Existence of Stationary Measure for 2D Incompressible Navier-Stokes with Random Kick Forces and White-in-time Noise

Kunyi (Mark) Ma (km5239) - *Honors Mathematics*

Supervision

– Vahagn Nersesyan (vn2111@nyu.edu)

NYU SHANGHAI
上海纽约大学

Introduction

The report serves as pedagogical summary and reflection of the author's combined interest in Navier-Stokes PDE Analysis and Probability, covering 3 readings [1], [2], [3], with a main focus on [3] Chapter 1 and 2. The goal is to show the existence of stationary measure for 2D Stochastic Navier-Stokes with random kick forcing and with regular white-in-time noise via Bogolyubov-Krylov argument. Contents to be covered includes

1. Stochastics.
    - Polish Space. Ulam's Theorem. Prokhorov's Theorem. Weak convergence of measures.
    - Markov Process. Markov Property. Kolmogorov-Chapman. Markov Semi-groups.
    - Bogolyubov-Krylov argument.
2. Navier-Stokes Equation.
    - Deterministic case: Leray-Projection. Existence and Uniqueness of Solutions (A priori estimate, energy balance). Navier-Stokes Process, the dissipativity property and continuity property.
    - Random Kick Force: Definiton of solution as Markov Chain. Second Moment Estimate.
    - White Noise: Definiton of Solution as Markov Process (using Itô Formula and Doob's Moment Inequality). Energy Balance and Second Moment Estimate. Second Exponential Moment Estimate.
3. Proof of Existence of Stationary Measure for Navier-Stokes Equations with Random Kick Force and with White Noise.

1 Stochastics

In this section, we define the mathematical object: Markov Process $(u_t, \mathbb{P}_v)$, taking values in a Polish space X, which later shall be used as function space to solutions of deterministic Navier-Stokes equations. Then we introduce Bogolyubov-Krylov argument to construct stationary measure using the family of measures generated by Markov-Semi-groups that is 'tight'.

1.1 Polish Space

Definition 1.1. *X is Polish space if it is complete, separable metric space. Let $\mathcal{T}$ be topology on X induced by its metric, then $(X, \mathcal{B}(X))$ is measurable Polish space for Borel σ-algebra $\mathcal{B}(X) := \sigma(\mathcal{T})$*

Now we study the set of probability measures $\mathcal{P}(X)$ defined on $(X, \mathcal{B}(X))$. There is beautiful interplay between topology and measure theory.

Theorem 1.1 (Ulam [4]). *$\forall\ \mu \in \mathcal{P}(X)$, $\forall\ \epsilon > 0$, $\exists\ K \subset X$ s.t. $\mu(K) \geq 1 - \epsilon$. In other words, any probability measure on $(X, \mathcal{B}(X))$ is regular.*

Proof. Since X is separable, we take $\{x_n\}_{n\geq 1}$ as countable dense subset of X, and because $\mu(X) = 1$, $\forall\ \epsilon > 0$ and $\forall\ m \in \mathbb{N}$, $\exists\ N = N(m) \in \mathbb{N}$ *s.t.*

$$\mu\left(X - \bigcup_{n=1}^{N} \overline{B}\left(x_n, \frac{1}{m}\right)\right) < \frac{\epsilon}{2^m} \implies \mu(X - K) \leq \epsilon \tag{1.1}$$

for $K = \bigcap_{m\geq 1} \bigcup_{n=1}^{N} \overline{B}\left(x_n, \frac{1}{m}\right)$. Recall for metric space X, a subset is compact if and only if it is complete and totally bounded. K satisfies completeness as closed subset of complete space X. For totally boundedness, $\forall\ \delta > 0$, $\exists\ m > \frac{1}{\delta}$ so $K \subset \bigcup_{n=1}^{N} \overline{B}(x_n, \delta)$. Hence K is compact subset depending on ϵ. □

One may further ask: given a family of measures $\{\mu_\alpha\} \subset \mathcal{P}(X)$, will there be one compact K that holds uniformly for all μ_α? In general, the answer is false. Yet as long as such compact K exists, known as 'tightness', we have relative compactness of $\{\mu_\alpha\}$ in the weak topology of $\mathcal{P}(X)$.

Definition 1.2. *Let $C_b(X) := \{f : X \to \mathbb{R} \mid f$ continous and bounded$\}$ with norm $\|f\|_\infty := \sup_{u\in X} |f(u)|$. Then $\{\mu_k\} \subset \mathcal{P}(X)$ converges weakly to $\mu \in \mathcal{P}(X)$ if $\forall\ f \in C_b(X)$, we have $\int f d\mu_k \to \int f d\mu$ as $k \to \infty$.*

Lemma 1.1 (Portmanteau theorem [4]). *$\mu_k \to \mu$ weakly in $\mathcal{P}(X)$ if and only if either one is satisfied:*

$$\liminf_{k\to\infty} \mu_k(G) \geq \mu(G) \qquad \forall\ G \text{ open } \subset X \tag{1.2}$$

$$\limsup_{k\to\infty} \mu_k(F) \leq \mu(F) \qquad \forall\ F \text{ closed } \subset X \tag{1.3}$$

Definition 1.3 (tightness). *Family $\{\mu_\alpha\} \subset \mathcal{P}(X)$ is tight if $\forall\ \epsilon > 0$, $\exists\ K \subset X$ s.t. $\mu_\alpha(K) \geq 1-\epsilon\ \forall\ \alpha$.*

Theorem 1.2 (Prokhorov [5]). *A family $\{\mu_\alpha\} \subset \mathcal{P}(X)$ is tight if and only if $\{\mu_\alpha\}$ is relatively compact in the weak topology, i.e., $\forall$ sequence $\{\mu_k\} \subset \{\mu_\alpha\}$, $\exists$ subsequence $\{\mu_{k_i}\}$ that converges weakly in $\overline{\{\mu_\alpha\}}$.*

Sketch of Proof. Relative compactness $\implies$ tightness. Take $\{x_n\}$ countable dense subset of X, then $\forall\ m \in N$, $\{B\left(x_n, \frac{1}{m}\right)\}_n$ forms open cover of X. We claim that $\exists\ N = N(m) \in \mathbb{N}$ *s.t.*

$$\mu_\alpha\left(X - \bigcup_{n=1}^{N} B\left(x_n, \frac{1}{m}\right)\right) < \frac{\epsilon}{2^m} \quad \forall\ \alpha$$

Assume false, then $\forall\ k \in \mathbb{N}$, we pick $\mu_k \in \{\mu_\alpha\}$ *s.t.* $\mu_k\left(\bigcup_{n=1}^{k} B\left(x_n, \frac{1}{m}\right)\right) \leq 1 - \frac{\epsilon}{2^m}$ by diagonalization argument. Now $\{\mu_k\}$ has weakly convergent subsequence $\{\mu_{k_j}\}$ with limit $\mu \in \overline{\{\mu_\alpha\}} \subset \mathcal{P}(X)$. Note $\forall\ N$, $\bigcup_{n=1}^{N} B\left(x_n, \frac{1}{m}\right)$ is open, so by Portmanteau theorem (1.2) we have

$$\begin{aligned}\mu(\bigcup_{n=1}^{N} B\left(x_n, \frac{1}{m}\right)) &\leq \liminf_{j\to\infty} \mu_{k_j}(\bigcup_{n=1}^{N} B\left(x_n, \frac{1}{m}\right)) \\ &\leq \liminf_{j\to\infty} \mu_{k_j}(\bigcup_{n=1}^{k_j} B\left(x_n, \frac{1}{m}\right)) \leq 1 - \frac{\epsilon}{2^m}\end{aligned}$$

Reach contradiction by taking $N \to \infty$ and $\mu(X) = 1$. The rest follows from (1.1).

Tightness $\implies$ Relative Compactness. The reverse direction is quite technical and heavy. So for simplicity, we restrict ourselves to the case that X is compact and show $\mathcal{P}(X)$ is compact as well. Notice for X compact, $C_b(X) = C(X)$. Denote $C^*(X)$ as Banach Dual space of $C(X)$, and consider a closed subspace $\Phi := \{\varphi \in C^*(X) \mid \|\varphi\| \leq 1,\ \varphi(\mathbb{I}) = 1,\ \varphi(f) \geq 0\ \forall\ f \in C(X) \text{ with } f \geq 0\}$. According to Riesz-Representation, the map $\mu \in \mathcal{P}(X) \to \varphi_\mu \in \Phi$ with $\varphi_\mu(f) = \int f d\mu$ is bijection and homeomorphism w.r.t. weak topology 1.2 on $\mathcal{P}(X)$ and weak* topology on Φ. Now using Banach-Alaoglu, the closed unit ball $\{\varphi \in C^*(X) \mid \|\varphi\| \leq 1\}$ is compact in weak* topology, hence Φ as its closed subset is also compact. Since homeomorphism preserves compactness, $\mathcal{P}(X)$ is compact. Hence any family of probability measures is automatically relative compact. Now for general case where X is not compact, we cite [5] the topology compactification fact that: For X separable metric space, there $\exists$ compact metric space Y and map $T : X \to Y$ $s.t.$ T is homeomorphism from X onto $T(X)$. To prove $\{\mu_\alpha\}$ is relatively compact, we take sequence $\{\mu_k\}$ and define push-forward measures $\{\nu_k\}$ for compactification of X

$$\nu_k(B) := \mu_k\left(T^{-1}(B)\right) \qquad \forall\ B \in \mathcal{B}(Y)$$

Since Y compact, $\mathcal{P}(Y)$ compact, so $\exists\ \nu \in \mathcal{P}(Y)$ as weak limit of subsequence $\{\nu_{k_j}\}$ in $\mathcal{P}(Y)$. Now we use tightness of $\overline{\mu_\alpha}$ to show concentration of ν on a large subset of $T(X) \subset Y$. $\forall\ m \in \mathbb{N}$, take $K_m \subset X$ compact $s.t.$ $\mu_\alpha(K_m) \geq 1 - \frac{1}{m}\ \forall\ \alpha$. Since $T(K_m)$ is compact subset of $T(X)$, hence closed, again by Portmanteau theorem (1.3) we have

$$\begin{aligned} \nu(T(K_m)) &\geq \limsup_{j\to\infty} \nu_{k_j}(T(K_m)) \\ &\geq \limsup_{j\to\infty} \mu_{k_j}(K_m) \geq 1 - \frac{1}{m} \end{aligned}$$

Define $E := \bigcup_{m\geq 1} T(K_m)$, so $E \in \mathcal{B}(Y)$ and $\nu(E) = 1$. Now we define

$$\mu(A) := \nu(T(A) \cap E) \qquad \forall\ A \in \mathcal{B}(X)$$

By requirement $\nu(E) = 1$, we have $\mu \in \mathcal{P}(X)$. Finally, we show $\{\mu_k\}$ has weak convergent subsequence $\{\mu_{k_j}\}$ to μ. Take any closed $C \subset X$, there $\exists\ F$ closed $\subset Y$ $s.t.$ $F \cap T(X) = T(C)$, $i.e.$, $C = T^{-1}(F)$. Notice hence $E \cap F = E \cap T(C)$. So

$$\begin{aligned} \limsup_{j\to\infty} \mu_{k_j}(C) &= \limsup_{j\to\infty} \nu_{k_j}(F) \\ &\leq \nu(F) = \nu(F \cap E) = \nu(T(C) \cap E) = \mu(C) \end{aligned}$$

Hence by Portmanteau theorem (1.3), μ_{k_j} converges weakly to μ. $\square$

1.2 Markov Process

A Markov Process $(u_t, \mathbb{P}_v)$ is set of X-valued random processes and family of probability measures satisfying the Markov Property. We construct from the basics [3]. Let time be $\mathcal{T}_+ = \mathbb{Z}_+$ or $\mathbb{R}_+$.

