Complex Geometry

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This is set of a notes taken by Mark Ma in for the Course Complex Analysis and Riemann Surfaces in Spring 2025 taught by Prof. Duong Phong. The main goal is to study analysis on complex manifolds, as intersection of differential, algebraic geometry and mathematical physics.

Contents

1 Manifolds

Intuitively speaking, a manifold is a space which looks locally like Euclidean Space E^n.

Definition 1.1 (Manifolds)**.** *Let $X = \bigcup_\mu X_\mu$. Charts*

$$\Phi_\mu : X_\mu \subset X \to E^n \qquad z \mapsto z_\mu$$

maps into E^n Euclidean Space with dimension n. For any $z \in X_\mu \cap X_\nu$, we have choice of two maps, and we want them to transit smoothly. It should be independent of the chart that we use. We need regularity on

$$z_\mu \overset{\Phi_\mu \circ \Phi_\nu^{-1}}{\to} z_\nu$$

1. *X is continuous manifold if $\Phi_\mu \circ \Phi_\nu^{-1} \in C^0$. This is manifold with least structure.*
2. *X is C^∞ manifold if $\Phi_\mu \circ \Phi_\nu^{-1} \in C^\infty$ is smooth. It admits infinite number of continuous derivatives. It might be the case that the map has zeros. Hence we want to impose invertibility*

$$\textbf{Jacobian}(\Phi_\mu \circ \Phi_\nu^{-1}) \neq 0$$

For $E^n = \mathbb{R}^n$,

$$z_\mu = (z_\mu^1, \cdots, z_\mu^n) \in \mathbb{R}^n \mapsto z_\nu = (z_\nu^1, \cdots, z_\nu^n) \in \mathbb{R}^n$$

That Jacobian is nonzero is equivalent to

$$\det(\frac{\partial z_\nu^k}{\partial z_\mu^m}) \neq 0$$

3. *Complex Manifolds. The local model in complex manifolds is $E^n = \mathbb{C}^n$.*

$$z_\mu = (z_\mu^1, \cdots, z_\mu^n) \in \mathbb{C}^n \overset{\Phi_\mu \circ \Phi_\nu^{-1}}{\to} z_\nu = (z_\nu^1, \cdots, z_\nu^n) \in \mathbb{C}^n$$

We want

(a) *The map to be holomorphic, i.e., each z_ν^k is holomorphic. But notice we're dealing with n-variable holomorphic functions.*

Definition 1.2 (holomorphic)**.** *Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$. f is holomorphic if for any $\zeta \in \Omega$,*

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(\zeta + h) - f(\zeta)}{h} \qquad \textit{exists}$$

In definition of holomorphicity, $h \in \mathbb{C}$.

Theorem 1.1. *$f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is holomorphic iff*

i. *We can write as power series, and furthermore, it is power series in variable z, no $\overline{z}$.*

$$f(z) = \sum_{m=0}^{\infty} c_m (z - \zeta)^m$$

ii. *$f \in C^1$ and satisfies the Cauchy-Riemann Equation*

$$\frac{\partial f}{\partial \overline{z}} = 0$$

i.e.

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0$$

Now we generalize to functions in n-variables.

Definition 1.3 (holomorphic)**.** *Let*

$$f : \Omega \subset \mathbb{C}^n \to \mathbb{C}$$

we take the perspective that power series has no $\overline{z}$. Introduce $M = (m_1, \cdots, m_n)$ and $z = (z^1, \cdots, z^n)$. Define

$$z^M := (z^1)^{m_1} \cdots (z^n)^{m_n}$$

We say f is holomorphic if one can write

$$f(z) = \sum_{m_i \geq 0, i=1,\cdots,n} c_M (z - \zeta)^M \qquad \forall\, \zeta \in \Omega$$

There is powerful theorem due to Hartogs.

Theorem 1.2 (Hartogs). *f is holomorphic iff f is holomorphic for each z^k with the other variables fixed.*

In other words, a function is holomorphic in all variables iff in each variables. It reduces the local theory in n-variables to one variable.

(b) The Jacobian to be non-vanishing

$$\det(\frac{\partial z_\nu^k}{\partial z_\mu^m}) \neq 0$$

1.1 Functions and Line Bundles

What we're interested in are functions, vector bundles and sections defined on complex manifolds.

