Onsager's Conjecture for admissible weak solutions Notes

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1 Problem Setting [1]

we consider the incompressible Euler equations

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = 0 \\ \\ \operatorname{div} v = 0, \end{cases} \tag{1.1}$$

in the periodic setting $x \in \mathbb{T}^3 = \mathbb{R}^3 \setminus \mathbb{Z}^3$, where v is a vector field representing the velocity of the fluid and p is the pressure. We study weak (distributional) solutions v which are Hölder continuous in space, i.e. such that

$$|v(x,t) - v(y,t)| \leq C|x-y|^\beta \quad \text{for all } t \in [0,T] \tag{1.2}$$

for some constant C which is independent of time t. On the other hand, we will write $v \in C^\beta(\mathbb{T}^3 \times [0,T])$ when v is Hölder continuous in the whole space-time. We wish to prove

Theorem 1.1. *Assume $e : [0,T] \to \mathbb{R}$ is a* strictly positive *smooth function. Then for any $0 < \beta < 1/3$ there exists a weak solution $v \in C^\beta(\mathbb{T}^3 \times [0,T])$ to (1.1) such that*

$$\int_{\mathbb{T}^3} |v(x,t)|^2 \, dx = e(t)\,.$$

Moreover, we have stronger version, namely the h-principle, saying any smooth strict subsolution can be suitably approximated by C^β solutions for any $\beta < 1/3$.

Definition 1.1. *A smooth strict subsolution of (1.1) on $\mathbb{T}^3 \times [0,T]$ is a smooth triple $(\bar{v}, \bar{p}, \bar{R})$ with $\bar{R}$ a symmetric 2-tensor, such that*

$$\begin{cases} \partial_t \bar{v} + \operatorname{div}(\bar{v} \otimes \bar{v}) + \nabla \bar{p} = -\operatorname{div} \bar{R} \\ \\ \operatorname{div} \bar{v} = 0, \end{cases} \tag{1.3}$$

and $\bar{R}(x,t)$ is positive definite for all (x,t). 'Smooth' comes from smootheness of the triple, 'subsolution' comes from the right-hand-side $-\operatorname{div} \bar{R}$ and that $\bar{R} \geq 0$ a.e., and 'strict' comes from the requirement $\bar{R} > 0$ a.e..

Theorem 1.2 (h-principle)**.** *Let $(\bar{v}, \bar{p}, \bar{R})$ be a smooth strict subsolution of the Euler equations on $\mathbb{T}^3 \times [0,T]$ and let $\beta < 1/3$. Then there exists a sequence (v_k, p_k) of weak solutions of (1.1) such that $v_k \in C^\beta(\mathbb{T}^3 \times [0,T])$,*

$$v_k \overset{*}{\rightharpoonup} \bar{v} \quad \text{and} \quad v_k \otimes v_k \overset{*}{\rightharpoonup} \bar{v} \otimes \bar{v} + \bar{R} \quad \text{in} \quad L^\infty$$

uniformly in time, and furthermore for all $t \in [0,T]$

$$\int_{\mathbb{T}^3} |v_k|^2 \, dx = \int_{\mathbb{T}^3} \left(|\bar{v}|^2 + \operatorname{tr} \bar{R} \right) dx. \tag{1.4}$$

2 Outline

2.1 Inductive Proposition

Proposition 2.1. *There is a universal constant M with the following property. Assume $0 < \beta < 1/3$ and*

$$1 < b < \frac{1-\beta}{2\beta}\,. \tag{2.1}$$

Then there exists an α_0 depending on β and b, such that for any $0 < \alpha < \alpha_0$ there exists an a_0 depending on β, b, α and M, such that for any $a \geq a_0$ the following holds: ***Given a strictly positive energy function*** *$e : [0, T] \to \mathbb{R}$ satisfying*

$$\sup_{t\in[0,T]} \left|\tfrac{d}{dt} e(t)\right| \leq 1 \tag{2.2}$$

and ***a triple*** $(v_q, \mathring{R}_q, p_q)$ ***solving the Euler-Reynolds system*** (1.3)*, namely such that*

$$\begin{cases} \partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q = \operatorname{div} \mathring{R}_q \\ \\ \operatorname{div} v_q = 0\,, \end{cases} \tag{2.3}$$

to which we add the constraints that

$$\operatorname{tr} \mathring{R}_q = 0 \tag{2.4}$$

and that

$$\int_{\mathbb{T}^3} p_q(x,t)\,dx = 0 \tag{2.5}$$

(which uniquely determines the pressure) and ***satisfying the estimates***

$$\left\|\mathring{R}_q\right\|_0 \leq \delta_{q+1}\lambda_q^{-3\alpha} \tag{2.6}$$

$$\|v_q\|_1 \leq M\delta_q^{1/2}\lambda_q \tag{2.7}$$

$$\|v_q\|_0 \leq 1 - \delta_q^{1/2} \tag{2.8}$$

$$\delta_{q+1}\lambda_q^{-\alpha} \leq e(t) - \int_{\mathbb{T}^3} |v_q|^2\,dx \leq \delta_{q+1} \tag{2.9}$$

where the size of the approximate solution v_q and the error $\mathring{R}_q$ are measured by a frequency λ_q and an amplitude δ_q given by

$$\lambda_q = 2\pi\lceil a^{(b^q)}\rceil \tag{2.10}$$

$$\delta_q = \lambda_q^{-2\beta} \tag{2.11}$$

where $\lceil x\rceil$ denotes the smallest integer $n \geq x$ (as required, $a > 1$ is a large parameter, $b > 1$ is close to 1 and both a and b are related to $0 < \beta < 1/3$). ***Then there exists a solution*** $(v_{q+1}, \mathring{R}_{q+1}, p_{q+1})$ ***to*** (2.3)-(2.5) ***satisfying the estimates*** (2.6)–(2.9) *with q replaced by $q + 1$.* ***Moreover****, we have*

$$\|v_{q+1} - v_q\|_0 + \frac{1}{\lambda_{q+1}}\|v_{q+1} - v_q\|_1 \leq M\delta_{q+1}^{1/2}. \tag{2.12}$$

2.2 Proof of Theorem 1.1

- First fix Hölder exponent $\beta < 1/3$, fix b satisfying (2.1) and then fix α smaller than threshold α_0. By Proposition 2.1, a_0 exists depending on β, b, α, M. But we're free to choose $a \geq a_0$. In particular, we first choose $a > 1$.

- **Claim**: We may further assume the energy profile satisfies

$$\inf_t e(t) \geq \delta_1 \lambda_0^{-\alpha}, \qquad \sup_t e(t) \leq \delta_1, \quad \text{and} \quad \sup_t e'(t) \leq 1\,, \tag{2.13}$$

provided the parameter a is chosen sufficiently large.

Proof. Note that the Euler equations are invariant under the transformation

$$v(x,t) \mapsto \Gamma v(x, \Gamma t) \quad \text{and} \quad p(x,t) \mapsto \Gamma^2 p(x, \Gamma t)\,.$$

so the stated problem reduces to finding a solution with the energy profile given by

$$\tilde{e}(t) = \Gamma^2 e(\Gamma t)\,,$$

Choose

$$\Gamma = \left(\frac{\delta_1}{\sup_t e(t)} \right)^{1/2},$$

so we have

$$\inf_t \tilde{e}(t) \geq \frac{\delta_1 \inf_t e(\Gamma t)}{\sup_t e(t)}, \qquad \sup_t \tilde{e}(t) \leq \delta_1, \qquad \text{and} \qquad \sup_t \tilde{e}'(t) \leq \left(\frac{\delta_1}{\sup_t e(t)} \right)^{3/2} \sup_t e'(\Gamma t).$$

If a is chosen sufficiently large, *i.e.*, λ_0 large and δ_1 small, we have

$$\sup_t \tilde{e}'(t) \leq \left(\frac{\delta_1}{\sup_t e(t)} \right)^{3/2} \sup_t e'(\Gamma t) \leq 1, \qquad \text{and} \qquad \frac{\inf_t e(\Gamma t)}{\sup_t e(t)} \geq \lambda_0^{-\alpha}\,.$$

□

- Apply Proposition 2.1 iteratively starting with $(v_0, R_0, p_0) = (0,0,0)$. Indeed the pair (v_0, R_0) trivially satisfies (2.6)–(2.8), whereas the estimate (2.9) and (2.2) follows as our assumption on energy profile (2.13). So the result of Proposition 2.1 says there exists sequence of solutions $(v_q, \mathring{R}_q, p_q)$ to (2.3)-(2.5) satisfying the estimates (2.6)–(2.9), along with (2.12).

- Note as $q \to \infty$, $\delta_q \to 0$, so (2.12) says v_q converges uniformly to some continuous v. Note the pressure is determined by

$$\Delta p_q = \nabla \cdot \nabla p_q = \operatorname{div} \operatorname{div}(-v_q \otimes v_q + \mathring{R}_q) \tag{2.14}$$

and(2.5) and thus p_q is also converging to some pressure p (for the moment only in L^r for every $r < \infty$). Since $\mathring{R}_q \to 0$ uniformly, the pair (v,p) solves the Euler equations. Now we show regularity of v.

- *Spatial Regularity.* Observe that using (2.12) we also infer for all $\beta' < \beta < 1/3$, by (A.3) [1]

$$\sum_{q=0}^{\infty} \|v_{q+1} - v_q\|_{\beta'} \lesssim \sum_{q=0}^{\infty} \|v_{q+1} - v_q\|_0^{1-\beta'} \|v_{q+1} - v_q\|_1^{\beta'} \lesssim \sum_{q=0}^{\infty} \delta_{q+1}^{\frac{1-\beta'}{2}} \left(\delta_{q+1}^{1/2} \lambda_{q+1} \right)^{\beta'} \lesssim \sum_{q=0}^{\infty} \lambda_{q+1}^{\beta' - 2\beta} < \infty$$

due to choice of a, $b > 1 \implies \sum_{q=0}^{\infty} \lambda_{q+1}^{-\epsilon} < \infty \; \forall\, \epsilon > 0$, so v_q is uniformly bounded in $C_t^0 C_x^{\beta'}$ for all $\beta' < \beta$.

- *Time Regularity.* Fix a smooth standard mollifier ψ *in space* and define $\psi_\ell(x) = \ell^{-3} \psi(x \ell^{-1})$. Let $q \in \mathbb{N}$, and consider $\tilde{v}_q := v * \psi_{2^{-q}}$. From standard mollification estimates (A.4) we have

$$\|\tilde{v}_q - v\|_0 \lesssim \|v\|_{\beta'} 2^{-q\beta'}, \tag{2.15}$$

[1] Throughout the manuscript we use the the notation $x \lesssim y$ to denote $x \leq Cy$, for a sufficiently large constant $C > 0$, which is independent of a, b, and q, but may change from line to line.

and thus $\tilde{v}_q - v \to 0$ uniformly as $q \to \infty$. Moreover, $\tilde{v}_q$ obeys the following equation

$$\partial_t \tilde{v}_q + \operatorname{div}(v \otimes v) * \psi_{2^{-q}} + \nabla p * \psi_{2^{-q}} = 0.$$

Next, since

$$-\Delta p * \psi_{2^{-q}} = \operatorname{div}\operatorname{div}(v \otimes v) * \psi_{2^{-q}}\,,$$

using Schauder's estimates, for any fixed $\varepsilon > 0$ we get

$$\|\nabla p * \psi_{2^{-q}}\|_0 \leq \|\nabla p * \psi_{2^{-q}}\|_\varepsilon \lesssim \|v \otimes v\|_{\beta'} 2^{q(1+\varepsilon-\beta')} \lesssim \|v\|_{\beta'}^2 2^{q(1+\varepsilon-\beta')}\,,$$

(where the constant in the estimate depends on ε but not on q). Similarly,

$$\|(v \otimes v) * \psi_{2^{-q}}\|_1 \lesssim \|v \otimes v\|_{\beta'} \, 2^{q(1-\beta')} \lesssim \|v\|_{\beta'}^2 \, 2^{q(1-\beta')}\,.$$

Hence

$$\|\partial_t \tilde{v}_q\|_0 = \|\operatorname{div}(v \otimes v) * \psi_{2^{-q}} + \nabla p * \psi_{2^{-q}}\|_0 \lesssim \|v\|_{\beta'}^2 2^{q(1+\varepsilon-\beta')}\,. \tag{2.16}$$

Next, for $\beta'' < \beta'$, again by standard interpolation (A.3), we conclude from (2.15) and (2.16) that

$$\begin{aligned}
\|\tilde{v}_q - \tilde{v}_{q+1}\|_{C^0_x C^{\beta''}_t} &\lesssim \left(\|\tilde{v}_q - v\|_0 + \|\tilde{v}_{q+1} - v\|_0\right)^{1-\beta''} \left(\|\partial_t \tilde{v}_q\|_0 + \|\partial_t \tilde{v}_{q+1}\|_0\right)^{\beta''} \\
&\lesssim \|v\|_{\beta'}^{1+\beta''} \, 2^{-q\beta'(1-\beta'')} 2^{q\beta''(1+\varepsilon-\beta')} = \|v\|_{\beta'}^{1+\beta''} \, 2^{-q(\beta'-(1+\varepsilon)\beta'')} \\
&\lesssim \|v\|_{\beta'}^{1+\beta''} \, 2^{-q\varepsilon}
\end{aligned}$$

with choice of $\varepsilon > 0$ sufficiently small in terms of β' and β'' so that that $\beta' - (1+\varepsilon)\beta'' \geq \varepsilon$. Thus, the series

$$v = \tilde{v}_0 + \sum_{q \geq 0} (\tilde{v}_{q+1} - \tilde{v}_q)$$

converges in $C^0_x C^{\beta''}_t$. Since we already know $v \in C^0_t C^{\beta'}_x$, we obtain that $v \in C^{\beta''}([0,T] \times \mathbb{T}^3)$ as desired, with $\beta'' < \beta' < \beta < 1/3$ arbitrary.

- Finally, since $\delta_{q+1} \to 0$ as $q \to \infty$, from (2.9) we have

$$\int_{\mathbb{T}^3} |v|^2 \, dx = e(t)\,,$$

which completes the proof of the theorem.

2.3 Stages

The majority of paper is devoted to the proof of Proposition 2.1. Note with conditions given for Proposition 2.1, we fix M, β, and b, and the proof lies in choosing threshold α_0 so that $\alpha < \alpha_0$ is sufficiently small. Then depending also on $\alpha < \alpha_0$, we can choose threshold a_0 so that $a \geq a_0$ is sufficiently large. Hence we're free to make assumptions on 'smallness' of α, and 'largeness' of a that, recalling (2.10), (2.11)

- α is small enough so we have

$$\lambda_q^{3\alpha} \leq \left(\frac{\delta_q}{\delta_{q+1}}\right)^{3/2} \leq \frac{\lambda_{q+1}}{\lambda_q}\,, \tag{2.17}$$

- which also require that a is large enough to absorb any constant appearing from the ratio $\lambda_q / a^{(b^q)}$, for which we have the elementary bounds

$$2\pi \leq \frac{\lambda_q}{a^{b^q}} \leq 4\pi \,. \tag{2.18}$$

The proof consists of three stages, in each of which we modify v_q. Roughly speaking, the stages are as follows:

(i) Mollification: $(v_q, \mathring{R}_q) \mapsto (v_\ell, \mathring{R}_\ell)$;

(ii) Gluing: $(v_\ell, \mathring{R}_\ell) \mapsto (\overline{v}_q, \overline{\mathring{R}}_q)$;

(iii) Perturbation: $(\overline{v}_q, \overline{\mathring{R}}_q) \mapsto (v_{q+1}, \mathring{R}_{q+1})$.

3 Mollification step $(v_q, \mathring{R}_q) \mapsto (v_\ell, \mathring{R}_\ell)$

The first stage is mollification: **we mollify v_q at length scale ℓ in order to handle the *loss of derivative* problem**, typical of convex integration schemes. To this aim, we fix a standard mollification kernel ψ *in space* and introduce the mollification parameter

$$\ell := \frac{\delta_{q+1}^{1/2}}{\delta_q^{1/2}\lambda_q^{1+3\alpha/2}} \,, \tag{3.1}$$

and define, recalling $\psi_\ell(x) = \ell^{-3}\psi(x\ell^{-1})$, and $f \mathring{\otimes} g$ is the traceless part of the tensor $f \otimes g$.

$$\begin{aligned} v_\ell &:= v_q * \psi_\ell \\ \mathring{R}_\ell &:= \mathring{R}_q * \psi_\ell + (v_q \mathring{\otimes} v_q) * \psi_\ell - v_\ell \mathring{\otimes} v_\ell \end{aligned}$$

$(v_\ell, \mathring{R}_\ell)$ obey the equation

$$\begin{cases} \partial_t v_\ell + \operatorname{div}(v_\ell \otimes v_\ell) + \nabla p_\ell = \operatorname{div} \mathring{R}_\ell \\ \\ \operatorname{div} v_\ell = 0 \,, \end{cases} \tag{3.2}$$

in view of (2.3). Observe, again

- choosing α sufficiently small and a sufficiently large we can assume

$$\lambda_q^{-3/2} \leq \ell = \frac{\delta_{q+1}^{1/2}}{\delta_q^{1/2}\lambda_q^{1+3\alpha/2}} \leq \lambda_q^{-1} \,, \tag{3.3}$$

  which will be applied repeatedly in order to simplify the statements of several estimates.

From (3.3), standard mollification estimates (A.4) and Proposition A.1 we obtain the following bounds[2]

Proposition 3.1.

$$\|v_\ell - v_q\|_0 \lesssim \delta_{q+1}^{1/2}\lambda_q^{-\alpha} \,, \tag{3.4}$$

$$\|v_\ell\|_{N+1} \lesssim \delta_q^{1/2}\lambda_q \ell^{-N} \qquad \forall N \geq 0 \,, \tag{3.5}$$

$$\left\|\mathring{R}_\ell\right\|_{N+\alpha} \lesssim \delta_{q+1}\ell^{-N+\alpha} \qquad \forall N \geq 0 \,. \tag{3.6}$$

$$\left| \int_{\mathbb{T}^3} |v_q|^2 - |v_\ell|^2 \, dx \right| \lesssim \delta_{q+1}\ell^{\alpha} \,. \tag{3.7}$$

Proof of Proposition 3.1. The bounds (3.4) and (3.5) follow from the estimate using (A.4), (2.7), (2.17)

$$\|v_\ell - v_q\|_0 = \|v_q * \psi_\ell - v_q\|_0 \leq \|v_q\|_1 \ell \lesssim \delta_q^{1/2}\lambda_q \ell \lesssim \delta_{q+1}^{1/2}\lambda_q^{-\alpha}$$

[2] In the following, when considering higher order norms $\|\cdot\|_N$ or $\|\cdot\|_{N+1}$, the symbol $\lesssim$ will imply that the constant in the inequality might also depend on N.

and again using (2.7)

$$\|v_\ell\|_{N+1} \le \|v_q\|_1 \|\psi_\ell\|_N \le \|v_q\|_1 \ell^{-N} \lesssim \delta_q^{1/2}\lambda_q \ell^{-N}\,.$$

Next, applying Proposition A.1, using (2.6), (2.7) to estimate size of $\|\mathring{R}_q\|_0$, $\|v_q\|_1$, and then assumptions (3.3), followed by (2.17)

$$\begin{aligned}\left\|\mathring{R}_\ell\right\|_{N+\alpha} &\lesssim \|\mathring{R}_q * \psi_\ell\|_{N+\alpha} + \|(v_q \mathring{\otimes} v_q) * \psi_\ell - v_\ell \mathring{\otimes} v_\ell\|_{N+\alpha} \\ &\lesssim \|\mathring{R}_q\|_0 \ell^{-N-\alpha} + \|v_q\|_1^2 \ell^{2-N-\alpha} \lesssim \delta_{q+1}\lambda_q^{-3\alpha}\ell^{-N-\alpha} + \delta_q\lambda_q^2\ell^2\ell^{-N-\alpha} \lesssim \delta_{q+1}\lambda_q^{-3\alpha}\ell^{-N-\alpha}\,,\end{aligned}$$

on the other hand, by (3.3) $\lambda_q^{-3\alpha} \le \ell^{2\alpha}$, from which (3.6) follows. Similarly, by Proposition A.1,

$$\left|\int_{\mathbb{T}^3} |v_q|^2 - |v_\ell|^2 \, dx\right| = \left|\int_{\mathbb{T}^3} (|v_q|^2)_\ell - |v_\ell|^2 \, dx\right| \lesssim \left\|(|v_q|^2)_\ell - |v_\ell|^2\right\|_0 \lesssim \|v_q\|_1^2 \ell^2\,,$$

which implies (3.7). □

4 Gluing Step $(v_\ell, \mathring{R}_\ell) \mapsto (\bar{v}_q, \mathring{\bar{R}}_q)$

We glue together exact solutions to the Euler equations in order to produce a new $\overline{v}_q$, close to v_q, whose associated Reynolds stress error $\mathring{\bar{R}}_q$ has support in pairwise disjoint temporal regions of length τ_q in time, where

$$\tau_q = \frac{\ell^{2\alpha}}{\delta_q^{1/2}\lambda_q}. \tag{4.1}$$

Note hence we have the CFL-like condition

$$\tau_q \|v_\ell\|_{1+\alpha} \overset{(3.5)}{\lesssim} \tau_q \delta_q^{1/2}\lambda_q \ell^{-\alpha} \lesssim \ell^\alpha \ll 1 \tag{4.2}$$

as long as a is sufficiently large.