Definition 1.4 (Markov Process)**.** *Fix measurable space $(\Omega, \mathcal{F})$ and measurable Polish space $(X, \mathcal{B}(X))$.*

- *Define on $(\Omega, \mathcal{F})$ a filtration $\{\mathcal{F}_t,\, t \in \mathcal{T}_+\}$, non-decreasing family of sub σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ in $\mathcal{T}_+$.*
- *Define on $(\Omega, \mathcal{F})$ a family of probability measures $\{\mathbb{P}_v,\, v \in X\}$ as parametrized by X Polish space. We also let mapping $v \mapsto \mathbb{P}_v(A)$ be universally measurable for any $A \in \mathcal{F}$, meaning $\forall\ a \in \mathbb{R}$, $\{v \in X \mid \mathbb{P}_v(A) > a\}$ belongs to the completion of $\mathcal{B}(X)$ w.r.t. any probability measure on $(X, \mathcal{B}(X))$.*
- *Define from $(\Omega, \mathcal{F}, \mathcal{F}_t, \{\mathbb{P}_v\})$ to $(X, \mathcal{B}(X))$ an X-valued random process $\{u_t,\, t \in \mathcal{T}_+\}$ adapted to the filtration $\mathcal{F}_t$, i.e., u_t is $\mathcal{F}_t$-measurable $\forall\ t \in \mathcal{T}_+$. We require $\{u_t\}$ to satisfy the Markov Property $\forall\ v \in X$, $\Gamma \in \mathcal{B}(X)$ and $s, t\ \in \mathcal{T}_+$.*

$$\mathbb{P}_v\{u_0 = v\} = 1 \tag{1.4}$$

$$\mathbb{P}_v\{u_{t+s} \in \Gamma \mid \mathcal{F}_s\} = P_t(u_s, \Gamma) \quad \textit{for } \mathbb{P}_v - \textit{almost every } \omega \in \Omega \tag{1.5}$$

where P_t is transition function of $(u_t, \mathbb{P}_v)$ defined as law of u_t under probability measure $\mathbb{P}_v$

$$P_t(v, \Gamma) := \mathbb{P}_v\{u_t \in \Gamma\}, \quad v \in X, \quad \Gamma \in \mathcal{B}(X) \tag{1.6}$$

The collection of above objects is a Markov Process, denoted by $(u_t, \mathbb{P}_v)$. If $\mathcal{T}_+ = \mathbb{Z}_+$, it is Markov Chain.

We have Kolmogorov-Chapman implying Markov Process generates an evolution in the space of probability measures.

Lemma 1.2 (Kolmogorov-Chapman relation [3])**.** *$\forall\ t, s \in \mathcal{T}_+$, $v \in X$, and $\Gamma \in \mathcal{B}(X)$, the transition function P_t satisfies the relation*

$$P_{t+s}(v, \Gamma) = \int_X P_t(v, dz)\, P_s(z, \Gamma) \tag{1.7}$$

Proof. By Towering Property, and then Markov Property,

$$P_{t+s}(v,\Gamma) = \mathbb{P}_v(u_{t+s}\in\Gamma) = \mathbb{E}_v\{\mathbb{I}_\Gamma(u_{t+s})\} = \mathbb{E}_v\{\mathbb{E}_v\{\mathbb{I}_\Gamma(u_{t+s}) \mid \mathcal{F}_t\}\} = \mathbb{E}_v\{\mathbb{P}_v\{u_{t+s}\in\Gamma \mid \mathcal{F}_t\}\}$$
$$= \mathbb{E}_v\{\mathbb{P}_{u_t}\{u_s\in\Gamma\}\} = \mathbb{E}_v\{P_s(u_t,\Gamma)\} = \int_\Omega P_s(u_t,\Gamma)\,d\mathbb{P}_v = \int_X P_s(z,\Gamma)\; u_{t_*}(\mathbb{P}_v)(dz)$$

where $u_{t_*}(\mathbb{P}_v)$ is image of $\mathbb{P}_v$ under u_t. But notice $\forall\ \Gamma\in\mathcal{B}(X)$, $u_{t_*}(\mathbb{P}_v)(\Gamma) = \mathbb{P}_v\{u_t\in\Gamma\} = P_t(v,\Gamma)$, hence $P_{t+s}(v,\Gamma) = \int_X P_s(z,\Gamma)\,P_t(v,dz)$ □

Now we introduce two linear operators associated to a given Markov Process $(u_t,\mathbb{P}_v)$, known as the Markov semigroups. They give rise to the notion of a 'stationary measure' for the Markov Process.

Definition 1.5 (Markov Semigroups)**.** *For each Markov Process $(u_t,\mathbb{P}_v)$, there correspond two families of linear operators acting in $L^\infty(X)$, space of bounded measurable functions, and $\mathcal{P}(X)$, space of probability measures. These families of linear operators are Markov Semigroups and defined in terms of transition functions $\forall\ t\in\mathcal{T}_+$*

$$\mathfrak{B}_t : L^\infty(X)\to L^\infty(X) \qquad \mathfrak{B}_t f(v) := \int_X P_t(v,dz)\,f(z) \quad \forall\ v\in X$$

$$\mathfrak{B}_t^* : \mathcal{P}(X)\to\mathcal{P}(X) \qquad \mathfrak{B}_t^*\mu(\Gamma) := \int_X P_t(v,\Gamma)\,\mu(dv) \quad \forall\ \Gamma\in\mathcal{B}(X)$$

Property 1.1. *(i) $\mathfrak{B}_t$ indeed forms a semigroup, i.e., $\mathfrak{B}_0 = Id$ and $\mathfrak{B}_{t+s} = \mathfrak{B}_s\circ\mathfrak{B}_t$. Same for $\mathfrak{B}_t^*$.*

Proof. Note by Markov Property $\mathbb{P}_v\{u_0 = v\} = 1$, so

$$\mathfrak{B}_0 f(v) = \int_X \mathbb{P}_v\{u_0\in dz\}f(z) = f(v), \qquad \mathfrak{B}_0^*\mu(\Gamma) = \int_X \mathbb{P}_v\{u_0\in\Gamma\}\mu(dv) = \mu(\Gamma)$$

Then by Kolmogorov-Chapman relation and Fubini, we have

$$\mathfrak{B}_{t+s}f(v) = \int_X P_{t+s}(v,dz)\,f(z) = \int_X\int_X P_s(v,dy)\,P_t(y,dz)\,f(z) = \int_X\left(\int_X P_t(y,dz)\,f(z)\right)P_s(v,dy)$$
$$= \int_X P_s(v,dy)\,\mathfrak{B}_t f(y) = \mathfrak{B}_s(\mathfrak{B}_t f(v))$$
$$\mathfrak{B}_{t+s}^*\mu(\Gamma) = \iint_{X\times X} P_t(v,dz)\,P_s(z,\Gamma)\,\mu(dv) = \int_X\left(\int_X P_t(v,dz)\,\mu(dv)\right)P_s(z,\Gamma)$$
$$= \int_X \mathfrak{B}_t^*\mu(dz)\,P_s(z,\Gamma) = \mathfrak{B}_s^*\circ\mathfrak{B}_t^*\mu(\Gamma)$$

□

(ii) $\mathfrak{B}_t$ *and* $\mathfrak{B}_t^*$ *satisfy the duality relation*

$$(\mathfrak{B}_t f, \mu) = (f, \mathfrak{B}_t^* \mu) \qquad \forall\ f \in L^\infty(X),\ \mu \in \mathcal{P}(X) \tag{1.8}$$

Proof. By Fubini

$$(\mathfrak{B}_t f, \mu) = \int_X \left(\int_X P_t(v, dz) f(z) \right) d\mu(v) = \int_X f(z) \int_X P_t(v, dz)\, \mu(dv) = (f, \mathfrak{B}_t^* \mu)$$

□

(iii) For $\lambda \in \mathcal{P}(X)$, *define its associated probability measure with* $(u_t, \mathbb{P}_v)$ *on* $(\Omega, \mathcal{F})$ *as*

$$\mathbb{P}_\lambda(\Gamma) := \int_X \mathbb{P}_v(\Gamma)\lambda(dv) \qquad \Gamma \in \mathcal{F} \tag{1.9}$$

then the $\mathbb{P}_\lambda$*-law of* u_t *coincides with* $\mathcal{B}_t^*\lambda$, *i.e.* $\forall\ t \in \mathcal{T}_+$,

$$\mathbb{P}_\lambda\{u_t \in \Gamma\} \equiv u_{t_*}(\mathbb{P}_\lambda)(\Gamma) = \mathfrak{B}_t^*\lambda(\Gamma) \qquad \forall\ \Gamma \in \mathcal{B}(X) \tag{1.10}$$

Proof.

$$\mathbb{P}_\lambda\{u_t \in \Gamma\} = \int_X \mathbb{P}_v(u_t \in \Gamma)\, \lambda(dv) = \int_X P_t(v, \Gamma)\, \lambda(dv) = \mathfrak{B}_t^* \lambda(\Gamma)$$

□

(iv) For the above $\lambda \in \mathcal{P}(X)$ *and* $\mathbb{P}_\lambda$, *define* $\mathbb{E}_\lambda$ *as associated mean value with probability measure* $\mathbb{P}_\lambda$. *Then if* $f \in L^\infty(X)$, *we have* $\mathbb{E}_\lambda f(u_t) = \int_X \mathfrak{B}_t f(v) \lambda(dv)$ *for any* $t \in \mathcal{T}_+$.

Proof. By duality relation

$$\mathbb{E}_\lambda f(u_t) = \int_\Omega f(u_t) d\mathbb{P}_\lambda = \int_\Omega f(v) u_{t_*}(\mathbb{P}_\lambda)(dv) = \int_X f(v) \mathfrak{B}_t^* \lambda(dv) = \int_X \mathfrak{B}_t f(v) \lambda(dv)$$

□

Now we're able to define stationary measure making use of (1.10).

Definition 1.6 (Stationary Measure)**.** *$\mu \in \mathcal{P}(X)$ is stationary for $(u_t, \mathbb{P}_v)$ if $\mathfrak{B}_t^* \mu = \mu \ \forall \ t \in \mathcal{T}_+$. In other words, the image of $\mathbb{P}_\mu$ under u_t coincides with μ, $\forall \ t \in \mathcal{T}_+$.*

In the following, we give an example of Markov Process suitable in our setting, and make sense of the notion 'stationary solution' for an equation.

Example 1.1. *Fix first a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a measurable space $(H, \mathcal{B}(H))$ for H separable Banach space (norm induces metric, hence Polish). We define $\eta_k : \Omega \to H$ as sequence of i.i.d. H-valued random variables for $k \geq 1$ and equip $(\Omega, \mathcal{F})$ with the natural filtration $\mathcal{F}_k := \sigma(\eta_1, \cdots, \eta_k)$ $\forall \ k \geq 1$, letting $\mathcal{F}_0 = \{\varnothing, \Omega\}$. Let $S : H \to H$ be a continuous mapping. Now pick $v \in H$, and define a sequence of H-valued random variables $v_k : \Omega \to H$ for $k \geq 0$ by*

$$v_0 = v, \qquad v_k = S(v_{k-1}) + \eta_k \ \forall \ k \geq 1 \tag{1.11}$$

The collection $(v_k, \mathbb{P})$ itself is a Markov Chain.
Moreover, we can define $(\tilde{\Omega}, \tilde{\mathcal{F}})$ as product of measurable spaces $(H, \mathcal{B}(H))$ and $(\Omega, \mathcal{F})$, i.e., $\tilde{\Omega} = H \times \Omega$, and $\tilde{\mathcal{F}} = \mathcal{B}(H) \otimes \mathcal{F}$. Equip it with filtration $\{\tilde{\mathcal{F}}_k, k \geq 0\}$ where $\tilde{\mathcal{F}}_k = \mathcal{B}(H) \otimes \mathcal{F}_k$. Impose on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_k)$ a family of probability measures $\mathbb{P}_v := \delta_v \otimes \mathbb{P}$, where δ_v is dirac measure for each given $v \in H$. Given $\{v_k\}_{k \geq 0}$ from (1.11), we define a sequence of H-valued random variables $u_k : \tilde{\Omega} \to H$ for $k \geq 0$ s.t.

$$u_k^{\tilde{\omega}} := v_k^{\omega} \qquad \forall \ \tilde{\omega} = (v, \omega) \in \tilde{\Omega}$$

If denote $P_k(v, \Gamma) \equiv \mathbb{P}\{v_k \in \Gamma\} \ \ \forall \ \Gamma \in \mathcal{B}(H)$ the law of v_k for $k \geq 0$ with fixed $v \in H$, the collection $(u_k, \mathbb{P}_v)$ defines a Markov Chain with transition function P_t.
Now we add randomness to initial data $v \in H$, letting v be H-valued random variable independent of $\{\eta_k\}$ with law μ, i.e., $v_*(\mathbb{P}) \equiv \mu \in \mathcal{P}(H)$. Then from (1.10), $u_{k_*}(\delta_v \otimes \mathbb{P}_\mu) \equiv u_{k_*}(\mathbb{P}_\mu) = v_{k_*}(\mathbb{P}_\mu) = \mathfrak{B}_k^* \mu$ for $k \geq 0$. If we have μ as stationary measure for Markov Process $(u_k, \mathbb{P}_v)$, then $\mathfrak{B}_k^* \mu = \mu$ for all k, hence $u_{k_*}(\mathbb{P}_\mu) = v_*(\mathbb{P})$ for all k. In this case, μ is called stationary measure for (1.11), and the sequence $\{v_k\}_{k \geq 0}$ is stationary solution. Notice $u_{k_*}(\mathbb{P}_\mu)$ is well-defined because u_k is only defined on $\{v\} \times \Omega$, and with dirac-delta at v, by definition of product measure $\delta_v \otimes \mathbb{P}_\mu\{u_k \in \Gamma\} = 1 \cdot \mathbb{P}_\mu\{u_k \in \Gamma\} = \mathbb{P}_\mu\{v_k \in \Gamma\}$. We shall see eventually S is our solution operator, and $\{v_k\}$ defines family of solutions to our stochastic