Definition 1.4 (Functions on Manifolds). *Let $E^n = \mathbb{C}^n$. $X = \bigcup_\mu X_\mu$.*

$$z_\mu = (z_\mu^1, \cdots, z_\mu^n) \in \mathbb{C}^n \overset{\Phi_\mu \circ \Phi_\nu^{-1}}{\to} z_\nu = (z_\nu^1, \cdots, z_\nu^n) \in \mathbb{C}^n$$

Let $f : X \to \mathbb{C}$. We understand f by restricting to the charts.

$$\left(f \circ \Phi_\mu^{-1}\right)(z_\mu) \equiv f|_{X_\mu} \qquad \left(f \circ \Phi_\nu^{-1}\right)(z_\nu) \equiv f|_{X_\nu}$$

On the overlap they agree with one another. Hence we may glue them together. The name of the game is how we glue the functions together. Denote

$$\varphi_\mu(z_\mu) := \left(f \circ \Phi_\mu^{-1}\right)(z_\mu) \qquad \varphi_\nu(z_\nu) := \left(f \circ \Phi_\nu^{-1}\right)(z_\nu)$$

To understand f on X, one try to understand restrictions to small charts

1. *$\varphi_\mu(z_\mu)$ on X_μ*
2. *$\varphi_\mu(z_\mu) = \varphi_\nu(z_\nu)$ on the overlap $X_\mu \cap X_\nu$ as the Gluing Rule.*

But is this gluing good enough?

Remark 1.1 (Differential Forms $n = 1$). *Take the case of $n = 1$ for simplicity. Then we have the point z corresponding to*

$$z_\mu \overset{\Phi_\mu}{\longleftarrow} z \overset{\Phi_\nu}{\longrightarrow} z_\nu$$

Then

$$\varphi_\mu(z_\mu) \longleftarrow f(z) \longrightarrow \varphi_\nu(z_\nu)$$

One differentiate and notice

$$\frac{\partial}{\partial z_\mu}\varphi_\mu(z_\mu) \neq \frac{\partial}{\partial z_\nu}\varphi_\nu(z_\nu)$$

What is the relation between the two quantities? Take the formula and differentiate

$$\begin{aligned} \varphi_\mu(z_\mu) &= \varphi_\nu(z_\nu) \\ \frac{\partial}{\partial z_\mu}\varphi_\mu(z_\mu) &= \frac{\partial}{\partial z_\mu}\left(\varphi_\nu(z_\nu)\right) \\ &= \frac{\partial}{\partial z_\nu}\varphi_\nu(z_\nu)\frac{\partial z_\nu}{\partial z_\mu} \end{aligned}$$

The quantities transit w.r.t. factor $\frac{\partial z_\nu}{\partial z_\mu}$. The differential of a scalar function is no longer a scalar function but a transition. We need to widen the construction by allowing a transition function.

One has the following definition of Line Bundles as generalization of differential forms.

Definition 1.5 (Line Bundles on Complex Manifold X). *Assume we have a chosen cover $X = \bigcup_\mu X_\mu$. A line bundle L is an assignment*

$$L \iff t_{\mu\nu}(z) \quad \text{invertible}(\neq 0) \text{ functions defined on } X_\mu \cap X_\nu \text{ satisfying co-cycle condition}$$

where the co-cycle condition writes

$$t_{\mu\nu}t_{\nu p} = t_{\mu p}$$

Remark 1.2 (Line Bundle)**.** *What makes a difference is how one glue these together. We have a new object* $t_{\mu\nu}$*. What are the* $t_{\mu\nu}$ *that we want to allow? We want*

1. $t_{\mu\nu} \neq 0$ *for invertibility issues.*
2. $t_{\mu\nu}$ *satisfies the co-cycle condition. Notice if there's another chart* X_ρ *then necessarily we want*

$$t_{\mu\rho}(z)\varphi_\rho(z) = \varphi_\mu(z_\mu) = t_{\mu\nu}(z)\varphi_\nu(z_\nu) = t_{\mu\nu}(z)t_{\nu\rho}(z)\varphi_\rho(z) \qquad \forall\ z \cap X_\mu \cap X_\nu \cap X_\rho$$

Hence we require co-cycle condition

$$t_{\mu\nu}t_{\nu p} = t_{\mu p} \qquad \forall\ z \cap X_\mu \cap X_\nu \cap X_\rho$$