4.1 Stability Estimate for Classical Exact Solutions

4.1.1 Classical solutions

For each i, let $t_i = i\tau_q$, and consider smooth solutions v_i of the Euler equations with t_i as initial time and v_ℓ at time t_i as initial value

$$\begin{cases} \partial_t v_i + \operatorname{div}(v_i \otimes v_i) + \nabla p_i = 0 \\ \operatorname{div} v_i = 0 \\ v_i(\cdot, t_i) = v_\ell(\cdot, t_i)\,. \end{cases} \tag{4.3}$$

defined over their own maximal interval of existence.

Proposition 4.1. *For any $\alpha > 0$ there exists a constant $c = c(\alpha) > 0$ with the following property. Given any initial data $v_0 \in C^\infty$, and $T \le c\|v_0\|_{1+\alpha}$, there exists a unique solution $v : \mathbb{R}^3 \times [-T, T] \to \mathbb{R}^3$ to the Euler equation*

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = 0 \\ \operatorname{div} v = 0\,, \\ v(\cdot, 0) = v_0 \end{cases}$$

Moreover, v obeys the higher-order bounds

$$\|v\|_{N+\alpha} \lesssim \|v_0\|_{N+\alpha} \ . \tag{4.4}$$

for all $N \geq 1$, where the implicit constant depends on N and $\alpha > 0$.

Proof of Proposition 4.1. The existence of a unique solution follows from the restriction $T \leq c\,\|v_0\|_{1+\alpha}$. The higher-order bounds (4.4) are obtained as follows: For any multi-index θ with $|\theta| = N$, let commutator

$$[\partial^\theta, v\cdot\nabla]v := \partial^\theta(v\cdot\nabla)v - v\cdot\nabla\left(\partial^\theta v\right)$$

we have

$$\partial_t\partial^\theta v + v\cdot\nabla\partial^\theta v + [\partial^\theta, v\cdot\nabla]v + \nabla\partial^\theta p = 0.$$

Using the equation for the pressure $-\Delta p = \nabla v\cdot\nabla v$ and Schauder estimates we obtain

$$\|\nabla\partial^\theta p\|_\alpha \lesssim \|\nabla p\|_{N+\alpha} \lesssim \|\nabla v\cdot\nabla v\|_{N-1+\alpha} \lesssim \|v\|_{1+\alpha}\|v\|_{N+\alpha}.$$

Therefore, after applying (C.3) to $[\partial^\theta, v\cdot\nabla]v$, we're left with

$$\|(\partial_t + v\cdot\nabla)\partial^\theta v\|_\alpha \lesssim \|v\|_{1+\alpha}\|v\|_{N+\alpha}.$$

Hence by applying (B.3)

$$\|v\|_{N+\alpha} \lesssim \|\partial^\theta v\|_\alpha \lesssim \|\partial^\theta v_0\|_\alpha + \int_0^T \|(\partial_t + v\,(\cdot,\tau)\cdot\nabla)\partial^\theta v\,(\cdot,\tau)\,\|_\alpha d\tau \lesssim \|v_0\|_{N+\alpha} + \int_0^T \|v\|_{1+\alpha}\|v\|_{N+\alpha}d\tau,$$

and Grönwall's inequality we recover (4.4). □

Corollary 4.1 (Length-scale for v_i)**.** *If a is sufficiently large, for $|t - t_i| \leq \tau_q$, we have*

$$\|v_i\|_{N+\alpha} \lesssim \delta_q^{1/2}\lambda_q\ell^{1-N-\alpha} \overset{(4.1)}{\lesssim} \tau_q^{-1}\ell^{1-N+\alpha} \qquad \textit{for any } N \geq 1. \tag{4.5}$$

Proof of Corollary 4.1. We apply Proposition 4.1 and using assumption $|t - t_i| \leq \tau_q$ with

$$(4.2) \quad \tau_q\,\|v_\ell\|_{1+\alpha} \lesssim \tau_q\delta_q^{1/2}\lambda_q\ell^{-\alpha} \lesssim \ell^\alpha \ll 1 \implies |t - t_i|\,\|v_\ell\|_{1+\alpha} \ll 1$$

to satisfy assumption for (B.3), from which the higher-order estimates of Proposition 4.1 says

$$\|v_i\|_{N+\alpha} \lesssim \|v_i(t_i)\|_{N+\alpha} = \|v_\ell(t_i)\|_{N+\alpha}$$

for any $N \geq 1$. From

$$(3.5) \quad \|v_\ell\|_{N+1} \lesssim \delta_q^{1/2}\lambda_q\ell^{-N}$$

we then deduce the estimate (4.5). □

4.1.2 Stability and estimates on $v_i - v_\ell$

We will now show that for $|t_i - t| \leq \tau_q$, v_i is close to v_ℓ and by the identity

$$v_i - v_{i+1} = (v_i - v_\ell) - (v_{i+1} - v_\ell),$$

the vector field v_i is also close to v_{i+1}.

Proposition 4.2. *For $|t-t_i| \le \tau_q$ and $N \ge 0$ we have*

$$\|v_i - v_\ell\|_{N+\alpha} \lesssim \tau_q \delta_{q+1} \ell^{-N-1+\alpha}\,, \tag{4.6}$$
$$\|\nabla(p_\ell - p_i)\|_{N+\alpha} \lesssim \delta_{q+1} \ell^{-N-1+\alpha}\,, \tag{4.7}$$
$$\|D_{t,\ell}(v_i - v_\ell)\|_{N+\alpha} \lesssim \delta_{q+1} \ell^{-N-1+\alpha}\,, \tag{4.8}$$

where we write

$$D_{t,\ell} = \partial_t + v_\ell \cdot \nabla \tag{4.9}$$

for the transport derivative.

Proof of Proposition 4.2. • Let us first consider (4.6) with $N=0$. From the system (3.2) that v_ℓ solves and the system (4.3) that v_i solves, we have

$$\partial_t(v_\ell - v_i) + (v_\ell \cdot \nabla)(v_\ell - v_i) = D_{t,\ell}(v_\ell - v_i) = (v_i - v_\ell)\cdot \nabla v_i - \nabla(p_\ell - p_i) + \operatorname{div} \mathring{R}_\ell. \tag{4.10}$$

In particular, using

$$\Delta(p_\ell - p_i) = \operatorname{div}\big(\nabla v_\ell (v_\ell - v_i)\big) + \operatorname{div}\big(\nabla v_i (v_\ell - v_i)\big) + \operatorname{div}\operatorname{div} \mathring{R}_\ell, \tag{4.11}$$

along with estimates

$$\text{(3.6)} \quad \left\|\operatorname{div} \mathring{R}_\ell\right\|_\alpha \lesssim \left\|\mathring{R}_\ell\right\|_{1+\alpha} \lesssim \delta_{q+1}\ell^{-1+\alpha}$$

$$\text{(3.5)} \quad \|\nabla v_\ell\|_\alpha \lesssim \|v_\ell\|_{1+\alpha} \lesssim \delta_q^{1/2}\lambda_q \ell^{-\alpha} \qquad \text{(4.5)} \quad \|\nabla v_i\|_\alpha \lesssim \|v_i\|_{1+\alpha} \lesssim \delta_q^{1/2}\lambda_q \ell^{-\alpha}$$

and Proposition C.1 (recall that $\partial_i\partial_j(-\Delta)^{-1}$ is given by $1/3\delta_{ij}$ + a Calderón-Zygmund operator), we conclude

$$\|\nabla(p_\ell - p_i)(\cdot,t)\|_\alpha \le \delta_q^{1/2}\lambda_q\ell^{-\alpha}\|v_i - v_\ell\|_\alpha + \delta_{q+1}\ell^{-1+\alpha}\,.$$

Thus, using (3.6) and the definition of $\tau_q = \frac{\ell^{2\alpha}}{\delta_q^{1/2}\lambda_q}$, we have

$$\|D_{t,\ell}(v_\ell - v_i)\|_\alpha = \left\|(v_i - v_\ell)\cdot\nabla v_i - \nabla(p_\ell - p_i) + \operatorname{div}\mathring{R}_\ell\right\|_\alpha \lesssim \delta_{q+1}\ell^{-1+\alpha} + \tau_q^{-1}\|v_\ell - v_i\|_\alpha \tag{4.12}$$

Note $D_{t,\ell} = \partial_t + v_\ell\cdot\nabla$, and we again have by combining $|t-t_i| \le \tau_q$ and

$$\text{(4.2)} \quad \tau_q \|v_\ell\|_{1+\alpha} \lesssim \tau_q\delta_q^{1/2}\lambda_q\ell^{-\alpha} \lesssim \ell^\alpha \ll 1 \implies |t-t_i|\,\|v_\ell\|_{1+\alpha} \ll 1$$

to satisfy assumptions for (B.3). Hence by having $D_{t,\ell}$ acting on $v_\ell - v_i$, we obtain from (B.3)

$$\|(v_\ell - v_i)(\cdot,t)\|_\alpha \lesssim 0 + \int_{t_i}^t \|D_{t,\ell}(v_\ell - v_i)\|_\alpha\, ds \lesssim |t-t_i|\,\delta_{q+1}\ell^{-1+\alpha} + \int_{t_i}^t \tau_q^{-1}\|(v_\ell - v_i)(\cdot,s)\|_\alpha\, ds.$$

Applying Grönwall's inequality and using the assumption $|t-t_i| \le \tau_q$ we obtain

$$\|v_i - v_\ell\|_\alpha \lesssim \tau_q\delta_{q+1}\ell^{-1+\alpha}\,, \tag{4.13}$$

i.e. (4.6) for the case $N=0$. Then, as a consequence of (4.12) we obtain (4.8) for the case $N=0$.

• Next, consider the case $N \ge 1$ and let θ be a multiindex with $|\theta| = N$. Commuting the derivative ∂^θ with the material derivative $D_{t,\ell} = \partial_t + v_\ell \cdot \nabla$ we have

$$\begin{aligned}\|D_{t,\ell}\partial^\theta(v_\ell - v_i)\|_\alpha &\lesssim \|\partial^\theta D_{t,\ell}(v_\ell - v_i)\|_\alpha + \|[v_\ell \cdot \nabla, \partial^\theta](v_\ell - v_i)\|_\alpha \\ &\overset{(C.3)}{\lesssim} \|\partial^\theta D_{t,\ell}(v_\ell - v_i)\|_\alpha + \|v_\ell\|_{N+\alpha}\|v_\ell - v_i\|_{1+\alpha} + \|v_\ell\|_{1+\alpha}\|v_\ell - v_i\|_{N+\alpha} \\ &\lesssim \|\partial^\theta D_{t,\ell}(v_\ell - v_i)\|_\alpha + \|v_\ell\|_{N+1+\alpha}\|v_\ell - v_i\|_\alpha + \|v_\ell\|_{1+\alpha}\|v_\ell - v_i\|_{N+\alpha}\,,\end{aligned}$$

where in the last inequality we used the standard interpolation inequalities on Hölder norms, cf. (A.1). On the other hand differentiating ∂^θ

$$(4.10) \quad D_{t,\ell}\,(v_\ell - v_i) = \partial_t(v_\ell - v_i) + (v_\ell \cdot \nabla)(v_\ell - v_i) = (v_i - v_\ell)\cdot \nabla v_i - \nabla(p_\ell - p_i) + \operatorname{div} \mathring{R}_\ell.$$

leads to

$$\begin{aligned}\|\partial^\theta D_{t,\ell}(v_\ell - v_i)\|_\alpha &\lesssim \|v_\ell - v_i\|_{N+\alpha}\|v_i\|_{1+\alpha} + \|v_\ell - v_i\|_\alpha\|v_i\|_{N+1+\alpha} + \|p_\ell - p_i\|_{N+1+\alpha} + \|\mathring{R}_\ell\|_{N+1+\alpha} \\ &\overset{(4.5)(4.13)(3.6)}{\lesssim} \tau_q^{-1}\ell^\alpha\|v_\ell - v_i\|_{N+\alpha} + \tau_q\delta_{q+1}\ell^{-1+\alpha}\tau_q^{-1}\ell^{-N+\alpha} + \|\nabla(p_\ell - p_i)\|_{N+\alpha} + \delta_{q+1}\ell^{-N-1+\alpha} \qquad (4.14)\\ &\lesssim \tau_q^{-1}\|v_\ell - v_i\|_{N+\alpha} + \delta_{q+1}\ell^{-N-1+\alpha} + \|\nabla(p_\ell - p_i)\|_{N+\alpha}\,. \qquad (4.15)\end{aligned}$$

Furthermore, from

$$(4.11) \quad \Delta(p_\ell - p_i) = \operatorname{div}\big(\nabla v_\ell(v_\ell - v_i)\big) + \operatorname{div}\big(\nabla v_i(v_\ell - v_i)\big) + \operatorname{div}\operatorname{div}\mathring{R}_\ell$$

we also obtain, using Corollary 4.1 and (4.13)

$$\begin{aligned}\|\nabla(p_\ell - p_i)\|_{N+\alpha} &\lesssim (\|v_\ell\|_{N+1+\alpha} + \|v_i\|_{N+1+\alpha})\|v_\ell - v_i\|_\alpha \\ &\quad + (\|v_\ell\|_{1+\alpha} + \|v_i\|_{1+\alpha})\|v_\ell - v_i\|_{N+\alpha} + \|\mathring{R}_\ell\|_{N+1+\alpha} \\ &\lesssim \delta_{q+1}\ell^{-N-1+\alpha} + \tau_q^{-1}\|v_\ell - v_i\|_{N+\alpha}\,. \qquad (4.16)\end{aligned}$$

Summarizing, for any multiindex θ with $|\theta| = N$ we obtain

$$\|D_{t,\ell}\partial^\theta(v_\ell - v_i)\|_\alpha \lesssim \delta_{q+1}\ell^{-N-1+\alpha} + \tau_q^{-1}\|v_\ell - v_i\|_{N+\alpha}.$$

Therefore, invoking once more (B.3) we deduce

$$\|(v_\ell - v_i)(\cdot,t)\|_{N+\alpha} \lesssim \tau_q\delta_{q+1}\ell^{-N-1+\alpha} + \int_{t_i}^t \tau_q^{-1}\|(v_\ell - v_i)(\cdot,s)\|_{N+\alpha}\,ds,$$

and hence, using Grönwall's inequality and the assumption $|t - t_i| \le \tau_q$ we obtain (4.6). From (4.16) and (4.15) we then also conclude (4.7) and (4.8).

□

4.1.3 Estimates on vector potentials

Define the vector potentials to the solutions v_i, *i.e.*, stream function as

$$z_i = \mathcal{B}v_i := (-\Delta)^{-1}\operatorname{curl} v_i, \qquad (4.17)$$

where $\mathcal{B}$ is the Biot-Savart operator, so that

$$\operatorname{div} z_i = 0 \qquad \text{and} \qquad \operatorname{curl} z_i = v_i. \qquad (4.18)$$

Our aim is to obtain estimates for the differences $z_i - z_{i+1}$. The heuristic is as follows: from Proposition 4.2 we obtain

$$(4.6) \quad \|v_i - v_\ell\|_{N+\alpha} \lesssim \tau_q \delta_{q+1} \ell^{-N-1+\alpha} \implies \|v_i - v_{i+1}\|_{N+\alpha} \lesssim \tau_q \delta_{q+1} \ell^{-N-1+\alpha}.$$

Since the characteristic length-scale of the vectorfields v_i is ℓ (cf. Corollary 4.1), we expect to gain a factor ℓ when passing to first order potentials.

Proposition 4.3. *For $|t - t_i| \leq \tau_q$, we have that*

$$\|z_i - z_{i+1}\|_{N+\alpha} \lesssim \tau_q \delta_{q+1} \ell^{-N+\alpha}, \tag{4.19}$$

$$\|D_{t,\ell}(z_i - z_{i+1})\|_{N+\alpha} \lesssim \delta_{q+1} \ell^{-N+\alpha}, \tag{4.20}$$

where $D_{t,\ell} = \partial_t + v_\ell \cdot \nabla$ is as in (4.9).

