

$$|\hat{f}(\xi)| \le \frac{1}{(2\pi)^{\frac{n}{2}}} \|f\|_{L^1}$$

And in fact

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$$

But we have no information about the decay rate.

Proposition 3.1. *$f \in \mathcal{S}(\mathbb{R}^n)$. Then*

(i) $\hat{f} \in C^\infty(\mathbb{R}^n)$.

Proof. $\hat{f} \sim \int e^{ix\cdot\xi} f(x)\, dx$ then

$$\partial_\xi^\beta \hat{f}(\xi) \sim \int (-i\xi)^\beta e^{ix\cdot\xi} f(x)\, dx$$

so

$$|\partial_\xi^\beta \hat{f}(\xi)| \le \int |\xi|^\beta |f|\, dx < \infty$$

□

(ii) $\partial_\xi^\beta \hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{-ix\cdot\xi}(-ix)^\beta f(x)\, dx = \hat{g}(\xi)$ *where* $g(x) = (-ix)^\beta f(x)$

(iii) $\widehat{\partial_x^\beta f}(\xi) = (i\xi)^\beta \hat{f}(\xi)$.

Proof.

$$\int e^{-ix\cdot}(\partial_{x_i} f)\, dx = \int \partial_{x_i}(e^{-ix\cdot\xi} f(x)) - \partial_{x_i}(e^{-ix\cdot\xi}) f(x)\, dx = \int \xi_i e^{-ix\cdot\xi} f(x)\, dx$$

Hence

$$\begin{aligned}
\widehat{\partial_x^\beta f}(\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{-ix\cdot\xi}(\partial_x^\beta) f(x)\, dx = \frac{1}{(2\pi)^{\frac{n}{2}}} \int (-1)^\beta \partial_x^\beta(e^{-ix\cdot\xi}) f(x)\, dx \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int (-1)^\beta (-i\xi)^\beta e^{-ix\cdot\xi} f(x)\, dx \\
&= (i\xi)^\beta \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{-ix\cdot\xi} f(x)\, dx = (i\xi)^\beta \hat{f}(\xi)
\end{aligned}$$

□

(iv) *Hence for $f \in \mathcal{S}(\mathbb{R}^n)$, $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. For any $q, r \in \mathbb{N}^n$

$$\begin{aligned}
\xi^q \partial_\xi^r \hat{f}(\xi) &= \xi^q \partial_\xi^r \int e^{-ix\cdot\xi} f(x)\, dx \\
&= \xi^q \int e^{-ix\cdot\xi}(-ix)^r f(x)\, dx \\
&= \frac{1}{|-i|^q} \int (-i\xi)^q e^{-ix\cdot\xi}(-ix)^r f(x)\, dx \\
&= \frac{1}{|-i|^q} \int \partial_x^q(e^{-ix\cdot\xi})(-ix)^r f(x)\, dx \\
&= \frac{1}{|i|^q} \int e^{-ix\cdot\xi} \partial_x^q((-ix)^r f(x))\, dx \\
\implies |\xi^q \partial_\xi^r \hat{f}(\xi)| &\le C \int |\partial_x^q((-ix)^r f(x))| dx < \infty
\end{aligned}$$

using $f \in \mathcal{S}(\mathbb{R}^n)$. □

Definition 3.4 (Fourier Inversion). $\check{f}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{ix\cdot\xi} f(\xi)\, d\xi = \hat{f}(-x)$

Lemma 3.2 (Fourier Inversion Formula). *(i) If $f \in \mathcal{S}(\mathbb{R}^n)$, $\check{\hat{f}}(x) = f$*

(ii) $\left\|\hat{f}\right\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$.

Proposition 3.2 (Plancherel). *$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ can be extended from $\mathcal{S}(\mathbb{R}^n)$ to all $L^2(\mathbb{R}^n)$ as a bounded linear operator.*

$$\|\mathcal{F}f\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$$

In fact, $\mathcal{F}$ extends, by density, to be an operator on $L^2(\mathbb{R}^n)$. This is unitary operator.

Proof. Use $\mathcal{S}(\mathbb{R}^n)$ dense in $L^2(\mathbb{R}^n)$ and BLT Theorem as the following □

Theorem 3.2 (BLT Theorem). *Let $(X, \|\cdot\|_X)$ be a normed linear space and $(Y, \|\cdot\|_Y)$ be a complete normed space (Banach Space). Suppose $M \subset X$ is a dense subspace of X, i.e., $\overline{M} = X$, and suppose*

$$T : M \to Y \textit{ is a bounded linear transformation}$$

i.e.,

$$\exists\ c_T > 0 \textit{ s.t. } \|Tx\|_Y \leq c_T \|x\|_X \textit{ for all } x \in M$$

Then there exists unique bounded linear transformation $\overline{T}$ s.t.

$$\overline{T} : X \to Y$$

with properties

(a) For any $x \in M$, $\overline{T}(x) = T(x)$

(b) $\|\overline{T}\|_{B(X,Y)} = \|T\|_{B(M,Y)}$

Proof. For any $x \in X$, $(x_n) \subset M$ s.t. $\|x_n - x\|_X \to 0$. Define $\overline{T}(x) = \lim_{n\to\infty} Tx_n$. □

3.1.2 Representation Formula

For $u_t = \Delta u$ for $x \in \mathbb{R}^n$ and $t > 0$, $u(x, t = 0) = f(x)$. Let's derive assuming $f \in \mathcal{S}(\mathbb{R}^n)$. Apply $\mathcal{F}$ to them, resulting in

$$\begin{aligned} \hat{u}_t(\xi, t) &= \widehat{\Delta u}(\xi, t) = -|\xi|^2 \hat{u}(\xi, t) \\ \hat{u}(\xi, t = 0) &= \hat{f}(\xi) \end{aligned}$$

This is family of Initial Value Problems for ODE's parametrized by $\xi \in \mathbb{R}^n$.

$$\begin{aligned} \hat{u}(\xi, t) &= e^{-|\xi|^2 t} \hat{f}(\xi) \\ \implies u(x, t) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-|\xi|^2 t}\, \hat{f}(\xi)\, d\xi \end{aligned}$$

But we want a clearer representation where we can read of positivity. Using Fubini

$$\begin{aligned} u(x, t) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi} e^{-|\xi|^2 t} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n_y} e^{-iy\cdot\xi} f(y)\, dy\, d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_y} \left(\int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} e^{-|\xi|^2 t}\, d\xi \right) f(y)\, dy \end{aligned}$$

Hence one obtain

$$u(x, t) = \int_{\mathbb{R}^n_y} K_t(x - y) f(y)\, dy = (K_t * f)(x, t)$$

where

$$K_t(x - y) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} e^{-|\xi|^2 t}\, d\xi$$

Compute by change of variables to $\zeta = \xi t^{\frac{1}{2}}$ so $d\zeta = t^{\frac{n}{2}} d\xi$.

$$\begin{aligned} K_t(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi} e^{-|\xi|^2 t}\, d\xi \\ &= \frac{1}{(2\pi)^n} \int e^{i\frac{1}{\sqrt{t}}x\cdot(t^{\frac{1}{2}}\xi)} e^{-|t^{\frac{1}{2}}\xi|^2} \frac{1}{t^{\frac{n}{2}}}\, d(t^{\frac{1}{2}}\xi) \\ &= \frac{1}{(2\pi)^n} \frac{1}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i(\frac{x}{\sqrt{t}})\cdot\xi} e^{-|\xi|^2}\, d\xi \end{aligned}$$

Let $\mu = \frac{x}{\sqrt{t}}$ so

$$e^{i\mu\cdot\xi - \xi\cdot\xi} = e^{-(\xi\cdot\xi - i\mu\cdot\xi)} = e^{-|\xi - \frac{i\mu}{2}|^2 - \frac{1}{4}|\mu|^2}$$

Here $|\xi|^2 = \xi\cdot\xi$. So

$$\begin{aligned} K_t(x) &= \frac{1}{(2\pi)^n} \frac{1}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-|\xi - \frac{i\mu}{2}|^2 - \frac{1}{4}|\mu|^2}\, d\xi \\ &= \frac{1}{(2\pi)^n} \frac{1}{t^{\frac{n}{2}}} e^{-\frac{1}{4}\frac{|x|^2}{t}} \int_{\mathbb{R}^n} e^{-|\xi - \frac{i\mu}{2}|^2}\, d\xi \\ &= \frac{1}{t^{\frac{n}{2}}} e^{-\frac{1}{4}\frac{|x|^2}{t}} \prod_{j=1}^n \left(\int_{\mathbb{R}} e^{-(\xi_j - \frac{i}{2}\mu_j)^2} \frac{d\xi_j}{2\pi} \right) \end{aligned}$$

where $\mu_j = \frac{x_j}{\sqrt{t}}$. By Cauchy's Theorem

$$\begin{aligned} K_t(x) &= \frac{1}{t^{\frac{n}{2}}} e^{-\frac{1}{4}\frac{|x|^2}{t}} \frac{1}{(2\pi)^n} \left(\int_{-\infty}^{\infty} e^{-z^2}\, dz\right)^n \\ K_t(x) &= \frac{1}{t^{\frac{n}{2}}} e^{-\frac{1}{4}\frac{|x|^2}{t}} \frac{1}{(2\pi)^n} \pi^{\frac{n}{2}} \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} \quad for\ t > 0 \end{aligned}$$

This is heat(diffusion) Kernel on $\mathbb{R}^n$ so

$$u(x,t) = (K_t * f)(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy \qquad for\ t > 0 \tag{31}$$

As with our study of the Poisson formula for the Dirichlet problem for

$$\begin{aligned} \Delta u &= 0 & x &\in B_a(0) \\ u|_{\partial B_a(0)} &= f(x) & x &\in \partial B_a(0) \end{aligned}$$

Proposition 3.3. *We have basic properties of* $K_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$

(i) $K_t(x) \in C^\infty$ *for all* $x \in \mathbb{R}^n$, $t > 0$

(ii) $(\frac{\partial}{\partial t} - \Delta) K_t(x) = 0$ *for* $t > 0$, $x \in \mathbb{R}^n$

(iii) $K_t(x) > 0$ *for all* $t > 0$ *and* $x \in \mathbb{R}^n$

(iv) For $t > 0$

$$\int_{\mathbb{R}^n} K_t(y)\, dy = 1 \tag{32}$$

Proof. For $t > 0$

$$\begin{aligned} &\int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|y|^2}{4t}}\, dy = \prod_{j=1}^n \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{y_j^2}{4t}}\, dy_j = \left(\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{\rho^2}{4t}}\, d\rho \right)^n \\ \implies &\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{\rho^2}{4t}}\, d\rho = \int_{\mathbb{R}} e^{-\sigma^2} \frac{1}{\sqrt{\pi}}\, d\sigma = 1 \end{aligned}$$

□

(v) *Fix $\delta > 0$, look at amount of mass outside the small neighborhood*

$$\lim_{t\to 0}\int_{|y-x|>\delta} K_t(x-y)\,dy = 0 \qquad uniformly\ in\ x\in\mathbb{R}^n \tag{33}$$

Proof. make change of variables $\xi = \frac{y-x}{\sqrt{4t}}$ for $t>0$ so $d\xi = \frac{1}{(4t)^{\frac{n}{2}}}dy$

$$\int_{|y-x|>\delta}\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4t}}\,dy = \frac{1}{(\pi)^{\frac{n}{2}}}\int_{|\xi|>\frac{\delta}{\sqrt{4t}}} e^{-|\xi|^2}\,d\xi \to 0\ as\ t\to 0$$

□

Theorem 3.3 (Representation Formula for Heat Equation)**.** *Let f be bounded in $\mathbb{R}^n$. Define u as in (31), then*

(i) *For fixed $t>0$, $u(x,t) = (K_t * f)(x) \in C^\infty(\mathbb{R}^n)$ smooth in space. In fact it is also smooth in t, so*

$$u(x,t)\in C^\infty(\mathbb{R}^n\times\{t>0\})$$

(ii) *$u_t = \Delta u$ for $t>0$ and $x\in\mathbb{R}^n$.*

(iii) *Suppose moreover f is continuous on $\mathbb{R}^n$*

$$u(x,t) = \begin{cases}\int_{\mathbb{R}^n} K_t(x-y,t)f(y)\,dy & for\ t>0\\ f(x) & for\ t=0,\ x\in\mathbb{R}^n\end{cases}$$

Then this is continuous on $\mathbb{R}^n\times[0,\infty)$. Thus,

$$\lim_{(x,t)\to(\xi,0)} u(x,t) = f(\xi)$$

for any $\xi\in\mathbb{R}^n$.

Proof. Since f bounded in $\mathbb{R}^n$, there exists $M>0$ s.t. $|f(x)|\le M$ for all $x\in\mathbb{R}^n$. Want to show (iii). Fix $\xi\in\mathbb{R}^n$, and let $(x,t)\to(\xi,0)$ for $t>0$, using (32)

$$u(x,t)-f(\xi) = \int_{\mathbb{R}^n} K_t(x-y)\left(f(y)-f(\xi)\right)dy$$

By continuity of f, given any $\varepsilon>0$, there is a $\delta=\delta(\varepsilon)>0$ s.t. if $|y-\xi|<2\delta(\varepsilon)$, then $|f(y)-f(\xi)|<\varepsilon$.

$$\begin{aligned}u(x,t)-f(\xi) &= \int_{|y-x|<\delta} K_t(x-y)\left(f(y)-f(\xi)\right)dy + \int_{|y-x|\ge\delta} K_t(x-y)\left(f(y)-f(\xi)\right)dy\\ &=: I_\delta(t,x)+J_\delta(t,x)\end{aligned}$$

Assume $|x-\xi|<\delta(\varepsilon)$. Then

$$|y-\xi| = |y-x+x-\xi| \le |y-x|+|x-\xi| \le 2\delta(\varepsilon)$$

Look at the first part $I_\delta(t,x)$.

$$|I_\delta(t,x)| \le \int_{|y-x|<\delta} K_t(x-y)|f(y)-f(\xi)|\,dy < \varepsilon$$

For second part $J_\delta(t,x)$, use (33)

$$|J_\delta(t,x)| \le \int_{|y-x|\ge\delta} K_t(x-y)|f(y)-f(\xi)|\,dy \le 2M\int_{|y-x|\ge\delta} K_t(x-y)\,dy \to 0\ as\ t\to 0\ uniformly\ in\ x$$

Hence

$$\limsup_{(x,t)\to(\xi,0)} |u(x,t)-f(\xi)| \le \varepsilon$$

Conclude by taking $\varepsilon\to 0$. □

Some notation

$$u(x,t) = (K_t * f)(x) =: e^{\Delta t}f(x)$$

In fact the above generalizes to the model

$$\begin{cases} u_t = D\Delta u & x\in\mathbb{R}^n\ t>0\\ u(x,0)=f(x) & t=0\end{cases} \qquad for\ D>0$$

for diffusion constant D with heat kernel

$$K_t(z) := \frac{1}{(4\pi Dt)^{\frac{n}{2}}}e^{-\frac{|z|^2}{4Dt}}$$

3.2 Properties of Solution to Heat Equation

Take (30)

$$\begin{cases} u_t = \Delta u & x \in \mathbb{R}^n \ t > 0 \\ u(x,0) = f(x) & t = 0 \end{cases}$$

3.2.1 Conservation Law and Dissipation

Theorem 3.4 (Conservation of L^1 norm)**.** *Assume $u(x,t)$ solution with nice decay properties as $|x| \to \infty$. One may take derivative outside*

$$\frac{d}{dt}\int_{\mathbb{R}^n} u(x,t)\,dx = \lim_{R\to\infty}\int_{|x|<R} \nabla_x \cdot (\nabla_x u(x,t))\,dx = \lim_{R\to\infty}\int_{|x|=R} \frac{\partial u}{\partial n}\,dS = 0$$

This is conservation law. And for $f \in L^1(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_t(x-y)f(y)\,dy\,dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_t(x-y)\,dx f(y)\,dy = \int_{\mathbb{R}^n} f(y)\,dy$$

But on the other hand, heat equation has dissipative character.

Theorem 3.5 (Dissipation of L^2 norm)**.**

$$\begin{aligned} u_t &= \Delta u \\ uu_t &= u\Delta u \\ \partial_t(\frac{u^2}{2}) &= u\Delta u = \nabla \cdot (u\nabla u) - |\nabla u|^2 \end{aligned}$$

Now integrate assuming that $u(x,t)$ and $\nabla u(x,t) \to 0$ as $|x| \to \infty$ sufficiently fast for any fixed $t > 0$

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^n} \frac{u^2}{2}\,dx &= \int_{\mathbb{R}^n} \nabla_x \cdot (u\nabla u) - |\nabla u|^2\,dx \\ \frac{d}{dt}\int_{\mathbb{R}^n} u^2\,dx &= -2\int_{\mathbb{R}^n} |\nabla u|^2\,dx \le 0 \end{aligned}$$

Hence energy $\|u\|_{L^2(\mathbb{R}^n)}(t)$ dissipates as $t \to \infty$. Alternatively one may view from the ODE on the Fourier Side

$$\begin{aligned} \partial_t \hat{u}(\xi,t) &= -|\xi|^2\hat{u}(\xi,t) \\ \hat{u}(\xi,t) &= e^{-|\xi|^2 t}\hat{u}(\xi,0) \\ \int_{\mathbb{R}^n} \hat{u}^2(\xi,t)\,d\xi &= \int_{\mathbb{R}^n} e^{-2|\xi|^2 t}\hat{u}^2(\xi,0)\,d\xi \\ \implies \frac{d}{dt}\int_{\mathbb{R}^n} \hat{u}^2(\xi,t)\,d\xi &= -2\int_{\mathbb{R}^n} |\nabla u(x,t)|^2\,dx \le 0 \end{aligned}$$

Hence using Plancherel we have

$$\int_{\mathbb{R}} |u(x,t)|^2\,dx \le \int_{\mathbb{R}} |f(x)|^2\,dx$$

3.2.2 Instantaneous Propagation and Comparison Principle

Proposition 3.4 (Instantaneous Propagation of Information)**.** *Consider IVP to heat equation (30). Take $f \ge 0$ for all $x \in \mathbb{R}^n$. Suppose also $f \in C^0_{bdd}(\mathbb{R}^n)$. If $f(x_0) > 0$ for some $x_0 \in \mathbb{R}^n$. Then $u(x,t) > 0$ for all $x \in \mathbb{R}^n$ and $t > 0$.*

Proof. Assume $f(x) \ge c > 0$ on some $B_\varepsilon(x_0)$ for $0 < \varepsilon \ll 1$. Then for any $x \in \mathbb{R}^n$ and $t > 0$

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} K_t(x-y)f(y)\,dy \ge \int_{B_\varepsilon(x_0)} K_t(x-y)f(y)\,dy \\ &\ge c\int_{B_\varepsilon(x_0)} \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4t}}\,dy \\ &\ge c\min_{y\in B_\varepsilon(x_0)}\left(\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4t}}\right)|B_\varepsilon(x_0)| \\ &\ge c\left(\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{(\mathrm{dist}(x,B_\varepsilon(x_0))+\varepsilon)^2}{4t}}\right)\frac{\omega_n}{n}\varepsilon^n > 0 \end{aligned}$$