Example 1.1 (Canonical Bundle K_X)**.** *Let* $X = \bigcup_\mu X_\mu$*. Take*

$$t_{\mu\nu} := \det\left((\frac{\partial z_\nu^\beta}{\partial z_\mu^\alpha})_{1 \leq \alpha,\beta \leq n} \right) \qquad \forall\ z \in X_\mu \cap X_\nu$$

Then with this choice we obtain a Line Bundle. This is known as the canonical Bundle, denoted K_X*.*

Example 1.2. *Line Bundle L is*

1. *a* C^∞ *bundle if* $t_{\mu\nu} \in C^\infty$
2. *a holomorphic bundle if* $t_{\mu\nu}$ *are holomorphic.*
3. *an antiholomorphic bundle if* $t_{\mu\nu}$ *are antiholomorphic, i.e.,* $\overline{t_{\mu\nu}}$ *are holomorphic.*

Remark 1.3. *From a Bundle L, we can generate many others.*

$$\begin{aligned} \overline{L} &\equiv \textit{bundle with transition functions } \overline{t_{\mu\nu}} \\ L^k &\equiv \textit{bundle with transition functions } t_{\mu\nu}^k \qquad \forall\ z \in \mathbb{Z} \\ K_X^{-1} &\equiv \textit{anticanonical bundle} \end{aligned}$$

Given Line Bundle L, one can associate it with sections. L is characterised by its space of sections.

Definition 1.6 (Sections of Line Bundles)**.** *Let L be a line bundle. A section* $\varphi \in \Gamma(X, L)$ *if*

1. $\varphi_\mu(z_\mu)$ *on* X_μ
2. *satisfying the gluing rule*

$$\varphi_\mu(z_\mu) = t_{\mu\nu}(z)\varphi_\nu(z_\nu) \text{ on } X_\mu \cap X_\nu$$

1.2 Vector-Valued Functions and Vector Bundles

Definition 1.7 (Vector-Valued Functions on Manifolds and Vector Bundles)**.** *On open set in* $\mathbb{C}^n$*, a vector-valued function* $f : \mathbb{C}^n \to \mathbb{C}^N$ *is*

$$f(z) = \begin{pmatrix} f^1(z) \\ \vdots \\ f^N(z) \end{pmatrix} = f^\alpha(z) \qquad 1 \leq \alpha \leq N$$

To generalize to manifold, we proceed as in the scalar case. We think about how to glue functions together. To understand f on $X = \bigcup_\mu X_\mu$*, one restrict to small charts*

1. $\varphi_\mu^\alpha(z_\mu)$ *on* X_μ*.*
2. *satisfying Gluing Rule*

$$\varphi_\mu^\alpha(z_\mu) = {t_{\mu\nu}}^\alpha{}_\beta(z)\varphi_\nu^\beta(z_\nu) \qquad \text{on } X_\mu \cap X_\nu$$

for any $1 \leq \alpha \leq N$*.*

A vector bundle L corresponds to

$${t_{\mu\nu}}^\alpha{}_\beta(z) \in GL(N, \mathbb{C}) \qquad \text{on } X_\mu \cap X_\nu$$

One can also write vector-valued functions in row vectors.

$$\psi(z) = (\psi_1(z), \cdots, \psi_N(z)) = \psi_\alpha(z)$$

We want to generalize this to vector bundles by the corresponding rule

$$\varphi_{\mu\alpha}(z_\mu) = \varphi_{\nu\beta}(z_\nu){t_{\nu\mu}}^\beta{}_\alpha(z) \qquad \forall\ z \in X_\mu \cap X_\nu$$

Now the transition function writes on the Right. This makes a difference due to $t_{\nu\mu}$ *right multiplication.*

Example 1.3 ($\Lambda^{1,0}(X)$)**.** *Given complex manifold* $X = \bigcup_\mu X_\mu$. *Define*

$$t_{\nu\mu}{}^{\beta}{}_{\alpha}(z) := \frac{\partial z_\nu^\beta}{\partial z_\mu^\alpha} \qquad \forall\ 1 \leq \alpha,\, \beta \leq N$$

This satisfies the co-cycle condition. The corresponding vector bundle if the bundle of $(1,0)$*-forms, denoted by* $\Lambda^{1,0}(X)$.

2 Connections and Curvature of Line Bundles

We think about differentiating sections of bundles.