Navier-Stokes system of interest. So the purpose is to find μ *as law of initial data which defines a stationary measure for* $(u_k, \mathbb{P}_v)$*, hence the stationary solutions* $\{v_k\}$*.*

1.3 Bogolyubov-Krylov

This is method to construct stationary distributions for Markov Processes that satisfy weak compactness condition [3]. Before proceeding, we make 2 assumptions on the Markov Processes we deal with.

$$\textbf{Feller Property} \quad \forall\ f \in C_b(X)\ \ and\ t \in \mathcal{T}_+,\ \ we\ have\ \mathfrak{B}_t f \in C_b(X) \tag{1.12}$$

$$\textbf{Time Continuity} \quad Trajectories\ u_t(\omega)\ \ with\ \omega \in \Omega\ are\ continuous\ in\ time\ if\ t \in \mathcal{T}_+ = \mathbb{R}_+ \tag{1.13}$$

Theorem 1.3 (Bogolyubov-Krylov [3])**.** *Let* $(\Omega, \mathcal{F}, \mathcal{F}_t)$ *be measurable space with filtration,* $(X, \mathcal{B}(X))$ *be measurable Polish space, and* $(u_t, \mathbb{P}_v)$ *be family of* X*-valued Markov Processes on* $(\Omega, \mathcal{F})$ *adapted to* $\mathcal{F}_t$*. Let* P_t *be the transition function and* $\mathfrak{B}_t$*,* $\mathfrak{B}_t^*$ *the associated Markov semi-groups. Let* $(u_t, \mathbb{P}_v)$ *satisfy* (1.12) *and* (1.13)*. Now fix* $\lambda \in \mathcal{P}(X)$ *and define family of measures* $\overline{\lambda}_t$ *for* $t \in \mathcal{T}_+$ *by*

$$\overline{\lambda}_t(\Gamma) := \frac{1}{t}\int_0^t (\mathfrak{B}_s^*\lambda)(\Gamma)ds = \frac{1}{t}\int_0^t \int_X P_s(u, \Gamma)\lambda(du)ds \qquad \Gamma \in \mathcal{B}(X) \tag{1.14}$$

and integral be replaced by sum over $s = 0, \ldots, t-1$ *if* $\mathcal{T}_+ = \mathbb{Z}_+$*. Then if in addition,* $\lambda \in \mathcal{P}(X)$ *is measure for which* $\{\overline{\lambda}_t,\ t \in \mathcal{T}_+\}$ *is tight, our Markov Process* $(u_t, \mathbb{P}_v)$ *has at least one stationary measure.*

Proof. We use Prokhorov Theorem (1.2). There $\exists\ \{t_n\} \subset \mathcal{T}_+$ increasing to ∞ so that $\{\overline{\lambda}_t\}$ converges weakly to some $\mu \in \mathcal{P}(X)$, *i.e.*,

$$(f, \overline{\lambda}_{t_n}) \to (f, \mu) \quad as\ n \to \infty \quad \forall\ f \in C_b(X) \tag{1.15}$$

We hope to show that μ is stationary measure for $(u_t, \mathbb{P}_v)$. Indeed $\forall\ f \in C_b(X)$ and $r \in \mathcal{T}_+$

$$
\begin{aligned}
(f, \mathfrak{B}_r^*\mu) &\overset{(1.8)}{=} (\mathfrak{B}_r f, \mu) \overset{(1.15)\ and\ Feller\,(1.12)}{=} \lim_{n\to\infty} (\mathfrak{B}_r f, \overline{\lambda}_{t_n}) \\
&= \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} (\mathfrak{B}_r f, \mathfrak{B}_s^*\lambda) ds \overset{semigroup\,(1.1)}{=} \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} (f, \mathfrak{B}_{s+r}^*\lambda) ds \\
&= \lim_{n\to\infty} \frac{1}{t_n} \int_r^{r+t_n} (f, \mathfrak{B}_s^*\lambda) ds = \lim_{n\to\infty} \frac{1}{t_n} \left(\int_0^{r+t_n} (f, \mathfrak{B}_s^*\lambda) ds - \int_0^r (f, \mathfrak{B}_s^*\lambda) ds \right) \\
&= \lim_{n\to\infty} \frac{1}{t_n} \left(\int_0^{t_n} (f, \mathfrak{B}_s^*\lambda) ds + \int_{t_n}^{r+t_n} (f, \mathfrak{B}_s^*\lambda) ds - \int_0^r (f, \mathfrak{B}_s^*\lambda) ds \right) \\
&= \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} (f, \mathfrak{B}_s^*\lambda) ds \overset{(1.15)}{=} (f, \mu)
\end{aligned}
$$

The above holds for any $f \in C_b(X)$ and $r \in \mathcal{T}_+$, hence μ is stationary measure for $(u_t, \mathbb{P}_v)$. □

Corollary 1.1. *Let $(u_t, \mathbb{P}_v)$ satisfy same conditions as Theorem 1.3. Suppose $\exists$ initial data $v \in X$, an increasing sequence of compact subsets $K_m \subset X$ and finite times $t_m \in \mathcal{T}_+$ such that*

$$
\sup_{t \geq t_m} \mathbb{P}_v\{u_t \notin K_m\} \to 0 \qquad as\ m \to \infty \tag{1.16}
$$

Then $\exists\ \lambda \in \mathcal{P}(X)$ s.t. $\overline{\lambda}_t$ as defined by (1.14) *is tight, so conditions for Theorem 1.3 is satisfied.*

Proof. Define $\lambda := \delta_v$, so that $\overline{\lambda}_t = \frac{1}{t}\int_0^t \int_X P_s(u, \cdot)\delta_v(du) ds = \frac{1}{t}\int_0^t P_s(v, \cdot) ds$. Now by (1.16), $\forall\ \epsilon > 0$, there exists $m \geq 1$ *s.t.*

$$
\sup_{t \geq t_m} \mathbb{P}_v\{u_t \notin K_m\} = 1 - \inf_{t \geq t_m} P_t(v, K_m) \leq \frac{\epsilon}{2} \implies P_t(v, K_m) \geq 1 - \frac{\epsilon}{2} \quad \forall\ t \geq t_m \tag{1.17}
$$

To deal with $\int_0^{t_m} P_s(v, \cdot) ds$, one observe that $[0, t_m]$ is compact in $\mathbb{R}_+$ w.r.t. standard topology, hence its image under the continuous mapping $s \mapsto P_s(v, \cdot)$, as guaranteed by (1.13), remains compact in its codomain, which is $\mathcal{P}(X)$ equipped with the weak topology. Apply Prokhorov Theorem 1.2 so that $\{P_s(v, \cdot)\}_{0 \leq s \leq t_m}$ is tight, hence there exists $K_0 \subset X$ such that

$$
P_s(v, K_0) \geq 1 - \frac{\epsilon}{2} \qquad \forall\ 0 \leq s \leq t_m \tag{1.18}
$$

Define $K = K_0 \cup K_m$ and using (1.17), (1.18), we see for fixed $\epsilon > 0$, $\overline{\lambda}_t(K) \geq 1 - \epsilon\ \forall\ t \in \mathcal{T}_+$, tightness.

□

2 Navier-Stokes Equation

In this section, we introduce the 2D Navier-Stokes Equation on $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ under Leray Decomposition, using the weak solution formulation. Then we introduce the language of Navier-Stoke Process and its dissipativity property that adapts to the Stochastic settings. With these preparations, we consider 2 cases of Stochastic Navier-Stokes: perturbation with random kick force and with white noise. For each case, we introduce the definitions of solutions under the Markov Process perspective, then prove certain moment estimates using Itô's Fomula and Doob's Moment Inequality.

2.1 Deterministic Case

Consider Incompressible $2D$-Navier Stokes Equations

$$\dot{u} + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = f\,(t, x) \tag{2.1}$$

$$\operatorname{div} u = 0 \tag{2.2}$$

- $u = (u^1, u^2)$ unknown velocity field, p pressure, $\nu > 0$ kinematic viscosity, f density of external force, $\langle u, \nabla \rangle = u^1\partial_1 + u^2\partial_2$ differential operator and $\operatorname{div} u = \partial_1 u^1 + \partial_2 u^2$ the divergence of velocity.
- We consider problem on $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, meaning $x = (x_1, x_2) \in \mathbb{R}^2$ and functions u, f, p are 2π-periodic *w.r.t.* x_i, $i = 1, 2$.

2.1.1 Leray Projection and Solution

Due to low regularity phenomenon observed in turbulence, we consider weak solutions to (2.1) (2.2) defined in distribution. To do so, we need Sobolev Space and make sense of the equation (2.1).

Definition 2.1 (H^m and $\dot{H}^m$)**.** *Heuristically, let H^m be Sobolev Space whose 'm'th order weak derivative lies in L^2. We may generalize to $m \in \mathbb{R}$ if notice the Fourier Series expansion for L^2 functions on $\mathbb{T}^2$*

$$u(x) = \sum_{s \in \mathbb{Z}^2} \hat{u}(s) e^{i\langle s, x \rangle} \;\in\; L^2\left(\mathbb{T}^2; \mathbb{R}^2\right)$$

$H^m \equiv H^m(\mathbb{T}^2;\mathbb{R}^2)$ is hence defined as closure of $C^\infty(\mathbb{T}^2;\mathbb{R}^2)$ w.r.t. the norm

$$\|u\|_{H^m} \equiv \|u\|_{H^m(\mathbb{T}^2;\mathbb{R}^2)} := \left(\sum_{s\in\mathbb{Z}^2} \left(1+|s|^2\right)^m |\hat{u}(s)|^2\right)^{\frac{1}{2}} \tag{2.3}$$

We also introduce $\dot{H}^m$ as subspace of H^m with zero-mean functions

$$\dot{H}^m \equiv \dot{H}^m\left(\mathbb{T}^2;\mathbb{R}^2\right) := \{u \in H^m\left(\mathbb{T}^2;\mathbb{R}^2\right) \mid \langle u\rangle := \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} u(x)dx = 0\}$$

with an equivalent norm as (2.3)

$$\|u\|_{\dot{H}^m} := \left(\sum_{s\neq 0} |s|^{2m}|\hat{u}(s)|^2\right)^{\frac{1}{2}}$$

We list 2 important ingredients for Sobolev space H^m with $m \in \mathbb{R}$ on $\mathbb{T}^2$: duality and Interpolation.

Proposition 2.1 (Duality Pairing)**.** *H^m and H^{-m} are dual w.r.t. L^2-scalar product $\langle\cdot,\cdot\rangle$*

$$\|u\|_{H^m} = \sup_{\substack{v\in C^\infty\left(\mathbb{T}^2;\mathbb{R}^2\right)\\ \|v\|_{H^{-m}}\le 1}} |\langle u,v\rangle| \quad \forall\ u \in C^\infty\left(\mathbb{T}^d;\mathbb{R}^n\right) \tag{2.4}$$

Proposition 2.2 (H^m Interpolation)**.** *$\forall\ a < b \in \mathbb{R}$, $0 \le \theta \le 1$ constant, then $\forall\ u \in H^b$*

$$\|u\|_{H^{(\theta a+(1-\theta)b)}} \le \|u\|_{H^a}^{\theta}\,\|u\|_{H^b}^{1-\theta} \tag{2.5}$$

Instead of working with NS system (2.1), (2.2), we think about formulating a new equation, where the pressure term is killed. In particular, we consider a function space where pressure term lies in

Definition 2.2 (∇H^{m+1})**.**

$$\nabla H^{m+1} := \left\{u \in H^m\left(\mathbb{T}^2;\mathbb{R}^2\right) \mid u = \nabla p \ for\ some\ p\ \in H^{m+1}\right\}$$

Note if $u = \nabla p \in \nabla H^{m+1}$ with $p \in H^{m+1}$, we have $\|u\|_{H^m} = \|\nabla p\|_{H^m} = \|p\|_{\dot{H}^{m+1}}$. But $\dot{H}^{m+1}$ is closed subspace of H^{m+1}, so ∇H^{m+1} is closed subspace of H^m as well.