Proof of Proposition 4.3. Set $\tilde{z}_i := \mathcal{B}(v_i - v_\ell)$ and observe that $z_i - z_{i+1} = \tilde{z}_i - \tilde{z}_{i+1}$. Hence, it suffices to estimate $\tilde{z}_i = \mathcal{B}(v_i - v_\ell) = (-\Delta)^{-1} \operatorname{curl}(v_i - v_\ell)$ in place of $z_i - z_{i+1}$. The estimate on $\|\nabla \tilde{z}_i\|_{N-1+\alpha}$ for $N \geq 1$ follows directly from

$$(4.6)\ \|v_i - v_\ell\|_{N+\alpha} \lesssim \tau_q \delta_{q+1} \ell^{-N-1+\alpha}$$

and the fact that $\nabla \mathcal{B}$ is a bounded operator on Hölder spaces:

$$\|\nabla \tilde{z}_i\|_{N-1+\alpha} = \|\nabla \mathcal{B}(v_i - v_\ell)\|_{N-1+\alpha} \|v_i - v_\ell\|_{N+\alpha} \lesssim \tau_q \delta_{q+1} \ell^{-N+\alpha}. \tag{4.21}$$

Next, observe that

$$\partial_t (v_i - v_\ell) + v_\ell \cdot \nabla (v_i - v_\ell) + (v_i - v_\ell) \cdot \nabla v_i + \nabla (p_i - p_\ell) + \operatorname{div} \mathring{R}_\ell = 0. \tag{4.22}$$

Since $v_i - v_\ell = \operatorname{curl} \tilde{z}_i$ with $\operatorname{div} \tilde{z}_i = 0$, we have[3]

$$\begin{aligned} v_\ell \cdot \nabla (v_i - v_\ell) &= \operatorname{curl}\big((v_\ell \cdot \nabla) \tilde{z}_i\big) + \operatorname{div}\big((\tilde{z}_i \times \nabla) v_\ell\big) \\ ((v_i - v_\ell) \cdot \nabla) v_i &= \operatorname{div}\big((\tilde{z}_i \times \nabla) v_i^T\big), \end{aligned}$$

so that we can write (4.22) as

$$\operatorname{curl}(\partial_t \tilde{z}_i + (v_\ell \cdot \nabla) \tilde{z}_i) = -\operatorname{div}\big((\tilde{z}_i \times \nabla) v_\ell + (\tilde{z}_i \times \nabla) v_i^T\big) - \nabla (p_i - p_\ell) - \operatorname{div} \mathring{R}_\ell. \tag{4.23}$$

Taking the curl of (4.23) the pressure term drops out. Using in addition that $\operatorname{div} \tilde{z}_i = \operatorname{div} v_i = 0$ and the identity $\operatorname{curl}\operatorname{curl} = -\Delta + \nabla \operatorname{div}$, we then arrive at

$$-\Delta\big(\partial_t \tilde{z}_i + (v_\ell \cdot \nabla) \tilde{z}_i\big) = -\nabla \operatorname{div}\left((\tilde{z}_i \cdot \nabla) v_\ell\right) - \operatorname{curl}\operatorname{div}\left((\tilde{z}_i \times \nabla) v_\ell + (\tilde{z}_i \times \nabla) v_i^T\right) - \operatorname{curl}\operatorname{div} \mathring{R}_\ell.$$

Consequently using (3.5) $\|v_\ell\|_{N+1} \lesssim \tau_q^{-1} \ell^{2\alpha} \ell^{-N}$ and (4.5) $\|v_i\|_{N+\alpha} \lesssim \tau_q^{-1} \ell^{1-N+\alpha}$

$$\begin{aligned} \|\partial_t \tilde{z}_i + (v_\ell \cdot \nabla) \tilde{z}_i\|_{N+\alpha} &\overset{(C.3)}{\lesssim} (\|v_i\|_{N+1+\alpha} + \|v_\ell\|_{N+1+\alpha}) \|\tilde{z}_i\|_\alpha \\ &\quad + (\|v_i\|_{1+\alpha} + \|v_\ell\|_{1+\alpha}) \|\tilde{z}_i\|_{N+\alpha} + \|\mathring{R}_\ell\|_{N+\alpha} \\ &\lesssim \tau_q^{-1} \|\tilde{z}_i\|_{N+\alpha} + \tau_q^{-1} \ell^{-N} \|\tilde{z}_i\|_\alpha + \delta_{q+1} \ell^{-N+\alpha}. \end{aligned} \tag{4.24}$$

Setting $N = 0$ and using (B.3) and Grönwall's inequality we obtain $\|\tilde{z}_i\|_\alpha \lesssim \tau_q \delta_{q+1} \ell^\alpha$, which together with

[3] Here we denote $[(z \times \nabla) v]^{ij} = \epsilon_{ikl} z^k \partial_l v^j = \begin{pmatrix} z^2 \partial_3 v^1 - z^3 \partial_2 v^1 & z^2 \partial_3 v^2 - z^3 \partial_2 v^2 & z^2 \partial_3 v^3 - z^3 \partial_2 v^3 \\ z^3 \partial_1 v^1 - z^1 \partial_3 v^1 & z^3 \partial_1 v^2 - z^1 \partial_3 v^2 & z^3 \partial_1 v^3 - z^1 \partial_3 v^3 \\ z^1 \partial_2 v^1 - z^2 \partial_1 v^1 & z^1 \partial_2 v^2 - z^2 \partial_1 v^2 & z^1 \partial_2 v^3 - z^2 \partial_1 v^3 \end{pmatrix}$ for vector fields z, v.

(4.21) gives (4.19). Using (4.19) into (4.24) we conclude

$$\|\partial_t \tilde{z}_i + (v_\ell \cdot \nabla)\tilde{z}_i\|_{N+\alpha} \lesssim \delta_{q+1}\ell^{-N+\alpha}\,.$$

Finally commuting the derivatives in the $N+\alpha$-norm with $D_{t,\ell}$ as in the proof of Proposition 4.2 and using again (4.19) we achieve (4.20).

□

4.2 Gluing Procedure

Now we glue the solutions v_i together in order to construct $\overline{v}_q$. The stability estimates above will be used in order to ensure that $\overline{v}_q$ remains an approximate solution to the Euler equations.

4.2.1 Partition of Unity and definition of $\overline{v}_q$

- Let
$$t_i = i\tau_q, \qquad I_i = [t_i + \tfrac{1}{3}\tau_q, t_i + \tfrac{2}{3}\tau_q] \cap [0,T], \qquad J_i = (t_i - \tfrac{1}{3}\tau_q, t_i + \tfrac{1}{3}\tau_q) \cap [0,T]\,.$$
Note that $\{I_i, J_i\}_i$is a decomposition of $[0,T]$ into pairwise disjoint intervals.

- We define a partition of unity $\{\chi_i\}_i$in time with the following properties:
  - The cut-offs form a partition of unity
  $$\sum_i \chi_i \equiv 1 \tag{4.25}$$
  - $\operatorname{supp}\chi_i \cap \operatorname{supp}\chi_{i+2} = \emptyset$ and moreover
  $$\begin{aligned} \operatorname{supp}\chi_i &\subset (t_i - \tfrac{2}{3}\tau_q, t_i + \tfrac{2}{3}\tau_q) \\ \chi_i(t) &= 1 \quad \text{ for } t \in J_i \end{aligned} \tag{4.26}$$
  - For any i and N we have
  $$\left\|\partial_t^N \chi_i\right\|_0 \lesssim \tau_q^{-N}\,. \tag{4.27}$$

- We define
$$\begin{aligned} \overline{v}_q &= \sum_i \chi_i v_i \\ \overline{p}_q^{(1)} &= \sum_i \chi_i p_i \end{aligned}$$
observe that
  (i) $\operatorname{div}\overline{v}_q = 0$.
  (ii) If $t \in I_i$, then $\chi_i + \chi_{i+1} = 1$ and $\chi_j = 0$ for $j \neq i, i+1$, therefore on I_i:
  $$\begin{aligned} \overline{v}_q &= \chi_i v_i + (1-\chi_i)v_{i+1} \\ \overline{p}_q^{(1)} &= \chi_i p_i + (1-\chi_i)p_{i+1} \end{aligned}$$

and

$$\begin{aligned}\partial_t\overline{v}_q+\operatorname{div}(\overline{v}_q\otimes\overline{v}_q)+\nabla\overline{p}_q^{(1)}&=\chi_i\partial_t v_i+(1-\chi_i)\partial_t v_{i+1}+\partial_t\chi_i(v_i-v_{i+1})\\&\quad+\operatorname{div}\left(\chi_i^2 v_i\otimes v_i+(1-\chi_i)^2 v_{i+1}\otimes v_{i+1}\right)\\&\quad+\chi_i(1-\chi_i)\operatorname{div}(v_i\otimes v_{i+1}+v_{i+1}\otimes v_i)\\&\quad+\chi_i\nabla p_i+(1-\chi_i)\nabla p_{i+1}\\&=\partial_t\chi_i(v_i-v_{i+1})-\chi_i(1-\chi_i)\operatorname{div}\left((v_i-v_{i+1})\otimes(v_i-v_{i+1})\right).\end{aligned}$$

(iii) If $t\in J_i$ then $\chi_i=1$ and $\chi_j=0$ for all $j\neq i$ for all $\tilde{t}$ sufficiently close to t (since J_i is open). Then for all $t\in J_i$ we have

$$\overline{v}_q=v_i,\quad \overline{p}_q^{(1)}=p_i,$$

and, from (4.3),

$$\partial_t\overline{v}_q+\operatorname{div}(\overline{v}_q\otimes\overline{v}_q)+\nabla\overline{p}_q^{(1)}=0.$$

4.2.2 New Reynolds Tensor $\mathring{\overline{R}}_q$

Definition 4.1 (Inverse Divergence Operator for symmetric tracefree 2-tensors)**.**

$$\begin{aligned}(\mathcal{R}f)^{ij}&=\mathcal{R}^{ijk}f^k\\ \mathcal{R}^{ijk}&=-\frac{1}{2}\Delta^{-2}\partial_i\partial_j\partial_k+\frac{1}{2}\Delta^{-1}\partial_k\delta_{ij}-\Delta^{-1}\partial_i\delta_{jk}-\Delta^{-1}\partial_j\delta_{ik}.\end{aligned}\tag{4.28}$$

when acting on vectors $f\in C^\infty\left(\mathbb{T}^3;\mathbb{R}^3\right)$ with zero mean on $\mathbb{T}^3$, i.e. $\int_{\mathbb{T}^3}f dx=0$.

Proposition 4.4. *The tensor $\mathcal{R}$ defined in (4.28) is symmetric, and we have*

$$\operatorname{div}(\mathcal{R}f)=f$$

for any f with zero mean on $\mathbb{T}^3$. So the above inverse divergence operator has the property that $\mathcal{R}f(x)$ is a symmetric trace-free matrix for each $x\in\mathbb{T}^3$, and $\mathcal{R}$ is an right inverse of the div *operator, i.e.* $\operatorname{div}(\mathcal{R}f)=f$*. When f does not obey $\int_{\mathbb{T}^3}f dx=0$, we overload notation and denote $\mathcal{R}f:=\mathcal{R}(f-\int_{\mathbb{T}^3}f dx)$.*

- We define

$$\begin{aligned}\mathring{\overline{R}}_q&=\partial_t\chi_i\mathcal{R}(v_i-v_{i+1})-\chi_i(1-\chi_i)(v_i-v_{i+1})\mathring{\otimes}(v_i-v_{i+1})\\ \overline{p}_q^{(2)}&=-\chi_i(1-\chi_i)|v_i-v_{i+1}|^2,\end{aligned}$$

  for $t\in I_i$ and $\mathring{\overline{R}}_q=0$, $\overline{p}_q^{(2)}=0$ for $t\notin\bigcup_i I_i$.

- We set $\overline{p}_q=\overline{p}_q^{(1)}+\overline{p}_q^{(2)}$
- It follows from the preceding discussion and Proposition 4.4 that
  - $\mathring{\overline{R}}_q$ is a smooth symmetric and traceless 2-tensor;
  - For all $(x,t)\in\mathbb{T}^3\times[0,T]$

$$\begin{cases}\partial_t\overline{v}_q+\operatorname{div}(\overline{v}_q\otimes\overline{v}_q)+\nabla\overline{p}_q=\operatorname{div}\mathring{\overline{R}}_q,\\ \\ \operatorname{div}\overline{v}_q=0;\end{cases}$$

  - $\operatorname{supp}\mathring{\overline{R}}_q\subset\mathbb{T}^3\times\bigcup_i I_i$.

4.2.3 Estimates on $\overline{v}_q$

Next, we estimate the various Hölder norms of $\overline{v}_q$.

Proposition 4.5. *The velocity field $\overline{v}_q$ satisfies the following estimates*

$$\|\overline{v}_q - v_\ell\|_\alpha \lesssim \delta_{q+1}^{1/2}\ell^\alpha \tag{4.29}$$

$$\|\overline{v}_q - v_\ell\|_{N+\alpha} \lesssim \tau_q\delta_{q+1}\ell^{-1-N+\alpha} \tag{4.30}$$

$$\|\overline{v}_q\|_{1+N} \lesssim \delta_q^{1/2}\lambda_q\ell^{-N} \tag{4.31}$$

for all $N \geq 0$.

Proof of Proposition 4.5. By definition

$$\overline{v}_q - v_\ell = \sum_i \chi_i(v_i - v_\ell).$$

Therefore Proposition 4.2 (4.6) $\|v_i - v_\ell\|_{N+\alpha} \lesssim \tau_q\delta_{q+1}\ell^{-N-1+\alpha}$ implies

$$\|\overline{v}_q - v_\ell\|_{N+\alpha} \lesssim \tau_q\delta_{q+1}\ell^{-1-N+\alpha}. \tag{4.32}$$

Note that using the definition of $\ell := \frac{\delta_{q+1}^{1/2}}{\delta_q^{1/2}\lambda_q^{1+3\alpha/2}}$ in (3.1) and $\tau_q := \frac{\ell^{2\alpha}}{\delta_q^{1/2}\lambda_q}$ in (4.1) and the comparison (3.3)

$$\delta_{q+1}^{1/2}\tau_q\ell^{-1} = \ell^{2\alpha}\lambda_q^{3\alpha/2} \leq \lambda_q^{-\alpha/2} \leq 1\,. \tag{4.33}$$

Therefore we obtain (4.29), and furthermore, for any $N \geq 0$

$$\|\overline{v}_q - v_\ell\|_{1+N+\alpha} \lesssim \delta_{q+1}\tau_q\ell^{-N-2+\alpha} = \delta_q^{1/2}\lambda_q(\ell\lambda_q)^{3\alpha}\ell^{-N} \leq \delta_q^{1/2}\lambda_q\ell^{-N}.$$

Then it also follows using (3.5) $\|v_\ell\|_{N+1} \lesssim \delta_q^{1/2}\lambda_q\ell^{-N}$ that

$$\|\overline{v}_q\|_{1+N} \lesssim \|v_\ell\|_{1+N} + \|v_\ell - \overline{v}_q\|_{1+N+\alpha} \lesssim \delta_q^{1/2}\lambda_q\ell^{-N}.$$

□

4.2.4 Estimates on stress tensor $\mathring{\overline{R}}_q$

We are now in a position to estimate the glued stress tensor $\mathring{\overline{R}}_q$:

Proposition 4.6. *The stress tensor $\mathring{\overline{R}}_q$ satisfies the following bounds for any $N \geq 0$:*

$$\left\|\mathring{\overline{R}}_q\right\|_{N+\alpha} \lesssim \delta_{q+1}\ell^{-N+\alpha} \tag{4.34}$$

$$\left\|(\partial_t + \overline{v}_q\cdot\nabla)\mathring{\overline{R}}_q\right\|_{N+\alpha} \lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q\ell^{-N-\alpha}. \tag{4.35}$$

Proof of Proposition 4.6. Recall that $v_i = \operatorname{curl} z_i$, so that we may write for $t \in I_i$:

$$\mathring{\overline{R}}_q = \partial_t\chi_i(\mathcal{R}\operatorname{curl})(z_i - z_{i+1}) - \chi_i(1-\chi_i)(v_i - v_{i+1})\mathring{\otimes}(v_i - v_{i+1}).$$

Note that $\mathcal{R}\operatorname{curl}$ is zero-order operator. Therefore from Propositions 4.2 (4.6) $\|v_i - v_\ell\|_{N+\alpha} \lesssim \tau_q\delta_{q+1}\ell^{-N-1+\alpha}$ and 4.3 (4.19) $\|z_i - z_{i+1}\|_{N+\alpha} \lesssim \tau_q\delta_{q+1}\ell^{-N+\alpha}$ for any $N \geq 0$ with $t \in I_i$, using Hölder product

$$\begin{aligned}\|\mathring{\overline{R}}_q\|_{N+\alpha} &\overset{(A.2)}{\lesssim} \tau_q^{-1}\|z_i - z_{i+1}\|_{N+\alpha} + \|v_i - v_{i+1}\|_{N+\alpha}\|v_i - v_{i+1}\|_\alpha \\ &\lesssim \delta_{q+1}\ell^{-N+\alpha} + \tau_q^2\delta_{q+1}^2\ell^{-2-N+2\alpha} \lesssim \delta_{q+1}\ell^{-N+\alpha}.\end{aligned}$$

Here we used again (4.33). Next, we calculate

$$
\begin{aligned}
D_{t,\ell}\mathring{\overline{R}}_q &= \partial_t^2\chi_i(\mathcal{R}\operatorname{curl})(z_i - z_{i+1}) \\
&\quad + \partial_t\chi_i(\mathcal{R}\operatorname{curl})D_{t,\ell}(z_i - z_{i+1}) + \partial_t\chi_i[v\cdot\nabla,\mathcal{R}\operatorname{curl}](z_i - z_{i+1}) \\
&\quad - \partial_t(\chi_i(1-\chi_i))(v_i - v_{i+1})\mathring{\otimes}(v_i - v_{i+1}) \\
&\quad - \chi_i(1-\chi_i))\Big((D_{t,\ell}(v_i - v_{i+1}))\mathring{\otimes}(v_i - v_{i+1}) - (v_i - v_{i+1})\mathring{\otimes}(D_{t,\ell}(v_i - v_{i+1}))\Big),
\end{aligned}
$$

where $[v\cdot\nabla,\mathcal{R}\operatorname{curl}]$ denotes the commutator. Hence, using Proposition C.3 and Propositions 4.2 and 4.3 we deduce

$$
\begin{aligned}
\|D_{t,\ell}\mathring{\overline{R}}_q\|_{N+\alpha} &\lesssim \tau_q^{-2}\|z_i - z_{i+1}\|_{N+\alpha} + \tau_q^{-1}\|D_{t,\ell}(z_i - z_{i+1})\|_{N+\alpha} \\
&\quad + \tau_q^{-1}\|v_\ell\|_\alpha\|z_i - z_{i+1}\|_{N+\alpha} + \tau_q^{-1}\|v_\ell\|_{N+\alpha}\|z_i - z_{i+1}\|_\alpha \\
&\quad + \tau_q^{-1}\|v_i - v_{i+1}\|_{N+\alpha}\|v_i - v_{i+1}\|_\alpha \\
&\quad + \|D_{t,\ell}(v_i - v_{i+1})\|_{N+\alpha}\|v_i - v_{i+1}\|_\alpha + \|v_i - v_{i+1}\|_{N+\alpha}\|D_{t,\ell}(v_i - v_{i+1})\|_\alpha \\
&\lesssim \tau_q^{-1}\delta_{q+1}\ell^{-N+\alpha} + (\tau_q^2\delta_{q+1}\ell^{-2})\tau_q^{-1}\delta_{q+1}\ell^{-N+2\alpha} \\
&\lesssim \tau_q^{-1}\delta_{q+1}\ell^{-N+\alpha}\,.
\end{aligned}
$$

Finally, we deduce using (4.30) $\|\overline{v}_q - v_\ell\|_{N+\alpha} \lesssim \tau_q\delta_{q+1}\ell^{-1-N+\alpha}$:

$$
\begin{aligned}
\Big\|(\partial_t + \overline{v}_q\cdot\nabla)\mathring{\overline{R}}_q\Big\|_{N+\alpha} &\lesssim \|(v_\ell - \overline{v}_q)\cdot\nabla\mathring{\overline{R}}_q\|_{N+\alpha} + \|D_{t,\ell}\mathring{\overline{R}}_q\|_{N+\alpha} \\
&\overset{(A.2)}{\lesssim} \|v_\ell - \overline{v}_q\|_{N+\alpha}\|\mathring{\overline{R}}_q\|_{1+\alpha} + \|v_\ell - \overline{v}_q\|_\alpha\|\mathring{\overline{R}}_q\|_{N+1+\alpha} + \|D_{t,\ell}\mathring{\overline{R}}_q\|_{N+\alpha} \\
&\lesssim \tau_q\delta_{q+1}^2\ell^{-N-2+2\alpha} + \tau_q^{-1}\delta_{q+1}\ell^{-N+\alpha} \\
&\lesssim \tau_q^{-1}\delta_{q+1}\ell^{-N+\alpha} = \delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q\ell^{-N-\alpha}
\end{aligned}
$$

again using (4.33). □

4.2.5 Estimates on energy difference between $\overline{v}_q$ and v_ℓ

To finish this section we show that $\overline{v}_q$ has approximately the same energy as v_ℓ:

Proposition 4.7. *The difference of the energies of $\overline{v}_q$ and v_ℓ satisfies*

$$
\left|\int_{\mathbb{T}^3}|\overline{v}_q|^2 - |v_\ell|^2 dx\right| \lesssim \delta_{q+1}\ell^\alpha \tag{4.36}
$$

Proof of Proposition 4.7. Observe that for $t\in I_i$

$$
\begin{aligned}
\overline{v}_q\otimes\overline{v}_q &= (\chi_i v_i + (1-\chi_i)v_{i+1})\otimes(\chi_i v_i + (1-\chi_i)v_{i+1}) \\
&= \chi_i v_i\otimes v_i + (1-\chi_i)v_{i+1}\otimes v_{i+1} - \chi_i(1-\chi_i)(v_i - v_{i+1})\otimes(v_i - v_{i+1}),
\end{aligned}
$$

so that, taking the trace:

$$
|\overline{v}_q|^2 - |v_\ell|^2 = \chi_i(|v_i|^2 - |v_\ell|^2) + (1-\chi_i)(|v_{i+1}|^2 - |v_\ell|^2) - \chi_i(1-\chi_i)|v_i - v_{i+1}|^2
$$

Next, recall that v_i and v_ℓ are smooth solutions of (4.3) and (3.2) respectively, therefore

$$
\begin{aligned}
\left|\frac{d}{dt}\int_{\mathbb{T}^3}|v_i|^2 - |v_\ell|^2\,dx\right| = \left|\int_{\mathbb{T}^3}\nabla v_\ell : \mathring{R}_\ell\,dx\right| &\lesssim \|\nabla v_\ell\|_0\|\mathring{R}_\ell\|_0 \\
&\lesssim \delta_q^{1/2}\lambda_q\delta_{q+1} \lesssim \tau_q^{-1}\delta_{q+1}\ell^\alpha,
\end{aligned}
$$

where we have used (3.6) and (3.5). Moreover, $v_i = v_\ell$ for $t = t_i$. Therefore, after integrating in time we deduce

$$\left| \int_{\mathbb{T}^3} |v_i|^2 - |v_\ell|^2 \, dx \right| \lesssim \delta_{q+1} \ell^\alpha.$$

Furthermore, using (4.6) $\|v_i - v_\ell\|_{N+\alpha} \lesssim \tau_q \delta_{q+1} \ell^{-N-1+\alpha}$ and (4.33) $\delta_{q+1}^{1/2} \tau_q \ell^{-1} = \ell^{2\alpha} \lambda_q^{3\alpha/2} \le \lambda_q^{-\alpha/2} \le 1$

$$\int_{\mathbb{T}^3} |v_i - v_{i+1}|^2 \, dx \lesssim \|v_i - v_{i+1}\|_\alpha^2 \lesssim \tau_q^2 \delta_{q+1}^2 \ell^{-2+2\alpha} \overset{(4.33)}{\lesssim} \delta_{q+1} \ell^{2\alpha},$$

Therefore

$$\left| \int |\overline{v}_q|^2 - |v_\ell|^2 dx \right| \lesssim \delta_{q+1} \ell^\alpha,$$

which concludes the proof. □

5 Perturbation Step $(\overline{v}_q, \overline{\mathring{R}}_q) \mapsto (v_{q+1}, \mathring{R}_{q+1})$ [2]

The gluing procedure can localize the Reynolds stress error $\overline{\mathring{R}}_q$ to small disjoint temporal regions, but it cannot completely eliminate the error. We will outline the construction of the perturbation w_{q+1}, where

$$v_{q+1} := w_{q+1} + \overline{v}_q \, ,$$

w_{q+1} is highly oscillatory and will be based on the Mikado flows, which are designed to cancel the low frequency error $\overline{R}_q$ and are Lie-advected by the mean flow of $\overline{v}_q$.