2.1 Differentiations of sections of Line Bundles

Fix a line bundle $L \to X$. $\varphi \in \Gamma(X, L)$ is given locally as

1. $\varphi_\mu(z_\mu)$ on X_μ
2. satisfying the gluing rule
$$\varphi_\mu(z_\mu) = t_{\mu\nu}(z)\varphi_\nu(z_\nu) \text{ on } X_\mu \cap X_\nu$$

Consider differentiating local expressions on X_μ

$$\frac{\partial}{\partial z_\mu}\varphi_\mu(z_\mu) = (\frac{\partial}{\partial z_\mu}t_{\mu\nu}(z))\varphi_\nu + t_{\mu\nu}(z)\frac{\partial}{\partial z_\mu}\varphi_\nu(z_\nu)$$

We do not want the first term. However, we may assume the Bundle L is holomorphic, i.e., the transition functions

$$t_{\mu\nu}(z) \qquad \text{are holomorphic} \implies \frac{\partial}{\partial \overline{z}_\mu}t_{\mu\nu}(z) = 0$$

In this case, we're differentiable in $\overline{z}$-direction

$$\begin{aligned}\frac{\partial}{\partial \overline{z}^j_\mu}\varphi_\mu(z_\mu) &= t_{\mu\nu}(z)\frac{\partial}{\partial \overline{z}^j_\mu}\varphi_\nu(z_\nu) \\ &= t_{\mu\nu}(z)\frac{\partial \overline{z}^k_\nu}{\partial \overline{z}^j_\mu}\frac{\partial}{\partial \overline{z}^k_\nu}\varphi_\nu(z_\nu)\end{aligned}$$

We have this new Gluing Rule. We have the additional piece

$$\frac{\partial \overline{z}^k_\nu}{\partial \overline{z}^j_\mu} \qquad \text{as transition function of } \Lambda^{0,1}$$

Recall that

$$(\frac{\partial z^k_\nu}{\partial z^j_\mu}) \qquad \text{is the transition function of a vector bundle denoted by } \Lambda^{1,0}$$

We make the claim that $\frac{\partial}{\partial \overline{z}^j_\mu}\varphi_\mu(z_\mu)$ is a section of the bundle

$$L \otimes \overline{\Lambda^{1,0}}$$

upon noticing L corresponds to $t_{\mu\nu}$ and $\overline{\Lambda^{1,0}}$ corresponds to $\frac{\partial \overline{z}^k_\nu}{\partial \overline{z}^j_\mu}$. This is a tensor product of the bundles. But what about the direction without bar? We need to make some choice. We have obtained

$$\varphi \in \Gamma(X, L) \overset{\overline{\nabla}}{\to} \overline{\nabla}\varphi = \frac{\partial}{\partial \overline{z}_\mu}\varphi_\mu(z_\mu) \in \Gamma(X, L \otimes \overline{\Lambda^{1,0}}) = \Gamma(X, L \otimes \Lambda^{0,1})$$

We need

$$\varphi \in \Gamma(X, L) \overset{\nabla}{\to} \nabla\varphi = \frac{\partial}{\partial z_\mu}\varphi_\mu(z_\mu) \in \Gamma(X, L \otimes \Lambda^{1,0})$$

One way of doing this is using Unitary Connection.

Definition 2.1 (Metric on Line Bundle). *A metric h on L is a section of $L^{-1} \otimes \overline{L}^{-1}$ satisfying*

$$h(z) > 0 \qquad \forall\ z$$

The transition functions of L^{-1} are $t_{\mu\nu}(z)^{-1}$ and those for $\overline{L}^{-1}$ are $\overline{t_{\mu\nu}(z)}^{-1}$. Hence gluing condition satisfies

$$\begin{aligned}h_\mu(z_\mu) &= t_{\mu\nu}(z)^{-1}\overline{t_{\mu\nu}(z)}^{-1}h_\nu(z_\nu) \\ h_\mu(z_\mu) &= |t_{\mu\nu}(z)|^{-2}h_\nu(z_\nu) > 0 \qquad \forall\ z \in X_\mu \cap X_\nu\end{aligned}$$

Here it makes sense to talk about positivity.

Now with the metric, we have notion of length, i.e., one can define the norm of $\varphi \in \Gamma(X, L)$.