We design a projection Π from H^m to the complement of ∇H^{m+1}, and then apply Π to our equation

(2.1). This is valid due to Leray's Decomposition of H^m into sum of H^m_σ and ∇H^{m+1}, see [1].

Definition 2.3 (H^m_σ)**.**

$$H^m_\sigma := \left\{u \in H^m\left(\mathbb{T}^2;\mathbb{R}^2\right) \mid \operatorname{div}\, u = 0 \textit{ on } \mathbb{T}^2\right\}$$

with div *in sense of distribution. We see H^m_σ is closed subspace of H^m due to continuity of* div *. Moreover, H^m_σ coincides with closure of $\{u \in C^\infty(\mathbb{T}^2;\mathbb{R}^2) \mid \operatorname{div} u = 0 \text{ on } \mathbb{T}^2\}$ w.r.t. norm $\|u\|_{H^m}$ in (2.3).*

Theorem 2.1 (Leray Decomposition [3])**.** *$\forall\ m \in \mathbb{R}$, H^m admits direct-sum decomposition*

$$H^m\left(\mathbb{T}^2;\mathbb{R}^2\right) = H^m_\sigma \oplus \nabla H^{m+1} \tag{2.6}$$

Proof. First note $0 \in H^m_\sigma \cap \nabla H^{m+1}$. For any function $u \in H^m = H^m(\mathbb{T}^2;\mathbb{R}^2)$, we first expand

$$u\left(x\right) = \sum_{s\in\mathbb{Z}^2} \hat{u}(s)e^{i\langle s,x\rangle} \quad \textit{for } \hat{u} = \left(\hat{u}^1, \hat{u}^2\right) \in \mathbb{C}^2$$

and observe for $s = (s_1, s_2)$ and $s^\perp = (-s_2, s_1)$ we have

$$\frac{1}{|s|^2}\left(\langle\hat{u}(s), s^\perp\rangle s^\perp + \langle\hat{u}(s), s\rangle s\right) = \frac{1}{|s|^2}\left(s_2^2\hat{u}^1 + s_1^2\hat{u}^1,\, s_1^2\hat{u}^2 + s_2^2\hat{u}^2\right) = \hat{u}(s)$$

Hence

$$u\left(x\right) = \sum_{s\in\mathbb{Z}^2}\frac{1}{|s|^2}\langle\hat{u}(s), s^\perp\rangle s^\perp e^{i\langle s,x\rangle} + \sum_{s\in\mathbb{Z}^2}\frac{1}{|s|^2}\langle\hat{u}(s), s\rangle s\, e^{i\langle s,x\rangle}$$

where the first term belongs to H^m_σ because for each $s \in \mathbb{Z}^2$ and $\operatorname{div} = \operatorname{div}_x$

$$\begin{aligned}\operatorname{div}\left(\frac{\langle\hat{u}(s), s^\perp\rangle s^\perp}{|s|^2}e^{i\langle s,x\rangle}\right) &= \operatorname{div}\left(\frac{1}{|s|^2}\begin{bmatrix} s_2^2\hat{u}^1 - s_1s_2\hat{u}^2 \\ -s_1s_2\hat{u}^1 + s_1^2\hat{u}^2\end{bmatrix} e^{i\langle s,x\rangle}\right) \\ &= \frac{ie^{\langle s,x\rangle}}{|s|^2}\left\{s_1\left(s_2^2\hat{u}^1 - s_1s_2\hat{u}^2\right) + s_2\left(-s_1s_2\hat{u}^1 + s_1^2\hat{u}^2\right)\right\} = 0\end{aligned}$$

and the second term is gradient of the function

$$p\left(x\right) = -i\sum_{s\in\mathbb{Z}^2}\frac{\langle\hat{u}(s), s\rangle}{|s|^2}e^{i\langle s,x\rangle}$$

thus belongs to ∇H^{m+1}. $\square$

Hence we define the Leray Projection Π

Definition 2.4 (Leray Projection). $\Pi : H^m(\mathbb{T}^2;\mathbb{R}^2) \to H^m_\sigma$ *is Leray Projection defined by*

$$u \mapsto \Pi u := \sum_{s\in\mathbb{Z}^2} \frac{1}{|s|^2} \langle \hat{u}(s), s^\perp \rangle s^\perp e^{i\langle s,x\rangle}$$

where Πu *depends only on* u *but not* m.

Now we're ready to define weak solutions to 2D Navier-Stokes Equations on $\mathbb{T}^2$. Let $T > 0$, function $f \in L^2(0,T;H^{-1})$, $u_0 \in L^2_\sigma(\mathbb{T}^2;\mathbb{R}^2)$. We define the space $\mathcal{H}$ as solution space (see [6]).

Definition 2.5 ($\mathcal{H}$).

$$\mathcal{H} := \left\{ u \in L^2\left(0,T;H^1_\sigma\right) \mid \dot{u} \in L^2\left(0,T;H^{-1}_\sigma\right) \right\} \tag{2.7}$$

endowed with norm $\|\cdot\|_{\mathcal{H}}$ *where* $\dot{u}$ *is understood in the sense of distributions. Note* $\mathcal{H}$ *is Hilbert Space.*

$$\|u\|_{\mathcal{H}} := \left(\int_0^T \left(\|u(t)\|^2_{H^1} + \|\dot{u}(t)\|^2_{H^{-1}} \right) dt \right)^{\frac{1}{2}}$$

and denote $\mathcal{D}'$ as space of $\mathbb{R}^2$-valued distributions on $(0,T)\times\mathbb{T}^2$.

Definition 2.6 (Solution). *A pair of functions* (u,p) *is called a solution to* (2.1) (2.2) *if* $u \in \mathcal{H}$, $p \in L^2(0,T;L^2)$ *and the equation* (2.1) *holds in* $\mathcal{D}'$. *If initial condition*

$$u(0,x) = u_0(x) \tag{2.8}$$

is also satisfied, then (u,p) *is called solution to Cauchy problem* (2.1) (2.2) (2.8).

Note that (2.1) (2.2) is not a system of evolution equations since time derivative of unknown function p is not included. Then our design to exclude pressure from the problem and obtain a nonlocal nonlinear PDE regarded as evolution equation in a Hilbert Space by using Π is of physical significance [3]. Formally apply Leray Projection Π to (2.1) which holds in $\mathcal{D}'$. Now by Leray Decomposition (2.6) with $m=-1$, we notice that $\Pi(\nabla p) = 0$ as $\nabla p \in \nabla L^2 \equiv \nabla H^0$. So the pressure term is killed under projection. Also note $\Pi\dot{u} = \dot{u}$. So using

$$Lu := -\Pi\Delta u \quad \textit{and} \quad B(u,v) := \Pi(\langle u,\nabla\rangle v) \tag{2.9}$$

and abbreviating $B(u) = B(u,u)$ we have

$$\dot{u} + \nu L\, u + B(u) = \Pi f(t) \tag{2.10}$$

We of course have definition of solution to (2.10).

Definition 2.7 (Solution under Π)**.** *$u \in \mathcal{H}$ is solution of* (2.10) *on* $(0,T) \times \mathbb{T}^2$ *if the equation* (2.10) *holds in $\mathcal{D}'$. If initial condition* (2.8) *is also satisfied, u is called solution to Cauchy problem* (2.10) (2.8)*.*

Note functions Δu and $\langle u,\, \nabla\rangle u$ are elements of $L^2(0,T;H^{-1})$ and thus their Leray Projections belong to $L^2(0,T;H_\sigma^{-1})$. So all terms in (2.10) makes sense in $\mathcal{D}'$. We briefly argue the equivalence of solutions.

Theorem 2.2 (Equivalence of Solutions)**.** *Let $(u,p) \in \mathcal{H} \times L^2(0,T;L^2)$ be solution in Definition 2.6 on $(0,T) \times \mathbb{T}^2$. Then $u \in \mathcal{H}$ is a solution in Definition 2.7. On the other hand, if $u \in \mathcal{H}$ is solution in 2.7, then $\exists\ p \in L^2(0,T;L^2)$ s.t. (u,p) is solution in 2.6.*

Proof. $\Longrightarrow$. Let $(u,p) \in \mathcal{H} \times L^2(0,T;L^2)$ be solution in 2.6. We integrate *w.r.t.* time

$$u(t) = u_0 + \int_0^t (\nu\Delta u - \langle u, \nabla\rangle u - \nabla p + f)\, ds, \quad 0 \le t \le T$$

with equality holds in H^{-1}. Apply Leray Projection Π, which is continuous in H^{-1}, and using Π commutes with integration in time

$$u(t) = u_0 + \int_0^t (-\nu L u - B(u) + \Pi f)\, ds, \quad 0 \le t \le T \implies 2.7 \textit{ holds in } \mathcal{D}'$$

$\Longleftarrow$. Let $u \in \mathcal{H}$ be solution in 2.7, then $\dot{u} \in L^2(0,T;H_\sigma^{-1})$. Consider

$$\nabla p(t) = g(t) := -\dot{u} + \nu\Delta u - \langle u, \nabla\rangle u + f$$

with $g \in L^2(0,T;H^{-1})$. It suffices to show also $g(t) \in \nabla L^2 := \{g(t) \in H^{-1} \mid g = \nabla p \ for\ some\ p \in L^2\}$ for *a.e.* $t \in [0,T]$, then we can find solution $p \in L^2(0,T;L^2)$. To show $g(t) \in \nabla L^2$, note

$$u(t) = u_0 + \int_0^t (-\nu L u - B(u) + \Pi f)\, ds, \quad 0 \le t \le T \implies \Pi\left(\int_0^t g(s)\, ds\right) = \int_0^t \Pi g(s)\, ds = 0 \quad for\ 0 \le t \le T$$

So $\Pi g(t) = 0$ *a.e.* t. Hence $g(t) \in \nabla L^2$ *a.e.* $t \in [0,T]$ by Leray Decomposition $H^{-1} = H_\sigma^{-1} \oplus \nabla L^2$. $\square$

We cite without proving the Uniqueness and Existence of Solution Theorem to Cauchy Problem (2.10), (2.8).

Theorem 2.3 (Existence and Uniqueness [3])**.** *For any $u_0 \in L^2_\sigma$ and $f \in L^2\left(0, T; H^{-1}\right)$, we have unique solution $u \in \mathcal{H}$ to Cauchy Problem* (2.10), (2.8) *satisfying*

$$\|u\,(t)\|^2_{L^2} + \nu \int_0^t \|\nabla u\,(s)\|^2_{L^2}\, ds \le \|u_0\|^2_{L^2} + \nu^{-1} \int_0^t \|f\,(s)\|^2_{H^{-1}}\, ds \quad 0 \le t \le T \tag{2.11}$$

The main tools in a priori estimates are multiplying by u on both sides and integrating in space

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2} &= \int_{\mathbb{T}^2} u(t)\, \dot{u(t)}\, dx = \int_{\mathbb{T}^2} u(t)\left(-\nu L\, u - B\,(u) + \Pi f\,(t)\right) dx \\
&\le -\nu\,\|\nabla u(t)\|^2_{L^2} + \int_{\mathbb{T}^2} u(t)\, f(t)\, dx \le -\nu\,\|\nabla u(t)\|^2_{L^2} + \|\nabla u(t)\|_{L^2}\,\|f(t)\|_{H^{-1}} \\
&\le -\frac{\nu}{2}\,\|\nabla u(t)\|^2_{L^2} + \frac{1}{2\nu}\,\|f(t)\|^2_{H^{-1}}
\end{aligned}$$

where we used Duality Pairing (2.4) and Young's Inequality. Integrating in time gives (2.11) and that $u \in L^2\left(0, T; H^1\right)$. To see $\dot{u} \in L^2\left(0, T; H^{-1}\right)$, it suffices to note by Sobolev Embedding $H^{1/2} \hookrightarrow L^4$

$$\begin{aligned}
\left|\langle B\,(u, v)\,, w\rangle\right| = \left|\langle B\,(u, w)\,, v\rangle\right| &= \left|\sum_{j,l=1}^{2} \int_{\mathbb{T}^2} u^j \left(\partial_j w^l\right) v^l dx\right| \le C_1 \int_{\mathbb{T}^2} |u|\,|\nabla w|\,|v|\, dx \\
&\le C_2\,\|\nabla w\|_{L^2}\,\|u\, v\|_{L^2} \le C_2\,\|\nabla w\|_{L^2}\,\|u\|_{L^4}\,\|v\|_{L^4} \le C_3\,\|w\|_{H^1}\,\|u\|_{H^{\frac{1}{2}}}\,\|v\|_{H^{\frac{1}{2}}}
\end{aligned}$$

By Duality Pairing (2.4)

$$\|B\,(u, v)\|_{H^{-1}} = \sup_{\|w\|_{H^1} \le 1} \frac{\left|\langle B\,(u, v)\,, w\rangle\right|}{\|w\|_{H^1}} \le C\,\|u\|_{H^{\frac{1}{2}}}\,\|v\|_{H^{\frac{1}{2}}}$$

So in our case, take $u = v = w$ to obtain by Sobolev Interpolation (2.5)

$$\|B(u)\|_{H^{-1}} \le C\,\|u\|^2_{H^{\frac{1}{2}}} \le C\,\|u\|_{L^2}\,\|u\|_{H^1}$$

The right hand side is bounded, hence writing $\dot{u} = \Pi f(t) - B(u) - \nu L u \ \in L^2(0, T; H^{-1})$ gives $u \in \mathcal{H}$. Note this is a sketch in a priori estimate, where the original method to construct the solution includes Galerkin Approximation and conclude using Banach-Alaoglu.