- First note that as a corollary of (2.9) $\delta_{q+1} \lambda_q^{-\alpha} \le e(t) - \int_{\mathbb{T}^3} |v_q|^2 \, dx \le \delta_{q+1}$ and $\left| \int_{\mathbb{T}^3} |v_q|^2 - |\overline{v}_q|^2 \right| \lesssim \delta_{q+1} \ell^\alpha$ as result from (3.7) $\left| \int_{\mathbb{T}^3} |v_q|^2 - |v_\ell|^2 \, dx \right| \lesssim \delta_{q+1} \ell^\alpha$ & (4.36) $\left| \int_{\mathbb{T}^3} |\overline{v_q}|^2 - |v_\ell|^2 dx \right| \lesssim \delta_{q+1} \ell^\alpha$, by choosing a sufficiently large we can ensure that

$$\frac{\delta_{q+1}}{2\lambda_q^\alpha} \le e(t) - \int_{\mathbb{T}^3} |\overline{v}_q|^2 \, dx \le 2\delta_{q+1} \, . \tag{5.1}$$

5.1 Mikado flows

Lemma 5.1 (Linear Algebra). *Denote by $\overline{B_{1/2}}(\mathrm{Id})$ the closed ball of radius $1/2$ around the identity matrix, in the space of symmetric 3×3 matrices. There exist mutually disjoint sets $\{\Lambda_i\}_{i=0,1} \subset \mathbb{S}^2 \cap \mathbb{Q}^3$ such that for each $\xi \in \Lambda_i$ there exist C^∞ smooth functions $\gamma_\xi : B_{1/2}(\mathrm{Id}) \to \mathbb{R}$ which obey*

$$R = \sum_{\xi \in \Lambda_i} \gamma_\xi^2(R)(\xi \otimes \xi)$$

for every symmetric matrix R satisfying $|R - \mathrm{Id}| \le 1/2$, and for each $i \in \{0, 1\}$.

- *For a sufficiently large geometric constant $C_\Lambda \ge 1$, to be chosen precisely in Section 5.3.3 below, we define the constant*

$$M = C_\Lambda \sup_{\xi \in \Lambda_i} \left(\|\gamma_\xi\|_{C^0} + \|\nabla \gamma_\xi\|_{C^0} \right) \, , \tag{5.2}$$

  which appears in (2.7).

- *Moreover, for $i \in \{0, 1\}$, and each $\xi \in \Lambda_i$, let us define $A_\xi \in \mathbb{S}^2 \cap \mathbb{Q}^3$ to be an orthogonal vector to ξ. Then for each $\xi \in \Lambda_i$, we have that $\{\xi, A_\xi, \xi \times A_\xi\} \subset \mathbb{S}^2 \cap \mathbb{Q}^3$ form an orthonormal basis for $\mathbb{R}^3$.*

- *Furthermore, similarly to the constant n_* of Proposition D.2, we label by n_* the smallest natural such that*

$$\{n_*\,\xi,\ n_*\,A_\xi,\ n_*\,\xi\times A_\xi\}\subset\mathbb{Z}^3 \tag{5.3}$$

for every $\xi\in\Lambda_i$ and for every $i\in\{0,1\}$. That is, n_* is the l.c.m. *of the denominators of the rational numbers ξ, A_ξ, and $\xi\times A_\xi$.*

(i) For $\varepsilon_\Lambda>0$, to be chosen later in terms of the set Λ_i, let $\Psi:\mathbb{R}^2\to\mathbb{R}$ be a C^∞ smooth function with support contained in a ball of radius ε_Λ around the origin. We normalize Ψ such that $\phi=-\Delta\Psi$ obeys

$$\int_{\mathbb{R}^2}\phi^2(x_1,x_2)\,dx_1dx_2=4\pi^2\,. \tag{5.4}$$

Moreover, as $\operatorname{supp}\Psi,\phi\subset\mathbb{T}^2$, we abuse notation and denote by Ψ,ϕ the $\mathbb{T}^2$-periodized versions of Ψ and ϕ.

(ii) Then, for any large $\lambda\in\mathbb{N}$ and every $\xi\in\Lambda_i$, we introduce the functions

$$\Psi_{(\xi)}(x):=\Psi_{\xi,\lambda}(x):=\Psi(n_*\lambda(x-\alpha_\xi)\cdot A_\xi,n_*\lambda(x-\alpha_\xi)\cdot(\xi\times A_\xi))\,, \tag{5.5a}$$
$$\phi_{(\xi)}(x):=\phi_{\xi,\lambda}(x):=\phi(n_*\lambda(x-\alpha_\xi)\cdot A_\xi,n_*\lambda(x-\alpha_\xi)\cdot(\xi\times A_\xi))\,, \tag{5.5b}$$

$\alpha_\xi\in\mathbb{R}^3$ are *shifts* whose purpose is to ensure that the functions $\{\Psi_{(\xi)}\}_{\xi\in\Lambda_i}$ have mutually disjoint support.

- Since n_*A_ξ and $n_*\xi\times A_\xi\in\mathbb{Z}^3$, and $\lambda\in\mathbb{N}$, the functions $\Psi_{(\xi)}$, $\phi_{(\xi)}:\mathbb{R}^3\to\mathbb{R}$ are $(\mathbb{T}/\lambda)^3$-periodic.
- By construction we have that $\{\xi,A_\xi,\xi\times A_\xi\}$ are an orthonormal basis or $\mathbb{R}^3$, and hence $\xi\cdot\nabla\Psi_{(\xi)}(x)=\xi\cdot\nabla\phi_{(\xi)}(x)=0$.
- From normalization of ϕ we have $\fint_{\mathbb{T}^3}\phi_{(\xi)}^2dx=1$ and $\fint_{(\mathbb{T}/\lambda)^3}\phi_{(\xi)}dx=0$, *i.e.*, $\phi_{(\xi)}$ zero mean on $(\mathbb{T}/\lambda)^3$.
- Since $\phi=-\Delta\Psi$ we have that $(n_*\lambda)^2\phi_{(\xi)}=-\Delta\Psi_{(\xi)}$.
- Last, we emphasize that the existence of the shifts α_ξ, which ensure that the supports of $\Psi_{(\xi)}$ are mutually disjoint for $\xi\in\Lambda_i$, is guaranteed by choosing ε_Λ sufficiently small solely in terms of the set Λ_i. Indeed, we can always ensure that the rational direction vectors in Λ_i give (periodized) straight lines which do not intersect, when shifted by suitably chosen vectors α_ξ.

(iii) With this notation, the *Mikado building blocks* $W_{(\xi)}:\mathbb{T}^3\to\mathbb{R}^3$ are defined as

$$W_{(\xi)}(x):=W_{\xi,\lambda}(x):=\xi\,\phi_{(\xi)}(x)\,. \tag{5.6}$$

Since $\xi\cdot\nabla\phi_{(\xi)}=0$, we immediately deduce that

$$\operatorname{div}W_{(\xi)}=0\qquad\text{and}\qquad\operatorname{div}\left(W_{(\xi)}\otimes W_{(\xi)}\right)=0\,. \tag{5.7}$$

- The Mikado flows are exact, smooth, pressure-less solutions of the stationary 3D Euler equations.
- By construction, the functions $W_{(\xi)}$ have zero mean on $\mathbb{T}^3$ and are in fact $(\mathbb{T}/\lambda)^3$-periodic.
- Moreover, by our choice of α_ξ we have that

$$W_{(\xi)}\otimes W_{(\xi')}\equiv0\qquad\text{whenever}\qquad\xi\neq\xi'\in\Lambda_i\,, \tag{5.8}$$

for $i\in\{0,1\}$, and our normalization of $\phi_{(\xi)}$ ensures that

$$\fint_{\mathbb{T}^3}W_{(\xi)}(x)\otimes W_{(\xi)}(x)\,dx=\xi\otimes\xi\,. \tag{5.9}$$

- Lastly, using (5.9), the definition of the functions γ_ξ in Lemma 5.1 and the L^2 normalization of the functions $\phi_{(\xi)}$ we have the spanning property of Mikado building blocks

$$\sum_{\xi \in \Lambda_i} \gamma_\xi^2(R) \fint_{\mathbb{T}^3} W_{(\xi)}(x) \otimes W_{(\xi)}(x) dx = R\,, \tag{5.10}$$

for every $i \in \{0,1\}$ and any symmetric matrix $R \in \overline{B}_{1/2}(\mathrm{Id})$.

We summarize properties (5.7)–(5.10) of the Mikado building blocks defined in (5.6) in the following result:

Lemma 5.2. *Given a symmetric matrix $R \in \overline{B}_{1/2}(\mathrm{Id})$ and $\lambda \in \mathbb{N}$, the Mikado flow*

$$\mathcal{W}(R,x) = \sum_{\xi \in \Lambda_i} \gamma_\xi(R)\, W_{\xi,\lambda}(x)$$

obeys

$$\operatorname{div} \mathcal{W} = 0, \quad \operatorname{div}(\mathcal{W} \otimes \mathcal{W}) = 0, \quad \int_{\mathbb{T}^3} \mathcal{W}\, dx = 0, \quad \fint_{\mathbb{T}^3} \mathcal{W} \otimes \mathcal{W}\, dx = R.$$

That is, $\mathcal{W}$ is a zero mean, presureless, solution of the stationary 3D Euler equations, which may be used to cancel the stress R.

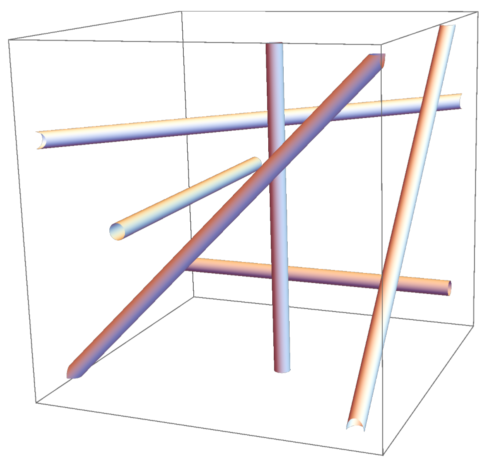

Figure 1: Example of a Mikado flow $\mathcal{W}$ restricted to one of the $(\mathbb{T}/\lambda)^3$ periodic boxes.

To conclude this section we note that $W_{(\xi)}$ may be written as the curl of a vector field, a fact which is useful in defining the incompressibility corrector in Section 5.3.2. Indeed, since $\xi \cdot \nabla \Psi_{(\xi)} = 0$, and since by definition we have that $-\frac{1}{(n_*\lambda)^2} \Delta \Psi_{(\xi)} = \phi_{(\xi)}$ we obtain

$$\operatorname{curl}\left(\frac{1}{(n_*\lambda)^2} \nabla \Psi_{(\xi)} \times \xi\right) = \operatorname{curl}\left(\frac{1}{(n_*\lambda)^2} \operatorname{curl}(\xi \Psi_{(\xi)})\right) = -\xi \left(\frac{1}{(n_*\lambda)^2} \Delta \Psi_{(\xi)}\right) = W_{(\xi)}\,. \tag{5.11}$$

For notational simplicity, we define $V_{(\xi)}$ the potential

$$V_{(\xi)} = \frac{1}{(n_*\lambda)^2} \nabla \Psi_{(\xi)} \times \xi \tag{5.12}$$

so that $\operatorname{curl} V_{(\xi)} = W_{(\xi)}$. With this notation we have the bounds for $N \geq 0$

$$\left\|W_{(\xi)}\right\|_N + \lambda_{q+1} \left\|V_{(\xi)}\right\|_N \lesssim \lambda_{q+1}^N. \tag{5.13}$$

5.2 Squiggling stripes and the stress tensor $\tilde{R}_{q,i}$

Recall that $\mathring{\overline{R}}_q$ is supported in the set $\mathbb{T}^3 \times \bigcup_i I_i$, whereas, from (4.26) it follows that $[0,T] \setminus \bigcup_i I_i = \bigcup_i J_i$, where the open intervals J_i have length $|J_i| = \frac{2}{3}\tau_q$ each, except for the first and last one, which might be shortened by the intersection with $[0,T]$, more precisely $J_i = (t_i - \frac{1}{3}\tau_q, t_i + \frac{1}{3}\tau_q) \cap [0,T]$.
We define smooth non-negative cut-off functions $\eta_i = \eta_i(x,t)$ with properties

(i) $\eta_i \in C^\infty(\mathbb{T}^3 \times [0,T])$ with $0 \le \eta_i(x,t) \le 1$ for all (x,t);

(ii) $\operatorname{supp} \eta_i \cap \operatorname{supp} \eta_j = \emptyset$ for $i \ne j$;

(iii) $\mathbb{T}^3 \times I_i \subset \{(x,t) : \eta_i(x,t) = 1\}$;

(iv) $\operatorname{supp} \eta_i \subset \mathbb{T}^3 \times I_i \cup J_i \cup J_{i+1} = \mathbb{T}^3 \times (t_i - \frac{1}{3}\tau_q, t_{i+1} + \frac{1}{3}\tau_q) \cap [0,T]$, we set $\tilde{I}_i = (t_i - \frac{1}{3}\tau_q, t_{i+1} + \frac{1}{3}\tau_q) \cap [0,T]$;

(v) There exists a positive geometric constant $c_0 > 0$ such that for any $t \in [0,T]$

$$2(2\pi)^3 \ge \sum_i \int_{\mathbb{T}^3} \eta_i^2(x,t)\, dx \ge c_0.$$

Lemma 5.3. *There exists cut-off functions $\{\eta_i\}_i$ with the properties (i)-(v) above and such that for any i and $n, m \ge 0$*

$$\|\partial_t^n \eta_i\|_m \le C(n,m)\tau_q^{-n}$$

where $C(n,m)$ are geometric constants depending only upon m and n.

Proof of Lemma 5.3. First of all we consider the sharp cutoffs $\tilde{\eta}_i$ defined by

$$\begin{aligned} \tilde{\eta}_i &= \mathbf{1}_{\tilde{\Omega}_i} \\ \tilde{\Omega}_i &= \left\{(x,t)\colon t_i + \tfrac{\tau_q}{6}(\sin(2\pi x_1) + \tfrac{1}{2}) \le t \le t_{i+1} + \tfrac{\tau_q}{6}(\sin(2\pi x_1) - \tfrac{1}{2})\right\} \end{aligned}$$

Next we fix a standard mollifier $\varkappa$ in time and the standard mollifier ψ in space already used so far. Hence we define η_i by mollifying $\tilde{\eta}_i$ in space and time as follows:

$$\eta_i(x,t) = \int \tilde{\eta}_i(y,s)\psi\left(\frac{x-y}{c_1}\right)\varkappa\left(\frac{t-s}{c_2\tau_q}\right) dy\, ds\,,$$

where c_1 and c_2 are positive geometric constants. One may check that a suitable choice of c_1 and c_2 yields the desired conclusions (see Figure 2). □

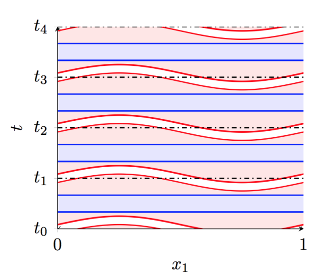

Figure 2: The support of $\mathring{\overline{R}}_q$ is given by the blue regions. The support of the cut-off functions η_i, which marks the region where the convex integration perturbation is supported, is given by the region between two consecutive red squiggling stripes.