Definition 2.2 (Norm on $\Gamma(X, L)$)**.**

$$|\varphi|^2 := \varphi_\mu \overline{\varphi}_\mu h_\mu \qquad on\ X_\mu$$

and notice

$$\varphi_\mu \overline{\varphi}_\mu h_\mu = \varphi_\nu \overline{\varphi}_\nu h_\nu \qquad on\ X_\mu \cap X_\nu$$

Now with a metric h, we can define the covariant derivative.

Definition 2.3 (Covariant Derivative)**.** *For simplicity we drop the index of μ*

$$\nabla_j \varphi := h^{-1} \partial_j (h\varphi) \tag{1}$$

Since $h\varphi$ is section of $L^{-1} \otimes \overline{L}^{-1}(L) = \overline{L}^{-1}$, which is anti-holomorphic. Then $\partial_j(h\varphi)$ is section of

$$\overline{L}^{-1} \otimes \Lambda^{1,0}$$

again h^{-1} as $L \otimes \overline{L}$ hits, we have

$$\nabla_j \varphi \in \Gamma(X, L \otimes \Lambda^{1,0})$$

Explicitly.

$$\begin{aligned}
\nabla_j \varphi &= h^{-1}(h\partial_j \varphi + (\partial_j h)\varphi) \\
&= \partial_j \varphi + (h^{-1}\partial_j h)\varphi \\
&= \partial_j \varphi + (\partial_j(\log(h)))\varphi
\end{aligned}$$

□

2.2 Curvature of Line Bundle w.r.t. metric

In summary, let $\partial_{\overline{j}} = \frac{\partial}{\partial \overline{z}^j_\mu}$

$$\begin{aligned}
\nabla_{\overline{j}} \varphi &= \partial_{\overline{j}} \varphi \\
\nabla_j \varphi &= h^{-1} \partial_j (h\varphi)
\end{aligned}$$

But these derivatives are more complicated than the usual ones. Here, partial derivatives do not commute, and we need to understand why and how. On Euclidean Spaces

$$\frac{\partial}{\partial x^m}\frac{\partial}{\partial x^\ell} f = \frac{\partial}{\partial x^\ell}\frac{\partial}{\partial x^m} f \implies [\frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^\ell}] f = 0$$

But here we have something more complicated.

$$\begin{aligned}
[\nabla_{\overline{j}}, \nabla_{\overline{k}}]\varphi &= 0 \\
[\nabla_j, \nabla_k]\varphi &= \nabla_j \nabla_k \varphi - \nabla_k \nabla_j \varphi \\
&= h^{-1}\partial_j(h(h^{-1}\partial_k(h\varphi))) - h^{-1}\partial_k(h(h^{-1}\partial_j(h\varphi))) \\
&= h^{-1}\partial_j\partial_k(h\varphi) - h^{-1}\partial_k\partial_j(h\varphi) \\
&= h^{-1}(\partial_j\partial_k - \partial_k\partial_j)(h\varphi) = 0 \\
[\nabla_j, \nabla_{\overline{k}}]\varphi &= \nabla_j \nabla_{\overline{k}} \varphi - \nabla_{\overline{k}} \nabla_j \varphi \\
&= h^{-1}\partial_j(h(\partial_{\overline{k}}\varphi)) - \partial_{\overline{k}}(h^{-1}\partial_j(h\varphi)) \\
&= h^{-1}\left((\partial_j h)(\partial_{\overline{k}}\varphi) + h\partial_j\partial_{\overline{k}}\varphi\right) - \partial_{\overline{k}}\left(h^{-1}(\partial_j h)\varphi + \partial_j\varphi\right) \\
&= h^{-1}(\partial_j h)\partial_{\overline{k}}\varphi - (\partial_{\overline{k}}(h^{-1}\partial_j h)\varphi + h^{-1}\partial_j h \partial_{\overline{k}}\varphi) \\
&= -(\partial_j\partial_{\overline{k}}(\log(h)))\varphi
\end{aligned}$$

Definition 2.4 (Curvature of Line Bundle)**.**

$$F_{\overline{k}j} := -\partial_j \partial_{\overline{k}}(\log(h)) \in \Gamma(X, \Lambda^{1,1})$$

is the curvature of L w.r.t. h.

Now we have the key formula in differential geometry.

$$[\nabla_j, \nabla_{\overline{k}}]\varphi = F_{\overline{k}j}\varphi \tag{2}$$

Observations.