2.1.2 Navier-Stokes Process

In settings of Stochastic Navier-Stokes, we impose zero-mean conditions to external force and expect to obtain zero mean solution in time. Hence we introduce a new set of function spaces.

Definition 2.8 (H, V, V^k, V^*)**.** *We begin by defining mean value of function $u \in H^m$ as*

$$\langle\, u \,\rangle := \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} u(x)\, dx \qquad \forall\ m \in \mathbb{Z}_{\geq 0}$$

If $m \in \mathbb{R}$, define $\langle\, u \,\rangle := \hat{u}\,(0)$ as zero order coefficient in Fourier Expansion. Let $u\,(t, x)$ solve (2.10)*, then integrating in time and taking mean value of both sides gives*

$$\langle u\,(t)\rangle + \int_0^t \langle -\Delta u + B\,(u)\rangle ds = \langle u\,(0)\rangle + \int_0^t \langle f\,(s)\rangle ds$$

Note $\int_0^t \langle -\Delta u + B\,(u)\rangle ds = 0$, so if $\langle f\,(t)\rangle = 0$ a.e. $t \in [0, T]$, we have $\langle u\,(t)\rangle$ constant. So if both $\langle f\,(t)\rangle$ and u_0 vanish, we achieve $\langle u\,(t)\rangle = 0\ \forall\ t \in [0, T]$. Now we introduce function spaces

- *$H := \{u \in L^2_\sigma\,(\mathbb{T}^2, \mathbb{R}^2) \mid \langle\, u \,\rangle = 0\}$, $V := H^1 \cap H$ and $V^k := H^k \cap H$. By Poincaré's, Sobolev norm on V^k is equivalent to $\langle\, L^k u,\, u \,\rangle^{\frac{1}{2}}$. We denote the norms $\|\cdot\|_k$ abusing notation.*

- *Let V^* be dual space of V w.r.t. scalar product in L^2.*

We introduce a setting suitable for defining solutions to random kick forces, known as the Navier-Stokes Process. Consider (2.10) with $f \in L^2_{loc}\,(\mathbb{R}_+, H^{-1})$ and $\nu = 1$ so (2.10) writes

$$\dot{u} + L\,u + B\,(u) = \Pi f\,(t) \tag{2.12}$$

By Theorem 2.3 $\forall\ \tau \in \mathbb{R}_+$, and any $u_0 \in H$, (2.10) has unique solution in $C\,(\mathbb{R}_\tau, H) \cap L^2_{loc}\,(\mathbb{R}_\tau, V)$ *s.t.* $u\,(\tau) = u_0$, $\mathbb{R}_\tau = [\tau, \infty)$. This follows from that fact that $\mathcal{H}$ continuously embedds into $C(0, T; L^2_\sigma)$.

Definition 2.9 (Navier-Stokes Process)**.** *Define resolving operator $S_{t,\tau} : H \to H$ by*

$$S_{t,\tau}\,(u_0) := u\,(t) \tag{2.13}$$

We see the family $\{S_{t,\tau},\, t \geq \tau \geq 0\}$ forms a process s.t.

$$S_{\tau,\tau} = \mathrm{Id}_H, \quad S_{t,\tau} = S_{t,s} \circ S_{s,\tau} \text{ for } t \geq s \geq \tau \geq 0 \tag{2.14}$$

We call $S_{t,\tau}$ the Navier-Stokes Process, or NS Process. Let $S_t = S_{t,0}$.

Proposition 2.3 (Dissipativity Property of NS Process). *Let $u_0 \in H$ and $f \in L^2_{loc}(\mathbb{R}_+, H^{-1})$. Then*

$$\|S_t(u_0)\|^2_{L^2(\mathbb{T}^2)} \leq e^{-\alpha_1 t} \|u_0\|^2_{L^2(\mathbb{T}^2)} + \int_0^t e^{-\alpha_1(t-s)} \|f(s)\|^2_{H^{-1}(\mathbb{T}^2)}\, ds \tag{2.15}$$

where $\alpha_1 > 0$ is first positive eigenvalue of the Laplacian. Note for $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, we have $\alpha_1 = 1$.

Proof. By (2.12), we have u solving

$$\dot{u} + L\,u + B(u) = f$$

dot with u taking L^2 inner-product, denoted $\langle \cdot, u(t)\rangle$, we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{T}^2)} + \|\nabla u(t)\|^2_{L^2(\mathbb{T}^2)} = \langle f(t)\,,\, u(t)\rangle \leq \frac{1}{2}\|f(t)\|^2_{H^{-1}} + \frac{1}{2}\|u(t)\|^2_{H^1}$$

Hence absorbing $\|u(t)\|_{H^1}$ into $\|\nabla u\|_{L^2}$, we have

$$\frac{d}{dt}\|u(t)\|^2_{L^2} + \|\nabla u(t)\|^2_{L^2} \leq \|f(t)\|^2_{H^{-1}}$$

Now notice, writing $u = \sum_{j=1}^{\infty}\langle u, e_j\rangle e_j$ where $-\Delta e_j = \alpha_j e_j$ eigenvalues and eigenvectors, we have

$$-\Delta u = \sum_{j=1}^{\infty} \alpha_j \langle u,\, e_j\rangle e_j \implies \|\nabla u\|^2_{L^2} = \langle -\Delta u,\, u\rangle = \sum_{j=1}^{\infty}\alpha_j\langle u, e_j\rangle^2 \geq \alpha_1 \|u\|^2_{L^2}$$

This is Poincaré's Inequality. So applying such estimate,

$$\frac{d}{dt}\|u(t)\|^2_{L^2} + \alpha_1\|u(t)\|^2_{L^2} \leq \|f(t)\|^2_{H^{-1}}$$

We multiply by $e^{\alpha_1 t}$ on both sides to see

$$\frac{d}{dt}\left(e^{\alpha_1 t}\|u(t)\|^2_{L^2}\right) = e^{\alpha_1 t}\frac{d}{dt}\|u(t)\|^2_{L^2} + \alpha_1\|u(t)\|^2_{L^2} e^{\alpha_1 t} \leq e^{\alpha_1 t}\|f(t)\|^2_{H^{-1}}$$

Integrating on both sides from 0 to t gives

$$e^{\alpha_1 t}\left\|u\left(t\right)\right\|_{L^2}^2-\left\|u\left(0\right)\right\|_{L^2}^2\leq\int_0^t e^{\alpha_1 s}\left\|f\left(s\right)\right\|_{H^{-1}}^2 ds \iff \left\|u\left(t\right)\right\|_{L^2}^2\leq e^{-\alpha_1 t}\left\|u\left(0\right)\right\|_{L^2}^2+\int_0^t e^{-\alpha_1(t-s)}\left\|f\left(s\right)\right\|_{H^{-1}}^2 ds$$

Take $u\left(t\right)=S_t\left(u_0\right)$ to conclude.

□

We also cite without proof the continuity property of S_t. In particular, we need the fact that S_t is continuous from H to V. Denote $S_t(u_0, f)$ to indicate dependence on force f, we have the following

Proposition 2.4 (Continuity Property of NS Process). *$\exists\ C>0$ s.t. $\forall\ 0<t\leq 1$, any $u_{0_1},\ u_{0_2}\in H$ and any $f_1,\ f_2\in L^2_{loc}(\mathbb{R}_+, H)$, we have*

$$\|S_t(u_{0_1}, f_1)-S_t(u_{0_2}, f_2)\|_1^2\leq C\int_0^t\|f_1-f_2\|_{L^2}^2 ds + A(t)t^{-3}\left(\|u_{0_1}-u_{0_2}\|_{L^2}^2+\int_0^t\|f_1-f_2\|_{H^{-1}}^2 ds\right)$$

for $A(t)=\exp\left(C\int_0^t\left(\|S_s(u_{0_1}, f_1)\|_1^2+\|S_s(u_{0_2}, f_2)\|_1^2+\|f_1\|_{L^2}^2+\|f_2\|_{L^2}^2\right)ds\right)$. $\|\cdot\|_k$ denotes norm in V^k.

2.2 Random Kick Force

We consider NS equations in which force has form $f\left(t,x\right)=h\left(x\right)+\eta\left(t,x\right)$, where h is deterministic but η is stochastic process. So the 2D stochastic Navier-Stokes has form

$$\dot{u}+\langle u,\nabla\rangle u-\nu\Delta u+\nabla p=h\left(x\right)+\eta\left(t,x\right) \tag{2.16}$$

$$\operatorname{div} u=0 \tag{2.17}$$

In this section, we consider η random kick force on $\mathbb{T}^2$, defined as

$$\eta\left(t,x\right)=\sum_{k=1}^{\infty}\eta_k\left(x\right)\delta\left(t-kT\right) \tag{2.18}$$

$T>0$ constant and $\{\eta_k\}$ *i.i.d.* random variables in H, defined on probability space $(\Omega,\mathcal{F},\mathbb{P})$. As in the deterministic case, we project using Π to space H and obtain

$$\dot{u}+\nu L\,u+B\left(u\right)=h+\sum_{k=1}^{\infty}\eta_k\delta\left(t-kT\right)\qquad\left(t,x\right)\in\mathbb{R}_+\times\mathbb{T}^2 \tag{2.19}$$

where as in (2.9)

$$Lu := -\Pi\Delta u \quad and \quad B(u) = B(u,u) := \Pi(\langle u, \nabla\rangle u)$$

2.2.1 Solution

We start by defining a 'solution' to (2.19). First consider the NS process to the deterministic equation. Let $S_t : H \to H$ be resolving operator for (2.10) with $f(t,x) = h$

$$\dot{u} + \nu L\, u + B(u) = h \tag{2.20}$$

i.e., $S_t(u_0) = u(t) \in C(\mathbb{R}_+, H) \cap L^2_{loc}(\mathbb{R}_+, V)$ is solution to (2.20) satisfying $u(0) = u_0$. Note that randomness is introduced per interval $J_k := [(k-1)\,T,\, kT)$, hence it's natural to define a filtration for (2.19) as non-decreasing family of σ-algebras $\{\mathcal{G}_t \mid t \geq 0\}$ of space $(\Omega, \mathcal{F})$ *s.t.* satisfies

$$\mathcal{G}_t = \mathcal{G}_{(k-1)T} \quad \forall\, t \in J_k = [(k-1)\,T,\, kT)$$

and that

$$\eta_k \textit{ is } \mathcal{G}_{kT} - \textit{measurable and independent of } \mathcal{G}_{(k-1)T} \quad \forall\, k \in \mathbb{N}_+$$

Recall that a random variable $v(t)$ is adapted to $\mathcal{G}_t$ for $t \geq 0$ if $v(t)$ is $\mathcal{G}_t$-measurable $\forall\, t \geq 0$. We then introduce the solution space analog of Definition 2.7 on each interval J_k as

$$\mathcal{H}(J_k) := \left\{u \in L^2(J_k, V) \mid \dot{u} \in L^2(J_k, V^*)\right\} \tag{2.21}$$

In view of continuous embedding of $\mathcal{H}$, we know $\mathcal{H}(J_k) \hookrightarrow C(J_k, H)$. Hence any $u \in \mathcal{H}(J_k)$ admits an extension to a continuous curve from $\overline{J}_k \to H$.