- **5.2.1 Cutoffs $\rho_{q,i}(x,t)$**

Define $\rho_q(t)$ which measures the remaining energy profile error after the gluing step, and after leaving ourselves room for adding a future velocity increment

$$\rho_q(t) := \frac{1}{3}\left(e(t) - \frac{\delta_{q+2}}{2} - \int_{\mathbb{T}^3} |\overline{v}_q|^2 \, dx\right)$$

and the last cutoff function combining η_i and ρ_q

$$\rho_{q,i}(x,t) := \frac{\eta_i^2(x,t)}{\sum_j \int_{\mathbb{T}^3} \eta_j^2(y,t)\, dy} \rho_q(t)$$

Lemma 5.4. *For any $N \geq 0$*

$$\frac{\delta_{q+1}}{8\lambda_q^\alpha} \leq |\rho_q(t)| \leq \delta_{q+1} \quad \textit{for all } t\,, \tag{5.14}$$

$$\|\rho_{q,i}\|_0 \leq \frac{\delta_{q+1}}{c_0}\,, \tag{5.15}$$

$$\|\rho_{q,i}\|_N \lesssim \delta_{q+1}\,, \tag{5.16}$$

$$\|\partial_t \rho_q\|_0 \lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q\,, \tag{5.17}$$

$$\|\partial_t \rho_{q,i}\|_N \lesssim \delta_{q+1}\tau_q^{-1}\,. \tag{5.18}$$

Proof of Lemma 5.4. Note that (5.14) is a trivial consequence of estimate (5.1)

$$\frac{\delta_{q+1}}{2\lambda_q^\alpha} \leq e(t) - \int_{\mathbb{T}^3} |\overline{v}_q|^2 \, dx \leq 2\delta_{q+1}$$

and the inequality $4\delta_{q+2} \leq \delta_{q+1}$. Note that by the definition of the cut-off functions η_i

$$c_0 \leq \sum_i \int_{\mathbb{T}^3} \eta_i^2(y,t)\, dy \tag{5.19}$$

and hence we obtain (5.15). Since $\left|\nabla^N \eta_j\right| \lesssim 1$, the bound (5.16) also follows. Finally, to prove (5.18) we first note that

$$\left|\frac{d}{dt}\int |\overline{v}_q(x,t)|^2 \, dx\right| = \left|2\int \nabla\overline{v}_q \cdot \mathring{\overline{R}}_q \, dx\right| \lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q$$

Thus

$$\|\partial_t \rho_q\|_0 \lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q$$

Then, since $\|\partial_t \eta_j\|_N \lesssim \tau_q^{-1}$ and $\delta_q^{1/2}\lambda_q \leq \tau_q^{-1}$, using (5.19), the estimate (5.18) follows. □

- **5.2.2 Flow Maps Φ_i**

Define the backward flows $\Phi_i$for the velocity field $\overline{v}_q$ as the solution of the transport equation

$$\begin{cases} (\partial_t + \overline{v}_q \cdot \nabla)\Phi_i = 0 \\ \\ \Phi_i\,(x,t_i) = x. \end{cases}$$

for all $(x,t) \in \operatorname{supp}(\eta_i) \subset \mathbb{T}^3 \times \tilde{I}_i$. It is convenient to denote the material derivative as $D_{t,q}$, that is

$$D_{t,q} = \partial_t + \overline{v}_q \cdot \nabla_x .$$

Lemma 5.5.

$$\|\nabla\Phi_i - \mathrm{Id}\|_0 \le \frac{1}{2} \qquad \textit{for } t \in \operatorname{supp}(\eta_i). \tag{5.20}$$

For any $t \in \tilde{I}_i$, $N \ge 0$

$$\left\|(\nabla\Phi_i)^{-1}\right\|_N + \|\nabla\Phi_i\|_N \lesssim \ell^{-N}, \tag{5.21}$$

$$\|D_{t,q}\nabla\Phi_i\|_N \lesssim \delta_q^{1/2}\lambda_q\ell^{-N} \tag{5.22}$$

Proof of Lemma 5.5. For every $t \in \tilde{I}_i$ we have $|t - t_i| \le 2\tau_q$, where (4.1) $\tau_q = \frac{\ell^{2\alpha}}{\delta_q^{1/2}\lambda_q}$, and using (4.31) $\|\bar{v}_q\|_{1+N} \lesssim \delta_q^{1/2}\lambda_q\ell^{-N}$, we have

$$\tau_q \|\nabla\bar{v}_q\|_0 \lesssim \ell^{2\alpha} \ll 1$$

Hence assumptions for (B.5) $\|\nabla\Phi(t) - \mathrm{Id}\|_0 \lesssim |t|\,[v]_1$ is satisfied, so we obtain

$$\|\nabla\Phi_i - \mathrm{Id}\|_0 \lesssim \tau_q\delta_q^{1/2}\lambda_q = \ell^{2\alpha} \le \frac{1}{2}$$

Hence $(\nabla\Phi_i)^{-1}$ is a well-defined object on $\tilde{I}_i$. Again from (4.31), (B.5) and (B.6) $[\Phi(t)]_N \lesssim |t|\,[v]_N\ \forall N \ge 2$ we obtain

$$\|\nabla\Phi_i\|_N \lesssim 1 + \tau_q\|D\overline{v}_q\|_N \lesssim 1 + \tau_q\delta_q^{1/2}\lambda_q\ell^{-N}.$$

Using the fact that (5.20) $\|\nabla\Phi_i - \mathrm{Id}\|_0 \le 1/2$, the estimate (5.21) follows (indeed it gives the slightly better estimate $\lesssim 1 + \ell^{-N+2\alpha}$, but the other is still enough for our purposes). Finally observe that

$$D_{t,q}\nabla\Phi_i = -\nabla\Phi_i D\overline{v}_q$$

In particular, by Hölder product inequality (A.2)

$$\|D_{t,q}\nabla\Phi_i\|_N \lesssim \|\nabla\Phi_i\|_0\|\overline{v}_q\|_{N+1} + \|\nabla\Phi_i\|_N\|\overline{v}_q\|_1 .$$

Thus (5.22) follows from (4.31) and (5.21). □

- **5.2.3 Stress Tensor $\tilde{R}_{q,i}$**

Since $\eta_i \equiv 1$ on $\mathbb{T}^3 \times I_i$, $\eta_i\eta_j \equiv 0$ for $i \ne j$, and since $\operatorname{supp}(\mathring{\overline{R}}_q) \subset \mathbb{T}^3 \times \cup_i I_i$, we have that

$$\sum_i \eta_i^2 \mathring{\overline{R}}_q = \mathring{\overline{R}}_q . \tag{5.23}$$

Moreover, the cutoff functions η_i already incorporate in them a temporal cutoff (recall that $\operatorname{supp}(\eta_i) \subset \mathbb{T}^3 \times \tilde{I}_i$), and thus it is convenient to define

$$R_{q,i} := \rho_{q,i}\mathrm{Id} - \eta_i^2\mathring{\overline{R}}_q$$

which is a stress supported in $\operatorname{supp}(\eta_i)$, and which obeys $\sum_i \mathring{R}_{q,i} = -\mathring{\overline{R}}_q$. For reasons which will become apparent only later (cf. (5.37)), we also define the symmetric tensor for all $(x,t) \in \operatorname{supp}(\eta_i)$

$$\tilde{R}_{q,i} := \frac{\nabla\Phi_i R_{q,i}(\nabla\Phi_i)^T}{\rho_{q,i}} = \mathrm{Id} + \left(\nabla\Phi_i\,\nabla\Phi_i^T - \mathrm{Id}\right) - \nabla\Phi_i\,\frac{\eta_i^2\mathring{\overline{R}}_q}{\rho_{q,i}}\,\nabla\Phi_i^T. \tag{5.24}$$

We summarize the following led by properties (ii)-(iv) of η_i,

- $\operatorname{supp} R_{q,i} \subset \operatorname{supp} \eta_i$ and on $\operatorname{supp} \eta_i$ we have $R_{q,i} = \rho_{q+1,i}\mathrm{Id} - \mathring{\overline{R}}_q$;
- $\operatorname{supp} \tilde{R}_{q,i} \subset \mathbb{T}^3 \times (t_i - \frac{1}{3}\tau_q, t_{i+1} + \frac{1}{3}\tau_q) = \mathbb{T}^3 \times \tilde{I}_i$;
- $\operatorname{supp} \tilde{R}_{q,i} \cap \operatorname{supp} \tilde{R}_{q,j} = \emptyset$ for all $i \neq j$.

Lemma 5.6. *For $a \gg 1$ sufficiently large,*

$$\left\| \tilde{R}_{q,i}(\cdot, t) - \mathrm{Id} \right\|_0 \lesssim \ell^\alpha \leq \frac{1}{2} \qquad \textit{for all} \qquad t \in \tilde{I}_i\,,$$

or equivalently, for all (x,t)

$$\tilde{R}_{q,i}(x,t) \in B_{1/2}(\mathrm{Id})\,,$$

where $B_{1/2}(\mathrm{Id})$ is the metric ball of radius $1/2$ around the identity Id in the space of 3 by 3 symmetric matrices. For $t \in \tilde{I}_i$ and any $N \geq 0$

$$\left\| \tilde{R}_{q,i} \right\|_N \lesssim \ell^{-N}\,, \tag{5.25}$$

$$\left\| D_{t,q} \tilde{R}_{q,i} \right\|_N \lesssim \tau_q^{-1} \ell^{-N} \tag{5.26}$$

Proof of Lemma 5.6. By definition we have

$$\begin{aligned}
\tilde{R}_{q,i} - \mathrm{Id} &= \nabla\Phi_i \left(\frac{R_{q,i}}{\rho_{q,i}} - \mathrm{Id} \right) \nabla\Phi_i^T + \nabla\Phi_i \nabla\Phi_i^T - \mathrm{Id} \\
&= \nabla\Phi_i \frac{\eta_i^2 \mathring{\overline{R}}_q}{\rho_{q,i}} \nabla\Phi_i^T + \nabla\Phi_i \nabla\Phi_i^T - \mathrm{Id}
\end{aligned}$$

Using (4.34) $\left\| \mathring{\overline{R}}_q \right\|_{N+\alpha} \lesssim \delta_{q+1} \ell^{-N+\alpha}$ we see that

$$\left| \frac{\eta_i^2 \mathring{\overline{R}}_q}{\rho_{q,i}} \right| \lesssim \frac{1}{\delta_{q+1}} \left| \mathring{\overline{R}}_q \right| \lesssim \ell^\alpha.$$

Consequently we obtain

$$|\tilde{R}_{q,i} - \mathrm{Id}| \lesssim \ell^\alpha$$

so that, recalling (3.1) $\ell := \frac{\delta_{q+1}^{1/2}}{\delta_q^{1/2} \lambda_q^{1+3\alpha/2}}$, so by choosing a sufficiently large, we ensure that $\tilde{R}_{q,i}(x,t)$ is contained in the ball of symmetric matrices $B_{1/2}(\mathrm{Id})$. Recalling property (iv) of η_i we see that $\rho_{q,i}$ is a function of t only on $\operatorname{supp} \mathring{\overline{R}}_q$, i.e.

$$\rho_{q,i}(x,t) = \frac{\eta_i^2(x,t)}{\sum_j \int_{\mathbb{T}^3} \eta_j^2(y,t)\,dy} \rho_q(t).$$

Thus,

$$\frac{R_{q,i}}{\rho_{q,i}} = \mathrm{Id} - \frac{\sum_j \int_{\mathbb{T}^3} \eta_j^2(y,t)\,dy}{\rho_q(t)} \mathring{\overline{R}}_q, \tag{5.27}$$

so that by (5.14) $\frac{\delta_{q+1}}{8\lambda_q^\alpha} \leq |\rho_q(t)| \leq \delta_{q+1}$ and (4.34) $\left\| \mathring{\overline{R}}_q \right\|_{N+\alpha} \lesssim \delta_{q+1} \ell^{-N+\alpha}$ we obtain

$$\left\| \frac{R_{q,i}}{\rho_{q,i}} \right\|_N \lesssim 1 + \frac{\lambda_q^\alpha}{\delta_{q+1}} \left\| \mathring{\overline{R}}_q \right\|_N \ell^{-N} \lesssim \ell^{-N}, \tag{5.28}$$

where we have applied the crude estimate $\lesssim 1 + \|\mathring{\overline{R}}_q\|_{N+\alpha} \lambda_q^\alpha \delta_{q+1}^{-1} \lesssim 1 + \ell^{-N+\alpha} \lambda_q^\alpha \lesssim \ell^{-N}$.

Therefore, using Lemma 5.5 and property (v) $\sum_i \int_{\mathbb{T}^3} \eta_i^2(x,t)\,dx \geq c_0$:

$$\left\|\tilde{R}_{q,i}\right\|_N = \left\|\frac{\nabla\Phi_i R_{q,i}(\nabla\Phi_i)^T}{\rho_{q,i}}\right\|_N \lesssim \|\nabla\Phi_i\|_N \|\nabla\Phi_i\|_0 + \left\|\frac{R_{q,i}}{\rho_{q,i}}\right\|_N \lesssim \|\nabla\Phi_i\|_N \|\nabla\Phi_i\|_0 + \ell^{-N}.$$

The estimate (5.25) then follows from (5.21).

Next, we observe that

$$D_{t,q}\rho_{q,i} = \partial_t \rho_{q,i} + \bar{v}_q \cdot \nabla \rho_{q,i}$$

and thus we can estimate

$$\|D_{t,q}\rho_{q,i}\|_N \lesssim \|\partial_t\rho_{q,i}\|_N + \|\rho_{q,i}\|_{N+1}\|\bar{v}_q\|_0 + \|\bar{v}_q\|_N\|\rho_{q,i}\|_1.$$

Recall that $\|\bar{v}_q\|_0 \leq \|v_\ell\|_0 + \|v_\ell - v_q\|_0 \lesssim 1 \lesssim \tau_q^{-1}$ and so from (4.31) we conclude $\|\bar{v}_q\|_N \leq \tau_q^{-1}\ell^{-N}$. Combining the latter estimate with (5.16) $\|\rho_{q,i}\|_N \lesssim \delta_{q+1}$ and (5.18) $\|\partial_t\rho_{q,i}\|_N \lesssim \delta_{q+1}\tau_q^{-1}$ we achieve

$$\|D_{t,q}\rho_{q,i}\|_N \lesssim \delta_{q+1}\tau_q^{-1}\ell^{-N}. \tag{5.29}$$

Differentiating (5.27) we have

$$D_{t,q}(\rho_{q,i}^{-1}R_{q,i}) = -\left(\partial_t \frac{\sum_j \int_{\mathbb{T}^3} \eta_j^2(y,t)\,dy}{\rho_q(t)}\right)\mathring{\bar{R}}_q - \frac{\sum_j \int_{\mathbb{T}^3} \eta_j^2(y,t)\,dy}{\rho_q(t)} D_{t,q}\mathring{\bar{R}}_q. \tag{5.30}$$

Thus we can estimate, using (4.34) $\left\|\mathring{\bar{R}}_q\right\|_{N+\alpha} \lesssim \delta_{q+1}\ell^{-N+\alpha}$ and (4.35) $\left\|(\partial_t + \bar{v}_q\cdot\nabla)\mathring{\bar{R}}_q\right\|_{N+\alpha} \lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q\ell^{-N-\alpha}$:

$$\begin{aligned}\|D_{t,q}(\rho_{q,i}^{-1}R_{q,i})\|_N &\lesssim \delta_{q+1}^{-1}\delta_q^{1/2}\lambda_q^{1+2\alpha}\|\mathring{\bar{R}}_q\|_N + \tau_q^{-1}\delta_{q+1}^{-1}\lambda_q^\alpha\|\mathring{\bar{R}}_q\|_N + \delta_{q+1}^{-1}\lambda_q^\alpha\|D_{t,q}\mathring{\bar{R}}_q\|_N \\ &\lesssim \delta_q^{1/2}\lambda_q^{1+2\alpha}\ell^{-N+\alpha} + \tau_q^{-1}\lambda_q^\alpha\ell^{-N+\alpha} + \lambda_q^\alpha\delta_q^{1/2}\lambda_q\ell^{-N-\alpha} \lesssim \tau_q^{-1}\ell^{-N}.\end{aligned} \tag{5.31}$$

Differentiating (5.24) we achieve

$$D_{t,q}\tilde{R}_{q,i} = D_{t,q}\nabla\Phi_i(\rho_{q,i}^{-1}R_{q,i})\nabla\Phi_i^T + \nabla\Phi_i D_{t,q}(\rho_{q,i}^{-1}R_{q,i})\nabla\Phi_i^T + \nabla\Phi_i(\rho_{q,i}^{-1}R_{q,i})(D_{t,q}\nabla\Phi_i)^T.$$

Thus we can estimate

$$\begin{aligned}\|D_{t,q}\tilde{R}_{q,i}\|_N \lesssim& \|D_{t,q}\nabla\Phi_i\|_N\|(\rho_{q,i}^{-1}R_{q,i})\|_0 + \|D_{t,q}\nabla\Phi_i\|_0\|(\rho_{q,i}^{-1}R_{q,i})\|_N \\ &+ \|D_{t,q}\nabla\Phi_i\|_0\|(\rho_{q,i}^{-1}R_{q,i})\|_0\|\nabla\Phi_i\|_N + \|D_{t,q}(\rho_{q,i}^{-1}R_{q,i})\|_N + \|D_{t,q}(\rho_{q,i}^{-1}R_{q,i})\|_0\|\nabla\Phi_i\|_N.\end{aligned}$$

Using (5.22), (5.31), (5.28) and (5.21), we conclude (5.26). □

- **5.2.4 Amplitudes $a_{(\xi,i)}(x,t)$**

Since $\tilde{R}_{q,i}$ obeys the conditions of Lemma 5.1 on $\mathrm{supp}(\eta_i)$, and since $\rho_{q,i}^{1/2}$ is a multiple of η_i, we may define the amplitude functions

$$a_{(\xi,i)}(x,t) = \rho_{q,i}(x,t)^{1/2}\,\gamma_\xi(\tilde{R}_{q,i}) \tag{5.32}$$

where the γ_ξ are the functions from Lemma 5.1. Note importantly that the amplitude functions already include a temporal cutoff, which shows that $\mathrm{supp}(a_{(\xi,i)}) \subset \mathrm{supp}(\eta_i)$. The amplitude functions $a_{(\xi)}$ inherit the C^N bounds, material derivative bounds from lemma 5.4, 5.6, and the product at the chain rules

$$\left\|a_{(\xi,i)}\right\|_N + \tau_q\left\|D_{t,q}a_{(\xi,i)}\right\|_N \lesssim \delta_{q+1}^{1/2}\ell^{-N} \quad \forall\ N \geq 0 \tag{5.33}$$

5.3 Perturbation v_{q+1}

5.3.1 Principal Part of the Velocity Increment $w_{q+1}^{(p)}(x,t)$

For the remainder of the paper we consider Mikado building blocks as defined in (5.6) with $\lambda = \lambda_{q+1}$, i.e.

$$W_{(\xi)}(x) = W_{\xi,\lambda_{q+1}}(x) = \xi\phi_{\xi,\lambda_{q+1}}(x) = \xi\phi(n_*\lambda_{q+1}(x-\alpha_\xi)\cdot A_\xi, n_*\lambda_{q+1}(x-\alpha_\xi)\cdot(\xi\times A_\xi)) .$$

Recall: for the index sets Λ_i of Lemma 5.1, we overload notation and write $\Lambda_i = \Lambda_0$ for i even, and $\Lambda_i = \Lambda_1$ for i odd. With this notation, we now define the principal part of the velocity increment as

$$w_{q+1}^{(p)}(x,t) = \sum_i \sum_{\xi\in\Lambda_i} a_{(\xi,i)}(x,t)(\nabla\Phi_i(x,t))^{-1} W_{(\xi)}(\Phi_i(x,t)) . \tag{5.34}$$

- We notice the presence of $(\nabla\Phi_i)^{-1}$. The reason for this modification is as follows. At time $t = t_i$, we have $\Phi_i(x,t_i) = x$, $\nabla\Phi_i = \mathrm{Id}$, and by (5.7) $\operatorname{div} W_{(\xi)} = 0$ we have that the vector field

$$U_{i,\xi} = (\nabla\Phi_i)^{-1} W_{(\xi)}(\Phi_i)$$

  is incompressible at $t = t_i$.

- We then notice that $U_{i,\xi}$ is *Lie-advected* by the flow of the incompressible vector field $\overline{v}_q$, in the sense that

$$D_{t,q} U_{i,\xi} = (U_{i,\xi}\cdot\nabla)\overline{v}_q = (\nabla\overline{v}_q)^T U_{i,\xi} . \tag{5.35}$$

  This implies directly that $D_{t,q}(\operatorname{div} U_{i,\xi}) = 0$, and thus the divergence free nature of $U_{i,\xi}$ is carried from $t = t_i$ to all t close to t_i. This shows that the function $w_{q+1}^{(p)}$ defined in (5.34) is to leading order in λ_{q+1} divergence-free (i.e. the incompressibility corrector will turn out to be small).