$\square$

Proposition 3.5 (Comparison Principle)**.** *Suppose* $f_1,\ f_2 \in L^1(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$*. Assume* $f_1(x) \le f_2(x)$ *and* $f_1(x_0) < f_2(x_0)$ *for some* x_0*. Let* $\{u_j(x,t)\}_{j=1}^2$ *solve the heat equation*

$$(\partial_t - \Delta)u_j = 0 \quad \textit{for } t > 0 \qquad \textit{with} \qquad u_j(x,0) = f_j(x)$$

Then $u_1(x,t) < u_2(x,t)$ *for* $x \in \mathbb{R}^n$ *and* $t > 0$*.*

Proof. Let $\delta(x,t) = u_2(x,t) - u_1(x,t)$, so

$$(\partial_t - \Delta)\delta = 0 \qquad \delta(x,0) = f_2(x) - f_1(x) \ge 0$$

and $\delta(x_0,0) > 0$. Hence by instantaneous propagation, $\delta(x,t) > 0$ for all $x \in \mathbb{R}^n$ and $t > 0$. □

3.2.3 Stability and Uniqueness

Proposition 3.6 (Stability)**.** *Suppose* $f_1,\ f_2 \in L^2(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$*. Let* $\{u_j(x,t)\}_{j=1}^2$ *solve the heat equation*

$$(\partial_t - \Delta)u_j = 0 \quad \textit{for } t > 0 \qquad \textit{with} \qquad u_j(x,0) = f_j(x)$$

Define $\delta(x,t) = u_2(x,t) - u_1(x,t)$*, then*

$$\partial_t \delta(x,t) = \Delta\delta(x,t) \qquad \textit{with} \qquad \delta(x,0) = f_2(x) - f_1(x)$$

Now

$$\begin{aligned} \delta\partial_t\delta &= \delta\Delta\delta \\ \partial_t(\frac{1}{2}\delta^2) &= \nabla\cdot(\delta\nabla\delta) - \nabla\delta\cdot\nabla\delta \\ \implies \frac{1}{2}\partial_t \int_{\mathbb{R}^n} \delta^2(x,t)\,dx &= -\int_{\mathbb{R}^n} |\nabla\delta(x,t)|^2\,dx \le 0 \end{aligned}$$

This contrasts with $\partial_t \int_{\mathbb{R}^n} \delta(x,t)\,ds = 0$*. Then for any* $t > 0$

$$\begin{aligned} \int_{\mathbb{R}^n} \delta^2(x,t)\,dx &\le \int_{\mathbb{R}^n} \delta^2(x,0)\,dx \\ \implies \int_{\mathbb{R}^n} |u_2(x,t) - u_1(x,t)|^2\,dx &\le \int_{\mathbb{R}^n} |f_2(x) - f_1(x)|^2\,dx \end{aligned}$$

Proposition 3.7 (Uniqueness)**.** *Let* $\{u_j(x,t)\}_{j=1}^2$ *solve the heat equation*

$$(\partial_t - \Delta)u_j = 0 \quad \textit{for } t > 0 \qquad \textit{with} \qquad u_j(x,0) = f(x)$$

Then

$$0 \le \int_{\mathbb{R}^n} |u_2(x,t) - u_1(x,t)|^2\,dx \le 0$$

for any $t > 0$*. So* $u_1(x,t) = u_2(x,t)$ *for any* $t > 0$ *and a.e.*$x \in \mathbb{R}^n$*.*

3.2.4 Semigroup Property of Heat Flow

For $u(x,t) = e^{\Delta t} f =: K_t * f$.

Lemma 3.3. *For* $t,\ s \ge 0$

$$e^{\Delta(t+s)} = e^{\Delta t}e^{\Delta s} = e^{\Delta s}e^{\Delta t}$$

Proof. For $f \in L^1(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$ nice,

$$t \mapsto e^{\Delta(t+s)} f \qquad u_t = \Delta u \qquad u(x,0) = e^{\Delta s} f$$

with solutions $u(t+s)$.

$$t \mapsto e^{\Delta t}(e^{\Delta s} f) \qquad u_t = \Delta u \qquad u(x,0) = e^{\Delta s} f$$

Hence by uniqueness, they give rise to the same solutions. □

Note

$$\begin{aligned} e^{\Delta(t+s)} f &= \int_{\mathbb{R}^n} K_{t+s}(x-y) f(y)\,dy \\ e^{\Delta t}e^{\Delta s} f &= \int_{\mathbb{R}^n} K_t(x-z) \int_{\mathbb{R}^n} K_s(z-y) f(y)\,dy\,dz \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_t(x-z)K_s(z-y)dz\,f(y)dy \\ \implies K_{t+s}(x-y) &= \int_{\mathbb{R}^n} K_t(x-z)K_s(z-y)dz \end{aligned}$$

This converts integral type information into pointwise information.

3.3 Heat Equation on Bounded Domains

Let $u_t = \Delta u$ for $t > 0$ and $x \in \Omega$ with boundary condition

- $u|_{\partial\Omega} = g(x)$ Dirichlet B.C. with prescribed temperature distribution.
- $\frac{\partial u}{\partial n}\big|_{\partial\Omega} = g(x)$ insulating B.C.
- $\frac{\partial u}{\partial n}\big|_{\partial\Omega} = -c(u(x) - c_0)$ Newton's Law of cooling.

3.3.1 Maximum Principle

Theorem 3.6 (Maximum Principle for Heat Equation)**.** *Ω bounded in $\mathbb{R}^n$, $0 < T < \infty$. $\Omega \times [0, T]$. Assume $\partial_t u = \Delta u$ in $\Omega \times (0, T)$. Claim: $u(x, t)$ attains its maximum either on $\Omega \times \{t = 0\}$ or on $\partial\Omega \times [0, T]$.*

Proof. Let $\varepsilon > 0$. Set $v^\varepsilon(x, t) := u(x, t) + \varepsilon|x|^2 \geq u(x, t)$. Then

$$(\partial_t - \Delta)v^\varepsilon = -2n\varepsilon < 0$$

Take $T' < T$. If the maximum value of v^ε occurs at an interior point of $\Omega \times (0, T')$. Then it is necessarily a critical point hence $\partial_t v^\varepsilon = 0$, and since it's maximum $0 \geq tr(Dv^\varepsilon) = \Delta v^\varepsilon$ at such (x_0, t_0) hence

$$(\partial_t - \Delta)v^\varepsilon(x_0, t_0) \geq 0$$

contradiction. Hence the maximum of v^ε occurs either on $\Omega \times \{t = 0\}$ or on $\partial\Omega \times [0, T']$.

$$\begin{aligned}\max_{\overline{\Omega}\times[0,T']} u &\leq \max_{\overline{\Omega}\times[0,T']} v^\varepsilon \\ &= \max_{\Omega\times\{t=0\}\cup\partial\Omega\times[0,T']} v^\varepsilon \\ &\leq \max_{\Omega\times\{t=0\}\cup\partial\Omega\times[0,T']} u + \varepsilon \max_{\Omega\times\{t=0\}\cup\partial\Omega\times[0,T']} |x|^2\end{aligned}$$

Let $\varepsilon \to 0$, $T' \to T$ so

$$\max_{\overline{\Omega}\times[0,T]} u \leq \max_{\Omega\times\{t=0\}\cup\partial\Omega\times[0,T]} u$$

□

3.3.2 Separation of Variables and Exponential Decay

Think about Initial Boundary Value Problem

$$\begin{cases} \partial_t u = \Delta u \qquad in\ \Omega_x \times (0, \infty)_t \\ u(x, 0) = f(x) \\ u|_{\partial\Omega} = 0 \end{cases} \tag{34}$$

One may try for separation of variables. Let $u(x, t) = F(x)G(t)$.

$$\begin{aligned} F(x)G'(t) &= \Delta F(x)G(t) \\ \implies \frac{G'(t)}{G(t)} &= \frac{\Delta F(x)}{F(x)} = \lambda \\ G'(t) &= \lambda G(t) \\ G(t) &= e^{\lambda t} \\ \implies &\begin{cases} \Delta F = \lambda F \\ F|_{\partial\Omega} = 0 \end{cases} \end{aligned}$$

We have an eigenvalue problem for Δ on a space with Dirichlet Boundary Value Problem.

Theorem 3.7. *There exists $\{F_j\}_{j\geq 1}$ s.t. $F_j \in C^\infty(\Omega)$ and corresponding λ_j eigenvalues (counting multiplicity) s.t. $0 > -\lambda_1 > -\lambda_2 \geq -\lambda_3 \geq \cdots$ with the first eigenvalue λ_1 simple (multiplicity 1) s.t.*

$$\begin{cases} \Delta F_j = -\lambda_j F_j \\ F_j|_{\partial\Omega} = 0 \end{cases}$$

One can arrange for $\{F_j\}$ to be orthonormal set

$$\int_\Omega F_j^2\, dx = 1 \qquad \int_\Omega F_j F_\ell\, dx = 0$$

In fact $\{F_j\}$ is orthonormal basis for $L^2(\Omega)$. Let $N < \infty$ and let $f \in L^2(\Omega)$, define

$$S_N[f](x) := \sum_{j=1}^{N} \langle F_j, f\rangle_{L^2(\Omega)} F_j(x)$$

where

$$\langle f, g\rangle_{L^2(\Omega)} := \int_\Omega fg\, dx$$

Then we have

$$\|f - S_N[f]\|_{L^2(\Omega)} \to 0 \qquad \text{as } N \to \infty$$

i.e., Δ Dirichlet Boundary Conditions on $\partial\Omega$ has an orthonormal basis of eigenvectors in $L^2(\Omega)$.

If $f \in$ Domian of Δ, i.e., f smooth and $f|_{\partial\Omega} = 0$, one get stronger convergence of partial sums to f. Let $u(x,t) := \sum_{j=1}^{\infty} c_j(t) F_j(x)$ so

$$\begin{aligned} \sum_{j=1}^{\infty} \dot{c}_j(t) F_j(x) &= \sum_{j=1}^{\infty} c_j(t) \Delta F_j(x) \\ &= \sum_{j=1}^{\infty} c_j(t)(-\lambda_j) F_j(x) \end{aligned}$$

Then take inner product in $L^2(\Omega)$ with F_m

$$\begin{aligned} &\dot{c}_m(t) = -\lambda_m c_m(t) \\ \implies &c_m(t) = e^{-\lambda_m t} c_m(0) \end{aligned}$$

Hence $u(x,t) := \sum_{j=1}^{\infty} c_j(t) F_j(x) = \sum_{j=1}^{\infty} e^{-\lambda_j t} c_j(0) F_j(x)$ This satisfies the PDE for $t > 0$. To match the initial data

$$\begin{aligned} f(x) = u(x,t)|_{t=0} &= \sum_{j=1}^{\infty} c_j(0) F_j(x) \\ \implies c_j(0) &= \langle F_j, f\rangle \\ \implies u(x,t) &= \sum_{j=1}^{\infty} e^{-\lambda_j t} \langle F_j, f\rangle_{L^2(\Omega)} F_j(x) \end{aligned}$$

One may estimate

$$\begin{aligned} \int_\Omega |u(x,t)|^2\, dx &= \int_\Omega \left(\sum_{j=1}^{\infty} e^{-\lambda_j t} \langle F_j, f\rangle_{L^2(\Omega)} F_j(x)\right)\left(\sum_{m=1}^{\infty} e^{-\lambda_m t} \langle F_m, f\rangle_{L^2(\Omega)} F_m(x)\right) dx \\ &= \sum_{m=1}^{\infty} e^{-2\lambda_m t} |\langle F_m, f\rangle|^2 \\ &\le e^{-2\lambda_1 t} \sum_{m=1}^{\infty} |\langle F_m, f\rangle|^2 \\ &= e^{-2\lambda_1 t} \int_\Omega |f|^2 \end{aligned}$$

where λ_1 is the largest eigenvalue of Δ with Dirichlet Boundary Condition on Ω

$$\|u(t)\|_{L^2(\Omega)} \le e^{-\lambda_1 t} \|f\|_{L^2(\Omega)}$$

Contrast with $(\partial_t - \Delta)u = 0$ with $u(x,0) = f(x)$ for $x \in \mathbb{R}^n$ where

$$\begin{aligned} u(x,t) = K_t * f &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy \\ |u(x,t)| &\le \frac{1}{(4\pi t)^{\frac{n}{2}}} \|f\|_{L^2(\mathbb{R}^n)} \end{aligned}$$

3.3.3 Backward Heat Equation

For forward heat equation $u_t = \partial_x^2 u$ with $u(x,0) = f(x)$ and $u|_{\partial\Omega} = 0$. Say we're dealing with $\Omega = [0,1]$.

$$\begin{aligned} &\left\| e^{\partial_x^2 t} f \right\|_{L^2([0,1])} \leq e^{-\lambda_1 t} \|f\|_{L^2([0,1])} \\ \implies &\left\| e^{\partial_x^2 t} f \right\|_{L^2} \leq \|f\|_{L^2} \end{aligned}$$

For any $\varepsilon > 0$, if $\|f\|_{L^2} < \varepsilon$ then

$$\left\| e^{\partial_x^2 t} f \right\|_{L^2} < \varepsilon \qquad \forall\ t > 0$$

We have stability result: start small, we stay small.
But for the backward heat equation

$$\begin{cases} u_t = -u_{xx} & (0,1) \times (0,\infty) \\ u(0,t) = 0 & u(1,t) = 0 \\ u(x,0) = f_n(x) = a\sin(2n\pi x) \end{cases}$$

Look at solutions of the form

$$\begin{aligned} u(x,t) &= U(t)\sin(2n\pi x) \\ U'(t)\sin(2n\pi x) &= 4n^2\pi^2 U(t)\sin(2n\pi x) \\ U'(t) &= 4n^2\pi^2 U(t) \\ U(t) &= e^{4n^2\pi^2 t} U(0) = a e^{4n^2\pi^2 t} \end{aligned}$$

Hence

$$u(x,T) = U(T)\sin(2n\pi x) = e^{4n^2\pi^2 T} a\sin(2n\pi x)$$

Fix $t = T$

$$\begin{aligned} \|u(T)\|_{L^2}^2 &= \int_0^1 e^{8n^2\pi^2 T} a^2 \sin^2(2n\pi x)\, dx \\ &= e^{8n^2\pi^2 T} a^2 \frac{1}{2} = e^{8n^2\pi^2 T} \|f_n\|_{L^2}^2 \end{aligned}$$

Instead of decaying, the solution grows exponentially.

4 Weak Solution and Ellipticity

Look at

$$\begin{cases} -\Delta u = f & x \in \Omega \\ u|_{\partial\Omega} = g \end{cases} \tag{35}$$

Want solution via Hilbert Space Methods.

- We want to reduce to homogeneous Dirichlet Boundary Conditions. Given g defined on $\partial\Omega$, we would like to extend g to a function $Ext(g)$ defined on all of Ω.
- Given $Ext(g)$, define $U(x) := u(x) - Ext(g)$. Ask: What PDE does U satisfy?

  $$-\Delta U = -\Delta u + \Delta Ext(g)$$

  we have

  $$\begin{cases} -\Delta U = f + \Delta Ext(g) \\ U|_{\partial\Omega} = 0 \end{cases}$$

- So if we can solve

  $$\begin{cases} -\Delta v = f \\ v|_{\partial\Omega} = 0 \end{cases} \quad \forall\ f \tag{36}$$

  Then we can solve for all f, g.

We now restrict to (36). We call v a classical solution to (36) if $v \in C^2(\Omega)$ and satisfies (36).

Definition 4.1 (Test Functions)**.** $C_0^\infty(\Omega) := \{$*functions* $u \in C^\infty(\Omega)$ *that have compact support in* Ω *(vanish on* $\partial\Omega$*)*$\}$.

Assume $v \in C^2(\Omega)$ classical solution, the Integration by Parts gives, for any $u \in C_0^\infty(\Omega)$

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega uf \, dx$$

We view the LHS as an inner product and the RHS as a linear functional. We need to develop Hilbert Space. Denote

$$(u, v)_D := \int_\Omega \nabla u \cdot \nabla v \qquad \phi_f(u) := \int_\Omega u\, f \, dx$$

Theorem 4.1 (Riesz Representation)**.** *Let H denote a Hilbert Space, which is complete normed linear space, i.e., Banach Space, where $\|\cdot\|_H := \sqrt{\langle\cdot,\cdot\rangle_H}$. Let H^* denote the set of all bounded linear functionals on H. $\phi \in H^*$ means $\phi : H \to \mathbb{C}$ or $\mathbb{R}$ s.t.*

$$\begin{aligned} &\textit{linearity} \qquad \phi(\alpha u + \beta v) = \alpha\phi(u) + \beta\phi(v) \\ &\textit{bounded} \qquad \exists M > 0 \ s.t.\ |\phi(u)| \le M \|u\|_H \qquad \forall\ u \in H \end{aligned}$$

Given any $\phi \in H^*$, there exists a unique $v_\phi \in H$ s.t.

$$\phi(u) = \langle v_\phi,\, u\rangle \qquad \forall\ u \in H$$

Furthermore $\|\phi\|_{H^*} = \|v_\phi\|_H$. Note it is trivial that for any $v \in H$, $\phi_v(u) := \langle v,\, u\rangle$ defines a bounded linear functional. The Riesz Representation tells us that all bounded linear functionals are of such type.

Note C_0^∞ is not a Hilbert space.