Definition 2.10 (Solution with random kick force)**.** *A random process $u(t)$, $t \geq 0$, is a solution of (2.19) if it is adapted to filtration $\mathcal{G}_t$ for (2.19), and a.e. trajectory of u satisfies the following $\forall\, k \in \mathbb{N}_+$.*

- *$u\big|_{J_k} \in \mathcal{H}(J_k)$ and satisfies (2.20). In particular, $u : J_k \to H$ is continuous curve which has limit at the right endpoint of J_k.*
- *Let point $t_k := k\,T$, and $u\left(t_k^-\right)$, $u\left(t_k^+\right)$ be left and right hand limits of u at the point t_k. We have*

relation

$$u\left(t_k^+\right) - u\left(t_k^-\right) = \eta_k \tag{2.22}$$

On intervals (t_{k-1}, t_k), the function u is solution to free Navier-Stokes, where at point t_k it sees instantaneous increment of size η_k, the kth kick. Now we try to interpret u in the language of Markov Process. Denoting $u_k = u(kT)$ for $k = 0, 1, \cdots$, we write

$$u_k = S_T(u_{k-1}) + \eta_k, \qquad k \geq 1 \tag{2.23}$$

and for $t = kT + \tau$ with $0 \leq \tau < T$, we have $u(t) = S_\tau(u_k)$. We cite without proof the Existence and Uniqueness theorem of Solution to (2.19), as direct consequence of Theorem 2.3.

Theorem 2.4 (Existence and Uniqueness [3]). *Let $u_0 \in H$ be $\mathcal{G}_0$-measurable random variable. Then* (2.19) *has solution satisfying initial condition* (2.8) $\forall\, \omega \in \Omega$. *The solution is unique in the sense that if $\tilde{u}$ is another random process as solution, then*

$$\mathbb{P}\{u(t) = \tilde{u}(t)\ \forall\ t \geq 0\} = 1$$

We recall example 1.1. Compare (2.23) to (1.11), we see the collection $(u_k, \mathbb{P})$ itself defines a Markov Chain in the exact same setting. Now taking randomness of initial data u_0 into consideration, we mimic the extension in example 1.1 and abuse of notation to define $(u_k, \mathbb{P}_v)$ on the extended space $(H \times \Omega, \mathcal{B}(H) \otimes \mathcal{F})$. Recall the definition of stationary measure 1.6, a measure μ is stationary for Markov Process $(u_k, \mathbb{P}_v)$ if $\forall\ k$, $\mathfrak{B}_k^* \mu = \mu$. With the Markov Process as defined by (2.23), we call such μ a stationary measure for equation (2.23), hence for (2.19). One goal of this Thesis is to show such μ for (2.19) exists. We naturally introduce the notion of stationary solution. A solution u to our equation (2.19) is stationary if the law of its associated trajectory $\{u_k,\ k \geq 0\}$ coincides with some stationary measure μ, *i.e.*, $u_{k_*}(\mathbb{P}) = \mu$ for any $k \geq 0$.

2.2.2 Moment Estimate

We establish moment estimates for solutions to (2.19). WLOG, let $\nu = 1$.

Proposition 2.5 (Moment Estimate). *Let $u_0 \in H$ be $\mathcal{G}_0$-measurable random variable (so Theorem 2.4* *holds). Also assume that for any $m \geq 1$*

$$\mathbb{E}\,\|u_0\|_{L^2}^m < \infty \qquad K_m := \mathbb{E}\,\|\eta_1\|_{L^2}^m < \infty \tag{2.24}$$

then there are positive constants $C_m = C_m(T)$ and $q < 1$ s.t.

$$\mathbb{E}\,\|u(kT)\|_{L^2}^m \leq q^{kT}\,\mathbb{E}\,\|u_0\|_{L^2}^m + C_m(K_m + 1) \qquad k \geq 1 \tag{2.25}$$

Proof. Recall $u_k = u(kT)$ where u_k satisfies (2.23). Notice by dissipative property of Navier-Stokes Process (2.15)

$$\|S_T(v)\|_{L^2} \leq e^{-\alpha_1 T/2}\,\|v\|_{L^2} + C_1\,\|h\|_{L^2} \qquad \forall\ v \in H$$

So taking $v = u_{k-1}$

$$\begin{aligned}\|u_k\|_{L^2} &\leq \|S_T(u_{k-1})\|_{L^2} + \|\eta_k\|_{L^2} \\ &\leq e^{-\frac{\alpha_1}{2}T}\,\|u_{k-1}\|_{L^2} + C_1\,\|h\|_{L^2} + \|\eta_k\|_{L^2}\end{aligned}$$

Hence taking mth power and letting $q < 1$ denote the power of negative exponential depending on first eigenvalue α_1 and $C_2 > 0$ only on m, we have

$$\|u_k\|_{L^2}^m \leq q^T\,\|u_{k-1}\|_{L^2}^m + C_2(\|\eta_k\|_{L^2}^m + \|h\|_{L^2}^m)$$

Iteratively apply to $u_{k-1}, \cdots, u_1$ gives

$$\|u_k\|_{L^2}^m \leq q^{kT}\,\|u_0\|_{L^2}^m + C_2 \sum_{\ell=1}^{k} q^{(k-\ell)T}(\|\eta_\ell\|_{L^2}^m + \|h\|_{L^2}^m)$$

Take mean on both sides to conclude. □

2.3 White Noise

Recall (2.16) (2.17). In this section, we consider η as white noise, *i.e.*, formal time derivative of Brownian Motion. We call η a white-in-time noise if

$$\eta\left(t,x\right)=\frac{\partial}{\partial t}\zeta\left(t,x\right)\quad where\quad \zeta\left(t,x\right)=\sum_{j=1}^{\infty}b_j\beta_j\left(t\right)e_j\left(x\right)\quad t\geq 0 \tag{2.26}$$

$\{e_j\}$ orthonormal basis in H, $b_j \geq 0$ constants *s.t.* $\mathfrak{B}:=\sum_{j=1}^{\infty}b_j^2<\infty$ and $\{\beta_j\}$ sequence of independent standard Brownian Motions. This ensures for almost every $\omega\in\Omega$, $\zeta^{\omega}(t,\cdot)$ is a continuous random process in H. We write $\zeta^{\omega}(t,\cdot)\in C(\mathbb{R}_+;H)$. So our equation of interest is

$$\dot{u}+\nu L\,u+B\left(u\right)=h+\frac{\partial}{\partial t}\zeta(t,x) \tag{2.27}$$

2.3.1 Solution

We begin with defining a solution to (2.27). Let us assume Brownian motions $\{\beta_j\}$ are defined on complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ with filtration $\mathcal{G}_t$ for $t\geq 0$, and σ-algebras $\mathcal{G}_t$ are complete w.r.t. $(\mathcal{F},\mathbb{P})$, *i.e.*, $\mathcal{G}_t$ contains all $\mathbb{P}$-nul sets $A\in\mathcal{F}$. In such case, the filtered probability space $(\Omega,\mathcal{F},\mathcal{G}_t,\mathbb{P})$ is said to satisfy the usual hypothesis.

Definition 2.11 (Solution with white noise)**.** *An H-valued random process $u(t)$, $t\geq 0$ is called a solution to* (2.27) *if*

- *$u(t)$ is adapted to filtration $\mathcal{G}_t$, and almost every trajectory belongs to*

$$\mathcal{X}:=C(\mathbb{R}_+;H)\cap L^2_{loc}(\mathbb{R}_+;V)$$

- (2.27) *holds in the sense that, with probability 1,*

$$u(t)+\int_0^t\left(\nu Lu+B(u)\right)ds=u(0)+th+\zeta(t),\qquad t\geq 0 \tag{2.28}$$

  where equality holds in space H^{-1}. Both sides of the equation lies in $C(\mathbb{R}_+;H^{-1})$, hence well-defined.

We sketch a construction of solution u. In principle, we simplify (2.27) into a Navier-Stokes system with random coefficients. Let $u = v + z$ where z solves

$$\dot{z} + \nu L z = \eta(t,x) \qquad t \geq 0 \tag{2.29}$$

Hence plugging into (2.27) we obtain

$$\dot{v} + \nu L v + B(v+z) = h \tag{2.30}$$

$$v(0) = u_0 \tag{2.31}$$

as Navier-Stokes system with random coefficients entering through stochastic process z. We call (2.29) stochastic Stokes Equation, and use the fact that z can be represented as stochastic convolution

Theorem 2.5 (Existence and Uniqueness of Stochastic Stokes solution)**.** (2.29) *has solution* $z(\cdot) \in \mathcal{X}$ *satisfying initial condition* $z(0) = 0$*.* z *is unique up to measure zero.* z *can be represented as stochastic convolution*

$$z(t) = \int_0^t e^{-\nu(t-s)L} d\zeta(s) \tag{2.32}$$

We sketch a proof for existence. This is essentially insightful, as it uses 2 main ingredients in moment estimating under stochastic integrals: Itô formula and Doobs (see [2]). Integration by parts for (2.32), we have

$$z(t) = \int_0^t e^{-\nu(t-s)L} d\zeta(s) = \zeta(t) - \nu \int_0^t L e^{-\nu(t-s)L} \zeta(s) ds \tag{2.33}$$

Let us consider sequence of processes $z_n(t)$ defined mimicing (2.33)

$$z_n(t) = \zeta^n(t) - \nu \int_0^t L e^{-\nu(t-s)L} \zeta^n(s) ds \tag{2.34}$$

with $\zeta^n(t) := \sum_{j=1}^n b_j \beta_j(t) e_j$. It's easy to see

$$z_n(t) + \nu \int_0^t L z_n(s) ds = \zeta^n(t)$$

it suffices to show that a subsequence z_{n_k} converges a.s. in space $\mathcal{X}_T := C(0,T;H) \cap L^2(0,T;V)$. We do so by showing z_n defines a Cauchy sequence in $L^2(\Omega, \mathcal{X}_T)$. Indeed let $z_{mn} = z_n - z_m$ for $m < n$, so z_{mn}

satisfies

$$\dot{z_{mn}} + \nu L z_{mn} = \sum_{j=m+1}^{n} b_j \dot{\beta}_j(t) e_j(x)$$

We need the Itô's formula

Theorem 2.6 (Itô [2], [3]). *let H be separable Hilbert Space and $F : [0,T] \times H \to \mathbb{R}$ be twice continuously differentiable bounded and uniformly continuous along with its derivative. Let $\{u_t,\, t \geq 0\}$ be continuous in H and locally square integrable in V. Moreover, u_t is Itô process in V^* with constant diffusion, i.e.*

$$u_t = u_0 + \int_0^t f_s ds + \sum_{j=1}^{\infty} \beta_j(t) g_j \qquad t \geq 0 \tag{2.35}$$

for $\{g_j\} \subset H$ s.t. $\sum_{j=1}^{\infty} \|g_j\|_H^2 < \infty$ and f_t $\mathcal{G}_t$-progressively measurable V^*-valued process such that

$$\mathbb{P}\{\int_0^T \|f_t\|_{V^*}^2 \, ds < \infty\} = 1 \tag{2.36}$$

If F further satisfies 2 conditions [3]

- *for any $T > 0$, $(\partial_u F)(t,u)$ defined initially on H admits a continuous extension to V^* for any $u \in V$*
- *For any sequence $\{w_k\} \subset V$ converging to $w \in V$ in topology of V and any $t \in \mathbb{R}_+$, $v \in V^*$, we have*

$$(\partial_u) F(t, w_k; v) \to (\partial_u F)(t, w; v) \qquad ask \to \infty$$

Then

$$F(t, u_t) = F(0, u_0) + \int_0^t A(s) ds + \sum_{j=1}^{\infty} \int_0^t B_j(s) d\beta_j(s) \qquad t \in [0,T] \tag{2.37}$$

where

$$A(t) = (\partial_t F)(t, u_t) + (\partial_u F)(t, u_t; f_t) + \frac{1}{2} \sum_{j=1}^{\infty} (\partial_u^2 F)(t, u_t; g_j) \tag{2.38}$$

$$B(t) = (\partial_u F)(t, u_t; g_j) \tag{2.39}$$

and $(\partial_u F)(u; v)$ and $(\partial_u^2 F)(u; v)$ denote the first and second derivatives of F w.r.t. to u at point $v \in H$.

Now Apply (2.37), (2.38), (2.39) to $F(t, z_{mn}(t)) = \|z_{mn}(t)\|_{L^2}^2$, we derive

$$\|z_{mn}(t)\|_{L^2}^2 = \int_0^t (-2\nu \|\nabla z_{mn}(s)\|_{L^2}^2 + F_{mn})ds + 2 \sum_{j=m+1}^{n} b_j \sum_0^t \langle z_{mn}, e_j \rangle d\beta_j \tag{2.40}$$

where $F_{mn} = \sum_{j=m+1}^n b_j^2$. Taking mean value and sup to (2.40), we have

$$\sup_{0 \le t \le T} \mathbb{E} \|z_{mn}(t)\|_{L^2}^2 + 2\nu \, \mathbb{E} \int_0^T \|z_{mn}(s)\|_1^2 \, dt \le C_4 T F_{mn} \tag{2.41}$$

On the other hand, we apply Doob's moment inequality.