- We also explain why $R_{q,i}$ isn't just normalized by $\rho_{q,i}$ but also conjugated with $\nabla\Phi_i$, and $(\nabla\Phi_i)^T$, in order to obtain $\tilde{R}_{q,i}$ (cf. (5.24)). Using the spanning property of the Mikado building blocks (5.10), the fact that they have mutually disjoint support (5.8), identity

$$\sum_i \rho_{q,i}(\nabla\Phi_i)^{-1}\tilde{R}_{q,i}(\nabla\Phi_i)^{-T} = \left(\sum_i \rho_{q,i}\right)\mathrm{Id} - \mathring{\overline{R}}_q , \tag{5.36}$$

  which is useful in cancelling the glued stress, and the fact that η_i have mutually disjoint supports, we get

$$\begin{aligned} w_{q+1}^{(p)}\otimes w_{q+1}^{(p)} &= \sum_i\sum_{\xi\in\Lambda_i} a_{(\xi,i)}^2(\nabla\Phi_i)^{-1}\left((W_{(\xi)}\circ\Phi_i)\otimes(W_{(\xi)}\circ\Phi_i)\right)(\nabla\Phi_i)^{-T} \\ &= \sum_i \rho_{q,i}(\nabla\Phi_i)^{-1}\left(\sum_{\xi\in\Lambda_i}\gamma_\xi^2(\tilde{R}_{q,i})\left((W_{(\xi)}\otimes W_{(\xi)})\circ\Phi_i\right)\right)(\nabla\Phi_i)^{-T} \\ &= \sum_i \rho_{q,i}(\nabla\Phi_i)^{-1}\tilde{R}_{q,i}(\nabla\Phi_i)^{-T} + \sum_i\sum_{\xi\in\Lambda_i} a_{(\xi,i)}^2(\nabla\Phi_i)^{-1}\left(\left(\mathbb{P}_{\neq 0}(W_{(\xi)}\otimes W_{(\xi)})\right)\circ\Phi_i\right)(\nabla\Phi_i)^{-T} \\ &= \left(\sum_i\rho_{q,i}\right)\mathrm{Id} - \mathring{\overline{R}}_q + \sum_i\sum_{\xi\in\Lambda_i} a_{(\xi,i)}^2(\nabla\Phi_i)^{-1}\left(\left(\mathbb{P}_{\geq\lambda_{q+1}/2}(W_{(\xi)}\otimes W_{(\xi)})\right)\circ\Phi_i\right)(\nabla\Phi_i)^{-T} \end{aligned} \tag{5.37}$$

  where we have denoted by $\mathbb{P}_{\neq 0} f(x) = f(x) - \fint_{\mathbb{T}^3} f(y)dy$, the projection of f onto its nonzero frequencies. We have also used that since $W_{(\xi)}\otimes W_{(\xi)}$ is $(\mathbb{T}/\lambda_{q+1})^3$-periodic, the identity $\mathbb{P}_{\neq 0}(W_{(\xi)}\otimes W_{(\xi)}) = \mathbb{P}_{\geq\lambda_{q+1}/2}(W_{(\xi)}\otimes W_{(\xi)})$ holds. The calculation (5.37) shows that by design, the low frequency part of $w_{q+1}^{(p)}\otimes w_{q+1}^{(p)}$ cancels the glued stress $\mathring{\overline{R}}_q$, modulo a multiple of the identity, which is then used to correct the energy profile and which contributes a pressure term to the equation.

5.3.2 Incompressibility corrector $w_{q+1}^{(c)}(x,t)$

Based on the definition (5.34) of the principal part of the velocity increment, we construct an incompressibility corrector. For any smooth vector field V, we have the identity

$$(\nabla\Phi_i)^{-1}\left((\operatorname{curl} V)\circ\Phi_i\right) = \operatorname{curl}\left((\nabla\Phi_i)^T(V\circ\Phi_i)\right).$$

Recalling identity

$$\text{(5.11)}\quad \operatorname{curl}\left(\frac{1}{(n_*\lambda)^2}\nabla\Psi_{(\xi)}\times\xi\right) = \operatorname{curl}\left(\frac{1}{(n_*\lambda)^2}\operatorname{curl}(\xi\Psi_{(\xi)})\right) = -\xi\left(\frac{1}{(n_*\lambda)^2}\Delta\Psi_{(\xi)}\right) = W_{(\xi)}$$

and the definition (5.12) $V_{(\xi)} = \frac{1}{(n_*\lambda)^2}\nabla\Psi_{(\xi)}\times\xi$, we may write $W_{(\xi)} = \operatorname{curl} V_{(\xi)}$ and thus the above identity shows that

$$(\nabla\Phi_i)^{-1}(W_{(\xi)}\circ\Phi_i) = \operatorname{curl}\left((\nabla\Phi_i)^T(V_{(\xi)}\circ\Phi_i)\right).$$

From the above identity and (5.34), it follows that if we define the incompressibility corrector as

$$w_{q+1}^{(c)}(x,t) := \sum_i\sum_{\xi\in\Lambda_i}\nabla a_{(\xi,i)}(x,t)\times\left((\nabla\Phi_i(x,t))^T(V_{(\xi)}(\Phi_i(x,t))\right) \tag{5.38}$$

then the total velocity increment w_{q+1} obeys

$$w_{q+1} = w_{q+1}^{(p)} + w_{q+1}^{(c)} = \operatorname{curl}\left(\sum_i\sum_{\xi\in\Lambda_i} a_{(\xi,i)}\,(\nabla\Phi_i)^T(V_{(\xi)}\circ\Phi_i)\right) \tag{5.39}$$

due to identity $\operatorname{curl} ab = a\operatorname{curl} b + \nabla a\times b$ for a scalar and b vector. Hence w_{q+1} is automatically incompressible.

5.3.3 Velocity inductive estimates

The velocity field at level $q+1$ is constructed as

$$v_{q+1} = \overline{v}_q + w_{q+1} = v_q + (v_\ell - v_q) + (\overline{v}_q - v_\ell) + w_{q+1}. \tag{5.40}$$

Corollary 5.1. *Assuming a is sufficiently large, the perturbations $w_{q+1}^{(p)}$, $w_{q+1}^{(c)}$ and w_{q+1} satisfy the following estimates*

$$\left\|w_{q+1}^{(p)}\right\|_0 + \frac{1}{\lambda_{q+1}}\left\|w_{q+1}^{(p)}\right\|_1 \le \frac{M}{8}\delta_{q+1}^{1/2} \tag{5.41a}$$

$$\left\|w_{q+1}^{(c)}\right\|_0 + \frac{1}{\lambda_{q+1}}\left\|w_{q+1}^{(c)}\right\|_1 \lesssim \delta_{q+1}^{1/2}\frac{\ell^{-1}}{\lambda_{q+1}} \tag{5.41b}$$

$$\left\|w_{q+1}\right\|_0 + \frac{1}{\lambda_{q+1}}\left\|w_{q+1}\right\|_1 \le \frac{M}{2}\delta_{q+1}^{1/2} \tag{5.41c}$$

Hence (2.12) *from Proposition 2.1 is satisfied*

$$\|v_{q+1}-v_q\|_0 + \frac{1}{\lambda_{q+1}}\|v_{q+1}-v_q\|_1 \le M\delta_{q+1}^{1/2}$$

so as bounds

$$\text{(2.7)}\quad \|v_{q+1}\|_1 \le M\delta_{q+1}^{1/2}\lambda_{q+1}$$

$$\text{(2.8)}\quad \|v_{q+1}\|_0 \le 1-\delta_{q+1}^{1/2}$$

Proof of Corollary 5.1. Recall (5.2) $M = C_\Lambda \sup_{\xi\in\Lambda_i}\left(\|\gamma_\xi\|_{C^0} + \|\nabla\gamma_\xi\|_{C^0}\right)$ and (5.32) $a_{(\xi,i)}(x,t) = \rho_{q,i}(x,t)^{1/2}\gamma_\xi(\tilde R_{q,i})$

$$\left\|w_{q+1}^{(p)}\right\|_0 = \left\|\sum_i \sum_{\xi\in\Lambda_i} \rho_{q,i}(x,t)^{1/2}\gamma_\xi(\tilde R_{q,i})(\nabla\Phi_i(x,t))^{-1} W_{(\xi)}(\Phi_i(x,t))\right\|_0$$

Using (5.15) $\|\rho_{q,i}\|_0 \le \frac{\delta_{q+1}}{c_0}$ and (5.20) $\|\nabla\Phi_i - \mathrm{Id}\|_0 \le \frac12\ \forall\, t\in\operatorname{supp}(\eta_i) \implies \left\|(\nabla\Phi_i)^{-1}\right\|_0 \lesssim 2$ on $\operatorname{supp}(\eta_i)$ and that η_i have disjoint supports, once a is sufficiently large we obtain

$$\begin{aligned}
\left\|w_{q+1}^{(p)}\right\|_0 &\le \frac{2|\Lambda_i|\,\|\phi\|_{C^0}}{c_0^{1/2}C_\Lambda} M\delta_{q+1}^{1/2} \le \frac{M}{8}\delta_{q+1}^{1/2}\\
\left\|w_{q+1}^{(p)}\right\|_1 &\le \frac{4|\Lambda_i|n_*\,\|\phi\|_{C^1}}{c_0^{1/2}C_\Lambda} M\delta_{q+1}^{1/2}\lambda_{q+1} \le \frac{M}{8}\delta_{q+1}^{1/2}\lambda_{q+1}
\end{aligned}$$

by choosing the parameter C_Λ from (5.2) to be large enough. Note that C_Λ only depends on the cardinality of Λ_i, on the universal constant c_0, the geometric integer n_*, and on the C^1 norm of the function ϕ, which in turn depends solely on the geometric constant ε_Λ.
For the incompressibility corrector

$$\left\|w_{q+1}^{(c)}(x,t)\right\|_0 = \left\|\sum_i \sum_{\xi\in\Lambda_i} \nabla a_{(\xi,i)}(x,t) \times \left((\nabla\Phi_i(x,t))^T (V_{(\xi)}(\Phi_i(x,t))\right)\right\|_0$$

we lose a factor of ℓ^{-1} from the gradient landing on $a_{(\xi,i)}$, but we gain a factor of λ_{q+1} because we have $V_{(\xi)}$ instead of $W_{(\xi)}$ (recall (5.12) $V_{(\xi)} = \frac{1}{(n_*\lambda)^2}\nabla\Psi_{(\xi)}\times\xi$ so that $\operatorname{curl} V_{(\xi)} = W_{(\xi)}$). Therefore, we may show that

$$\left\|w_{q+1}^{(c)}\right\|_0 + \frac{1}{\lambda_{q+1}}\left\|w_{q+1}^{(c)}\right\|_1 \lesssim \delta_{q+1}^{1/2}\frac{\ell^{-1}}{\lambda_{q+1}}\,.$$

We note that by choosing α to be sufficiently small in therms of b and β, we have

$$\frac{\ell^{-1}}{\lambda_{q+1}} = \frac{\delta_q^{1/2}\lambda_q^{1+3\alpha/2}}{\delta_{q+1}^{1/2}\lambda_{q+1}} = \frac{\lambda_q^{1-\beta+3\alpha/2}}{\lambda_{q+1}^{1-\beta}} \le 2\lambda_q^{3\alpha/2-(b-1)(1-\beta)} \le \lambda_q^{-(b-1)(1-\beta)/2} \ll 1\,, \tag{5.43}$$

and thus by choosing a sufficiently large we may ensure that the velocity increment

$$\|w_{q+1}\|_0 + \frac{1}{\lambda_{q+1}}\|w_{q+1}\|_1 \le \frac{M}{2}\delta_{q+1}^{1/2}$$

By writing v_{q+1} as

$$v_{q+1} = \overline{v}_q + w_{q+1} = v_q + (v_\ell - v_q) + (\overline{v}_q - v_\ell) + w_{q+1}$$

and using velocity error estimate from mollification step (3.4) $\|v_\ell - v_q\|_0 \lesssim \delta_{q+1}^{1/2}\lambda_q^{-\alpha}$, and estimate from gluing step (4.29) $\|\overline{v}_q - v_\ell\|_\alpha \lesssim \delta_{q+1}^{1/2}\ell^\alpha$ we obtain

$$\|v_{q+1} - v_q\|_0 + \frac{1}{\lambda_{q+1}}\|v_{q+1} - v_q\|_1 \le M\delta_{q+1}^{1/2}$$

Combining requirement on the original size of v_q (2.7) $\|v_q\|_1 \le M\delta_q^{1/2}\lambda_q$, (2.8) $\|v_q\|_0 \le 1-\delta_q^{1/2}$ we have

$$\begin{aligned}
&(2.7)\quad \|v_{q+1}\|_1 \le M\delta_{q+1}^{1/2}\lambda_{q+1}\\
&(2.8)\quad \|v_{q+1}\|_0 \le 1-\delta_{q+1}^{1/2}
\end{aligned}$$

□

5.4 Reynolds Stress $\mathring{R}_{q+1}$

Recall that the pair $(\overline{v}_q, \overline{\mathring{R}}_q)$ solves the Euler-Reynolds system (2.3), and that v_{q+1} is defined in (5.40). In this subsection we define the new Reynolds stress $\mathring{R}_{q+1}$, and show that it obeys the estimate

$$\left\|\mathring{R}_{q+1}\right\|_\alpha \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-4\alpha}}\,. \tag{5.44}$$

- The above bound immediately implies the desired estimate (2.6) $\left\|\mathring{R}_{q+1}\right\|_0 \leq \delta_{q+2}\lambda_{q+1}^{-3\alpha}$ at level $q+1$, upon noting that the following parameter inequality holds (after taking α sufficiently small and a sufficiently large)

$$\frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-4\alpha}} \leq \frac{\delta_{q+2}}{\lambda_{q+1}^{4\alpha}}\,. \tag{5.45}$$

The remaining power of $\lambda_{q+1}^{-\alpha}$ is used to absorb the implicit constant in (5.44).

In order to define $\mathring{R}_{q+1}$, we write

$$\begin{aligned}\operatorname{div}\mathring{R}_{q+1} - \nabla p_{q+1} = \ &\underbrace{D_{t,q}w_{q+1}^{(p)}}_{\operatorname{div}(R_{\text{transport}})} + \underbrace{\operatorname{div}(w_{q+1}^{(p)}\otimes w_{q+1}^{(p)} + \overline{\mathring{R}}_q)}_{\operatorname{div}(R_{\text{oscillation}})+\nabla p_{\text{oscillation}}} + \underbrace{w_{q+1}\cdot\nabla\overline{v}_q}_{\operatorname{div}(R_{\text{Nash}})}\\ &+ \underbrace{D_{t,q}w_{q+1}^{(c)} + \operatorname{div}\left(w_{q+1}^{(c)}\otimes w_{q+1} + w_{q+1}^{(p)}\otimes w_{q+1}^{(c)}\right)}_{\operatorname{div}(R_{\text{corrector}})+\nabla p_{\text{corrector}}} - \nabla\overline{p}_q\,.\end{aligned} \tag{5.46}$$

The various traceless symmetric stresses present implicitly in (5.46) are defined using the inverse divergence operator $\mathcal{R}$ (4.28)

$$\begin{aligned}(\mathcal{R}f)^{ij} &= \mathcal{R}^{ijk}f^k\\ \mathcal{R}^{ijk} &= -\frac{1}{2}\Delta^{-2}\partial_i\partial_j\partial_k + \frac{1}{2}\Delta^{-1}\partial_k\delta_{ij} - \Delta^{-1}\partial_i\delta_{jk} - \Delta^{-1}\partial_j\delta_{ik}.\end{aligned}$$

and by recalling the identity (5.37)

$$w_{q+1}^{(p)}\otimes w_{q+1}^{(p)} = \left(\sum_i \rho_{q,i}\right)\mathrm{Id} - \overline{\mathring{R}}_q + \sum_i\sum_{\xi\in\Lambda_i} a_{(\xi,i)}^2(\nabla\Phi_i)^{-1}\left(\left(\mathbb{P}_{\geq\lambda_{q+1}/2}(W_{(\xi)}\otimes W_{(\xi)})\right)\circ\Phi_i\right)(\nabla\Phi_i)^{-T}$$

(for the oscillation error) as

$$R_{\text{transport}} = \mathcal{R}\left(D_{t,q}w_{q+1}^{(p)}\right) \tag{5.47a}$$

$$R_{\text{oscillation}} = \sum_i\sum_{\xi\in\Lambda_i}\mathcal{R}\operatorname{div}\left(a_{(\xi,i)}^2(\nabla\Phi_i)^{-1}\left(\left(\mathbb{P}_{\geq\lambda_{q+1}/2}(W_{(\xi)}\otimes W_{(\xi)})\right)\circ\Phi_i\right)(\nabla\Phi_i)^{-T}\right) \tag{5.47b}$$

$$R_{\text{Nash}} = \mathcal{R}\left(w_{q+1}\cdot\nabla\overline{v}_q\right) \tag{5.47c}$$

$$R_{\text{corrector}} = \mathcal{R}\left(D_{t,q}w_{q+1}^{(c)}\right) + \left(w_{q+1}^{(c)}\mathring{\otimes}w_{q+1}^{(c)} + w_{q+1}^{(c)}\mathring{\otimes}w_{q+1}^{(p)} + w_{q+1}^{(p)}\mathring{\otimes}w_{q+1}^{(c)}\right) \tag{5.47d}$$

while the pressure terms are given by $p_{\text{oscillation}} = \sum_i \rho_{q,i}$ and $p_{\text{corrector}} = 2w_{q+1}^{(c)}\cdot w_{q+1}^{(p)} + |w_{q+1}^{(c)}|^2$. With this notation we have $p_{q+1} = \overline{p}_q - p_{\text{oscillation}} - p_{\text{corrector}}$ and

$$\mathring{R}_{q+1} = R_{\text{transport}} + R_{\text{oscillation}} + R_{\text{Nash}} + R_{\text{corrector}}\,. \tag{5.48}$$

5.4.1 Inverse divergence and stationary phase bounds

Prior to estimating the above stresses, it is convenient to adapt the stationary phase bounds from Beltrami flows to Mikado flows.

- We decompose the function $\phi_{(\xi)}$ which defines $W_{(\xi)} = \xi\phi_{(\xi)}$ in (5.6) as a Fourier series. Recall that $\phi_{(\xi)}$ defined in (5.5) is $(\mathbb{T}/\lambda_{q+1})^3$ periodic and has zero mean. Additionally, the function ϕ is C^∞ smooth. Therefore, we may decompose

$$\phi_{(\xi)}(x) = \phi_{\xi,\lambda_{q+1}}(x) = \sum_{k\in\mathbb{Z}^3\setminus\{0\}} f_\xi(k) e^{i\lambda_{q+1}k\cdot(x-\alpha_\xi)} \tag{5.49}$$

  where the complex numbers $f_\xi(k)$ are the Fourier series coefficients of the C^∞ smooth, mean-zero $\mathbb{T}^3$ periodic function $z \mapsto \phi(n_* z\cdot A_\xi, n_* z\cdot(\xi\times A_\xi))$. The shift $x\mapsto x-\alpha_\xi$ has no effect on the estimates. Moreover, the Fourier coefficients decay arbitrarily fast. For any $m\in\mathbb{N}$ we have $|f_\xi(k)| = |f_\xi(k)e^{i\lambda_{q+1}k\cdot\alpha_\xi}| \le C|k|^{-m}$, where the constant C depends on m and on geometric parameters of the construction, such as n_*, the sets Λ_i, the shifts α_ξ, and norms of the bump function $\phi(x_1,x_2)$. Thus, C is independent of λ_{q+1}, or any other q-dependent parameter.

- A similar Fourier series decomposition applies to the function $\frac{1}{n_*\lambda_{q+1}}\nabla\Psi_{(\xi)} = (\nabla\Psi)_{(\xi)}$ which is used in (5.12) to define the potential $V_{(\xi)} = \frac{1}{(n_*\lambda)^2}\nabla\Psi_{(\xi)}\times\xi$. For this function we also obtain that its Fourier series coefficients decay arbitrarily fast, with constants that are bounded independently of q (and hence λ_{q+1}).

- Therefore, for a smooth function $a(x,t)$, in order to estimate $\mathcal{R}(a\, W_{(\xi)}\circ\Phi_i)$, we use identity (5.49), and apply Lemma E.1 for each k individually, and then sum in k using the fast decay of the Fourier coefficients $f_\xi(k)$. Without giving all the details, we summarize this procedure as follows. Let $a\in C^0([0,T];C^{m,\alpha}(\mathbb{T}^3))$ be such that $\mathrm{supp}(a)\subset\mathrm{supp}(\eta_i)$, which ensures that the phase Φ_i obeys the conditions of Lemma E.1 by (5.20) $\|\nabla\Phi_i - \mathrm{Id}\|_0 \le \frac12$ for $t\in\mathrm{supp}(\eta_i)$. Also using (5.21) $\left\|(\nabla\Phi_i)^{-1}\right\|_N + \|\nabla\Phi_i\|_N \lesssim \ell^{-N}$, we obtain from Lemma E.1 that

$$\left\|\mathcal{R}\left(a\left(W_{(\xi)}\circ\Phi_i\right)\right)\right\|_{C^\alpha} + \lambda_{q+1}\left\|\mathcal{R}\left(a\left(V_{(\xi)}\circ\Phi_i\right)\right)\right\|_{C^\alpha} \lesssim \frac{\|a\|_{C^0}}{\lambda_{q+1}^{1-\alpha}} + \frac{\|a\|_{C^{m,\alpha}} + \|a\|_{C^0}\,\ell^{-m-\alpha}}{\lambda_{q+1}^{m-\alpha}}, \tag{5.50}$$

  where the implicit constant is independent of q.