4.1 Weak Derivatives and Sobolev Space

Note for $u \in C^1(\Omega)$, let $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega u_{x_i}\phi \, dx = \int_\Omega (u\phi)_{x_i} - u\phi_{x_i} \, dx = -\int_\Omega u\, \phi_{x_i} \, dx$$

We wish to generalize this to arbitrary derivatives. For any $\alpha \in \mathbb{N}$ with $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$

$$\int_\Omega \partial_x^\alpha u(x)\ \phi(x) \, dx = (-1)^{|\alpha|} \int_\Omega u(x)\partial_x^\alpha \phi(x) \, dx \qquad \forall \phi \in C_0^\infty$$

Definition 4.2 (Weak Derivative)**.** *Let $u, v \in L^1_{loc}(\Omega)$, i.e., for any $K \subset \mathbb{R}^n$ compact, $\int_K |u| \, dx < \infty$. We say $\partial_x^\alpha u = v$ in the weak sense, or v is the α-th weak derivative of u provided*

$$(-1)^{|\alpha|} \int_\Omega u \partial_x^\alpha \phi(x) \, dx = \int_\Omega v\, \partial_x^\alpha \phi \, dx \qquad \forall \phi \in C_0^\infty(\Omega) \tag{37}$$

Lemma 4.1 (Uniqueness of weak derivative)**.** *If u has an α-th weak derivative, then it is unique up to a measure zero set.*

Proof. If both v_1, $v_2 \in L^1_{loc}(\Omega)$ satisfies

$$\int_\Omega u\, D^\alpha \phi\, dx = (-1)^{|\alpha|} \int_\Omega v_1\, \phi\, dx = (-1)^{|\alpha|} \int_\Omega v_2\, \phi\, dx \quad for\ any\ \phi \in C_0^\infty(\Omega)$$

Then $\int_\Omega (v_1 - v_2)\, \phi\, dx = 0$ for any test function ϕ. Hence $v_1 = v_2$ a.e. $\square$

Example 4.1. *Look at $u(x)$ s.t. $u(x) = \begin{cases} x & 0 \le x \le 1 \\ 1 & 1 \le x \le 2 \end{cases}$ Now define $v(x) := \begin{cases} 1 & 0 \le x \le 1 \\ 0 & 1 < x \le 2 \end{cases}$ This is our natural candidate for the weak derivative of u. The claim is: $u' = v$ in the weak sense.*

Proof. Clearly $v \in L^1(0,2)$. Now for any $\phi \in C_0^\infty(0,2)$.

$$\begin{aligned}
-\int_0^2 u(x)\phi'(x)\, dx &= -\int_0^1 u(x)\phi'(x)\, dx - \int_1^2 u(x)\phi'(x)\, dx \\
&= -\int_0^1 x\phi'(x)\, dx - \int_1^2 \phi'(x)\, dx \\
&= -\int_0^1 x\phi'(x)\, dx + \phi(1) \\
&= -\int_0^1 \left((x\phi(x))' - \phi(x)\right)\, dx + \phi(1) \\
&= -\left(\phi(1) - 0 - \int_0^1 \phi(x)\, dx\right) + \phi(1) \\
&= \int_0^1 \phi(x)\, dx
\end{aligned}$$

$\square$

Example 4.2. *Suppose $u(x) = \begin{cases} x & 0 < x \le 1 \\ 2 & 1 \le x < 2 \end{cases}$ Claim: u' does not exist in the weak sense.*

Proof. Want to show there is not $v \in L^1_{loc}(0,2)$ s.t.

$$-\int_0^2 u\phi'\, dx = \int_0^2 v\phi\, dx$$

Suppose there is such $v \in L^1_{loc}(0,2)$. We probe the suspicious point $x = 1$ with a sequence $\phi_m(x) \in C_0^\infty(0,2)$. Choose $\phi_m(x) \in [0,1]$ with $\phi_m(0) = 0 = \phi_m(2)$ where $\phi_m(x) \to 0$ as $m \to \infty$ for all $x \in (0,2) \setminus \{1\}$ and $\phi_m(1) = 1$ for any m. If so,

$$\begin{aligned}
\int_0^2 v\phi_m\, dx &= -\int_0^2 u\phi_m'\, dx \\
&= -\int_0^1 x\phi_m'\, dx - 2\int_0^1 \phi_m'\, dx \\
&= -\phi_m(1) + \int_0^1 \phi_m(x)\, dx - 2\phi_m(2) + 2\phi_m(1) \\
&= \phi_m(1) + \int_0^1 \phi_m(x)\, dx
\end{aligned}$$

$$\implies \int_0^2 v\phi_m\, dx - \int_0^1 \phi_m(x)\, dx = \phi_m(1)$$

But LHS goes to 0 as $m \to \infty$, yet $\phi_m(1) = 1$. $\square$

Definition 4.3 (Sobolev Space)**.** $1 \le p \le \infty$.

$$W^{k,p}(\Omega) := \{f \in L^1_{loc}(\Omega) \mid \partial^\alpha f \ exists\ in\ the\ weak\ sense\ \forall\ |\alpha| \le k,\ \partial^\alpha f \in L^p(\Omega)\}$$

For $p = 2$, we often write $H^k(\Omega) = W^{k,2}(\Omega)$. Norms on $W^{k,p}(\Omega)$

$$\|u\|_{W^{k,p}(\Omega)} = \|u\|_{k,p} := \begin{cases} \left(\sum_{|\alpha| \le k} \int_\Omega |\partial^\alpha u|^p\, dx\right)^{\frac{1}{p}} & 1 \le p < \infty \\ \sum_{|\alpha| \le k} \operatorname*{ess\,sup}_\Omega |D^\alpha u| & p = \infty \end{cases}$$

We say $u_m \to u$ in $W^{k,p}(\Omega)$ if

$$\|u_m - u\|_{W^{k,p}(\Omega)} \to 0 \qquad as\ m \to \infty$$

Definition 4.4 ($W_0^{k,p}(\Omega)$)**.** *$W_0^{k,p}(\Omega)$:= closure of $C_0^\infty(\Omega)$ w.r.t. $W^{k,p}(\Omega)$ norm, i.e., $u \in W^{k,p}(\Omega)$ iff there exists $\{u_m\} \subset C_0^\infty(\Omega)$ s.t.*

$$\|u_m - u\|_{W^{k,p}(\Omega)} \to 0 \qquad as\ m \to \infty$$

Roughly speaking, $W_0^{k,p}(\Omega)$ consists of all $u \in W^{k,p}(\Omega)$ s.t. $\partial^\alpha u|_{\partial\Omega} = 0$ for any $|\alpha| \leq k-1$.

Of particular interest to us is the space $H_0^1(\Omega) = W_0^{1,2}(\Omega)$.

Theorem 4.2. *$H^1(\Omega)$ is Hilbert space with norm defined by*

$$\|u\|^2_{H^1(\Omega)} := \int_\Omega |\nabla u|^2 + |u|^2 \, dx$$

with inner product

$$(f,g)_{H^1(\Omega)} := \int_\Omega \overline{\nabla f} \cdot \nabla g + \overline{f} g \, dx$$

$H_0^1(\Omega)$ as a closed subspace of $H^1(\Omega)$ is also a Hilbert Space w.r.t. $\|u\|^2_{H^1(\Omega)}$.

Lemma 4.2 (Smooth Approximation)**.** *For Ω bounded open subset with $\partial\Omega$ smooth, one has smooth approximations*

- *For any $u \in H^1(\Omega) = W^{1,2}(\Omega)$, there exists $\{u_n\} \subset C^\infty(\Omega)$ s.t. $\|u_n - u\|_{H^1(\Omega)} \to 0$. Here*

$$\|u\|^2_{H^1(\Omega)} = \sum_{|\alpha| \leq 1} \int_\Omega |\partial^\alpha u(x)|^2 \, dx$$

- *For any $u \in H_0^1(\Omega) = W_0^{1,2}(\Omega)$, there exists $\{u_n\} \subset C_0^\infty(\Omega)$ s.t. $\|u_n - u\|_{H^1(\Omega)} \to 0$ as $n \to \infty$.*

More generally, for $0 \leq s \in \mathbb{N}_0$

$$\|f\|^2_{H^s(\Omega)} := \sum_{|\alpha| \leq s} \int_\Omega |\partial^\alpha f(x)|^2 \, dx$$

4.2 Weak Solutions

Recall (36)

$$\begin{cases} -\Delta v = f \\ v|_{\partial\Omega} = 0 \end{cases} \qquad \forall\ f$$

We first study

$$\begin{cases} (-\Delta + 1)v = f & x \in \Omega \\ v|_{\partial\Omega} = 0 \end{cases} \tag{38}$$

If v is a classical solution, then for any $\phi \in C_0^\infty$

$$\int (-\Delta + 1) v \, \phi = \int_\Omega f \, \phi$$

$$\int_\Omega \nabla v \cdot \nabla \phi + v\phi = \int_\Omega f \, \phi$$

Definition 4.5. *We say v is a weak solution to (38) if $v \in H_0^1(\Omega)$ and for all $u \in H_0^1(\Omega)$*

$$\int_\Omega \nabla v \cdot \nabla u + vu = \int_\Omega f \, u$$

Equivalently, v is a weak solution to (38) if

$$(v,u)_{H_0^1(\Omega)} = \phi_f(u) \qquad \forall u \in H_0^1(\Omega)$$

Existence of unique solution. Note $u \mapsto \phi_f(u) := \int_\Omega f\, u \, dx$ is a bounded linear functional on $H_0^1(\Omega)$. Linearity is trivial. To see boundedness

$$|\phi_f(u)| = |\int_\Omega f u \, dx| \leq \|f\|_{L^2(\Omega)} \, \|u\|_{H_0^1(\Omega)}$$

using Cauchy Schwarz. Thus there exists v_f by Riesz Representation. $\square$

Claim: work with $H := (H_0^1(\Omega), \|u\|_D := \int_\Omega |\nabla u|^2\, dx)$ and $(f,g)_D := \int_\Omega \nabla f \cdot \nabla g\, dx$.

Definition 4.6. *Let $f \in L^2(\Omega)$. We say $v \in H_0^1(\Omega)$ is a weak solution to (36) if for all $u \in H_0^1(\Omega)$*

$$\int_\Omega \nabla v \cdot \nabla u = \int_\Omega f\, u$$

The goal is to use functional analysis techniques to construct weak solutions using Riesz Theorem 4.1. Recall we've defined for any u, $v \in H_0^1(\Omega)$

$$(u,v)_D := \int_\Omega \nabla u \cdot \nabla v \qquad \phi_f(u) := \int_\Omega u\, f\, dx$$

Then v is a weak solution to the Dirichlet problem iff for any $u \in H_0^1(\Omega)$

$$(u,v)_D = \phi_f(u) \qquad \forall u \in H_0^1(\Omega)$$

Now is $(u,v)_D$ an inner product? Notice inner product requires $(u,u) = 0$ iff $u = 0$.

Lemma 4.3. $\|u\|_D := \sqrt{(u,u)_D}$ *is a norm on $H_0^1(\Omega)$. In fact, it is equivalent to the standard norm on $H_0^1(\Omega)$, i.e., there exists c_1, $c_2 > 0$ independent of u s.t. for any $u \in H_0^1(\Omega)$*

$$c_2 \|u\|_{H^1(\Omega)} \leq \|u\|_D \leq c_1 \|u\|_{H^1(\Omega)}$$

It follows that

$$\left(H_0^1(\Omega), \|\cdot\|_{H^1}\right) \cong \left(H_0^1(\Omega), \|\cdot\|_D\right)$$

as Hilbert space.

Assume the above lemma, then there exists unique $v_f \in H_0^1$ s.t.

$$\phi_f(u) = (v_f,\, u)_D$$

from Riesz Representation Theorem provided that the functional $\phi_f \in (H_0^1(\Omega))^*$ is bounded.

Proof of Lemma 4.3. We want to show that

$$c_2' \|u\|_{H^1(\Omega)}^2 \leq \|u\|_D^2 \leq c_1' \|u\|_{H^1(\Omega)}^2$$

The second inequality indeed holds for $c_1 = 1$. We essentially want to show $\int_\Omega |u|^2 \leq C \int_\Omega |\nabla u|^2$ for any $u \in H_0^1(\Omega)$. □

Theorem 4.3 (Poincaré Inequality)**.** *Consider any domain Ω bounded between 2 planes. There is a constant $P_\Omega > 0$ s.t. for all $u \in H_0^1(\Omega)$, $\|u\|_{L^2(\Omega)} \leq p_\Omega \|u\|_D$.*

Poincaré Inequality clearly implies equivalence of norms between

$$\left(H_0^1(\Omega), \|\cdot\|_{H^1}\right) \cong \left(H_0^1(\Omega), \|\cdot\|_D\right)$$

And furthermore, it implies that the mapping from $u \mapsto \phi_f(u) := \int_\Omega f\, u\, dx$ is a bounded linear functional on the Hilbert space with Dirchlet inner product norm.

Proof of boundedness of ϕ_f.

$$\begin{aligned} |\phi_f(u)| &= |\int_\Omega f u\, dx| \\ &\leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \\ &\leq C \|f\|_{L^2(\Omega)} \|u\|_D \end{aligned}$$

□

It all boils down to proving the Poincaré Inequality.

Proof of Poincaré (4.3). After possible rotation of coordinates. We may assume that Ω lies between 2 hyperplanes $\{x_1 = -a\}$ and $\{x_1 = a\}$ for some $a \geq 0$. The distance in between in $2a$. Let $w \in C_0^\infty(\Omega)$. We prove for such w and then extend to all $H_0^1(\Omega)$. Namely let $u \in H_0^1(\Omega)$, then there exists $\{u_j\}_{j\geq 1} \subset C_0^\infty(\Omega)$ s.t. $\|u_j - u\|_D \to 0$. So we first do for C_0^∞ and let u be 0 outside Ω. Let $u(x_1, x')$ where $x' = (x_2, \cdots, x_n)$. Do

$$
\begin{aligned}
u(x_1, x') &= \int_{-a}^{x_1} \partial_{x_i} u(s, x')\, ds \\
u^2(x_1, x') &= \left(\int_{-a}^{x_1} \partial_{x_i} u(s, x')\, ds\right)^2 \leq \left(\int_{-a}^{x_1} 1\, ds\right)\left(\int_{-a}^{x_1} (\partial_{x_i} u(s, x'))^2\, ds\right) \\
|u(x_1, x')|^2 &\leq (x_1 + a)\left(\int_{-a}^{x_1} (\partial_{x_i} u(s, x'))^2\, ds\right) \\
\int_{-a}^{a} |u(x_1, x')|^2\, d(x_1) &\leq \int_{-a}^{a} (x_1 + a)\, d(x_1)\left(\int_{-a}^{x_1} (\partial_{x_i} u(s, x'))^2\, ds\right) \\
&= 2a^2\left(\int_{-a}^{x_1} (\partial_{x_i} u(s, x'))^2\, ds\right) \\
\int_{\mathbb{R}^{n-1}} \int_{-a}^{a} |u(x_1, x')|^2\, d(x_1)\, dx' &\leq 2a^2 \int_\Omega |\nabla u|^2\, dx \\
\int_\Omega |u|^2\, dx &\leq 2a^2 \int_\Omega |\nabla u|^2\, dx
\end{aligned}
$$

Then use this to extend to $H_0^1(\Omega)$ by density. $\square$

Now to generalize a little bit

$$
L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j}
$$

for $a_{ij} = a_{ji} \in C^\infty(\Omega)$ and symmetric. L is uniformly elliptic if there exists $0 < \lambda_- \leq \lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_+$ for λ_- and λ_+ independent of x. What if we want to solve for

$$
\begin{cases} Lv = f & for\ x \in \Omega \\ \quad v|_{\partial\Omega} = 0 \end{cases} \tag{39}
$$

Now for $u \in C_0^\infty(\Omega)$, if v is a classical solution $v \in C^2(\Omega)$ s.t. $v|_{\partial\Omega} = 0$.

$$
\begin{aligned}
u\left(-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} v\right) &= uf \\
\sum_{ij} \int_\Omega \frac{\partial u}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} v\, dx &= \int_\Omega f u\, dx
\end{aligned}
$$

Definition 4.7.

$$
(u, v)_{D,a} = \sum_{i,j=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} v\, dx
$$

Now note

$$
(u, u)_{D,a} = \int_\Omega \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j}\, dx \geq \lambda_- \int_\Omega \sum_{i,j=1}^{n} |\frac{\partial u}{\partial x_i}|^2\, dx
$$

so one may define

$$
\|u\|_{D,a} := \sqrt{(u, u)_{D,a}}
$$

Notice

$$
\begin{gathered}
(\nabla u)^T a(x)(\nabla u) \geq \lambda_- |\nabla u|^2 \\
\lambda_- \|u\|_D^2 \leq (u, u)_{D,a} \leq \lambda_+ \|u\|_D^2
\end{gathered}
$$

Then

$$
\left(H_0^1(\Omega), \|u\|_{D,a}\right)
$$

is a Hilbert Space.

Theorem 4.4. *Given any $f \in L^2(\Omega)$, there exists unique $v_f \in H_0^1(\Omega)$ s.t. solving the weak Dirichlet Problem in the sense of Definition 4.6.*

4.3 Elliptic Regularity

We will show that if $f \in L^2(\Omega)$, then $v_f \in H_0^1(\Omega) \cap H^2(\Omega)$. In fact, if $f \in H^m(\Omega)$ then $v_f \in H^{m+2}(\Omega) \cap H_0^1(\Omega)$. Then if $f \in H^m(\Omega)$ for any m, we have $v_f \in H^m(\Omega)$ for any m, hence $v_f \in C^\infty(\Omega)$. Always 2 degrees smoother in the Sobolev sense. Then in the classical sense.

$$v_f = (-\Delta)^{-1} f \qquad in \quad \Omega$$

Idea: If $v_f \in H_0^1(\Omega)$, then

$$\int_\Omega \nabla u \cdot \nabla v_f = \int_\Omega uf \qquad \forall\, u \in H_0^1(\Omega)$$

Instead of u, put in second order derivative formally $u = \frac{\partial^2 v_f}{\partial x_i^2}$, so we have

$$\int_\Omega \nabla \frac{\partial^2 v_f}{\partial x_i^2} \cdot \nabla v_f = \int_\Omega \frac{\partial^2 v_f}{\partial x_i^2} f$$

We integrate by parts then taking $\varepsilon = \frac{1}{2}$

$$\begin{aligned}
-\sum_j \int_\Omega \partial_{x_j} \frac{\partial}{\partial x_i} v_f \partial_{x_j} \frac{\partial}{\partial x_i} v_f &= \int_\Omega \frac{\partial^2 v_f}{\partial x_i^2} f \\
\sum_j \int_\Omega \left(\frac{\partial^2}{\partial x_i x_j} v_f\right)^2 dx &= \int_\Omega \left|\frac{\partial^2 v_f}{\partial x_i^2} f\right| \le \int \left|\frac{\partial^2 v_f}{\partial x_i^2}\right| |f| \\
&\le \frac{\varepsilon}{2} \int_\Omega \left|\frac{\partial^2 v_f}{\partial x_i^2}\right|^2 + \frac{1}{2\varepsilon} \int_\Omega |f|^2 \\
\sum_i \sum_j \int_\Omega \left(\frac{\partial^2}{\partial x_i x_j} v_f\right)^2 dx &\le \frac{\varepsilon}{2} \int_\Omega \sum_i \sum_j \left|\frac{\partial^2}{\partial x_i x_j} v_f\right|^2 + \frac{n}{2\varepsilon} \int_\Omega |f|^2 \\
\frac{1}{2} \sum_i \sum_j \int_\Omega \left(\frac{\partial^2}{\partial x_i x_j} v_f\right)^2 dx &\sim \|v_f\|_{H^2(\Omega)} \le n \int_\Omega |f|^2
\end{aligned}$$

This suggests $v_f \in H^2(\Omega)$.

Theorem 4.5 (Interior Regularity)**.** $v_f \in H^2(\Omega)$

Proof. To justify interior regularity, let $V \subset \Omega$ be arbitrary open set. Want to show that $v_f \in H^2(V)$. Now take any $V \subset W \subset \Omega$ s.t. $\overline{W} \subset \Omega$.

Definition 4.8 (Difference Quotient)**.**

$$(D_k^h f)(x) := \frac{f(x + he_k) - f(x)}{h}$$

Note for f smooth

$$D_k^{-h} D_k^h f(x) \to \frac{\partial^2}{\partial x_k^2} f$$

as $h \to \infty$.