Theorem 2.7 (Doob's Moment Inequality [3]). *let $\{M_t, \ t \ge 0\}$ be non-negative submartingale, then for any $p \in (1, \infty)$ and fixed $T > 0$, we have*

$$\mathbb{E}(\sup_{0 \le t \le T} M_t^p) \le \left(\frac{p}{p-1}\right)^p \mathbb{E} \, M_T^p \tag{2.42}$$

Apply (2.42) to (2.40) results in

$$\mathbb{E}(\sup_{0 \le t \le T} \|z_{mn}(t)\|_{L^2}^2) \le F_{mn} T + C_5 \, \mathbb{E} \int_0^T \sum_{j=1}^n b_j^2 \int_{\mathbb{T}^2} z_{mn}^2(t) e_j^2 dx dt \tag{2.43}$$

Note $F_{mn} \to 0$ as $m, n \to \infty$, so combining (2.41) and (2.43) we coclude

$$\mathbb{E}(\sup_{0 \le t \le T} \|z_{mn}(t)\|_{L^2}^2) + \mathbb{E} \int_0^T \|z_{mn}(s)\|_1^2 \, dt \to 0 \qquad \text{as } m, \ n \to \infty$$

This concludes sketch of Theorem 2.5. Now we cite main theorem on existence and uniqueness of solution to (2.27)

Theorem 2.8 (Existence and Uniqueness[3]). *For any $\nu > 0$ and any $\mathcal{G}_0$-measurable random variable $u_0(x)$, (2.27) has solution $u(t)$, $t \ge 0$ satisfying initial condition*

$$u(0) = u_0 \qquad \text{almost surely} \tag{2.44}$$

If $\tilde{u}(t)$ is another solution to (2.27) and (2.44) then they agree almost surely. Furthermore, $u(t)$ has following properties

- *Almost all trajectories $u(t)$ are continuous in range H and locally square integrable in range V.*

- *The process $u(t)$ can be written in form*

$$u(t) = u_0 + \int_0^t f(s)ds + \zeta(t) \qquad t \geq 0$$

where $f(t)$ is V^*-valued $\mathcal{G}_t$-progressively measurable process such that

$$\mathbb{P}\{\int_0^T \|f_t\|_{V^*}^2 \, ds < \infty\} = 1$$

Notice they match assumptions on Infinite dimensional Itô formulas Theorem 2.6 in order to construct the solution.

To understand the solution under a Markovian perspective, let $\tilde{\Omega} = H \times \Omega$ where $(\Omega, \mathcal{F}, \mathcal{G}_t, \mathbb{P})$ is complete filtered probability space as defined. Also let

$$\tilde{F}_t = \mathcal{B}(H) \otimes \mathcal{G}_t, \qquad \mathbb{P}_v = \delta_v \otimes \mathbb{P}$$

where $v \in H$. Writing $\tilde{\omega} = (v, w) \in \tilde{\Omega}$ and we define $t \mapsto u_t(\tilde{w}) \equiv u^{\omega}(t; v)$, where $u^{\omega}(t; v)$ denotes solution of (2.27) as in Theorem 2.8 for $u_0 = v$ a deterministic initial data. So adding randomness to u_0, we define a family of Markov Process $(u_t, \mathbb{P}_v)$, $v \in H$, associated with (2.27). Notice to justify the Markov process for $\{u_t\}$, one needs to rewrite the law of $u^{\omega}(t)$ in terms of measures of randomness for ζ and initial data u_0. To do so, fix $T > 0$, let $\mathcal{C}_T = C(0, T; H)$, $m_{\zeta,T} = \zeta|_{[0,T]_*}(\mathbb{P})$ and $\mathcal{F}_{\zeta,T}$ as $m_{\zeta,T}$-completion of $\mathcal{B}(\mathcal{C}_T)$. We have proposition that follows from construction of solution to (2.27)

Proposition 2.6 ([3])**.** *There exists measurable mapping*

$$U : (H \times \mathcal{C}_T, \mathcal{B}(H) \otimes \mathcal{F}_{\zeta,T}) \mapsto (\mathcal{X}_T, \mathcal{B}(\mathcal{X}_T))$$

which is locally Lipschitz continuous in $u_0 \in H$ and for $m_{\zeta,T}$-almost every ω in $\zeta^{\omega} \in \mathcal{C}_T$ such that

$$u^{\omega} = U(u_0, \zeta^{\omega}) \qquad for\ m_{\zeta,T} - almost\ every\ \omega\ and\ all\ u_0 \in H$$

The restriction of U to $H \times C(0, T; V)$ is locally Lipschitz continuous in both variables.

Now denote U_t the restriction of U to time $t \in [0,T]$, for any u_0 random initial data independent of ζ^ω, the law of $u^\omega(t)$, $u^\omega(t)_*(\mathbb{P})$ can be written as image of product measure $u_{0_*}(\mathbb{P}) \otimes m_{\zeta,T}$ under the mapping U_t

$$u^\omega(t)_*(\mathbb{P}) \equiv \mathbb{P}\{u^\omega(t) \in \cdot\} = u_{0_*}(\mathbb{P}) \otimes m_{\zeta,t}\{U_t \in \cdot\} \equiv U_{t_*}(u_{0_*}(\mathbb{P}) \otimes m_{\zeta,t}) \tag{2.45}$$

We unpack Markov Property and using (2.45) for law of $u^\omega(t)$ to justify $(u_t, \mathbb{P}_v)$ defines a Markov Process. Recall a measure $\mu \in \mathcal{P}(H)$ is stationary for (2.27) if μ is invariant under $\mathfrak{B}_t^*$ associated with $(u_t, \mathbb{P}_v)$ for any $t \geq 0$. A solution $u(t)$ to (2.27) is stationary if for any $t \geq 0$, the law $u_t^\omega(\mathbb{P})$ coincides with some stationary measure μ. Notice $\mathfrak{B}_t^* u_{0_*}(\mathbb{P})$ coincides with RHS of (2.45).

2.3.2 Moment Estimate

One naturally asks: Does energy balance still hold for solutions to (2.27)? We provide an analogue result of (2.11).

Proposition 2.7 (Moment Estimate)**.** *Under same hypothesis as Theorem 2.8, assume initial data u_0 has finite second moment $\mathbb{E}\,\|u_0\|_{L^2} < \infty$. Then for any $\nu > 0$, the following energy balance holds for solution $u(t)$ of problem* (2.27) (2.44)

$$\mathbb{E}\,\|u(t)\|_{L^2}^2 + 2\nu\,\mathbb{E}\int_0^t \|\nabla u(s)\|_{L^2}^2\,ds = \mathbb{E}\,\|u_0\|_{L^2}^2 + \mathfrak{B}t + 2\,\mathbb{E}\int_0^t \langle u, h\rangle ds \qquad t \geq 0 \tag{2.46}$$

Moreover, for $\alpha_1 > 0$ first eigenvalue of Stokes operator L, we have

$$\mathbb{E}\,\|u(t)\|_{L^2}^2 \leq e^{-\nu\alpha_1 t}\,\mathbb{E}\,\|u_0\|_{L^2}^2 + (\nu\alpha_1)\mathfrak{B} + (\nu\alpha_1)^{-2}\,\|h\|_{L^2}^2 \qquad t \geq 0 \tag{2.47}$$

Proof. We use a version of Itô's Formula Theorem 2.6 to functional $F(u) = \|u\|_{L^2}^2$, where we consider evaluating u at $t \wedge \tau_n$ for $\tau_n := \inf\{t \geq 0 \mid \|u(t)\|_{L^2} > n\}$ stopping time. We calculate

$$\partial_u F(u; v) = 2\langle u, v\rangle \qquad \partial_u^2 F(u; v) = 2\,\|v\|_{L^2}^2$$

and observe

$$\|u(t \wedge \tau_n)\|_{L^2}^2 = \|u_0\|_{L^2}^2 + 2\int_0^{t\wedge\tau_n} (\langle u, h - \nu L u\rangle + \mathfrak{B})\,ds + 2\sum_{j=1}^\infty b_j \int_0^{t\wedge\tau_n} \langle u, e_j\rangle d\beta_j(s)$$

Now taking mean value on both sides to obtain

$$\mathbb{E}\left\|u(t\wedge\tau_n)\right\|_{L^2}^2+2\nu\,\mathbb{E}\int_0^{t\wedge\tau_n}\left\|\nabla u(s)\right\|_{L^2}^2ds=\mathbb{E}\left\|u_0\right\|_{L^2}^2+\mathfrak{B}\,\mathbb{E}(t\wedge\tau_n)+2\,\mathbb{E}\int_0^{t\wedge\tau_n}\langle u,h\rangle ds$$

passing to the limit at $n\to\infty$ and using monotone convergence theorem we have (2.46) Also note that Poincaré's Inequality gives

$$\left\|\nabla u\right\|_{L^2}^2\geq\alpha_1\left\|u\right\|_{L^2}^2$$

while Young's Inequality gives

$$2|\langle u,h\rangle|\leq\nu\alpha_1\left\|u\right\|_{L^2}^2+\frac{1}{\nu\alpha_1}\left\|h\right\|_{L^2}^2$$

substituting these into (2.46) and using Gronwall's we have (2.7). □

We now establish an estimate for second exponential moment with assumption on stronger initial data.

Proposition 2.8 (Exponential Moment Estimate)**.** *Under hypothesis of Theorem 2.8, there is constant* $c>0$ *not depending on* ν,h *and* $\{b_j\}$ *such that if* $\varkappa>0$ *satisfies*

$$\varkappa\sup_{j\geq1}b_j^2\leq c \tag{2.48}$$

Then the following holds: For any $\mathcal{G}_0$*-measurable* H*-valued random variable* $u_0(x)$ *such that*

$$\mathbb{E}\exp\left(\varkappa\nu\left\|u_0\right\|_{L^2}^2\right)<\infty \tag{2.49}$$

then the corresponding solution to (2.27), (2.44) *satisfies*

$$\mathbb{E}\exp\left(\varkappa\nu\left\|u(t)\right\|_{L^2}^2\right)\leq e^{-\varkappa\nu^2t}\,\mathbb{E}\exp\left(\varkappa\nu\left\|u_0\right\|_{L^2}^2\right)+K(\nu,\varkappa,\mathfrak{B},h) \tag{2.50}$$

where the constants are

$$K(\nu,\varkappa,\mathfrak{B},h)=\nu^{-1}R\exp(\varkappa R/\alpha_1),\qquad R=C\nu^{-1}\left\|h\right\|_{L^2}^2+\mathfrak{B}+\nu \tag{2.51}$$

where $C>0$ *is absolute constant.*

Proof. The proof mimics that of Theorem 2.7. Apply Itô's to functional $F : H \to \mathbb{R}$ as defined by $F(u) = \exp\left(\varkappa\nu \|u\|_{L^2}^2\right)$. We calculate

$$\partial_u F(u;v) = 2\varkappa\nu F(u)\langle u, v\rangle \qquad \partial_u^2 F(u;v) = 2\varkappa\nu F(u)\left(\|v\|_{L^2}^2 + 2\varkappa\nu\langle u, v\rangle^2\right)$$

Hence evaluating $t \wedge \tau_n$ gives

$$F(u(t\wedge\tau_n)) = F(u_0) + \int_0^{t\wedge\tau_n} A(s)ds + M(t\wedge\tau_n)$$

for $M(t)$ stochastic integral, hence mean value zero, and

$$A(t) = 2\varkappa\nu F(u(t))\left(-\nu\langle u(t), Lu(t)\rangle + \langle u(t), h\rangle + \frac{1}{2}\mathfrak{B} + \varkappa\nu\sum_{j=1}^{\infty} b_j^2\langle u(t), e_j\rangle^2\right)$$

Taking Mean value on both sides gives

$$\mathbb{E}\, F(u(t\wedge\tau_n)) = \mathbb{E}\, F(u_0) + \mathbb{E}\int_0^{t\wedge\tau_n} A(s)ds \tag{2.52}$$

Now we try to estimate size of $A(t)$. Note if (2.48) holds with $c \le (8\alpha_1)^{-1}$ so the last term in $A(t)$ get absorbed. Using Young's, and rewriting equivalent norm for V, *i.e.* $\|u\|_V^2 = \langle u, Lu\rangle$ we have

$$\begin{aligned} A(t) &\le \varkappa\nu F(u(t))\left(-\nu\|u\|_V^2 + C_1\nu^{-1}\|h\|_{L^2}^2 + \mathfrak{B}\right) \\ &\le -\varkappa\nu^2 F(u(t)) + \varkappa\nu^2 K(\nu, \varkappa, \mathfrak{B}, h) \end{aligned}$$

substituting into (2.52) and using Fatou to pass to limit as $n \to \infty$, we obtain

$$\mathbb{E}\exp\left(\varkappa\nu\|u(t)\|_{L^2}^2\right) + \varkappa\nu^2\int_0^t \mathbb{E}\exp\left(\varkappa\nu\|u(s)\|_{L^2}^2\right) \le \varkappa\nu^2 K(\nu, \varkappa, \mathfrak{B}, h)$$

conclude by Gronwall's. □

3 Existence of Stationary Measure for (2.19) and (2.27)

Now, with Bogolyubov-Krylov argument Theorem 1.3, and its corollary 1.1 at hand, and with 2D NS equations with random kick force (2.19), and with white noise (2.27) as the equations of which we consider the stationary measure, we prove their Existence in the section. Recall we've already made sense of the Solutions in a Markovian Perspective, and defined the notion of stationary measure in each setting.