- Recalling that $W_{(\xi)}\otimes W_{(\xi)} = (\xi\otimes\xi)\phi_{(\xi)}^2$, and using that the function $\mathbb{P}_{\ge\lambda_{q+1}/2}\phi_{(\xi)}^2$ is also zero mean $(\mathbb{T}/\lambda_{q+1})^3$-periodic, a similar argument shows that

$$\left\|\mathcal{R}\left(a\,\left(\left(\mathbb{P}_{\ge\lambda_{q+1}/2}(W_{(\xi)}\otimes W_{(\xi)})\right)\circ\Phi_i\right)\right)\right\|_{C^\alpha} \lesssim \frac{\|a\|_{C^0}}{\lambda_{q+1}^{1-\alpha}} + \frac{\|a\|_{C^{m,\alpha}} + \|a\|_{C^0}\,\ell^{-m-\alpha}}{\lambda_{q+1}^{m-\alpha}} \tag{5.51}$$

  holds. The above estimate is useful for estimating the oscillation error.

5.4.2 Estimate for $\mathring{R}_{q+1}$

In this section we show that the stresses defined in (5.48) obey (5.44)

$$\left\|\mathring{R}_{q+1}\right\|_\alpha \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-4\alpha}}$$

The Nash error and the corrector error are in a sense lower order, and they can be treated similarly (or using similar bounds) to the transport and oscillation errors. Because of this, we omit the details for estimating R_{Nash} and $R_{\mathrm{corrector}}$.

- **Transport error.** Recalling the definition of (5.34)

$$w_{q+1}^{(p)} = \sum_i \sum_{\xi\in\Lambda_i} a_{(\xi,i)}(x,t)(\nabla\Phi_i(x,t))^{-1}W_{(\xi)}(\Phi_i(x,t))$$

and the Lie-advection identity (5.35) $D_{t,q}U_{i,\xi} = (U_{i,\xi}\cdot\nabla)\overline{v}_q = (\nabla\overline{v}_q)^T U_{i,\xi}$, we obtain that the transport stress $R_{\text{transport}} = \mathcal{R}\left(D_{t,q}w_{q+1}^{(p)}\right)$ in (5.47a) is given by

$$R_{\text{transport}} = \sum_i \sum_{\xi\in\Lambda_i} \mathcal{R}\left(a_{(\xi,i)}(\nabla\overline{v}_q)^T(\nabla\Phi_i)^{-1}W_{(\xi)}(\Phi_i)\right) + \mathcal{R}\left(\left(D_{t,q}a_{(\xi,i)}\right)(\nabla\Phi_i)^{-1}W_{(\xi)}(\Phi_i)\right)\,. \tag{5.52}$$

In order to bound the terms in (5.52) we use (5.50) to gain a factor of $\lambda_{q+1}^{-1+\alpha}$ from the operator $\mathcal{R}$ acting on the highest frequency term $W_{(\xi)}\circ\Phi_i$. The derivatives of $a_{(\xi,i)}$, $\nabla\overline{v}_q$, and $(\nabla\Phi_i)^{-1}$ are estimated using (5.33), (4.31), and (5.21) respectively. These bounds show that each additional spacial derivatives costs a power of ℓ^{-1}. We obtain from (5.50) that

$$\|R_{\text{transport}}\|_{C^\alpha} \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-\alpha}}\left(1+\frac{\ell^{-m-\alpha}}{\lambda_{q+1}^{m-1}}\right) + \frac{\delta_{q+1}^{1/2}\tau_q^{-1}}{\lambda_{q+1}^{1-\alpha}}\left(1+\frac{\ell^{-m-\alpha}}{\lambda_{q+1}^{m-1}}\right)\,.$$

Recalling (5.43), we have that $(\ell\lambda_{q+1})^{-1}\leq\lambda_q^{-(b-1)(1-\beta)/2}$, and thus upon taking the parameter m in to be sufficiently large (in terms of β and b), we obtain that $R_{\text{transport}}$ indeed is bounded by the right side of (5.44), as desired.

- **Oscillation error.** For $R_{\text{oscillation}} = \sum_i\sum_{\xi\in\Lambda_i}\mathcal{R}\,\mathrm{div}\left(a_{(\xi,i)}^2(\nabla\Phi_i)^{-1}\left(\left(\mathbb{P}_{\geq\lambda_{q+1}/2}(W_{(\xi)}\otimes W_{(\xi)})\right)\circ\Phi_i\right)(\nabla\Phi_i)^{-T}\right)$ defined in (5.47b), the main observation is that when the div operator lands on the highest frequency term, namely $\left(\mathbb{P}_{\geq\lambda_{q+1}/2}(W_{(\xi)}\otimes W_{(\xi)})\right)\circ\Phi_i$, due to certain cancellations this term vanishes. Since by construction we have $(\xi\cdot\nabla)\phi_{(\xi)}=0$ it also follows that $(\xi\cdot\nabla)\mathbb{P}_{\geq\lambda_{q+1}/2}(\phi_{(\xi)}^2)=0$. Therefore,

$$\begin{aligned}
&\mathrm{div}\left(a_{(\xi,i)}^2(\nabla\Phi_i)^{-1}\left(\left(\mathbb{P}_{\geq\lambda_{q+1}/2}(W_{(\xi)}\otimes W_{(\xi)})\right)\circ\Phi_i\right)(\nabla\Phi_i)^{-T}\right)\\
&= \mathrm{div}\left(a_{(\xi,i)}^2(\nabla\Phi_i)^{-1}(\xi\otimes\xi)(\nabla\Phi_i)^{-T}\left(\left(\mathbb{P}_{\geq\lambda_{q+1}/2}(\phi_{(\xi)}^2)\right)\circ\Phi_i\right)\right)\\
&= \left(\left(\mathbb{P}_{\geq\lambda_{q+1}/2}(\phi_{(\xi)}^2)\right)\circ\Phi_i\right)\mathrm{div}\left(a_{(\xi,i)}^2(\nabla\Phi_i)^{-1}(\xi\otimes\xi)(\nabla\Phi_i)^{-T}\right)\\
&\quad+\underbrace{a_{(\xi,i)}^2(\nabla\Phi_i)^{-1}(\xi\otimes\xi)(\nabla\Phi_i)^{-T}\left((\nabla\Phi_i)^T\left(\nabla\mathbb{P}_{\geq\lambda_{q+1}/2}(\phi_{(\xi)}^2)\right)\circ\Phi_i\right)}_{=0}.
\end{aligned}$$

The above identity shows that

$$R_{\text{oscillation}} = \sum_i\sum_{\xi\in\Lambda_i}\mathcal{R}\left(\left(\left(\mathbb{P}_{\geq\lambda_{q+1}/2}(W_{(\xi)}\otimes W_{(\xi)})\right)\circ\Phi_i\right)\mathrm{div}\left(a_{(\xi,i)}^2(\nabla\Phi_i)^{-1}(\xi\otimes\xi)(\nabla\Phi_i)^{-T}\right)\right)\,,$$

at which point we may appeal to the stationary phase estimate (5.51) combined with the bounds (5.33) $\|a_{(\xi,i)}\|_N + \tau_q\|D_{t,q}a_{(\xi,i)}\|_N \lesssim \delta_{q+1}^{1/2}\ell^{-N}$ and (5.21) $\|(\nabla\Phi_i)^{-1}\|_N + \|\nabla\Phi_i\|_N \lesssim \ell^{-N}$ to obtain

$$\|R_{\text{oscillation}}\|_{C^\alpha} \lesssim \frac{\delta_{q+1}\ell^{-1}}{\lambda_{q+1}^{1-\alpha}}\left(1+\frac{\ell^{-m-\alpha}}{\lambda_{q+1}^{m-1}}\right) \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-5\alpha/2}}\,.$$

Here we have again taken m sufficiently large, and have recalled the definition of $\ell = \frac{\delta_{q+1}^{1/2}}{\delta_q^{1/2}\lambda_q^{1+3\alpha/2}}$ in (3.1). Thus the oscillation error is also bounded by the right side of (5.44).

5.5 Energy Increment

To conclude the proof of Proposition 2.1, it remains to show that (2.9) holds with q replaced by $q+1$. In order to prove this bound we show that

$$\left| e(t) - \int_{\mathbb{T}^3} |v_{q+1}(x,t)|^2 dx - \frac{\delta_{q+2}}{2} \right| \lesssim \frac{\delta_{q+1}^{1/2} \delta_q^{1/2} \lambda_q^{1+2\alpha}}{\lambda_{q+1}} \tag{5.53}$$

holds. Recalling the parameter estimate (5.45) $\frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-4\alpha}} \le \frac{\delta_{q+2}}{\lambda_{q+1}^{4\alpha}}$, and taking a sufficiently large to absorb all the implicit constants, it is clear that (5.53) implies the bound (2.9) at level $q+1$.

Proof of (5.53). The principal observation is the following. Taking the trace of (5.37)

$$w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} = \left(\sum_i \rho_{q,i} \right) \mathrm{Id} - \mathring{\overline{R}}_q + \sum_i \sum_{\xi \in \Lambda_i} a_{(\xi,i)}^2 (\nabla \Phi_i)^{-1} \left(\left(\mathbb{P}_{\ge \lambda_{q+1}/2} (W_{(\xi)} \otimes W_{(\xi)}) \right) \circ \Phi_i \right) (\nabla \Phi_i)^{-T}$$

since $\mathring{\overline{R}}_q$ is traceless we obtain

$$\begin{aligned} \int_{\mathbb{T}^3} |w_{q+1}^{(p)}|^2 dx &= 3 \sum_i \int_{\mathbb{T}^3} \rho_{q,i} dx \\ &\quad + \sum_i \sum_{\xi \in \Lambda_i} \int_{\mathbb{T}^3} a_{(\xi,i)}^2 \mathrm{tr} \left((\nabla \Phi_i)^{-1} (\xi \otimes \xi) (\nabla \Phi_i)^{-T} \right) \left(\left(\mathbb{P}_{\ge \lambda_{q+1}/2} (W_{(\xi)} \otimes W_{(\xi)}) \right) \circ \Phi_i \right) dx \,. \end{aligned}$$

The second term in the above identity can be made arbitrarily small, since it is the L^2 inner product of a function whose oscillation frequency is $\lesssim \ell^{-1}$ (cf. (5.33) $\left\| a_{(\xi,i)} \right\|_N + \tau_q \left\| D_{t,q} a_{(\xi,i)} \right\|_N \lesssim \delta_{q+1}^{1/2} \ell^{-N}$ and (5.21) $\left\| (\nabla \Phi_i)^{-1} \right\|_N + \| \nabla \Phi_i \|_N \lesssim \ell^{-N}$) and a function which is λ_{q+1} periodic and zero mean. On the other hand, by the design of the functions $\rho_{q,i} = \frac{\eta_i^2(x,t)}{\sum_j \int_{\mathbb{T}^3} \eta_j^2(y,t)\, dy} \rho_q(t)$, where $\rho_q(t) = \frac{1}{3} \left(e(t) - \frac{\delta_{q+2}}{2} - \int_{\mathbb{T}^3} |\overline{v}_q|^2 \, dx \right)$ we have

$$3 \sum_i \int_{\mathbb{T}^3} \rho_{q,i} dx = 3 \rho_q(t) = e(t) - \frac{\delta_{q+2}}{2} - \int_{\mathbb{T}^3} |\overline{v}_q(x,t)|^2 dx \,.$$

Since $v_{q+1} = \overline{v}_q + w_{q+1}$, the above identity implies that

$$e(t) - \int_{\mathbb{T}^3} |v_{q+1}(x,t)|^2 dx - \frac{\delta_{q+2}}{2} = -2 \int_{\mathbb{T}^3} \overline{v}_q \cdot w_{q+1} dx - 2 \int_{\mathbb{T}^3} w_{q+1}^{(p)} \cdot w_{q+1}^{(c)} dx - \int_{\mathbb{T}^3} |w_{q+1}^{(c)}|^2 dx \,.$$

The corrector terms in the above give estimates consistent with (5.53) by appealing to

$$\text{(5.41a)} \quad \left\| w_{q+1}^{(p)} \right\|_0 + \frac{1}{\lambda_{q+1}} \left\| w_{q+1}^{(p)} \right\|_1 \le \frac{M}{8} \delta_{q+1}^{1/2} \qquad \text{(5.41b)} \quad \left\| w_{q+1}^{(c)} \right\|_0 + \frac{1}{\lambda_{q+1}} \left\| w_{q+1}^{(c)} \right\|_1 \lesssim \delta_{q+1}^{1/2} \frac{\ell^{-1}}{\lambda_{q+1}}$$

and (5.43) $\frac{\ell^{-1}}{\lambda_{q+1}} \ll 1$. For the first term on the right side of the above we recall (cf. (5.39)) that w_{q+1} may be written as the curl of a vector field whose size is $\delta_{q+1}^{1/2} \lambda_{q+1}^{-1}$

$$w_{q+1} = w_{q+1}^{(p)} + w_{q+1}^{(c)} = \mathrm{curl} \left(\sum_i \sum_{\xi \in \Lambda_i} a_{(\xi,i)} (\nabla \Phi_i)^T (V_{(\xi)} \circ \Phi_i) \right)$$

Integrating by parts the curl and using (4.31) $\|\overline{v}_q\|_{1+N} \lesssim \delta_q^{1/2} \lambda_q \ell^{-N}$ with $N=0$ we conclude the proof of (5.53), and hence of Proposition 2.1. □

6 An h-principle

In order to prove Theorem 1.2, let us first state an already-known theorem

Theorem 6.1. *Let $(\bar v, \bar p, \bar R)$ be a smooth strict subsolution of the Euler equations on $\mathbb{T}^3 \times [0,T]$ and fix $0 < \gamma < 1$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, and for any sufficiently large λ depending on ε_0 and $(\bar v, \bar p, \bar R)$, we have the following: There exists a smooth solution (v, p, R) of (1.3)*

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = -\operatorname{div} R \\ \\ \operatorname{div} v = 0, \end{cases}$$

satisfying the estimates

$$\begin{aligned} \|v - \bar v\|_{H^{-1}} &\le C\lambda^{-1} \\ \|v\|_0 + \lambda^{-1}\|v\|_1 &\le C \\ \left\|v \otimes v + R - \bar v \otimes \bar v - \bar R\right\|_{H^{-1}} &\le C\lambda^{\gamma-1} \\ \|\mathring{R}\|_0 &\le C\lambda^{\gamma-1} \\ \|\operatorname{tr} R\|_0 &\le \varepsilon\,, \end{aligned}$$

where C depends solely on $(\bar v, \bar p, \bar R)$, and $\mathring{R}$ is the traceless part of R. Moreover setting

$$e(t) := \int_{\mathbb{T}^3} |\bar v|^2 + \operatorname{tr} \bar R \, dx \tag{6.1}$$

for any $t \in [0,T]$ we have

$$\frac{\varepsilon}{2} \le e(t) - \int_{\mathbb{T}^3} |v|^2 dx \le \varepsilon\,.$$

We now prove Theorem 1.2.

Theorem 6.2 (h-principle Theorem 1.2)**.** *Let $(\bar v, \bar p, \bar R)$ be a smooth strict subsolution of the Euler equations on $\mathbb{T}^3 \times [0,T]$ and let $\beta < 1/3$. Then there exists a sequence (v_k, p_k) of weak solutions of*

$$(1.1) \quad \begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = 0 \\ \\ \operatorname{div} v = 0, \end{cases}$$

such that $v_k \in C^\beta(\mathbb{T}^3 \times [0,T])$,

$$v_k \overset{*}{\rightharpoonup} \bar v \quad \text{and} \quad v_k \otimes v_k \overset{*}{\rightharpoonup} \bar v \otimes \bar v + \bar R \quad \text{in} \quad L^\infty$$

uniformly in time, and furthermore for all $t \in [0,T]$

$$(1.4) \quad \int_{\mathbb{T}^3} |v_k|^2 \, dx = \int_{\mathbb{T}^3} \left(|\bar v|^2 + \operatorname{tr} \bar R\right) dx.$$

Proof of Theorem 1.2.

- Fix $k \ge 1$ and let $\varepsilon_k < \varepsilon_0$. We'll later use ε_k to iterate, satisfying $\forall\ \varepsilon_k < \varepsilon_0$ assumption. We apply Theorem 6.1 with $\gamma = \alpha$ and $\lambda = \lambda_0$, where here (α, λ_0) are given in the statement of Proposition 2.1, and where we take a sufficiently large such that λ_0 is sufficiently large (in terms of ε_k

and $(\bar{v}, \bar{p}, \bar{R})$), so that the hypothesis of Theorem 6.1 is satisfied. We obtain (v, p, R) satisfying

$$\|v - \bar{v}\|_{H^{-1}} \leq C\lambda_0^{-1} \tag{6.2}$$

$$\|v\|_0 + \lambda_0^{-1}\|v\|_1 \leq C \tag{6.3}$$

$$\left\|v \otimes v + R - \bar{v} \otimes \bar{v} - \bar{R}\right\|_{H^{-1}} \leq C\lambda_0^{\alpha-1} \tag{6.4}$$

$$\|\mathring{R}\|_0 \leq C\lambda_0^{\alpha-1} \tag{6.5}$$

$$\|\operatorname{tr} R\|_0 \leq \varepsilon_k\,, \tag{6.6}$$

and the function $e(t) = \int_{\mathbb{T}^3} |\bar{v}|^2 + \operatorname{tr} \bar{R}\, dx$ as defined by (6.1) obeys

$$\frac{\varepsilon_k}{2} \leq e(t) - \int_{\mathbb{T}^3} |v|^2 \, dx \leq \varepsilon_k\,. \tag{6.7}$$

- Analogous to the proof of Theorem 1.1, we set

$$\Gamma = \frac{\delta_1^{1/2}}{\varepsilon_k^{1/2}}$$

and rescale (v, p, R) to obtain

$$\widetilde{v}_0(x,t) := \Gamma v(x, \Gamma t), \qquad \widetilde{p}_0(x,t) := \Gamma^2 p(x, \Gamma t) \quad \text{and} \quad \widetilde{R}_0(x,t) := \Gamma^2 R(x, \Gamma t)\,,$$

so that $(\widetilde{v}_0, \widetilde{p}_0, \widetilde{R}_0)$ also solves (1.3). Moreover, we have the estimates

$$\|\widetilde{v}_0\|_0 + \lambda_0^{-1}\|\widetilde{v}_0\|_1 \leq \frac{C\delta_1^{1/2}}{\varepsilon_k^{1/2}} \tag{6.8}$$

$$\|\mathring{\widetilde{R}}_0\|_0 \leq \frac{C\delta_1}{\varepsilon_k \lambda_0^{1-\alpha}}\,.$$

Choosing α sufficiently small and choosing a sufficiently large depending on ε_k, C, and M, we obtain

$$\frac{C\delta_1^{1/2}}{\varepsilon_k^{1/2}} \leq \min(M\delta_0^{1/2}, 1 - \delta_0) \quad \text{and} \quad \frac{C}{\varepsilon_k \lambda_0^{1-\alpha}} \leq \lambda_0^{-3\alpha}\,.$$

from which we obtain (2.6), (2.7), and (2.8).