Take ζ smooth cutoff s.t. $0 \le \zeta \le 1$, $\zeta \equiv 1$ on V and $\zeta \equiv 0$ on $\Omega \setminus W$. Let

$$u := -D_k^{-h} \zeta^2 D_k^h v_f \in H_0^1(\Omega) \qquad for\ |h| \ll 1$$

Hence

$$-\int_\Omega \nabla (D_k^{-h} \zeta^2 D_k^h v_f) \cdot \nabla v_f \, dx = \int_\Omega (D_k^{-h} \zeta^2 D_k^h v_f) f \, dx$$

notice LHS writes by commuting ∇ with D_k^{-h} and throwing to the other side

$$\begin{aligned}
-\int D_k^{-h} \nabla (\zeta^2 D_k^h v_f) \cdot \nabla v_f \, dx &= \int \nabla (\zeta^2 D_k^h v_f) \cdot D_k^h \nabla v_f \, dx \\
&= \int \nabla (\zeta^2 D_k^h v_f) \cdot \nabla (D_k^h v_f) \, dx \\
&= \int \zeta^2 \nabla (D_k^h v_f) \cdot \nabla D_k^h v_f \, dx + 2 \int \zeta \nabla \zeta D_k^h v_f \cdot \nabla D_k^h v_f \, dx \\
&= \int \zeta^2 |\nabla (D_k^h v_f)|^2 \, dx + 2 \int \zeta \nabla \zeta D_k^h v_f \cdot \nabla D_k^h v_f \, dx
\end{aligned}$$

Hence throwing this to RHS gives

$$\begin{aligned}
\int \zeta^2 |\nabla (D_k^h v_f)|^2\, dx &= -\int D_k^{-h}(\zeta^2 D_k^h v_f) f\, dx - 2\int \zeta \nabla\zeta D_k^h v_f \cdot \nabla D_k^h v_f\, dx \\
&= \mathbf{I} + \mathbf{II} \\
\mathbf{II} &\le 2\int_\Omega |\nabla\zeta||D_k^h v_f|\zeta|\nabla D_k^h v_f|\, dx \\
&\le \frac{1}{\varepsilon_1}\int_\Omega |\nabla\zeta|^2 |D_k^h v_f|^2\, dx + \varepsilon_1 \int_\Omega \zeta^2 |\nabla D_k^h v_f|^2\, dx \\
\mathbf{I} &\le \frac{\varepsilon_2}{2}\int_\Omega |D_k^{-h}(\zeta^2 D_k^h v_f)|^2\, dx + \frac{1}{2\varepsilon_2}\int_\Omega |f|^2\, dx
\end{aligned}$$

$$\int \zeta^2 |\nabla (D_k^h v_f)|^2\, dx \le \varepsilon_1 \int_\Omega \zeta^2 |\nabla D_k^h v_f|^2\, dx + \frac{\varepsilon_2}{2}\int_\Omega |D_k^{-h}(\zeta^2 D_k^h v_f)|^2\, dx + \frac{1}{\varepsilon_1}\int_\Omega |\nabla\zeta|^2 |D_k^h v_f|^2\, dx + \frac{1}{2\varepsilon_2}\int_\Omega |f|^2\, dx$$

$$(1-\varepsilon_1)\int \zeta^2 |\nabla (D_k^h v_f)|^2\, dx \le \frac{\varepsilon_2}{2}\int_\Omega |D_k^{-h}(\zeta^2 D_k^h v_f)|^2\, dx + \frac{1}{\varepsilon_1}\int_\Omega |\nabla\zeta|^2 |D_k^h v_f|^2\, dx + \frac{1}{2\varepsilon_2}\int_\Omega |f|^2\, dx$$

Choose $\varepsilon_1 = \frac{1}{2}$

$$\frac{1}{2}\int \zeta^2 |\nabla (D_k^h v_f)|^2\, dx \le \frac{\varepsilon_2}{2}\int_\Omega |D_k^{-h}(\zeta^2 D_k^h v_f)|^2\, dx + 2\int_\Omega |\nabla\zeta|^2 |D_k^h v_f|^2\, dx + \frac{1}{2\varepsilon_2}\int_\Omega |f|^2\, dx$$

Lemma 4.4. *$\chi(x) D_k^h F(x) \in H_0^1(\Omega)$ for all $|h| \ll 1$. In particular,*

$$\int_\Omega \chi(x)^2 |D_k^h F(x)|^2\, dx \le c\int_\Omega |\nabla F(x)|^2\, dx$$

RHS independent of $h \ll 1$.

Proof.

$$h D_k^h F(x) = F(x - he_k) - F(x) = \int_0^1 \frac{d}{dt} F(x - the_k)\, dt = -\int_0^1 \nabla_x F(x - the_k) e_k\, dth$$

Divide by h on both sides. ∎

Apply Lemma to first the second terms on RHS so

$$\frac{1}{2}\int \zeta^2 |\nabla (D_k^h v_f)|^2\, dx \le C\frac{\varepsilon_2}{2}\int_\Omega |\nabla(\zeta^2 D_k^h v_f)|^2\, dx + 2C\int_\Omega |\nabla v_f|^2\, dx + \frac{1}{2\varepsilon_2}\int_\Omega |f|^2\, dx$$

Note

$$\int_\Omega |\nabla(\zeta^2 D_k^h v_f)|^2\, dx \le \int_\Omega \zeta^4 |\nabla D_k^h v_f|^2 + 4\zeta^2 |\nabla\zeta||\nabla D_k^h v_f||D_k^h v_f| + 4\zeta^2|\nabla\zeta|^2 |D_k^h v_f|^2\, dx$$

Conclude with same trick, one obtain

$$\int \zeta^2 |\nabla (D_k^h v_f)|^2\, dx \le C\left(\int_\Omega |\nabla v_f|^2\, dx + \int_\Omega |f|^2\, dx\right)$$

Now integrating over V

$$\int_V |\nabla (D_k^h v_f)|^2\, dx \le \int \zeta^2 |\nabla (D_k^h v_f)|^2\, dx \le C\left(\int_\Omega |\nabla v_f|^2\, dx + \int_\Omega |f|^2\, dx\right)$$

since the bound is uniformly in h for $|h| \ll 1$. We take $h \to 0$ to obtain

$$\nabla_x \frac{\partial v_f}{\partial x_k}$$

exists in L^2 on LHS. Since $k = 1, \cdots, n$ arbitrary, we have $v_f \in H^2(\Omega)$. ∎

Iterating this argument we obtain $v \in H^m(\Omega)$ for higher order Sobolev Spaces.
Now we'd like to address the existence of classical solutions from weak solutions defined via Definition 4.6. Note we've already proved

- Given any $f \in L^2(\Omega)$, there exists unique $v_f \in H_0^1(\Omega)$ s.t. v_f is a weak solution.

- If $f \in H^m(\Omega)$, then $v_f \in H^{m+2}(\Omega) \cap H_0^1(\Omega)$ for any $m \in \mathbb{N}$.

$$v_f = (-\Delta_{Dir,\Omega})^{-1} f$$

We're left to prove for $f \in C^\infty(\Omega)$, then $v_f \in C^\infty(\Omega)$. To prove this, we use the Sobolev Lemma.

Lemma 4.5 (Sobolev Lemma)**.** $H^s(\Omega) \subset C^k(\Omega)$ *if* $s > k + \frac{n}{2}$ *pointwise regularity.*

For simplicity, we prove this with $\Omega = \mathbb{R}^n$. Recall

$$\|f\|^2_{H^s(\mathbb{R}^n)} = \sum_{|\alpha| \le s} \int_{\mathbb{R}^n} |\partial^\alpha f(x)|^2 \, dx$$

and the Fourier Transform

$$\hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{-ix\cdot\xi} f(x)\, dx$$
$$f(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{ix\cdot\xi} \hat{f}(x)\, dx$$

Note $\mathcal{S}$ is dense in H^s

$$\|f\|^2_{H^s(\mathbb{R}^n)} = \sum_{|\alpha| \le s} \int_{\mathbb{R}^n} |\partial^\alpha f(x)|^2 \, dx = \sum_{|\alpha| \le s} \int_{\mathbb{R}^n} |\widehat{\partial^\alpha f}(\xi)|^2 \, d\xi = \sum_{|\alpha| \le s} \int_{\mathbb{R}^n} |(i\xi)^\alpha \hat{f}(\xi)|^2 \, d\xi$$

Observation: there exists $C_+, C_- > 0$ depending only on n, s s.t.

$$C_-(1+|\xi|^2)^s \le \sum_{|\alpha|\le s} |(i\xi)^\alpha|^2 \le C_+(1+|\xi|^2)^s$$

Hence

$$C_- \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi \le \|f\|^2_{H^s(\mathbb{R}^n)} \le C_+ \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi$$

and so

$$\left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}$$

is an equivalent norm for $H^s(\mathbb{R}^n)$.

Proof of 4.5 on $\mathbb{R}^n$*.* Take $f \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\begin{aligned}
\partial_x^\alpha f(x) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{ix\cdot\xi} (i\xi)^\alpha \hat{f}(x)\, dx \\
|\partial_x^\alpha f(x)| &\le \frac{1}{(2\pi)^{\frac{n}{2}}} \int |\xi|^{|\alpha|} |\hat{f}(x)| d\xi \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int \frac{|\xi|^{|\alpha|}}{(1+|\xi|^2)^{\frac{s}{2}}} |\hat{f}(x)| (1+|\xi|^2)^{\frac{s}{2}} d\xi \\
&\le \frac{1}{(2\pi)^{\frac{n}{2}}} \left(\int \frac{|\xi|^{2|\alpha|}}{(1+|\xi|^2)^s} d\xi \right)^{\frac{1}{2}} \left(\int |\hat{f}(x)|^2 (1+|\xi|^2)^s d\xi \right)^{\frac{1}{2}} \\
&\le C_n \left(\int \frac{|\xi|^{2|\alpha|}}{(1+|\xi|^2)^s} d\xi \right)^{\frac{1}{2}} \|f\|_{H^s(\mathbb{R}^n)}
\end{aligned}$$

Fix k. Then we ask a condition on s so that $|\partial_x^\alpha f(x)| \le C_{n,k} \|f\|_{H^s(\mathbb{R}^n)}$. This is equivalent to observing when does the integral $\left(\int \frac{|\xi|^{2|\alpha|}}{(1+|\xi|^2)^s} d\xi \right)^{\frac{1}{2}}$ converge. Since $|\alpha| \le k$, we want to know when

$$\int \frac{|\xi|^{2k}}{(1+|\xi|^2)^s} d\xi$$

converges. We go to polar coordinates

$$\int \frac{|\xi|^{2k}}{(1+|\xi|^2)^s} d\xi = \int_{\mathbb{S}^{n-1}} d\theta \int_0^\infty \frac{r^{2k}}{(1+r^2)^s} r^{n-1} \, dr$$

near 0 there's no singular. It suffices to estimate

$$\int_1^\infty \frac{r^{2k}}{(1+r^2)^s} r^{n-1}\,dr \le \int_1^\infty r^{2k-2s+n-1}\,dr$$

This converges if $2k-2s+n-1<-1$ hence $2k+n<2s$ so $k+\frac{n}{2}<s$. Hence if $s>k+\frac{n}{2}$, for $f\in\mathcal{S}(\mathbb{R}^n)$, we have

$$|\partial^\alpha f(x)| \le C_{s,n}\,\|f\|_{H^s(\mathbb{R}^n)}$$

This is the Sobolev Inequality. Now let $F\in H^s(\mathbb{R}^n)$. Then there exists a sequence $F_j\subset\mathcal{S}(\mathbb{R}^n)$ s.t.

$$\|F_j-F\|_{H^s(\mathbb{R}^n)}\to 0$$

as $s\to\infty$. This is Cauchy in $H^s(\mathbb{R}^n)$. Hence by Sobolev inequality on $\mathbb{R}^n$

$$|\partial^\alpha F_j-\partial^\alpha F_\ell| \le C_{k,n}\,\|F_j-F_\ell\|_{H^s(\mathbb{R}^n)}\to 0$$

as $j,\ell\to\infty$. Hence $\{\partial^\alpha F_j(x)\}$ is Cauchy in $\left(C^0(\mathbb{R}^n),\|\cdot\|_\infty\right)$ for all $|\alpha|\le k$. Thus $\partial^\alpha F_j(x)$ converges uniformly to $\partial^\alpha F(x)$ due to uniqueness of limits in weak derivatives. Moreover

$$|\partial^\alpha F(x)| \le C_{s,n}\,\|F\|_{H^s(\mathbb{R}^n)}$$

for any $s>k+\frac{n}{2}$. □

Definition 4.9. *A complete orthonormal sequence in $L^2(\Omega)$ is a sequence $\phi_j\subset L^2(\Omega)$ s.t. $\langle\phi_j,\phi_\ell\rangle=\delta_{j\ell}$, and for any $f\in L^2(\Omega)$, define*

$$S_N[f](x) := \sum_{j=1}^N \langle\phi_j,\, f\rangle_{L^2(\Omega)}\phi_j$$

we have

$$\|S_N[f]-f\|_{L^2(\Omega)}\to 0 \qquad \text{as } N\to\infty$$

Theorem 4.6. *For $\Omega\subset\mathbb{R}^n$ open and $\partial\Omega$ smooth, $L^2(\Omega)$ has a complete orthonormal sequence of eigenfunctions of*

$$\begin{cases} -\Delta\phi_j=\lambda_j\phi_j & x\in\Omega \\ \phi_j|_{\partial\Omega}=0 \end{cases}$$

where $\phi_j\in C^\infty(\Omega)$.

Proof. Let $f\in L^2(\Omega)$, there exists a unique $v_f\in H_0^1(\Omega)$ s.t.

$$\int_\Omega \nabla u\cdot\nabla v_f\,dx = \int_\Omega uf \qquad \forall\ u\in H_0^1(\Omega)$$

Let $T:L^2(\Omega)\to H_0^1(\Omega)\subset L^2(\Omega)$ s.t. $T(f):=v_f$.

- First argue T is linear. This follows from uniqueness of solutions.
- Second argue T is bounded. Write

  $$\int_\Omega \nabla u\cdot\nabla(Tf)\,dx = \int_\Omega uf\,dx$$

  Take $u=Tf$ so

  $$\begin{aligned}\int_\Omega |\nabla(Tf)|^2\,dx = \int_\Omega Tff\,dx &\le \left(\int_\Omega |Tf|^2\right)^{\frac12}\left(\int_\Omega |f|^2\right)^{\frac12} \\ &\le \left(\int_\Omega |\nabla(Tf)|^2\right)^{\frac12}\left(\int_\Omega |f|^2\right)^{\frac12}\end{aligned}$$

  by recalling Poincare for $Tf\in H_0^1$. Hence

  $$\left(\int_\Omega |\nabla(Tf)|^2\right)^{\frac12} \le \left(\int_\Omega |f|^2\right)^{\frac12}$$

  Again by Poincare

  $$\frac{1}{C}\left(\int_\Omega |Tf|^2\,dx\right)^{\frac12} \le \left(\int_\Omega |\nabla(Tf)|^2\right)^{\frac12} \le C\left(\int_\Omega |f|^2\right)^{\frac12}$$

  So T is a bounded linear operator on $L^2(\Omega)$.

- Thirdly claim $T : L^2(\Omega) \to L^2(\Omega)$ is self-adjoint operator, i.e., for any $f,\ g \in L^2(\Omega)$

$$\int_\Omega fTg\,dx = (f, Tg)_{L^2(\Omega)} = (Tf, g)_{L^2(\Omega)} = \int_\Omega Tfg\,dx$$

Recall

$$\int_\Omega \nabla u \cdot \nabla(Tf)\,dx = \int_\Omega uf\,dx$$

For $g \in L^2(\Omega)$, $Tg \in H_0^1(\Omega)$

$$\int_\Omega \nabla(Tg) \cdot \nabla(Tf)\,dx = \int_\Omega (Tg)f\,dx$$

But one may interchange f and g and conclude $\int_\Omega (Tg)f = \int_\Omega g(Tf)$.

- Fourth, we claim $T : f \to Tf = v_f$ from $L^2(\Omega)$ to $L^2(\Omega)$ is also a compact operator, i.e., if $\{f_j\}$ is a bounded sequence in $L^2(\Omega)$, that is there is a constant M s.t.

$$\|f_j\|_{L^2(\Omega)} \le M \qquad \forall\ j$$

Then the sequence $\{Tf_j\}$ has a subsequence $\{Tf_{j_k}\}_k$ that converges in $L^2(\Omega)$. The heart of this is Rellich's compactness theorem. Since

$$\begin{aligned}
(u, Tf_j)_D &= \int_\Omega uf_j\,dx \\
\int_\Omega |Tf_j|^2\,dx = (Tf_j, Tf_j)_D &= \int_\Omega Tf_jf_j\,dx \\
&\le \left(\int_\Omega |Tf_j|^2\right)^{\frac{1}{2}} \left(\int_\Omega |f_j|^2\right)^{\frac{1}{2}} \le C_\Omega M^2 < \infty
\end{aligned}$$

So $\{Tf_j\}$

$$\int_\Omega |Tf_j|^2 + |\nabla(Tf_j)|^2\,dx \le C < \infty$$

is uniformly bounded in $H_0^1(\Omega)$.

Theorem 4.7 (Rellich Compactness)**.** *For Ω bounded and $\partial\Omega$ C^1, $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, i.e., for any sequence $\{u_j\} \subset H_0^1(\Omega)$ uniformly bounded in H_0^1, there exists a subsequence $\{u_{j_k}\}$ that converges in $L^2(\Omega)$.*

Proof for $d = 1$. Idea for $n = 1$ is to use Arzela-Ascoli. We want to show $H_0^1(a, b) \subset\subset L^2(a, b)$. Take f smooth, say $f \in C_0^\infty(a, b)$. WLOG let $a = 0$, $b = 1$. Since smooth, we apply FTC so that

$$\begin{aligned}
f(x) &= \int_0^x f'(s)\,ds \le (\int_0^x 1ds)^{\frac{1}{2}} \left(\int_0^x (f'(s))^2\right)^{\frac{1}{2}} \\
&\le \sqrt{x}\,\|f'\|_{L^2(0,1)} \\
\implies |f(x)| &\le \sqrt{x}\,\|f'\|_{L^2(0,1)} \qquad \forall\ 0 \le x \le 1 \\
|f(x) - f(y)| &\le \int_y^x |f'(s)|\,ds \le (\int_y^x 1ds)^{\frac{1}{2}} \left(\int_y^x (f'(s))^2\right)^{\frac{1}{2}} \\
&\le |x - y|^{\frac{1}{2}}\,\|f'\|_{L^2(0,1)}
\end{aligned}$$

This is true for C_0^∞, so we extend to H_0^1 by density. Now let $\{f_j\}$ be bounded in $H_0^1(0, 1)$, we want to show there exists $\{f_{j_k}\}$ subsequence s.t. f_{j_k} converges in $L^2(0, 1)$. Hence for $M > 0$ independent of j

$$\begin{aligned}
|f_j(x)| &\le M \\
|f_j(x) - f_j(y)| &\le \sqrt{|x - y|}M
\end{aligned}$$

Since $\{f_j\}_j$ is uniformly bounded family of equi-continuous functions in $C^0(0, 1)$, by Arzela-Ascoli, the sequence $\{f_j\}$ is precompact, so there exists a convergent subsequence $\{f_{j_k}\} \subset \{f_j\}$ uniformly to a limiting function $f_* \in C^0(0, 1)$, i.e.

$$\max_{0\le x\le 1} |f_{j_k}(x) - f_*(x)| \to 0 \qquad as\ k \to \infty$$

But this is more than enough to bound

$$\int_0^1 |f_{j_k}(x) - f_*(x)|^2\,dx \le \max_{0\le x\le 1} |f_{j_k}(x) - f_*(x)| \to 0$$

$\square$

Theorem 4.8 (Rellich-Kondrachov Theorem). *$\partial\Omega$ C^1. Let $1 \le p < n$. Then $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$ for $1 \le q < p^* = \frac{np}{n-p}$.*

Apply Rellich, there exists subsequence $\{Tf_{j_k}\}$ s.t. converges in $L^2(\Omega)$.