3.1 Random Kick Force

Theorem 3.1 (Existence of Stationary Measure for (2.19)). *Let $h \in H$ be deterministic function in H and $\{\eta_k\}$ be sequence of i.i.d random variables in H with finite second moment $\mathbb{E}\|\eta_1\|_{L^2} < \infty$. Then* (2.19) *has at least one stationary measure. Moreover, every stationary measure $\mu \in \mathcal{P}(H)$ satisfies*

$$\int_H \|u\|_{L^2} \mu(du) < \infty \tag{3.1}$$

Proof. First, we show existence of stationary measure using Corollary 1.1, showing it holds for initial data $v = 0$. Recall (2.23) $u_k = S_T(u_{k-1}) + \eta_k$, so it suffices to construct for any $\epsilon > 0$ two compact sets K^1, $K^2 \subset H$ such that

$$\mathbb{P}\{S_T(u_{k-1}) \notin K^1\} \leq \epsilon, \qquad \mathbb{P}\{\eta_k \notin K^2\} \leq \epsilon$$

where $u_k = u(kT)$ for u solution as defined in Theorem 2.4 with $u_0 = 0$. If so, take $K = K^1 + K^2$, we see $\mathbb{P}\{u_k \notin K\} \leq 2\epsilon$ for any $k \geq 1$. Notice H is Polish space, so by Ulam's Theorem 1.1, for single random variable η_k, existence of K^2 is given for free. To construct K^1, recall (2.25), since initial data is $u_0 = v = 0$, we have

$$\mathbb{E}\|u_k\|_{L^2} \leq C \qquad for\ all\ k \geq 0$$

In other words, for given $\epsilon > 0$, there exists R_ϵ such that, by Chebyshev's inequality

$$\mathbb{P}\{|u_{k-1}| > R_\epsilon\} \leq R_\epsilon^{-1}\,\mathbb{E}\|u_{k-1}\|_{L^2} \leq \epsilon \qquad \forall\ k \geq 1$$

Then recall continuity property of S_T 2.4, S_T is continuous from H to V, hence for closed ball $B_H(R_\epsilon)$

in H of radius R_ϵ, we have compactness of ball $K^1 := S_T(B_H(R_\epsilon))$ via continuity, and that

$$\mathbb{P}\{S_T(u_{k-1}) \notin K^1\} \leq \epsilon$$

Now we prove (3.1). Take any stationary measure $\mu \in \mathcal{P}(H)$ to (2.19), we have $\mathfrak{B}_k^*\mu = \mu$ for any $k \geq 0$. Under duality perspective (1.8), we have for any $f \in L^\infty(H)$, $(\mathfrak{B}_k f, \mu) = (f, \mu)$ for any $k \geq 0$. Now fix a constant $R > 0$ and take function $f_R : H \to \mathbb{R}$ defined by

$$f_R(u) := \begin{cases} \|u\|_{L^2} & \|u\|_{L^2} \leq R \\ R & \|u\|_{L^2} > R \end{cases}$$

Then for any $k \geq 0$, we have for P_k transition function associated with $(u_k, \mathbb{P}_v)$

$$\int_H f_R(u)\mu(du) = \int_H \int_H P_k(u, dv) f_R(v)\mu(du) \tag{3.2}$$

We estimate RHS of (3.2). For $\|u\|_{L^2} \leq \rho$, by (2.25) with $m = 1$, we have for fixed $u \in H$

$$\int_H P_k(u, dv) f_R(v) \leq \mathbb{E}\, \|u_k\|_{L^2} \leq q^k \rho + C \tag{3.3}$$

where u_k is trajectory of (2.23) with $u_0 = u$, and $C > 0$, $q < 1$ constants independent of u, k, R. Substituting (3.3) into (3.2) and using $f_R \leq R$ for any $R > 0$ for $\|u\|_{L^2} > \rho$, where $\rho > 0$, we have

$$\int_H f_R(u)\mu(du) \leq R\mu(H \setminus B_H(\rho)) + q^k \rho + C$$

pass $k \to \infty$, then $\rho \to \infty$, we have

$$\int_H f_R(u)\mu(du) \leq C$$

Now apply Fatou's Lemma and push $R \to \infty$ to conclude. □

In fact, (3.1) easily generalizes to, for any stationary measure $\mu \in \mathcal{P}(H)$ to (2.19)

$$\int_H \|u\|_s^m \mu(du) < \infty \qquad \forall\, m \geq 1$$

and

$$\int_H \exp(\varkappa_s \|u\|_s^{p_s})\mu(du) < \infty$$

under suitable modifications of hypothesis for u_0 and η_1.

3.2 White Noise

Theorem 3.2 (Existence of Stationary Measure for (2.27)). *Under hypothesis of Theorem 2.8, the stochastic Navier-Stokes system* (2.27) *with arbitrary* $\nu > 0$ *has a stationary measure. Moreover, any stationary measure* $\mu_\nu \in \mathcal{P}(H)$ *satisfies relations*

$$\int_H \left(\nu \|u\|_1^2 + \exp\left(\varkappa\nu \|u\|_{L^2}^2\right)\right) \mu_\nu(du) \le C\left(\mathfrak{B} + \nu^{-1} \|h\|_{L^2}^2\right) \tag{3.4}$$

$$\nu \int_H \|u\|_1^2 \mu_\nu(du) = \frac{\mathfrak{B}}{2} + \int_H \langle u, h\rangle \mu_\nu(du) \tag{3.5}$$

where positive constants $\varkappa$, C *do not depend on* ν. *(Recall* $\|\cdot\|_1$ *denotes norm in* $V = H^1 \cap H$*).*

Proof. First, we show existence of stationary measure to (2.27) via Bogolyubov-Krylov 1.3. Denote $u(t,x)$ the solution to (2.27) with initial data $u_0 = 0$, and λ_t the law of $u(t)$ regarded as random variable in H. Thus by (2.45), $\lambda_t = \mathfrak{B}_t^*\delta_0$ for $t \ge 0$. The existence of stationary measure will be established if the family $\{\overline{\lambda}_t, t \ge 0\}$ is tight, which is defined as (1.14) with $X = H$.

$$\overline{\lambda}_t(\Gamma) := \frac{1}{t}\int_0^t (\mathfrak{B}_s^*\delta_0)(\Gamma)ds = \frac{1}{t}\int_0^t \int_H P_s(u,\Gamma)\delta_0(du)ds \qquad \Gamma \in \mathcal{B}(H)$$

Since for $d = 2$, H^1 compactly embeds into L^2, the embedding $V \subset H$ is compact. We take $B_V(R)$ closed ball of radius R in V as compact subset of H, hence by definition of tightness, it suffices to prove

$$\sup_{t\ge 0}\overline{\lambda}_t\left(H \setminus B_V(R)\right) \to 0 \qquad \textit{as } R \to \infty \tag{3.6}$$

The tool we use is (2.46) with $u_0 = 0$ and Young's Inequality

$$|\langle u, h\rangle| \le C(\|u\|_{L^2}^2 + \nu^{-1} \|h\|_{L^2}^2)$$

we have

$$\mathbb{E}\int_0^t \|\nabla u(s)\|_{L^2}^2\, ds \leq t(\mathfrak{B} + \nu^{-1}\|h\|_{L^2}^2) \qquad t \geq 0$$

Combining this with Chebyshev

$$\begin{aligned}\overline{\lambda}_t(H \setminus B_V(R)) &= \frac{1}{t}\int_0^t \lambda_s(H \setminus B_V(R))ds = \frac{1}{t}\int_0^t \mathbb{P}\{\|u(s)\|_1 > R\}ds \\ &\leq \frac{1}{tR^2}\,\mathbb{E}\int_0^t \|\nabla u(s)\|_{L^2}^2\, ds \leq \frac{1}{R^2}\left(\mathfrak{B} + \nu^{-1}\|h\|_{L^2}^2\right)\end{aligned}$$

conclude by pushing $R \to \infty$. Now we show that any stationary measure μ_ν satisfies (3.4) and (3.5). Recall we've shown second exponential moment estimate (2.50). By arguing similarly as in the argument to (3.1), one proves second exponential part to (3.4). Now let u_0 be H-valued random variable independent of ζ whose law coincide with μ_ν, our stationary measure, and let $u(t)$ be solution to (2.27), (2.44). Then $u(t)$ is stationary solution, *i.e.*, $\mathbb{E}\|u(t)\|_{L^2}^2 = \mathbb{E}\|u_0\|_{L^2}^2$. Hence (2.46) gives

$$2\nu t\,\mathbb{E}\|u\|_1^2 = \mathfrak{B}t + 2t\,\mathbb{E}\langle u, h\rangle$$

dividing by $2t$ gives (3.5). Now combine (3.5) with Schwartz to deal with $\mathbb{E}\langle u, h\rangle$ and use Friedrich's Inequality to bound L^2-norm of u introduced by Schwartz, then absorb it to the LHS, we arrive at $\|u\|_1^2$ term estimate in (3.4). □

To this end, we've shown the existence of stationary measures to (2.19) and (2.27), and proved basic moment estimates integrating against the stationary measures.

4 Conclusion and Discussions

We've introduced basic Polish space theory, definition of Markov Processes taking value in Polish space, hence Bogolyubov-Krylov as main tools to show existence of stationary measure. On the other hand, we introduced deterministic 2D NS equation on torus, and with random kick forces, or white noise. We made sense of their solutions, in particular, the Markovian perspectives for the stochastic case, and cited the existence and uniqueness of solution theorems. We derived some basics moment estimates for the 2 stochastic cases, mainly focusing on second moment and second exponential moment. Finally we showed existence of stationary measure to both stochastic NS equations using Bogolyubov-Krylov and its corollary, and proved basic moment estimates integrating against the stationary measure.

Our presentation omits details in some steps, yet hope to keep the overall construction and logic flow clear. The major proofs follow from the book [3], very nice material as adviced by Prof. Nersesyan. The author's understanding for the materials are built upon materials cited in the reference section.

Of course, there are large amount of materials skipped during the construction, and the author wish to address their importance in his future studies. Topics include Random Dynamics Systems and Ergodic Theory associated with stochastic Navier-Stokes, Uniqueness and Mixing of the Stationary Measures, and the Inviscid limit.

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References

[1] J. Bedrossian and V. Vicol, *The mathematical analysis of the incompressible Euler and Navier-Stokes equations : an introduction*, ser. Graduate studies in mathematics. American Mathematical Society, 2022. [Online]. Available: https://cir.nii.ac.jp/crid/1130012225924137482

[2] L. C. Evans, *An introduction to stochastic differential equations.* American Mathematical Soc., 2012, vol. 82.

[3] S. Kuksin and A. Shirikyan, *Mathematics of Two-Dimensional Turbulence*, ser. Cambridge Tracts in Mathematics. Cambridge University Press, 2012.

[4] R. M. Dudley, *Real Analysis and Probability*, 2nd ed., ser. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2002.

[5] K. R. Parthasarathy, *Probability measures on metric spaces.* American Mathematical Soc., 2005, vol. 352.

[6] L. Evans, *Partial Differential Equations*, ser. Graduate studies in mathematics. American Mathematical Society, 2010. [Online]. Available: https://books.google.com/books?id=Xnu0o_EJrCQC