If in addition we set

$$\tilde{e}(t) = \Gamma^2 e(\Gamma t)$$

then from (6.7) we obtain

$$\frac{\delta_1}{2} \leq \tilde{e}(t) - \int_{\mathbb{T}^3} |\widetilde{v}_0|^2 \, dx \leq \delta_1\,,$$

and hence we obtain (2.9) for $q = 0$. Letting a be sufficiently large, we also obtain (2.2). Applying Proposition 2.1 and arguing as was done in the proof of Theorem 1.1 we obtain a solution $(\widetilde{v}, \widetilde{p})$ to the Euler equations satisfying

$$\int_{\mathbb{T}^3} |\widetilde{v}|^2 \, dx = \tilde{e}(t)\,. \tag{6.9}$$

Moreover, by (2.12) we have the estimate

$$\|\widetilde{v} - \widetilde{v}_0\|_0 \lesssim \delta_1^{1/2}\,. \tag{6.10}$$

- Lastly, we define (v_k, p_k) by the rescaling back

$$v_k := \Gamma^{-1}\widetilde{v}(x, \Gamma^{-1}t) \quad \text{and} \quad p_k := \Gamma^{-2}\widetilde{p}(x, \Gamma^{-1}t)\,.$$

Then (v_k, p_k) is a solution to the Euler equations, satisfying (1.4) as a consequence of rescaling (6.9). The sequence v_k is uniformly bounded in C^0 since

$$\|v_k\|_0 \le \Gamma^{-1}(\|\widetilde{v}\|_0 + \|\widetilde{v} - \widetilde{v}_0\|_0) \lesssim \varepsilon_k^{1/2}\delta_1^{-1/2}(\delta_1^{1/2} + C\delta_1^{1/2}\varepsilon_k^{-1/2}) \lesssim \varepsilon_0^{1/2} + C.$$

Thus $(v_k \otimes v_k)$ is also uniformly bounded in C^0. By Banach-Alaoglu, v_k and $v_k \otimes v_k$ have weak$-*$ convergent subsequences.

- Moreover, by rescaling (6.10) and using (6.2) we have

$$\|v_k - \overline{v}\|_{H^{-1}} \lesssim \|v_k - v\|_0 + \|v - \overline{v}\|_{H^{-1}} \lesssim \Gamma^{-1}\delta_1^{1/2} + C\lambda_0^{-1} \lesssim \varepsilon_k^{1/2} + C\lambda_0^{-1} \lesssim \varepsilon_k^{1/2} \tag{6.11}$$

by choosing a (and thus λ_0) sufficiently large in terms of ε_k. Moreover, from (6.4)–(6.6), (6.8), and (6.10) we obtain

$$\begin{aligned} \left\|v_k \otimes v_k - v \otimes v - \bar{R}\right\|_{H^{-1}} &\lesssim \|v_k \otimes v_k - v \otimes v\|_0 + \|R\|_0 + \left\|v \otimes v + R - \bar{v} \otimes \bar{v} - \bar{R}\right\|_{H^{-1}} \\ &\lesssim \Gamma^{-2}\,\|\widetilde{v} \otimes \widetilde{v} - \widetilde{v}_0 \otimes \widetilde{v}_0\|_0 + \left\|\mathring{R}\right\|_0 + \|\mathrm{tr}\, R\|_0 + C\lambda_0^{\alpha-1} \\ &\lesssim \varepsilon_k\delta_1^{-1/2}(C\delta_1^{1/2}\varepsilon_k^{-1/2} + \delta_1^{1/2}) + \varepsilon_k + C\lambda_0^{\alpha-1} \lesssim C\varepsilon_k^{1/2}. \end{aligned} \tag{6.12}$$

Since the H^{-1} topology uniquely captures the weak$-*$ limit, the theorem is completed upon passing $\varepsilon_k \to 0$ in (6.11)–(6.12).

□

A Hölder spaces

$m = 0, 1, 2, \dots$, $\alpha \in (0,1)$, and θ is a multi-index. We introduce the usual (spatial) Hölder norms.

Definition A.1 (Hölder Norms). *(i) Supremum norm* $\|f\|_0 := \sup_{\mathbb{T}^3\times[0,1]} |f|$

(ii) Hölder seminorms

$$[f]_m = \max_{|\theta|=m} \|D^\theta f\|_0 \,,$$
$$[f]_{m+\alpha} = \max_{|\theta|=m} \sup_{x\neq y, t} \frac{|D^\theta f(x,t) - D^\theta f(y,t)|}{|x-y|^\alpha} \,,$$

where D^θ *are* space derivatives *only.*

(iii) Hölder norms

$$\|f\|_m = \sum_{j=0}^m [f]_j$$
$$\|f\|_{m+\alpha} = \|f\|_m + [f]_{m+\alpha}.$$

Moreover, we write $[f(t)]_\alpha$ *and* $\|f(t)\|_\alpha$ *when the time* t *is fixed and the norms are computed for the restriction of* f *to the* t*-time slice.*

Theorem A.1 (Standard Interpolation Inequality). *(i) for* $r \geq s \geq 0$, $\varepsilon > 0$

$$[f]_s \leq C\big(\varepsilon^{r-s}[f]_r + \varepsilon^{-s}\|f\|_0\big) \tag{A.1}$$

(ii) for $r \geq 0$

$$[fg]_r \leq C\big([f]_r\|g\|_0 + \|f\|_0[g]_r\big) \tag{A.2}$$

(iii) From (A.1) *with* $\varepsilon = \|f\|_0^{1/r}[f]_r^{-1/r}$ *we obtain the standard interpolation inequality for* $r \geq s \geq 0$

$$[f]_s \leq C\|f\|_0^{1-s/r}[f]_r^{s/r}. \tag{A.3}$$

Theorem A.2 (Standard Mollification Estimate). *Given Standard radial smooth mollifier* ψ in space $\mathbb{R}^3$ *and define* $\psi_\ell(x) = \ell^{-3}\psi(x\ell^{-1})$ *, then* $\forall\ r \in (0,1]$

$$\|f * \psi_\ell - f\|_0 \leq C\|f\|_r \ell^r \tag{A.4}$$

for constant C *depending on* r.

Proposition A.1 (Quadratic Commutator Estimate). *Let* $f, g \in C^\infty(\mathbb{T}^3 \times \mathbb{T})$ *and* ψ *a standard radial smooth and compactly supported kernel. For any* $r \geq 0$ *we have the estimate*

$$\Big\|(f*\psi_\ell)(g*\psi_\ell) - (fg)*\psi_\ell\Big\|_r \leq C\ell^{2-r}\|f\|_1\|g\|_1 \,,$$

where the constant C *depends only on* r.

B Estimates for transport equations

We recall some well known results regarding smooth solutions of the *transport equation*:

$$\begin{cases} \partial_t f + v \cdot \nabla f = g, \\ \\ f(\cdot, 0) = f_0, \end{cases} \tag{B.1}$$

where $v = v(t, x)$ is a given smooth vector field. We will consider solutions on the entire space $\mathbb{R}^3$ and treat solutions on the torus simply as periodic solution in $\mathbb{R}^3$.

Proposition B.1 (Standard Estimates for solutions to Transport Equation)**.** *Assume* $|t|\, \|v\|_1 \le 1$. *Then, any solution* f *of* (B.1) *satisfies*

$$\|f(t)\|_0 \le \|f_0\|_0 + \int_{t_0}^{t} \|g(\cdot, \tau)\|_0 \, d\tau \,, \tag{B.2}$$

$$\|f(t)\|_\alpha \le 2 \left(\|f_0\|_\alpha + \int_{t_0}^{t} \|g(\cdot, \tau)\|_\alpha \, d\tau \right) , \tag{B.3}$$

for all $0 \le \alpha \le 1$, *and, more generally, for any* $N \ge 1$ *and* $0 \le \alpha < 1$

$$[f(t)]_{N+\alpha} \lesssim [f_0]_{N+\alpha} + |t|\, [v]_{N+\alpha} [f_0]_1 + \int_0^t \Big([g(\tau)]_{N+\alpha} + (t - \tau)[v]_{N+\alpha} [g(\tau)]_1 \Big) \, d\tau. \tag{B.4}$$

Define $\Phi(t, \cdot)$ *to be the inverse of the flux* X *of* v *starting at time* t_0 *as the identity (i.e.* ${}^{d}/_{dt} X = v(X, t)$ *and* $X(x, t_0) = x$*). Under the same assumptions as above we have:*

$$\|\nabla \Phi(t) - \mathrm{Id}\|_0 \lesssim |t|\, [v]_1 \,, \tag{B.5}$$

$$[\Phi(t)]_N \lesssim |t|\, [v]_N \qquad \forall N \ge 2 \,. \tag{B.6}$$

C Potential theory estimates

We recall the definition of the standard class of periodic Calderón-Zygmund operators. Let K be an $\mathbb{R}^3$ kernel which obeys the properties

- $K(z) = \Omega\left(\frac{z}{|z|}\right) |z|^{-3}$, for all $z \in \mathbb{R}^3 \setminus \{0\}$
- $\Omega \in C^\infty(\mathbb{S}^2)$
- $\int_{|\hat{z}|=1} \Omega(\hat{z}) d\hat{z} = 0.$

From the $\mathbb{R}^3$ kernel K, use Poisson summation to define the periodic kernel

$$K_{\mathbb{T}^3}(z) = K(z) + \sum_{\ell \in \mathbb{Z}^3 \setminus \{0\}} \left(K(z + \ell) - K(\ell) \right).$$

Then the operator

$$T_K f(x) = p.v. \int_{\mathbb{T}^3} K_{\mathbb{T}^3}(x - y) f(y) dy$$

is a $\mathbb{T}^3$-periodic Calderón-Zygmund operator, acting on $\mathbb{T}^3$-periodic functions f with zero mean on $\mathbb{T}^3$. We first have boundedness of periodic Calderón-Zygmund operators on periodic Hölder spaces

Proposition C.1. *Fix* $\alpha \in (0, 1)$. *Periodic Calderón-Zygmund operators are bounded on the space of zero mean* $\mathbb{T}^3$*-periodic* C^α *functions.*

Second, we have simple consequence of classical stationary phase techniques.

Proposition C.2 (Updated Version Lemma E.1)**.** *Let $\alpha \in (0,1)$ and $N \geq 1$. Let $a \in C^\infty(\mathbb{T}^3)$, $\Phi \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ be smooth functions and assume that*

$$\hat{C}^{-1} \leq |\nabla \Phi| \leq \hat{C}$$

holds on $\mathbb{T}^3$. Then

$$\left| \int_{\mathbb{T}^3} a(x) e^{ik\cdot\Phi}\, dx \right| \lesssim \frac{\|a\|_N + \|a\|_0 \|\Phi\|_N}{|k|^N}\,, \tag{C.1}$$

and for the operator $\mathring{R}$ defined in (4.28), we have

$$\left\| \mathcal{R}\left(a(x) e^{ik\cdot\Phi}\right)\right\|_\alpha \lesssim \frac{\|a\|_0}{|k|^{1-\alpha}} + \frac{\|a\|_{N+\alpha} + \|a\|_0 \|\Phi\|_{N+\alpha}}{|k|^{N-\alpha}}\,,$$

where the implicit constant depends on $\hat{C}$, α and N, but not on k.

Proposition C.3 (Commutators involving singular integrals)**.** *Let $\alpha \in (0,1)$ and $N \geq 0$. Let T_K be a Calderón-Zygmund operator with kernel K. Let $b \in C^{N+1,\alpha}(\mathbb{T}^3)$ a vectorfield. Then we have*

$$\|[T_K, b\cdot\nabla] f\|_{N+\alpha} \lesssim \|b\|_{1+\alpha} \|f\|_{N+\alpha} + \|b\|_{N+1+\alpha} \|f\|_\alpha$$

for any $f \in C^{N+\alpha}(\mathbb{T}^3)$, where the implicit constant depends on α, N and K.

D Beltrami Flows

Given $\xi \in \mathbb{S}^2 \cap \mathbb{Q}^3$ let $A_\xi \in \mathbb{S}^2 \cap \mathbb{Q}^3$ obey

$$A_\xi \cdot \xi = 0, \quad A_{-\xi} = A_\xi\,.$$

We define the complex vector

$$B_\xi = \tfrac{1}{\sqrt{2}}\left(A_\xi + i\xi \times A_\xi\right)\,.$$

By construction, the vector B_ξ has the properties

$$|B_\xi| = 1, \quad B_\xi \cdot \xi = 0, \quad i\xi \times B_\xi = B_\xi, \quad B_{-\xi} = \overline{B_\xi}\,.$$

This implies that for any $\lambda \in \mathbb{Z}$, such that $\lambda\xi \in \mathbb{Z}^3$, the function

$$W_{(\xi)}(x) := W_{\xi,\lambda}(x) := B_\xi e^{i\lambda\xi\cdot x} \tag{D.1}$$

is $\mathbb{T}^3$ periodic, divergence free, and is an eigenfunction of the curl operator with eigenvalue λ. That is, $W_{(\xi)}$ is a complex Beltrami plane wave. The following lemma states a useful property for linear combinations of complex Beltrami plane waves.

Proposition D.1. *Let Λ be a given finite subset of $\mathbb{S}^2 \cap \mathbb{Q}^3$ such that $-\Lambda = \Lambda$, and let $\lambda \in \mathbb{Z}$ be such that $\lambda\Lambda \subset \mathbb{Z}^3$. Then for any choice of coefficients $a_\xi \in \mathbb{C}$ with $\overline{a}_\xi = a_{-\xi}$ the vector field*

$$W(x) = \sum_{\xi\in\Lambda} a_\xi B_\xi e^{i\lambda\xi\cdot x} \tag{D.2}$$

is a real-valued, divergence-free Beltrami vector field $\operatorname{curl} W = \lambda W$*, and thus it is a stationary solution of the Euler equations*

$$\operatorname{div}(W \otimes W) = \nabla \frac{|W|^2}{2}. \tag{D.3}$$

Furthermore, since $B_\xi \otimes B_{-\xi} + B_{-\xi} \otimes B_\xi = 2\mathbb{P}(B_\xi \otimes B_{-\xi}) = \mathrm{Id} - \xi \otimes \xi$*, we have*

$$\fint_{\mathbb{T}^3} W \otimes W \, dx = \frac{1}{2} \sum_{\xi \in \Lambda} |a_\xi|^2 \left(\mathrm{Id} - \xi \otimes \xi\right) . \tag{D.4}$$

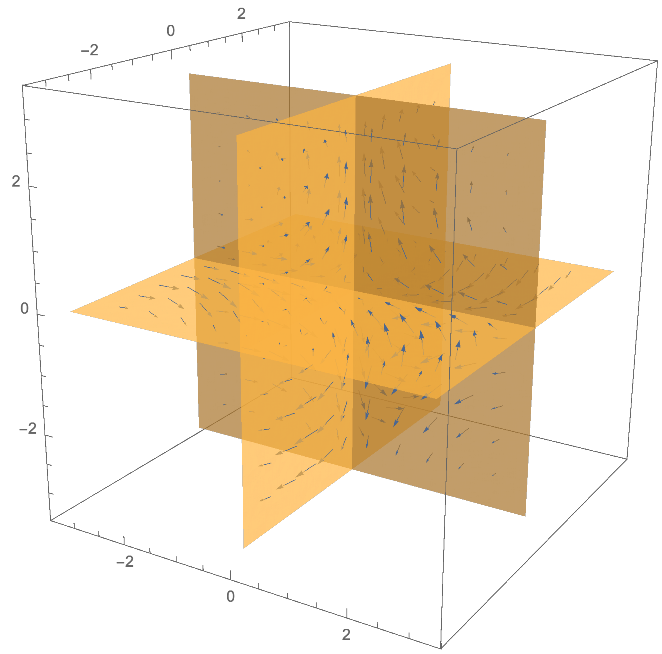

Figure 3: Example of a Beltrami flow $W(x)$ as defined in (D.2).

Proposition D.2. *There exists a sufficiently small* $c_* > 0$ *with the following property. Let* $B_{c_*}(\mathrm{Id})$ *denote the closed ball of symmetric* 3×3 *matrices, centered at* Id*, of radius* c_**. Then, there exist pairwise disjoint subsets*

$$\Lambda_\alpha \subset \mathbb{S}^2 \cap \mathbb{Q}^3 \qquad \alpha \in \{0, 1\} ,$$

and smooth positive functions

$$\gamma_\xi^{(\alpha)} \in C^\infty\left(B_c(\mathrm{Id})\right) \qquad \alpha \in \{0, 1\},\, \xi \in \Lambda_\alpha ,$$

such that the following hold. For every $\xi \in \Lambda_\alpha$ *we have* $-\xi \in \Lambda_\alpha$ *and* $\gamma_\xi^{(\alpha)} = \gamma_{-\xi}^{(\alpha)}$*. For each* $R \in B_{c_*}(\mathrm{Id})$ *we have the identity*

$$R = \frac{1}{2} \sum_{\xi \in \Lambda_\alpha} \left(\gamma_\xi^{(\alpha)}(R)\right)^2 \left(\mathrm{Id} - \xi \otimes \xi\right) . \tag{D.5}$$

We label by n_* *the smallest natural number such that* $n_* \Lambda_\alpha \subset \mathbb{Z}^3$ *for all* $\alpha \in \{1, 2\}$*.*

It is sufficient to consider index sets Λ_0 and Λ_1 in Proposition D.2 to have 12 elements. Moreover, by abuse of notation, for $j \in \mathbb{Z}$ we denote $\Lambda_j = \Lambda_{j \bmod 2}$. Also, it is convenient to denote by M a *geometric constant* such that

$$\sum_{\xi \in \Lambda_\alpha} \left\| \gamma_\xi^{(\alpha)} \right\|_{C^1(B_{c_*}(\mathrm{Id}))} \leq M \tag{D.6}$$

holds for $\alpha \in \{0, 1\}$ and $\xi \in \Lambda_\alpha$. This parameter is universal.

E Stationary Phase Lemma

The operator $\mathcal{R}$ which acts on vector fields v with $\int_{\mathbb{T}^3} v dx = 0$ as

$$(\mathcal{R}v)^{k\ell} = (\partial_k \Delta^{-1} v^\ell + \partial_\ell \Delta^{-1} v^k) - \frac{1}{2}\left(\delta_{k\ell} + \partial_k \partial_\ell \Delta^{-1}\right) \operatorname{div} \Delta^{-1} v \tag{E.1}$$

for $k, \ell \in \{1, 2, 3\}$. The above inverse divergence operator has the property that $\mathcal{R}v(x)$ is a symmetric trace-free matrix for each $x \in \mathbb{T}^3$, and $\mathcal{R}$ is an right inverse of the div operator, i.e. $\operatorname{div}(\mathcal{R}v) = v$. When v does not obey $\int_{\mathbb{T}^3} v dx = 0$, we overload notation and denote $\mathcal{R}v := \mathcal{R}(v - \int_{\mathbb{T}^3} v dx)$. Note that $\nabla\mathcal{R}$ is a Calderón-Zygmund operator.

The following lemma makes rigorous the fact that $\mathcal{R}$ obeys the same elliptic regularity estimates as $|\nabla|^{-1}$.

Lemma E.1. *Let $\lambda\xi \in \mathbb{Z}^3$, $\alpha \in (0, 1)$, and $m \geq 1$. Assume that $a \in C^{m,\alpha}(\mathbb{T}^3)$ and $\Phi \in C^{m,\alpha}(\mathbb{T}^3; \mathbb{R}^3)$ are smooth functions such that the phase function Φ obeys*

$$C^{-1} \leq |\nabla\Phi| \leq C$$

on $\mathbb{T}^3$, for some constant $C \geq 1$. Then, with the inverse divergence operator $\mathcal{R}$ defined in (E.1) we have

$$\left\| \mathcal{R}\left(a(x) e^{i\lambda\xi\cdot\Phi(x)}\right) \right\|_{C^\alpha} \lesssim \frac{\|a\|_{C^0}}{\lambda^{1-\alpha}} + \frac{\|a\|_{C^{m,\alpha}} + \|a\|_{C^0} \|\nabla\Phi\|_{C^{m,\alpha}}}{\lambda^{m-\alpha}},$$

where the implicit constant depends on C, α and m (in particular, not on the frequency λ).

References

[1] T. Buckmaster, C. D. Lellis, L. S. J. au2, and V. Vicol. Onsager's conjecture for admissible weak solutions, 2017.

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