- Now we've verified that T is self-adjoint and compact, by Hilbert-Schmidt theorem(Spectral Theorem for self-adjoint compact operators), there exists $\phi_j \in L^2(\Omega)$ s.t. $\{\phi_j\}_{j\ge1}$ is orthonormal basis for $L^2(\Omega)$ s.t.

$$T\phi_j = \alpha_j \phi_j$$

and $\alpha_j \to 0$ as $j \to \infty$. Recall $T\phi_j \in H_0^1(\Omega)$ satisfies for all $u \in H_0^1(\Omega)$

$$(u, T\phi_j)_D = (u, \phi_j)_{L^2(\Omega)}$$
$$(u, \alpha_j\phi_j)_D = \alpha_j(u, \phi_j)_D = (u, \phi_j)_{L^2(\Omega)}$$

Suppose there is a 0 eigenvalue $\alpha_{j_0} = 0$. Then $(u, \phi_{j_0})_{L^2(\Omega)} = 0$ for all $u \in H_0^1(\Omega)$. We may take $u = \phi_{j_0}$ then $\|\phi_{j_0}\|_2 = 0$. But ϕ_{j_0} has to be nonzero otherwise ϕ_{j_0} by definition cannot be eigenfunction. Thus there are no zero eigenvalues. So for any $u \in H_0^1(\Omega)$

$$(u, \phi_j)_D = (u, \frac{1}{\alpha_j}\phi_j)_{L^2(\Omega)}$$

Hence ϕ_j are weak solutions to $-\Delta\phi_j = \frac{1}{\alpha_j}\phi_j$. Define $\lambda_j = \frac{1}{\alpha_j} \to \infty$ as $j \to \infty$. By Elliptic Regularity, $\phi_j \in C^\infty(\Omega)$.

□

Example 4.3.

- *For $-\frac{d^2}{dx^2}$ on $[0,1]$ with Dirichlet Boundary Conditions. Fourier Sine Series $\{\sqrt{2}\sin(2\pi n)\}_{n\ge1}$ are complete. Eigenvalues are $(2\pi n)^2$.*
- *For $-\Delta$ on $L^2_{rad}(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2) \mid f = f(r)\ r = |x|\}$. Bessel Series s.t. $J_0(0) = 1$ and $J_0'(0) = 0$*

$$J_0''(r) + \frac{1}{r}J_0'(r) + J_0(r) = 0$$

are complete. Denote α_n as nth zeros of $J_0(r)$. Eigenvalues are α_n^2.

5 Wave Equation

Example 5.1 (Transport Equation)**.** *For one-dimensional transport equation with $c > 0$*

$$\partial_t u + c\partial_x u = 0$$

one has solution for $F \in C^1(\mathbb{R})$

$$u(x,t) = F(x - ct)$$

A generalization to $\mathbb{R}^n$ is, for $c \in \mathbb{R}^n$

$$\partial_t u + c \cdot \nabla u = 0$$

where

$$u(x,t) = F(x - ct)$$

Definition 5.1 (Wave Equation)**.**

$$(\partial_t^2 - c^2\Delta_x)u := \Box u = 0$$

where

$$\Box := \partial_t^2 - c^2\Delta_x$$

5.1 Wave Equation in $n = 1$

Example 5.2. *For $n = 1$,*

$$\Box u = (\partial_t^2 - c^2\partial_x^2)u = (\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0$$

Do a change of variables $(x,t) \mapsto (\xi, \eta)$ s.t.

$$\xi := x + ct \qquad \eta := x - ct$$

Thus

$$\begin{aligned}
\frac{\partial}{\partial t} &= \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = c(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}) \\
\frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \\
\left(\frac{\partial}{\partial t}\right)^2 &= c^2(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta})^2 \\
\left(\frac{\partial}{\partial x}\right)^2 &= (\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta})^2
\end{aligned}$$

Hence

$$\partial_t^2 - c^2\partial_x^2 = c^2(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta})^2 - c^2(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta})^2 = -4c^2\partial_\xi\partial_\eta$$

And

$$\Box u = 0 \iff \partial_\xi(\partial_\eta u) = 0$$

Note $\partial_\eta u$ is independent of ξ so

$$\partial_\eta u = g(\eta)$$

Then do the same again

$$u(\xi, \eta) = \int^\eta g(\xi')d\xi' + F(\xi)$$

Hence any solution of the wave equation is of the form

$$u(x,t) = F(x + ct) + G(x - ct)$$

Definition 5.2 (IVP)**.** *For $c > 0$ fixed number (speed of propagation)*

$$\begin{cases} \Box u = 0 \qquad x \in \mathbb{R} \; t \in \mathbb{R} \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \tag{40}$$

with prescribed initial conditions.

Solve IVP for d'Alembert's Solution. We know that

$$u(x,t) = F(x+ct) + G(x-ct)$$

Then using $u(x,0) = f(x)$ we have

$$F(x) + G(x) = f(x)$$

Also using $u_t(x,0) = g(x)$ and since we have

$$u_t(x,t) = cF'(x+ct) - cG'(x-ct)$$

we obtain

$$g(x) = u_t(x,0) = cF'(x) - cG'(x)$$

Assume we may differentiate

$$\begin{cases} F'(x) + G'(x) = f'(x) \\ F'(x) - G'(x) = \frac{1}{c}g(x) \end{cases}$$

So we have

$$\begin{aligned} F'(x) &= \frac{1}{2}\left(f'(x) + \frac{1}{c}g(x)\right) \\ G'(x) &= \frac{1}{2}\left(f'(x) - \frac{1}{c}g(x)\right) \end{aligned}$$

upon integration we have

$$\begin{aligned} F(\xi) &= \frac{1}{2}f(\xi) + \frac{1}{2c}\int_0^{\xi} g(s)ds + C_1 \\ G(\eta) &= \frac{1}{2}f(\eta) - \frac{1}{2c}\int_0^{\eta} g(s)ds + C_2 \end{aligned}$$

Thus we may write down solution to wave equation and verify the initial conditions

$$\begin{aligned} u(x,t) &= \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)ds + C_1 + \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)ds + C_2 \\ u(x,0) &= f(x) + C_1 + C_2 = f(x) \implies C_1 + C_2 = 0 \\ u(x,t) &= \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds \end{aligned}$$

□

Theorem 5.1. *Let $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$, then*

$$u(x,t) = \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds \tag{41}$$

solves the Initial Value Problem (40) for 1 dimension wave equation.

5.1.1 Properties of Wave Equation

Definition 5.3 (Domain of Dependence/Domain of Influence)**.** *Take any $(x_0, t_0) \in \mathbb{R} \times \mathbb{R}$.*

- *One has 2 characteristic lines connecting (x_0,t_0) with $(x_0 - ct_0, 0)$ and $(x_0 + ct_0, 0)$*

$$x - x_0 = c(t-t_0) \qquad x - x_0 = -c(t-t_0) \qquad 0 \le t \le t_0$$

  The three points form the backward characteristic cone $C^-(x_0,t_0)$ emanating from (x_0,t_0). This means to determine the value $u(x_0,t_0)$, one only need information for initial condition on $[x_0 - ct_0, x_0 + ct_0] \subset \mathbb{R}$. $C^-(x_0,t_0)$ is the domain of dependence for the point (x_0,t_0).

- *If one alternatively choose (x_0,t_0) and look at the future $T > t_0$. The $C_T^+(x_0,t_0)$ denotes forward triangle connecting the three points (x_0,t_0) with the 2 points that intersects the line $t = T$ using the same characteristic lines*

$$x - x_0 = c(t-t_0) \qquad x - x_0 = -c(t-t_0) \qquad t_0 \le t \le T$$

  Here $C_T^+(x_0,t_0)$ is the domain of influence for (x_0,t_0).

The fact that the size of domain of influence grows at finite speed is Huygen's Principle.

Remark 5.1. *Domain of Dependence, Domain of Influence and Huygen's Principle holds as well for variable coefficient PDEs for example*

$$\partial_t^2 u = c^2(x)\partial_x^2 u$$

for $c_1^2 \geq c^2(x) \geq c_0^2 > 0$

Theorem 5.2 (Conservation of Energy)**.**

$$\begin{aligned}
u_{tt} &= c^2 u_{xx} \\
u_t u_{tt} &= c^2 u_t u_{xx} \\
\partial_t(\frac{u_t^2}{2}) &= c^2\left(\partial_x(u_t u_x) - (\partial_x u_t)u_x\right) \\
&= c^2\left(\partial_x(u_t u_x) - \partial_t(\frac{u_x^2}{2})\right) \\
\partial_t\left(\frac{u_t^2}{2} + c^2\frac{u_x^2}{2}\right) &= c^2\partial_x(u_t u_x)
\end{aligned}$$

This take the form

$$\partial_t\mathcal{E} + \partial_x\mathcal{J} = 0$$

where $\mathcal{E} := \frac{u_t^2}{2} + c^2\frac{u_x^2}{2}$ *is the conserved energy density and* $\mathcal{J} := -c^2 u_t u_x$ *is current. Upon integrating w.r.t. spatial domain, we obtain*

$$\int_{\mathbb{R}} \partial_t\left(\frac{u_t^2}{2} + c^2\frac{u_x^2}{2}\right) = c^2\ u_t u_x|_{-\infty}^{\infty} = 0 \implies \int_{\mathbb{R}} \frac{u_t^2}{2}(x,t) + c^2\frac{u_x^2}{2}(x,t)\,dx = \int_{\mathbb{R}} \frac{u_t^2}{2}(x,0) + c^2\frac{u_x^2}{2}(x,0)\,dx \qquad \forall\, t \geq 0$$

This is conservation Law. If moreover we give initial data $u(x,0) = f(x)$ *and* $u_t(x,0) = g(x)$ *we integrate in spatial dimension so*

$$\int_{-\infty}^{\infty} \frac{u_t^2}{2} + c^2\frac{u_x^2}{2}\,dx = \int_{-\infty}^{\infty} \frac{1}{2}g^2(x) + \frac{c^2}{2}(f')^2\,dx$$

This is conservation of Energy

5.1.2 Geometric Interpretation of Wave Equation

Consider $C_-(x_0,t_0)$ the domain of dependence for (x_0,t_0). Rewrite

$$u_{tt} - c^2 u_{xx} = 0 \iff \partial_x(-c^2 u_x) - \partial_t(-u_t) = 0$$

Recall Green's Theorem for (P,Q) vector fields over Ω that

$$\iint_{\Omega}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \int_{\partial\Omega} Pdx + Qdy$$

Thus apply Green's Theorem we obtain

$$\begin{aligned}
0 &= \iint_{C_-(x_0,t_0)} \partial_x(-c^2 u_x) - \partial_t(-u_t)\,dxdt \\
&= \int_{\partial C_-(x_0,t_0)} -u_t\,dx - c^2 u_x\,dt
\end{aligned}$$

Let **I** denote the line segment pointing from $(x_0 - ct_0, 0)$ to $(x_0 + ct_0, 0)$, **II** pointing from $(x_0 + ct_0, 0)$ to (x_0, t_0) and finally **III** pointing from (x_0, t_0) to $(x_0 - ct_0, 0)$. Then we write down contour integral explicitly

$$\begin{aligned}
\int_{\mathbf{I}} -u_t dx - c^2 u_x dt &= -\int_{x_0 - ct_0}^{x_0 + ct_0} u_t(x,0)\,dx \\
\int_{\mathbf{II}} -u_t dx - c^2 u_x dt &= \int_{x_0 + ct_0}^{x_0} -u_t(x, t_0 - \frac{1}{c}(x - x_0))dx - c^2\int_{x_0 + ct_0}^{x_0} u_x(x, t_0 - \frac{1}{c}(x - x_0))\frac{dt}{dx}dx
\end{aligned}$$

Notice $\frac{dt}{dx} = -\frac{1}{c}$ and

$$\begin{aligned}
u_x(x, t_0 - \frac{1}{c}(x - x_0)) &= \frac{\partial}{\partial x}\left(u(x, t_0 - \frac{1}{c}(x - x_0))\right) + \frac{1}{c}u_t(x, t_0 - \frac{1}{c}(x - x_0)) \\
cu_x(x, t_0 - \frac{1}{c}(x - x_0)) &= c\frac{\partial}{\partial x}\left(u(x, t_0 - \frac{1}{c}(x - x_0))\right) + u_t(x, t_0 - \frac{1}{c}(x - x_0))
\end{aligned}$$

so

$$
\begin{aligned}
-u_t(x, t_0 - \frac{1}{c}(x-x_0)) + cu_x(x, t_0 - \frac{1}{c}(x-x_0)) &= c\frac{\partial}{\partial x}\left(u(x, t_0 - \frac{1}{c}(x-x_0))\right) \\
\int_{\mathbf{II}} -u_t dx - c^2 u_x dt &= c\int_{x_0+ct}^{x_0} \frac{\partial}{\partial x}\left(u(x, t_0 - \frac{1}{c}(x-x_0))\right) dx \\
&= c\left(u(x_0, t_0) - u(x_0 + ct_0, 0)\right)
\end{aligned}
$$

For the third one

$$
\int_{\mathbf{III}} -u_t dx - c^2 u_x dt = -c\left(u(x_0 - ct_0, 0) - u(x_0, t_0)\right)
$$

so summing up gives

$$
\begin{aligned}
0 &= -\int_{x_0-ct}^{x_0+ct} u_t(x,0)\, dx + c\left(u(x_0,t_0) - u(x_0+ct_0,0)\right) + -c\left(u(x_0-ct_0,0) - u(x_0,t_0)\right) \\
u(x_0,t_0) &= \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} u_t(x,0)dx + \frac{1}{2}u(x_0+ct_0,0) + \frac{1}{2}u(x_0-ct_0,0) \\
&= \frac{1}{2}f(x_0+ct_0) + \frac{1}{2}f(x_0-ct_0) + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(s)ds
\end{aligned}
$$

If we're given initial data $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.

5.2 Wave Equation in $n = 3$

Definition 5.4 (IVP)**.** *For $c > 0$*

$$
\begin{cases}
\Box u = (\partial_t^2 - c^2\Delta_x)u = 0 & x \in \mathbb{R}^3 \ t \in \mathbb{R} \\
u(x,0) = f(x) \\
u_t(x,0) = g(x)
\end{cases}
\tag{42}
$$

with prescribed initial conditions.

Recall for $f \in \mathcal{S}(\mathbb{R}^n)$ Definition 3.2, the Fourier Transform

$$
\hat{f}(\xi) = (\mathcal{F}f)(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-iy\cdot\xi} f(y)\, dy
$$

and the Fourier Inversion

$$
\check{f}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int e^{ix\cdot\xi} f(\xi)\, d\xi = \hat{f}(-x)
$$

Note $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is unitary operator in L^2, i.e.

$$
\|f\|_{L^2}^2 = \left\|\hat{f}\right\|_{L^2}^2
$$

and that $\mathcal{S}$ is dense in L^2. Hence $\mathcal{F}$ extends to a unitary operator on all of $L^2(\mathbb{R}^n)$ as in Proposition 3.2.

Lemma 5.1. *One has convolution properties. For $(f * g)(x) = \int f(x-y)g(y)\, dy$*

- *$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$*
- *$(f * g)(x) = (\hat{f}\hat{g})^{\vee}(x)$*
- *Denote $f = \check{F}$ and $g = \check{G}$ so $(\check{F} * \check{G}) = (FG)^{\vee}$*

Derive formula for IVP (42) using Fourier Transform. Do Fourier Transform in space. Assume $f, g \in \mathcal{S}(\mathbb{R}^3)$

$$
\begin{aligned}
\partial_t^2 \hat{u}(\xi,t) - c^2\mathcal{F}(\Delta_x u)(\xi,t) &= 0 \\
\hat{u}(\xi,0) &= \hat{f}(\xi) \\
\partial_t \hat{u}(\xi,0) &= \hat{g}(\xi)
\end{aligned}
$$

Note $\mathcal{F}(\partial_x^\alpha w) = (i\xi)^\alpha \hat{w}(\xi)$, so

$$
(\partial_t^2 + c^2|\xi|^2)\hat{u}(\xi,t) = 0
$$

Now for each fixed $\xi \in \mathbb{R}^n$, we have that

$$u(\xi, t) = \cos(c|\xi|t)\hat{f}(\xi) + \frac{\sin(c|\xi|t)}{c|\xi|}\hat{g}(\xi)$$

Now inverting Fourier Transform

$$u(x,t) = \mathcal{F}^{-1}\left(\cos(c|\xi|t)\hat{f}(\xi)\right) + \mathcal{F}^{-1}\left(\frac{\sin(c|\xi|t)}{c|\xi|}\hat{g}(\xi)\right)$$

How to proceed?

Lemma 5.2.

$$\mathcal{F}^{-1}\left(A(\xi)\hat{F}(\xi)\right) = \frac{1}{(2\pi)^{\frac{n}{2}}}\left(A^{\vee} * F\right)(x)$$

Proof.

$$\begin{aligned}
\mathcal{F}^{-1}\left(A(\xi)\hat{F}(\xi)\right) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int e^{ix\cdot\xi}A(\xi)\hat{F}(\xi)\,d\xi \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int e^{ix\cdot\xi}A(\xi)\left(\frac{1}{(2\pi)^{\frac{n}{2}}}\int e^{-iy\cdot\xi}F(y)\,dy\right)d\xi \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int\left(\int\frac{e^{i(x-y)\cdot\xi}}{(2\pi)^{\frac{n}{2}}}A(\xi)\,d\xi\right)F(y)\,dy \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int\check{A}(\xi)(x-y)F(y)\,dy \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\left(A^{\vee} * F\right)(x)
\end{aligned}$$

□

Thus

$$u(x,t) = \frac{1}{(2\pi)^{\frac{n}{2}}}\mathcal{F}^{-1}\left(\cos(c|\xi|t)\right) * f(x,t) + \frac{1}{(2\pi)^{\frac{n}{2}}}\mathcal{F}^{-1}\left(\frac{\sin(c|\xi|t)}{c|\xi|}\right) * g(x,t) \tag{43}$$

But how does (43) help? We know for $n = 1$ the solution (41) is beautiful. We need to compute

$$\begin{aligned}
\mathcal{F}^{-1}\left(\frac{\sin(c|\xi|t)}{c|\xi|}\right) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int e^{ix\cdot\xi}\frac{\sin(c|\xi|t)}{c|\xi|}\,d\xi \\
\left(\mathcal{F}^{-1}\left(\frac{\sin(c|\xi|t)}{c|\xi|}\right) * g\right)(x,t) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\left(\int e^{i(x-y)\cdot\xi}\frac{\sin(c|\xi|t)}{c|\xi|}\,d\xi\right)g(y)\,dy
\end{aligned}$$

□

Let's derive a theory for tempered distributions.

5.2.1 Tempered Distributions

Definition 5.5. *$T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{R}$ or $\mathbb{C}$ is a linear functional if for any ϕ, $\psi \in \mathcal{S}$*

$$T(\alpha\phi + \beta\psi) = \alpha T(\phi) + \beta T(\psi) \qquad \forall\ \alpha,\ \beta \in \mathbb{R} \text{ or } \mathbb{C}$$

Definition 5.6 (Convergence)**.** *We say $\{\phi_j\}_{j\geq 1} \subset \mathcal{S}$ converges to $\phi \in \mathcal{S}$ if*

$$\sup_{x\in\mathbb{R}^n}|x^{\alpha}\partial_x^{\beta}(\phi_j - \phi)| \to 0 \qquad \text{as } j \to \infty \qquad \forall\ \alpha,\ \beta \in \mathbb{N}_0^n$$

Definition 5.7 (Tempered Distribution)**.** *A tempered distribution is a continuous linear functional on $\mathcal{S}$, i.e. for any $\phi_j \to \phi$ in $\mathcal{S}$*

$$T(\phi_j) \to T(\phi) \qquad \text{as } j \to \infty$$

Denote $\mathcal{S}'(\mathbb{R}^n)$ as the space of tempered distributions.

Example 5.3. *Let $f(x)$ be such that*

$$|f(x)| \leq C(1+|x|)^N \qquad \forall\ x \in \mathbb{R}^1$$

known as polynomially bounded functions. Here C and N are fixed. To this function, we can naturally associate a tempered distribution $i_f \in \mathcal{S}'(\mathbb{R}^n)$ s.t.

$$i_f[\psi] := \int_{\mathbb{R}} f(x)\psi(x)\,dx \qquad \forall\ \psi \in \mathcal{S}(\mathbb{R})$$

This is clearly linear. One may also easily check that i_f is a tempered distribution. Assume that $\psi_j \to \psi$ in $\mathcal{S}$, then for any $m \in \mathbb{N}_0^n$

$$\sup_{x\in\mathbb{R}}|x^m(\psi_j(x)-\psi(x))| \to 0 \qquad as\ j\to\infty$$

We want to show $|i_f[\psi_j]-i_f[\psi]| \to 0$.

$$\begin{aligned}
|i_f[\psi_j]-i_f[\psi]| = |i_f[\psi_j-\psi]| &\leq \int |f(x)||\psi_j-\psi|dx \\
&\leq C\int (1+|x|)^N|\psi_j(x)-\psi(x)|\,dx \\
&= C\int \frac{1}{(1+|x|)^2}(1+|x|)^{N+2}|\psi_j(x)-\psi(x)|\,dx \\
&\leq C\left(\int \frac{1}{(1+|x|)^2}\,dx\right)\cdot \sup_{z\in\mathbb{R}}(1+|z|)^{N+2}|\psi_j(z)-\psi(z)| \to 0
\end{aligned}$$

By assumption that $\psi_j \to \psi$ in $\mathcal{S}$.

Example 5.4. *For any $\psi \in \mathcal{S}(\mathbb{R}^n)$, define*

$$\delta_z[\psi] := \psi(z)$$

This is Dirac Delta Distribution at z. This is clearly linear and continuous. Hence $\delta \in \mathcal{S}'(\mathbb{R}^n)$.

To study Fourier Transform on Tempered Distributions $\mathcal{S}'(\mathbb{R}^n)$, note the following.

Lemma 5.3 (Duality Relation)**.** *For $f,\ g \in \mathcal{S}(\mathbb{R}^n)$, then*

$$\int \hat{f}g\,dx = \int f\hat{g}\,dx$$

Motivated by such Lemma, if $f,\ g \in \mathcal{S}(\mathbb{R}^n)$

$$i_{\hat{f}}[g] = i_f[\hat{g}]$$

Now we extend the Fourier Transform to $\mathcal{S}'(\mathbb{R}^n)$ via the following

Definition 5.8 (Fourier Transform on $\mathcal{S}'(\mathbb{R}^n)$)**.** *If $T \in \mathcal{S}'(\mathbb{R}^n)$, then*

$$\hat{T}(g) := T[\hat{g}] \qquad \forall\ g \in \mathcal{S}(\mathbb{R}^n)$$

In other words, we look for $\hat{T} \in \mathcal{S}'(\mathbb{R}^n)$ s.t.

$$T[\hat{\psi}] = \hat{T}[\psi] \qquad \forall\ \psi \in \mathcal{S}$$

Example 5.5. *One compute examples for $\mathcal{S}'(\mathbb{R}^n)$*

- *$T = i_1$ then*

  $$T[\psi] = i_1(\psi) = \int_{\mathbb{R}^n} \psi(x)\,dx$$

  Compute

  $$\hat{T}[\psi] := \int 1\hat{\psi}(x)\,dx$$

  Now if we write via Fourier inversion

  $$\psi(y) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int e^{ix\cdot y}\hat{\psi}(y)\,dy$$

  in particular

  $$\psi(0) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int \hat{\psi}(y)\,dy$$

  Then

  $$\hat{T}[\psi] = (2\pi)^{\frac{n}{2}}\psi(0) = (2\pi)^{\frac{n}{2}}\delta_0(\psi)$$

  Hence

  $$\hat{i_1} = \hat{1} = (2\pi)^{\frac{n}{2}}\delta_0(\psi)$$

- $f(x) = e^{ix\cdot a}$ *for* $a \in \mathbb{R}^n$ *and* $x \in \mathbb{R}^n$, *let* $T = i_f$ *then* T *acts as Fourier inversion*

$$\begin{aligned} T[\hat{\psi}] &= \int_{\mathbb{R}^n} e^{ix\cdot a}\hat{\psi}(x)\,dx \\ &= (2\pi)^{\frac{n}{2}}\psi(a) \\ \widehat{i_{e^{ix\cdot a}}} = \widehat{e^{ix\cdot a}} &= (2\pi)^{\frac{n}{2}}\delta_a \end{aligned}$$

- *Look at, for* $T = i_f$ *where* $f = e^{ix\cdot a}$

$$\begin{aligned} \check{T}[\psi] &= (\mathcal{F}^{-1}T)(\psi) = T[\check{\psi}] \\ &= \int e^{ix\cdot a}(\mathcal{F}^{-1}\psi)(x)\,dx \\ &= \int e^{-ix\cdot(-a)}(\mathcal{F}^{-1}\psi)(x)\,dx \\ &= (2\pi)^{\frac{n}{2}}\psi(-a) = (2\pi)^{\frac{n}{2}}\delta_{-a}(\psi) \end{aligned}$$

Hence for $T = i_{e^{ix\cdot a}}$

$$\mathcal{F}^{-1}T = \check{T} = (2\pi)^{\frac{n}{2}}\delta_{-a}$$

i.e.

$$(e^{ix\cdot a})^{\vee} = (2\pi)^{\frac{n}{2}}\delta(x+a)$$

- *For* $T = \delta_b \in \mathcal{S}'(\mathbb{R}^n)$ *with* $b \in \mathbb{R}^n$

$$\begin{aligned} [\mathcal{F}\delta_b](\psi) &= \delta_b[\mathcal{F}\psi] = \delta_b[\hat{\psi}] \qquad \forall\ \psi \in \mathcal{S} \\ &= \delta_b\left(\frac{1}{(2\pi)^{\frac{n}{2}}}\int e^{-ix\cdot\xi}\psi(x)\,dx\right) \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int e^{-ix\cdot b}\psi(x)\,dx = i_F[\psi] \end{aligned}$$

For

$$F(x) = \frac{e^{-ix\cdot b}}{(2\pi)^{\frac{n}{2}}}$$

So

$$\widehat{\delta_b} = \frac{e^{-ix\cdot b}}{(2\pi)^{\frac{n}{2}}}$$

- *Let* $H(s)$ *be Heaviside where* $H = 1$ *for* $s \geq 0$ *and* 0 *otherwise. Let*

$$f(x) = H(a - |x|)$$

and $T = i_f$. *Then*

$$\begin{aligned} \hat{T}[\psi] = T[\hat{\psi}] &= \int H(a-|x|)\hat{\psi}(x)\,dx \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int H(a-|x|)\int e^{-iy\cdot x}\psi(y)\,dy\,dx \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int \left(\int_{-a}^{a} e^{-iy\cdot x}\,dx\right)\psi(y)\,dy \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int \left(\frac{e^{-iy\cdot x}}{-iy}\Bigg|_{x=-a}^{x=a}\right)\psi(y)\,dy \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int \left(\frac{2e^{iy\cdot a} - 2e^{-iy\cdot a}}{2iy}\right)\psi(y)\,dy \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int \frac{2\sin(ay)}{y}\psi(y)\,dy \end{aligned}$$

So we obtain

$$\hat{T}(\psi) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int \frac{2\sin(ay)}{y}\psi(y)\,dy$$

Thus

$$\hat{T} = \widehat{H}(a - |\cdot|)(z) = \frac{1}{(2\pi)^{\frac{n}{2}}}\frac{2\sin(az)}{z}$$

and

$$H(a-|x|) = \mathcal{F}^{-1}\left(\frac{2\sin(ay)}{(2\pi)^{\frac{n}{2}}y}\right)(x) \tag{44}$$

5.2.2 Application to Wave Equation

Definition 5.9 (Convolutions)**.** *For $|F(x)| \le C(1+|x|)^N$ for $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\phi_x(y) := \phi(x-y)$*

$$(F * \phi)(x) = \int F(y)\phi(x-y)\,dy = i_F[\phi_x]$$

Then fro $T \in \mathcal{S}'(\mathbb{R}^n)$, define

$$T * \phi(x) := T[\phi_x]$$

Example 5.6. *For $T = \delta_a$*

$$(\delta_a * f)(x) \equiv \delta_a[f_x] = f(x-y)|_{y=a} = f(x-a)$$

Example 5.7 ($n=1$ compute (43))**.** *For $n = 1$*

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left(\cos(c|\xi|t)\right) * f(x,t) + \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left(\frac{\sin(c|\xi|t)}{c|\xi|}\right) * g(x,t) = \boldsymbol{f} + \boldsymbol{g} \\
\boldsymbol{f} &= \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left(\frac{1}{2}e^{ic|\xi|t}\right) * f(x,t) + \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left(\frac{1}{2}e^{-ic|\xi|t}\right) * f(x,t) \\
&= \frac{1}{2}\delta_{-ct} * f(x,t) + \frac{1}{2}\delta_{+ct} * f(x,t) \\
&= \frac{1}{2}\int \delta(x+ct-y)f(y)\,dy + \frac{1}{2}\int \delta(x-ct-y)f(y)\,dy = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right)
\end{aligned}$$

For term $\boldsymbol{g}$, use (44) so that

$$\begin{aligned}
H(ct-|x|) &= \mathcal{F}^{-1}\left(\frac{2\sin(cty)}{(2\pi)^{\frac{1}{2}}y}\right)(x) \\
\frac{1}{2c}H(ct-|x|) &= \frac{1}{(2\pi)^{\frac{1}{2}}}\mathcal{F}^{-1}\left(\frac{\sin(cty)}{cy}\right)(x)
\end{aligned}$$

Hence we compute

$$\begin{aligned}
\boldsymbol{g} &= \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left(\frac{\sin(cty)}{cy}\right)(x) * g \\
&= \frac{1}{2c}H(ct-|x|) * g(x,t) = \frac{1}{2c}\int H(ct-|x-y|)g(y)\,dy \\
&= \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds
\end{aligned}$$

What about finally for $n = 3$? Recall (43).

$$u(x,t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\mathcal{F}^{-1}\left(\cos(c|\xi|t)\right) * f(x,t) + \frac{1}{(2\pi)^{\frac{3}{2}}}\mathcal{F}^{-1}\left(\frac{\sin(c|\xi|t)}{c|\xi|}\right) * g(x,t) = \mathbf{f} + \mathbf{g}$$

Note formally for $\mathbf{f}$

$$\frac{\partial}{\partial t}\mathcal{F}^{-1}\left(\frac{\sin(c|\xi|t)}{c|\xi|}\right) = \mathcal{F}^{-1}\left(\cos(c|\xi|t)\right)$$

Hence it suffices to compute $\mathbf{g}$. The key is to compute $\widehat{\delta}_{|\cdot|-R} \in \mathcal{S}'(\mathbb{R}^3)$ so that

$$\widehat{\delta_{|\cdot|-R}}[\psi] = \int_{|x|=R} \hat{\psi}(x)\,dS(x) \qquad \forall\ \psi \in \mathcal{S}(\mathbb{R}^3)$$

Theorem 5.3 ($n = 3$)**.**

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\mathcal{F}_\xi^{-1}\left(\frac{\sin(c|\xi|t)}{c|\xi|}\right)(x,t) = \frac{\delta_{|x|-ct}}{4\pi c^2 t} \tag{45}$$

Proof of (45). Compute

$$\begin{aligned}
\widehat{\delta}_{|\cdot|-R}[\psi] = \delta_{|\cdot|-R}[\hat{\psi}] &= \int_{|y|=R} \hat{\psi}(y)\,dS(y) \\
&= \int_{|y|=R} \frac{1}{(2\pi)^{\frac{3}{2}}}\left(\int e^{-i\eta\cdot y}\psi(\eta)\,d\eta\right) dS(y) \\
&= \int_{\mathbb{R}^3}\left(\int_{|y|=R} e^{-i\eta\cdot y}\,dS(y)\right)\psi(\eta)\,d\eta\frac{1}{(2\pi)^{\frac{3}{2}}}
\end{aligned}$$

To compute the integral inside, go to spherical coordinates with polar angle θ and azimuthal angle ϕ so that

$$dS(y) = R^2 \sin(\theta) d\theta d\phi$$

Also for η fixed, pick spherical coordinates where η is the north pole direction. Hence

$$\eta \cdot y = |\eta||y| \cos(\theta)$$

The the integral writes

$$\begin{aligned}
\int_{|y|=R} e^{-i\eta\cdot y}\, dS(y) &= \int_0^{2\pi} d\phi \int_0^{\pi} e^{-i|\eta||y|\cos(\theta)} R^2 \sin(\theta)\, d\theta \\
&= 2\pi R^2 \int_0^{\pi} e^{-i|\eta|R\cos(\theta)} \frac{iR|\eta|}{iR|\eta|} \sin(\theta)\, d\theta \\
&= 2\pi R^2 \int_0^{\pi} \frac{\partial}{\partial\theta} \left(e^{-i|\eta|R\cos(\theta)} \right) \frac{1}{iR|\eta|}\, d\theta \\
&= \frac{2\pi R}{i|\eta|} 2i \left(\frac{e^{iR|\eta|} - e^{-iR|\eta|}}{2i} \right) \\
&= \frac{4\pi R}{|\eta|} \sin(R|\eta|)
\end{aligned}$$

Hence

$$\begin{aligned}
\widehat{\delta}_{|\cdot|-R}[\psi] &= \int_{\mathbb{R}^3} \left(\frac{4\pi R}{|\eta|} \sin(R|\eta|) \right) \psi(\eta)\, d\eta \frac{1}{(2\pi)^{\frac{3}{2}}} \\
\frac{1}{4\pi R} \widehat{\delta}_{|\cdot|-R}[\psi] &= \int_{\mathbb{R}^3} \frac{\sin(R|\eta|)}{|\eta|} \frac{1}{(2\pi)^{\frac{3}{2}}} \psi(\eta)\, d\eta \\
\implies \frac{1}{4\pi R} \widehat{\delta}_{|\cdot|-R} &= \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{\sin(R|\eta|)}{|\eta|} \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}} \mathcal{F}^{-1} \left(\frac{\sin(R|\cdot|)}{|\cdot|} \right)
\end{aligned}$$

So let $R = ct$

$$\begin{aligned}
\frac{1}{4\pi ct} \delta_{|\cdot|-ct} &= \frac{1}{(2\pi)^{\frac{3}{2}}} \mathcal{F}^{-1} \left(\frac{\sin(ct|\cdot|)}{|\cdot|} \right) \\
\frac{1}{4\pi c^2 t} \delta_{|\cdot|-ct} &= \frac{1}{(2\pi)^{\frac{3}{2}}} \mathcal{F}^{-1} \left(\frac{\sin(ct|\xi|)}{c|\xi|} \right) (x,t)
\end{aligned}$$

□

Thus

$$\begin{aligned}
u(x,t) &= \mathbf{f} + \frac{1}{4\pi c^2 t} \delta_{|x|-ct} * g \\
&= \mathbf{f} + \frac{1}{4\pi c^2 t} \int_{|x-y|=ct} g(y)\, dy \\
&= \frac{\partial}{\partial t} \left(\frac{1}{4\pi c^2 t} \int_{|x-y|=ct} f(y)\, dy \right) + \frac{1}{4\pi c^2 t} \int_{|x-y|=ct} g(y)\, dy
\end{aligned} \tag{46}$$

How to make it look a little bit more like the solution at $n = 1$? Let $y = x + ct\omega$ where $|\omega| = 1$. Do a change of variables

$$dS(y) = (ct)^2 dS(\omega)$$

so

$$\begin{aligned}
\frac{1}{4\pi c^2 t} \int_{|x-y|=ct} g(y)\, dS(y) &= \frac{(ct)^2}{4\pi c^2 t} \int_{|\omega|=1} g(x + ct\omega)\, dS(\omega) \\
&= \frac{t}{4\pi} \int_{|\omega|=1} g(x + ct\omega)\, dS(\omega)
\end{aligned}$$

$$u(x,t) = \frac{t}{4\pi} \int_{|\omega|=1} g(x + ct\omega)\, dS(\omega) + \frac{\partial}{\partial t} \left(\frac{t}{4\pi} \int_{|\omega|=1} f(x + ct\omega)\, dS(\omega) \right) \tag{47}$$

It is easy to verify both (46) and (47) satisfies the IVP (42).

5.2.3 Properties of Wave Equation in $n = 3$

- We have finite propagation speed c, and sharp arrival and departure of signals. This is known as Strong Huygen's Principle.
- Conservation of Energy.
$$\mathcal{E}(t) := \int_{\mathbb{R}^3} \frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\,dx = \mathcal{E}(0)$$
if u solves (42).
- Diffraction of Waves.
$$|u(x,t)| \leq \frac{C}{t} \qquad t \gg 1$$

Let's carry out the derivative in (47).

$$\begin{aligned}
u(x,t) &= \frac{t}{4\pi}\int_{|\omega|=1} g(x+ct\omega)\,dS(\omega) + \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_{|\omega|=1} f(x+ct\omega)\,dS(\omega)\right)\\
&= \frac{t}{4\pi}\int_{|\omega|=1} g(x+ct\omega)\,dS(\omega) + \frac{1}{4\pi}\int_{|\omega|=1} f(x+ct\omega)\,dS(\omega) + \frac{t}{4\pi}\int_{|\omega|=1} \nabla_x f(x+ct\omega)\cdot c\omega\,dS(\omega)\\
&= \frac{t}{4\pi}\frac{1}{(ct)^2}\int_{|x-y|=ct} g(y)\,dS(y) + \frac{1}{4\pi}\frac{1}{(ct)^2}\int_{|x-y|=ct} f(y)\,dS(y) + \frac{t}{4\pi}\frac{1}{(ct)^2}\int_{|x-y|=ct} \nabla_y f(y)\cdot\frac{y-x}{t}\,dS(y)\\
&= \frac{1}{4\pi c^2t^2}\left(\int_{|x-y|=ct} (tg(y) + f(y) + \nabla_y f(y)\cdot(y-x))\,dS(y)\right) \qquad (48)
\end{aligned}$$

This is loss of smoothness. In physics, this is focusing effect.

Theorem 5.4 (Domain of Dependence)**.** *Let $(x_0,t_0) \in \mathbb{R}^3 \times [0,\infty)$. The backward cone of (x_0,t_0) writes*
$$c(t_0 - t) = |x_0 - x|$$
So according to (48) $u(x_0,t_0)$ depends only on $g, f, \nabla f$ on the Surface of sphere of radius ct_0
$$\partial B(x_0, ct_0) := \{x \in \mathbb{R}^3 \mid |x - x_0| = ct_0\}$$

Theorem 5.5 (Domain of Influence)**.** *Let $(x_0, 0) \in \mathbb{R}^3 \times \{0\}$. What parts of solution does the point influence? It's the forward light cone, which is essentially the union of spheres of ascending radius with center $x_0 \in \mathbb{R}^3$. Assume that f, g have support inside $B_\rho(0)$. To see what is influenced, we need to see the union of all spheres arising from each point in $B_\rho(0)$. Notice $u \equiv 0$ in the inner region of the spheres as the energy just radiates out.*

In particular, if $\operatorname{supp}(f,g) = B_\rho(0)$
$$t_{arrivial\ of\ signal}(x_0) = \frac{\operatorname{dist}(x_0, B_\rho(0))}{c}$$
and
$$t_{departure\ of\ signal}(x_0) = \frac{\operatorname{dist}(x_0, B_\rho(0)) + 2\rho}{c}$$

Theorem 5.6 (Conservation of Energy)**.**

$$\begin{aligned}
u_t u_{tt} &= u_t c^2 \Delta u\\
\frac{1}{2}\partial_t(u_t^2) &= c^2\left(\nabla\cdot(u_t\nabla u) - \nabla u_t\cdot\nabla u\right)\\
\partial_t(\frac{1}{2}u_t^2) &= c^2\left(\nabla\cdot(\partial_t u\nabla u) - \partial_t\frac{\nabla u\cdot\nabla u}{2}\right)
\end{aligned}$$
$$\partial_t\left(\frac{1}{2}u_t^2 + c^2\frac{|\nabla u|^2}{2}\right) + \nabla\cdot(-c^2u_t\nabla u) = 0$$

Again energy density $\mathcal{E} := \frac{1}{2}u_t^2 + c^2\frac{|\nabla u|^2}{2}$ and current $\mathcal{J} := -c^2u_t\nabla u$ writes
$$\partial_t\mathcal{E} + \nabla_x\cdot\mathcal{J} = 0$$

Hence integrating in spatial dimensions gives

$$\int_{\mathbb{R}^3} \partial_t \left(\frac{1}{2} u_t^2 + c^2 \frac{|\nabla u|^2}{2} \right) dx = c^2 \int_{\mathbb{R}^3} \nabla \cdot (u_t \nabla u)\, dx = 0$$

further implies

$$\implies \int_{\mathbb{R}^3} \left(\frac{1}{2} u_t^2 + c^2 \frac{|\nabla u|^2}{2} \right)(t,x)\, dx = \int_{\mathbb{R}^3} \left(\frac{1}{2} u_t^2 + c^2 \frac{|\nabla u|^2}{2} \right)(0,x)\, dx = \int_{\mathbb{R}^3} \left(\frac{1}{2} g^2(x) + c^2 \frac{|\nabla f(x)|^2}{2} \right) dx$$

where $u(0,x) = f$ and $u_t(0,x) = g$ denotes initial data.

Theorem 5.7 (Diffraction and Amplitude Decay)**.**

$$\sup_{x \in \mathbb{R}^3} |u(x,t)| \leq \frac{C_{data}}{t}$$

is uniform in x. This is attenuation. In $n = 3$, $\frac{1}{t}$ is the attenuation rate.

According to expression of solution (48), the term g has the slowest decay in t and is the only term that we should worry about.

Lemma 5.4. *For* $\operatorname{supp}(g) \subset B_\rho(0)$*, we have estimate*

$$\frac{1}{4\pi c^2 t} \int_{|x-y|=ct} g(y)\, dS(y) \leq \frac{1}{4\pi c^2 t} \|g\|_{L^\infty} 4\pi\rho^2 = \frac{C_{data}}{t} \qquad \forall\ x$$

Proof.

$$\int_A h = \int_A \mathbb{1}_{\{h \neq 0\}} h$$

Hence

$$|\int_A h| \leq \|h\|_{L^\infty} |\{h \neq 0\} \cap A|$$

Now suppose $\operatorname{supp}(h) \subset B_\rho(0) = \{x \mid |x| \leq \rho\}$ and $A = \{y \mid |x - y| = ct\}$. We have

$$\int_{|x-y|=ct} h\, dS(y) \leq \|h\|_{L^\infty} |B_\rho(0) \cap \{|x-y| = ct\}| \leq 4\pi\rho^2 \|h\|_\infty$$

Hence applying to

$$h = g$$

we have

$$\frac{1}{4\pi c^2 t} \int_{|x-y|=ct} g(y)\, dS(y) \leq \frac{1}{4\pi c^2 t} \|g\|_{L^\infty} 4\pi\rho^2$$

□

Remark 5.2. *Why do we not have diffraction in $n = 1$?*

$$u(x,t) = \frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$$

If fix x and let t get large, then $f(x \pm ct)$ are 0 due to compact support. On the other hand

$$\frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds \to \frac{1}{2c} \int_{\operatorname{supp}(g)} g(y)\, dy$$

so u is identically constant for t large enough.

5.3 Wave Equation in $n = 2$

We use Method of Descent to derive solution to $n = 2$.

Definition 5.10 (IVP)**.** *For $c > 0$*

$$\begin{cases} \Box u = (\partial_t^2 - c^2(\partial_{x_1}^2 + \partial_{x_2}^2))u = 0 & x \in \mathbb{R}^2 \ t \in \mathbb{R} \\ u(x_1, x_2, 0) = f(x_1, x_2) \\ u_t(x_1, x_2, 0) = g(x_1, x_2) \end{cases} \tag{49}$$

with prescribed initial conditions.

Solution to IVP (49)*.* We view IVP at $n = 2$ as special case for IVP at $n = 3$. Take special initial condition that is translational invariant in x_3

$$v(x_1, x_2, x_3, 0) = f(x_1, x_2)$$
$$v_t(x_1, x_2, x_3, 0) = g(x_1, x_2)$$

Where v solves

$$\Box v = (\partial_t^2 - c^2\Delta_x)v = 0 \qquad x \in \mathbb{R}^3$$

Hence we have

$$\Box_{n=3} \frac{\partial}{\partial x_3} v = 0$$
$$\frac{\partial}{\partial x_3} v\Big|_{t=0} = 0$$
$$\frac{\partial}{\partial t}\frac{\partial}{\partial x_3} v\Big|_{t=0} = 0$$

Denote $V := \frac{\partial}{\partial x_3} v$. By uniqueness we know $V \equiv 0$ then

$$\frac{\partial}{\partial x_3} v = 0 \implies v = v(x_1, x_2)$$

This gives us a $2d$ solution. Now we plug into solution to $3d$ (46)

$$\begin{aligned} u(x_1, x_2, t) &= v(x_1, x_2, x_3 = 0, t) \\ &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\int_{(x_1-y_1)^2+(x_2-y_2)^2+y_3^2=c^2t^2} f(y_1, y_2)\, dy\right) + \frac{1}{4\pi c^2 t}\int_{(x_1-y_1)^2+(x_2-y_2)^2+y_3^2=c^2t^2} g(y_1, y_2)\, dy \end{aligned}$$

We parametrize the sphere about $(x_1, x_2, 0)$. We look at the upper hemi-sphere.

$$y_3 = \sqrt{c^2t^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2}$$

and the lower hemi-sphere

$$y_3 = -\sqrt{c^2t^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2}$$

Thus $y_3 = y_3(y_1, y_2)$. Introduce notations and first consider the upper hemi-sphere

$$x' = (x_1, x_2)$$
$$y' = (y_1, y_2)$$
$$y_3 = \sqrt{c^2t^2 - |x' - y'|^2}$$

Notice the Surface element writes

$$dS(y) = \sqrt{1 + (\frac{dy_3}{dy_1})^2 + (\frac{dy_3}{dy_2})^2} dy_1 dy_2$$

due to tangent vectors

$$(1, 0, \frac{\partial y_3}{\partial y_1}) \qquad (0, 1, \frac{\partial y_3}{\partial y_2})$$

and calculating their cross product. Thus our solution writes via parametrization

$$\begin{aligned} &\frac{1}{4\pi c^2 t}\int_{|x'-y'|^2+y_3^2=c^2t^2} g(y_1, y_2)\, dy \\ &= \frac{1}{4\pi c^2 t}\int_{|x'-y'|^2 \le c^2t^2} g(y')\sqrt{1 + (\frac{dy_3}{dy_1})^2 + (\frac{dy_3}{dy_2})^2} dy' \end{aligned}$$

Notice

$$\frac{dy_3}{dy_j} = \frac{x_j - y_j}{\sqrt{c^2t^2 - |x' - y'|^2}} \qquad j = 1, 2$$

Now look at

$$1 + (\frac{dy_3}{dy_1})^2 + (\frac{dy_3}{dy_2})^2 = 1 + \frac{|x' - y'|^2}{c^2t^2 - |x' - y'|^2} = \frac{c^2t^2}{c^2t^2 - |x' - y'|^2}$$
$$\sqrt{1 + (\frac{dy_3}{dy_1})^2 + (\frac{dy_3}{dy_2})^2} = \frac{ct}{\sqrt{c^2t^2 - |x' - y'|^2}}$$

Hence

$$\begin{aligned}&\frac{1}{4\pi c^2 t}\int_{|x'-y'|^2+y_3^2=c^2t^2} g(y_1, y_2)\,dy \\ &= \frac{1}{4\pi c^2 t}\int_{|x'-y'|\leq ct} g(y')\frac{ct}{\sqrt{c^2t^2 - |x' - y'|^2}}\,dy'\end{aligned}$$

Now do the same for lower hemi-sphere. Hence we add them up and obtain

$$\frac{1}{4\pi c^2 t}\int_{(x_1-y_1)^2+(x_2-y_2)^2+y_3^2=c^2t^2} g(y_1, y_2)\,dy = \frac{1}{2\pi c}\int_{|x'-y'|\leq ct}\frac{g(y')}{\sqrt{c^2t^2 - |x' - y'|^2}}\,dy'$$

Hence we summarize solution at $n = 2$.

$$u(x_1, x_2, t) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi c}\int_{|x'-y'|\leq ct}\frac{f(y')}{\sqrt{c^2t^2 - |x' - y'|^2}}\,dy'\right) + \frac{1}{2\pi c}\int_{|x'-y'|\leq ct}\frac{g(y')}{\sqrt{c^2t^2 - |x' - y'|^2}}\,dy' \tag{50}$$

□

Suppose f, g are supported in $B_\rho(0) \subset \mathbb{R}^2$. Suppose we're at (x_{10}, x_{20}) far away from $B_\rho(0)$, then initially the supports of f, g do not overlap

$$\{|x_1 - y_1|^2 + |x_2 - y_2|^2 \leq c^2t^2\}$$

But for t large enough, we have contribution from f, g. Wait longer, we overlap more. But for t even larger, eventually the ball of ct radius centered at (x_{10}, x_{20}) completely covers $B_\rho(0)$. We can in fact show the uniform estimate

$$\sup_{x\in\mathbb{R}^2}|u(x,t)| \leq \frac{C_{data}}{\sqrt{t}}$$

5.4 Inhomogeneous Wave Equation

Consider inhomogeneous Wave Equation in $n = 1$.

Definition 5.11 (Inhomogeneous IVP)**.** *For $c > 0$ fixed number (speed of propagation)*

$$\begin{cases}\Box u = w(x,t) & x \in \mathbb{R}\ t \in \mathbb{R} \\ u(x, 0) = f(x) \\ u_t(x, 0) = g(x)\end{cases} \tag{51}$$

with prescribed initial conditions.

We think of writing solution

$$u(x,t) = u_0(x,t) + u_1(x,t)$$

where u_0 solves

$$\begin{cases}\Box u_0 = 0 & x \in \mathbb{R}\ t \in \mathbb{R} \\ u_0(x, 0) = f(x) \\ u_{0t}(x, 0) = g(x)\end{cases}$$

purely initial value problem and u_1 solves

$$\begin{cases}\Box u_1 = w(x,t) & x \in \mathbb{R}\ t \in \mathbb{R} \\ u_1(x, 0) = 0 \\ u_{1t}(x, 0) = 0\end{cases} \tag{52}$$

purely forced problem.

5.4.1 Solving Purely Forced IVP

To solve (52) we use Duhamel's Principle.

$$\begin{cases} \frac{\partial}{\partial t} u = Au + w(t) \\ u(0) = 0 \end{cases}$$

for A independent of t. Then write

$$\begin{aligned}
\frac{\partial}{\partial t} u - Au &= w(t) \\
\partial_t (e^{-At} u(t)) &= e^{-At} w(t) \\
e^{-At} u(t) &= \int_0^t e^{-As} w(s)\, ds \\
u(t) &= \int_0^t e^{A(t-s)} w(s)\, ds = \int_0^t U(t,s) w(s)\, ds
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial}{\partial t} U(t,s) &= AU(t,s) \\
U(t,s)|_{t=0} &= w(s)
\end{aligned}$$

Recall for $A = \Delta$

$$\begin{aligned}
u(x,t) &= \int_0^t e^{\Delta(t-s)} w(s)\, ds \\
&= \int_0^t \int \frac{e^{-\frac{(x-y)^2}{4(t-s)}}}{(4\pi(t-s))^{\frac{n}{2}}} w(y,s)\ dyds
\end{aligned}$$

Now the Analogous Duhamel's Principle for Wave Equation is, taking $u = u_1$ in (52)

Lemma 5.5. *Define*

$$u(x,t) = \int_0^t U(x,t,s)\, ds$$

s.t. for each $s \geq 0$

$$\begin{aligned}
\Box_{x,t} U(x,t,s) &= 0 \\
U(x,t,s)|_{t=s} &= 0 \\
\partial_t U(x,t,s)|_{t=s} &= w(x,s)
\end{aligned}$$

Then u solves purely forced problem (52)*.*

Proof. Using assumptions on $U(x,t,s)$ we have

$$\begin{aligned}
\partial_t u(x,t) &= U(x,t,s)|_{s=t} + \int_0^t U_t(x,t,s)\, ds = \int_0^t U_t(x,t,s)\, ds \\
\partial_t^2 u(x,t) &= \partial_t U(x,t,s)|_{s=t} + \int_0^t U_{tt}(x,t,s)\, ds \\
&= w(x,s) + \int_0^t c^2 \Delta_x U(x,t,s)\, ds \\
&= w(x,s) + c^2 \Delta_x u(x,t)
\end{aligned}$$

Hence

$$(\partial_t^2 - c^2 \Delta) u(x,t) = w(x,t)$$

□

Now we wish to construct U.

$n = 1$ U construction. Write

$$U(x,t,s) = \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} w(y,s)\, dy$$

Indeed

$$U(x,t,s)|_{t=s} = 0 \qquad \forall\ s \geq 0$$

and

$$\begin{aligned}
\partial_t U(x,t,s) &= \frac{1}{2}\left(w(x+c(t-s),s) + w(x-c(t-s),s)\right)\\
\partial_t^2 U(x,t,s) &= \frac{c}{2}w_1(x+c(t-s),s) - \frac{c}{2}w_1(x-c(t-s),s)
\end{aligned}$$

where w_1 denotes partial derivative w.r.t. first coordinate.

$$\begin{aligned}
\partial_x U(x,t,s) &= \frac{1}{2c}\left(w(x+c(t-s),s) - w(x-c(t-s),s)\right)\\
\partial_x^2 U(x,t,s) &= \frac{1}{2c}\left(w_1(x+c(t-s),s) - w_1(x-c(t-s),s)\right)\\
\implies \partial_t^2 U(x,t,s) &= c^2\partial_x^2 U(x,t,s)
\end{aligned}$$

Then verify

$$\begin{aligned}
U_t(x,t,s)|_{t=s} &= \left.\frac{1}{2c}\left(cw(x+c(t-s),s) - (-c)w(x-c(t-s),s)\right)\right|_{t=s}\\
&= w(x,s)
\end{aligned}$$

Now apply Duhamel

$$u(x,t) = \frac{1}{2c}\int_0^t ds \int_{x-c(t-s)}^{x+c(t-s)} w(y,s)\,dy$$

□

$n=3$ U *construction.* Write

$$U(x,t,s) = \frac{1}{4\pi c^2(t-s)}\int_{|x-y|=c(t-s)} w(y,s)\,dS(y)$$

So

$$\begin{aligned}
u(x,t) &= \int_0^t \frac{1}{4\pi c^2(t-s)}\int_{|x-y|=c(t-s)} w(y,s)\,dS(y)\,ds\\
&= \int_0^t \frac{1}{4\pi c}\int_{|x-y|=c(t-s)} \frac{w(y,s)}{|x-y|}\,dS(y)\,ds\\
&= \frac{1}{4\pi c}\int_{|x-y|<ct} \frac{w(y,t-\frac{|x-y|}{c})}{|x-y|}\,dy
\end{aligned}$$

Here $t-\frac{|x-y|}{c}$ is the retarded time. □

5.4.2 Properties of Inhomogeneous Wave Equation

Consider wave equation with inhomogeneous media

$$\partial_t^2 u = \nabla\cdot(c^2(x)\nabla u) - q(x)u$$

assuming $c_2 \geq c(x) \geq c_1 > 0$ and $q(x) \geq 0$. Also assume that $u \in C^2$ solution. Then we conclude that

- signals propagate with speed $\leq c_2$.
- If data f, g have compact support. Then for all $t>0$, $u(x,t)$ has compact support.

Theorem 5.8 (Conservation of Energy)**.** *Write*

$$\begin{aligned}
u_{tt} &= \nabla\cdot(c^2(x)\nabla u) - q(x)u\\
u_t u_{tt} &= u_t\nabla\cdot(c^2(x)\nabla u) - q(x)u_t u\\
\partial_t(\frac{1}{2}u_t^2) &= \nabla\cdot(u_t c^2(x)\nabla u) - \nabla u_t\cdot c^2(x)\nabla u - q(x)\partial_t(\frac{1}{2}u^2)\\
&= \nabla\cdot(u_t c^2(x)\nabla u) - \frac{1}{2}\partial_t\left(\nabla u\cdot c^2(x)\nabla u\right) - q(x)\partial_t(\frac{1}{2}u^2)\\
0 &= \partial_t\left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2(x)|\nabla u|^2\right) + \nabla\cdot\left(-u_t c^2(x)\nabla u(x)\right) + q(x)\partial_t(\frac{1}{2}u^2)
\end{aligned}$$

Hence conservation of energy writes

$$\partial_t \mathcal{E}(u_t, u) + \nabla_x \cdot \mathcal{J}(u_t, u) = 0$$

where

$$\begin{aligned}
\mathcal{E}(u_t, u) &= \frac{1}{2}u_t^2 + \frac{1}{2}c^2(x)|\nabla u|^2 + \frac{1}{2}q(x)u^2 \\
\mathcal{J}(u_t, u) &= -u_t c^2(x)\nabla u
\end{aligned}$$

For such equation we have domain of dependence competing with each other

- two line with slope $\pm\frac{1}{c_2}$ crossing (x_0, t_0).
- two line with slope $\pm\frac{1}{c_1}$ crossing (x_0, t_0).

For any $s < t$ fixed, consider

- $\Sigma_s = \{(x, s) \in \mathbb{R}^{n+1} \mid x \text{ lies within lines with slope } \pm\frac{1}{c_2}\} = \{(x, s) \in \mathbb{R}^{n+1} \mid |x - x_0| \leq c_2(t_0 - s)\}$
- and $\Sigma_t = \{(x, t) \in \mathbb{R}^{n+1} \mid x \text{ lies within lines with slope } \pm\frac{1}{c_2}\} = \{(x, t) \in \mathbb{R}^{n+1} \mid |x - x_0| \leq c_2(t_0 - t)\}$.
- and $\Sigma_0 = \{(x, 0) \in \mathbb{R}^{n+1} \mid |x - x_0| \leq c_2 t_0\}$

Hence for

$$Q(t, s) = \{|x - x_0| \leq c_2(t_0 - \eta) \mid s \leq \eta \leq t\}$$

whose boundary we write

$$\partial Q(t, s) = \Sigma_t \cup F_{s,t} \cup \Sigma_s$$

where

$$F_{s,t} = \{|x - x_0| = c_2(t_0 - \eta) \mid s \leq \eta \leq t\}$$

We use Gauss Theorem

$$\begin{aligned}
(\nabla, \partial_t) \cdot (\mathcal{J}, \mathcal{E}) = 0 \\
0 &= \iint (\nabla, \partial_t) \cdot (\mathcal{J}, \mathcal{E})\, dxdt \\
&= \iint_{\partial Q(t,s)} (\mathcal{J}, \mathcal{E}) \cdot (n_x, n_t)\, dS \\
&= \int_{\Sigma_t} (\mathcal{J}, \mathcal{E}) \cdot (0, 1)\, dS + \int_{\Sigma_s} (\mathcal{J}, \mathcal{E}) \cdot (0, -1)\, dS \\
&\quad + \iint_{F_{s,t}} (\mathcal{J}, \mathcal{E}) \cdot (n_x, n_t)\, dS \\
\int_{\Sigma_s} \mathcal{E}\, dx &= \int_{\Sigma_t} \mathcal{E}\, dx + \int_{F_{s,t}} (\mathcal{J}, \mathcal{E}) \cdot (n_x, n_t)\, dS
\end{aligned}$$

Notice the localized energy

$$\begin{aligned}
E(\Sigma_s) &:= \int_{|x-x_0| \leq c_2(t_0 - s)} \mathcal{E}(\partial_s u(x, s), u(x, s))\, dx \\
&= \int_{|x-x_0| \leq c_2(t_0 - t)} \mathcal{E}(\partial_t u(x, t), u(x, t))\, dx + \iint_{F_{s,t}} (\mathcal{J}, \mathcal{E}) \cdot (n_x, n_t)\, dS \\
&= E(\Sigma_t) + \iint_{F_{s,t}} (\mathcal{J}, \mathcal{E}) \cdot (n_x, n_t)\, dS
\end{aligned}$$

we claim that

$$\iint_{F_{s,t}} (\mathcal{J}, \mathcal{E}) \cdot (n_x, n_t)\, dS \geq 0 \tag{53}$$

If so we have our energy inequality

$$E(\Sigma_s) \geq E(\Sigma_t) \geq 0 \tag{54}$$

Proof for (53). On $F_{s,t}$ we have plane

$$c_2(t-\tilde{t}) - |x-x_0| = 0 \qquad \forall\ s \le \tilde{t} \le t$$

Thus normal vectors to the plane write

$$(n_x, n_{\tilde{t}}) = \frac{\left(\frac{x-x_0}{|x-x_0|}, c_2\right)}{\sqrt{1+c_2^2}}(\pm 1)$$

We choose our normal vectors as

$$(n_x, n_{\tilde{t}}) = \frac{\left(\frac{x-x_0}{|x-x_0|}, c_2\right)}{\sqrt{1+c_2^2}}$$

Therefore

$$\iint_{F_{s,t}} (\mathcal{J}, \mathcal{E}) \cdot (n_x, n_{\tilde{t}})\, dS = \iint_{F_{s,t}} \frac{\mathcal{J} \cdot \frac{x-x_0}{|x-x_0|}}{\sqrt{1+c_2^2}} + c_2 \mathcal{E}\, dS_{x,\tilde{t}}$$

We would like to show

$$\mathcal{J} \cdot \frac{x-x_0}{|x-x_0|} + c_2 \mathcal{E} \ge 0$$

Notice

$$\begin{aligned}
2\mathcal{J} \cdot \frac{x-x_0}{|x-x_0|} + 2c_2\mathcal{E} &= -2c^2(x) u_t \nabla u \cdot \frac{x-x_0}{|x-x_0|} + c_2\left(u_t^2 + c^2(x)|\nabla u|^2 + qu^2\right) \\
&\ge -2c^2(x)|u_t||\nabla u| + c_2\left(u_t^2 + c^2(x)|\nabla u|^2 + qu^2\right) \\
&\ge -2c(x)c_2|u_t||\nabla u| + c_2\left(u_t^2 + c^2(x)|\nabla u|^2 + qu^2\right) \\
&= c_2\left(-2|u_t|c(x)|\nabla u| + u_t^2 + c^2(x)|\nabla u|^2 + qu^2\right) \\
&\ge c_2\left(-u_t^2 - c^2(x)|\nabla u|^2 + u_t^2 + c^2(x)|\nabla u|^2 + qu^2\right) \\
&\ge c_2 qu^2 \ge 0
\end{aligned}$$

Hence we're done. □

Remark 5.3 (Consequences of the Energy Inequality)**.** *Fix some* $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$*. Let* $u(x,t)$*,* $u_t(x,t)$ *be* 0 *on* $\Sigma_0 = \{(x,0) \in \mathbb{R}^{n+1} \mid |x_0 - x| \le c_2 t_0\}$*. Then from* (54)

$$0 \le E(\Sigma_s) \le E(\Sigma_0) = 0 \qquad \forall\ 0 \le s \le t_0$$

Hence $E(\Sigma_s) = 0$ *for all* $0 < s < t_0$*. Recall definition for localized energy*

$$E(\Sigma_t) = \frac{1}{2}\int_{\Sigma_t} u_t^2 + |\nabla u|^2 c^2(x) + qu^2\, dx$$

This implies that $u_t = 0$*,* $\nabla u = 0$ *and* $u = 0$ *on all* Σ_s *for* $0 \le s \le t_0$*. Hence* $u(x_0, t_0) = 0$*.*

6 Schrödinger Equation

Let $\psi(x,t)$ where x denotes position and t as time. ψ is called wave function.

$$\psi : \mathbb{R}^n_x \times \mathbb{R}^1_t \to \mathbb{C}$$

for $n = 1, 2, 3$. Let

$$|\psi(x,t)|^2 := \overline{\psi(x,t)}\psi(x,t)$$

Then $|\psi(x,t)|^2\, dx$ is some probability measure. Let $\Omega \subset \mathbb{R}^n$ we have physical meaning

$$\int_\Omega |\psi(x,t)|^2\, dx = \mathbb{P}(\text{ quantum particle in } \Omega \text{ at time } t)$$

We prescribe $\psi(x, t=0) = \psi_0(x)$ and $\int_{\mathbb{R}^n} |\psi_0(x)|^2\, dx = 1$. The data $\psi_0(x)$ evolves as t increases according to Schrödinger Equation

$$ih\partial_t\psi = (-\frac{h^2}{2m}\Delta + V(x))\psi(x)$$

where h is Planck's constant over 2π. m is mass of the particle. $V(x)$ the potential. For example Hydrogen has

$$V(x) = \frac{-e^2}{|x|}$$

Coulomb potential. Now rewrite

$$i\partial_t\psi = (-\Delta + V(x))\psi$$

We further study the case of free Schrödinger Equation with $V(x) = 0$.

6.1 Free Schrödinger Equation

We study the Free Schrödinger Equation governing a free particular.

$$\begin{cases} i\partial_t\psi = -\Delta\psi \\ \psi(x,0) = \psi_0(x) \in \mathcal{S}(\mathbb{R}^n) \end{cases} \tag{55}$$

Solve using Fourier.

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{-ix\cdot\xi} f(x)\, dx$$
$$\check{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{ix\cdot\xi} f(x)\, dx = \hat{f}(-\xi)$$

So for $|\xi|^2 = \xi \cdot \xi$ we have

$$\begin{cases} i\partial_t\hat{\psi} = |\xi|^2\hat{\psi} \\ \hat{\psi}(\xi,0) = \hat{\psi_0}(\xi) \in \mathcal{S}(\mathbb{R}^n) \end{cases}$$

so

$$\begin{aligned} \hat{\psi}(\xi,t) &= e^{-i|\xi|^2 t}\hat{\psi}_0(\xi) \\ \psi(x,t) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{ix\cdot\xi}\hat{\psi}(\xi,t)\, d\xi \\ \psi(x,t) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{ix\cdot\xi} e^{-i|\xi|^2 t}\hat{\psi}_0(\xi)\, d\xi \\ &= \int_{\mathbb{R}^n} S(x-y,t)\psi_0(y)\, dy \qquad \forall\ x \in \mathbb{R}^n \quad t \in \mathbb{R} \end{aligned}$$

where

$$S(z,t) = \frac{e^{\frac{i|z|^2}{4t}}}{(4\pi i t)^{\frac{n}{2}}}$$

This is Schrödinger Kernel. □

To rewrite, we claim

$$\psi(x,t) = \mathcal{F}^{-1}\left(e^{-i|z|^2 t}\hat{\psi}_0(z)\right) = \left(\mathcal{F}^{-1}(e^{-i|\xi|^2 t}) * \psi_0\right)(x,t)$$

Recall that if $f \in L^1_{loc}(\mathbb{R}^n)$ then we define

$$T[\phi] := \int f\phi\, dx$$

and its Fourier Transform is

$$\hat{T}[\phi] \equiv T[\hat{\phi}] = \int f\hat{\phi}\, dx \qquad \forall\ \phi \in \mathcal{S}(\mathbb{R}^n)$$

Lemma 6.1. *For*

$$g_{a,b}(x) = e^{(-a+ib)|x|^2}$$

with a, $b \in \mathbb{R}$, we have

$$\hat{g}_{a,b}(z) = \frac{1}{(a-ib)^{\frac{n}{2}}}\frac{1}{2^{\frac{n}{2}}}e^{-\frac{|z|^2}{4(a-ib)}}$$

provided either

- $a > 0$, $b \in \mathbb{R}$
- $a = 0$, $b \in \mathbb{R}$ *and* $b \neq 0$

If so, for $a = 0$ and $b = -t$ we have

$$\begin{aligned}
g_{0,-t}(x) &= e^{-it|x|^2}\\
\hat{g}_{0,-t}(\xi) &= \frac{1}{(i2t)^{\frac{n}{2}}}e^{-\frac{|\xi|^2}{4it}}\\
\mathcal{F}^{-1}\left(e^{-i|\xi|^2 t}\right)(y,t) &= \frac{1}{(4\pi i t)^{\frac{n}{2}}}e^{\frac{-i|y|^2}{4it}}\\
\psi(x,t) &= \frac{1}{(4\pi i t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{-|x-y|^2}{4it}}\psi_0(y)\, dy = [S(t)\psi_0](x,t)
\end{aligned}$$

Lemma 6.2. *If $t > 0$ and $x \in \mathbb{R}^n$ with $\psi_0 \in \mathcal{S}(\mathbb{R}^n)$, the above expression for $\psi(x,t)$ satisfies*

$$(i\partial_t + \Delta)\psi = 0$$

This leave the question in which sense does this object attain the initial condition.

Lemma 6.3.

$$\|\psi(\cdot,t) - \psi_0(\cdot)\|^2_{L^2} \to 0$$

Proof.

$$\begin{aligned}
\|\psi(\cdot,t) - \psi_0(\cdot)\|^2_{L^2} &= \int_{\mathbb{R}} |\psi(x,t) - \psi_0(x)|^2\, dx\\
&= \int_{\mathbb{R}} |\hat{\psi}(\xi,t) - \hat{\psi}_0(\xi)|^2\, d\xi\\
&= \int_{\mathbb{R}} |e^{-i|\xi|^2 t} - 1|^2 |\hat{\psi}_0(\xi)|^2\, d\xi \to 0 \qquad \textit{as } t \to 0
\end{aligned}$$

Using DCT. ∎

There's huge difference between Schrödinger $i\psi_t = -\Delta\psi$ and Heat Equation $u_t = \Delta u$.

- heat $u : \mathbb{R}^n_x \times \mathbb{R}^+_t \to \mathbb{R}$ and Schrödinger $\psi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$.

Example 6.1. *Consider $i\psi_t = -\psi_{xx}$. Write $\psi = U + iV$ for U, V $\mathbb{R}$-valued. Then*

$$\begin{aligned}
i(U_t + iV_t) &= -(U_{xx} + iV_{xx})\\
U_t &= -V_{xx}\\
V_t &= U_{xx}
\end{aligned}$$

Then

$$\begin{aligned}
(U_t)_t &= -(V_{xx})_t\\
&= -(V_t)_{xx}\\
U_{tt} &= -U_{xxxx}\\
V_{tt} &= -V_{xxxx}
\end{aligned}$$

This is in fact wave equation. But it supports infinite propagation speed.

6.2 Properties of Solutions

Theorem 6.1. *The Schrödinger evolution is unitary in L^2.*

$$\|\psi(\cdot,t)\|_{L^2}^2 = \|\psi_0(\cdot)\|_{L^2}^2 \qquad \forall\ t \in \mathbb{R}$$

Proof.

$$\begin{aligned}
\|\psi(\cdot,t)\|_{L^2}^2 &= \int |\psi(x,t)|^2\,dx = \int |\hat{\psi}(\xi,t)|^2\,d\xi \\
&= \int e^{-i|\xi|^2 t}|\hat{\psi}_0(\xi)|^2\,d\xi = \int |\hat{\psi}_0(\xi)|^2\,d\xi = \|\psi_0\|_{L^2}^2
\end{aligned}$$

□

Proof. Alternatively we can do in pure physical space.

$$\begin{aligned}
i\psi_t &= (-\Delta + V(x))\psi \\
i\psi_t\overline{\psi} &= (-\Delta + V)\psi\overline{\psi} \\
-i\overline{\psi}_t\psi &= (-\Delta + V)\overline{\psi}\psi \\
i(\psi_t\overline{\psi} + \overline{\psi}_t\psi) &= -\Delta\psi\overline{\psi} + \Delta\overline{\psi}\psi \\
&= \nabla\cdot(\psi\nabla\overline{\psi} - \overline{\psi}\nabla\psi) \\
\partial_t|\psi|^2 + \nabla\cdot(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi}) &= 0 \\
\frac{d}{dt}\int_{\mathbb{R}^n}|\psi|^2 &= 0
\end{aligned}$$

□

On the other hand, look at

Theorem 6.2 (Dispersive Decay)**.**

$$|\psi(x,t)| = |(S_t * \psi)(x,t)| \leq \frac{1}{(4\pi|t|)^{\frac{n}{2}}}\|\psi_0\|_{L^1(\mathbb{R}^n)}$$

Dispersion is waves of different wave length travel at different speed. It is why the wave packets pull apart.

The solution is decaying but the area underneath stays the same. This is dispersive decay.

Definition 6.1. *Let*

- $\int_A |\psi(x,t)|^2\,dx$ *resembles the probability that the particle position is in the set A at time t.*
- $\int_M |\hat{\psi}(\xi,t)|^2\,d\xi$ *resemble the probability that the particle momentum is in the set M at time t.*
- *Mean position of a particle at time t*

$$\langle X\rangle(t) \equiv \int_{\mathbb{R}^n} x|\psi(x,t)|^2\,dx$$

- *Mean momentum*

$$\langle \Xi\rangle(t) \equiv \int_{\mathbb{R}^n} \xi|\hat{\psi}(\xi,t)|^2\,d\xi$$

- *Variance in position*

$$\langle |X|^2\rangle(t) \equiv \int_{\mathbb{R}^n} |x|^2|\psi(x,t)|^2\,dx$$

  This is uncertainty in position.

- *Variance in momentum*

$$\langle |\Xi|^2\rangle(t) \equiv \int_{\mathbb{R}^n} |\xi|^2|\hat{\psi}(\xi,t)|^2\,d\xi$$

  This is uncertainty in momemtum. (But usually we substract the mean).

Theorem 6.3 (Uncertainty Relation)**.** *Suppose $\int_{\mathbb{R}^n}|\psi_0|^2dx = 1$ and ψ satisfies the free Schrödinger Equation (55) with data ψ_0. Then*

$$\langle |X|^2\rangle(t)\langle|\Xi|^2\rangle(t) \geq (\frac{n}{2})^2$$

Lemma 6.4 (Weighted Sobolev)**.** *Let f be such that $xf(x) \in L^2(\mathbb{R}^n)$, i.e., has it finite variance. Let $\nabla f \in L^2(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} |f|^2 \, dx \leq \frac{2}{n} \left(\int |xf|^2 \, dx \right)^{\frac{1}{2}} \left(\int |\nabla f|^2 \, dx \right)^{\frac{1}{2}}$$

Proof.

$$x \cdot \nabla |f|^2 = x \cdot \nabla (f\overline{f}) = x\overline{f} \cdot \nabla f + xf \cdot \nabla \overline{f}$$
$$\int x \cdot \nabla |f|^2 = \int x\overline{f} \cdot \nabla f + \int xf \cdot \nabla \overline{f}$$

But IBP on LHS we have

$$-n \int |f|^2 = \int x \cdot \nabla |f|^2$$

Using Hölder we have

$$-n \int |f|^2 \leq 2 (\int |x|^2 |f|^2)^{\frac{1}{2}} (\int |\nabla f|^2)^{\frac{1}{2}}$$

□

Proof of (6.3)*.* Using Plancherel we have $\widehat{\nabla f} = i\xi \hat{f}(\xi)$

$$\begin{aligned} 1 &= \int |\psi_0|^2 = \int |\psi(x,t)|^2 \, dx \\ &\leq \frac{2}{n} \left(\int |x\psi|^2 \, dx \right)^{\frac{1}{2}} \left(\int |\nabla \psi|^2 \, dx \right)^{\frac{1}{2}} \\ &= \frac{2}{n} \langle |X|^2 \rangle^{\frac{1}{2}}(t) \langle |\Xi|^2 \rangle^{\frac{1}{2}}(t) \end{aligned}$$

□

Now think of

$$i\partial_t \psi = -\Delta \psi$$

with

$$\psi(x, 0) = f_L(x) = e^{-\frac{|x|^2}{2L^2}}$$

We have solution

$$\psi(x,t) = \frac{1}{(2n)^{\frac{n}{2}}} \int e^{-ix\cdot\xi} e^{-i|\xi|^2 t} \hat{f}_L(\xi) \, d\xi$$

But the RHS is complex Gaussian. Then take inverse transform

$$\psi(x,t) = \frac{1}{(1 + \frac{2it}{L^2})^{\frac{n}{2}}} e^{-\frac{|x|^2}{2L^2(1+\frac{2it}{L^2})}}$$

Introduce

$$\psi(x,0) = f_{L,\xi_0}(x) = e^{i\xi_0 x} e^{-\frac{|x|^2}{2L^2}}$$

This is Wave Packet, with wave length $\frac{2\pi}{|\xi_0|}$. This is giving it a push (kick) with ξ_0. We want to study how it evolves.

Lemma 6.5. *Let $\psi(x,t)$ be any solution of*

$$i\partial_t \psi = -\Delta \psi$$

For any $\xi_0 \in \mathbb{R}^n$ define

$$\tilde{\psi}(x,t) := \mathcal{G}_{\xi_0}(\psi)(x,t) = \psi(x - 2\xi_0 t, t) e^{i\xi_0(x - \xi_0 t)}$$

Then

$$i\partial_t \tilde{\psi} = -\Delta \tilde{\psi}$$

with initial data

$$\tilde{\psi}(x,0) = e^{i\xi_0 \cdot x} \psi(x,0)$$

Lemma 6.6.

$$\psi(x,t) = \frac{1}{(1 + \frac{2it}{L^2})^{\frac{n}{2}}} e^{-\frac{|x-2\xi_0 t|^2}{2L^2(1+\frac{2it}{L^2})}} e^{i\xi_0(x-\xi_0 t)}$$

Very very concentrated things will spread out much faster. Things propagate outwards, decay and then spread.