1.
$$F = \sum_{kj} F_{\overline{k}j} dz^j \wedge d\overline{z}^k$$

is a $(1,1)$-form, i.e.
$$F \in \Gamma(X, \Lambda^{1,1} := \Lambda^{1,0} \otimes \Lambda^{0,1})$$

where $\Lambda^{0,1} = \overline{\Lambda^{1,0}}$.

2. $dF = 0$, since
$$\begin{aligned}
F &= \overline{\partial}(\partial(\log(h))) \\
&= \overline{\partial}(\partial_j(\log(h))dz^j) \\
&= d\overline{z}^k \partial_{\overline{k}}(\partial_j(\log(h)))dz^j \\
&= \partial_j \partial_{\overline{k}} \log(h) d\overline{z}^k \wedge dz^j \\
F &= \overline{\partial}(\partial \log(h)) \\
dF &= (\partial + \overline{\partial})(\overline{\partial}\partial(\log(h))) = 0
\end{aligned}$$

Since
$$\partial\overline{\partial} + \overline{\partial}\partial = \partial^2 = \overline{\partial}^2 = 0$$

Hence there is closed form but not exact. This is example of De-Rham Cohomology.

Definition 2.5 (Curvature Form)**.**
$$F := \sum_{k,j} iF_{\overline{k}j} dz^j \wedge d\overline{z}^k = -i\partial\overline{\partial}\log(h)$$
$$\overline{F} = \sum_{k,j} (-i) F_{\overline{j}k} d\overline{z}^j \wedge dz^k = F$$

Hence the curvature form is taking Real Values.

Property 2.1. *1. F is closed form, i.e., $dF = 0$.*

2. F clearly depends on the metric, but $[F]_{\mathrm{dR}}$ de Rham Cohomology class is independent of h.

3. $[F]_{\mathrm{dR}}$ is hence an invariant of the bundle L. We call $[F]_{\mathrm{dR}}$ the first Chern Class of L
$$c_1(L) := [F]_{\mathrm{dR}}$$

Remark 2.1. *Whenever you have a closed form, you can consider the De Rham Cohomology Class.*

2.2.1 de Rham Cohomology

We need some Background on De Rham Cohomology.

Definition 2.6 (p-form)**.** *Let X be a smooth, differentiable, compact manifolds. A p-form φ is an expression of the type*
$$\varphi = \frac{1}{p!} \sum c_{i_1,\cdots,i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

There is a basic operator, d called the De-Rham Exterior differential

Definition 2.7 (De-Rham Exterior differential)**.** *Let f be any function on X, denote $f \in \Lambda^0$*
$$df := \sum_j \frac{\partial f}{\partial u^j} du^j \qquad (u^1, \cdots, u^n) \text{ are local coordinates for } X$$

Then
$$d : \Lambda^0 \to \Lambda^1 \qquad f \mapsto df$$

d can be extended s.t.
$$d : \Lambda^p \to \Lambda^{p+1} \qquad \varphi \mapsto d\varphi := \frac{1}{p!} \sum (dc_{i_1,\cdots,i_p}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

Property 2.2. *As consequence of* $\frac{\partial^2}{\partial u^j \partial u^m} f = \frac{\partial^2}{\partial u^m \partial u^j} f$

$$d^2 = 0$$

Definition 2.8 (de Rham Cohomology)**.** *Let F be a p-form which is closed, i.e., $dF = 0$. Then*

$$[F]_{\mathrm{dR}} := F/\{\textit{exact forms } d\psi \textit{ where } \psi \in \Lambda^{p-1}\}$$

Remark 2.2. *$d^2 = 0$ implies that*

$$\{d\psi\} = \textit{exact forms} \subset \{F \mid dF = 0\} = \textit{closed forms}$$

We consider the de Rham cohomology group

$$H^p_{\mathrm{dR}}(X) := \{F \mid dF = 0\}/\{d\psi\}$$

The group structure is addition. This is in fact a vector space.

2.2.2 de Rham Cohomology on Complex Manifolds

Now given a complex structure. We consider the 1-dim case for simplicity. Let X be a complex manifold, and let z be a local holomorphic coordinate, i.e.

$$z = x + iy$$

One can view $u = (x, y)$ as the real coordinates.

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

But

$$dz = dx + idy \qquad d\overline{z} = dx - idy$$

Adding yields

$$\begin{aligned}
dx &= \frac{1}{2}(dz + d\overline{z}) \\
dy &= \frac{1}{2i}(dz - d\overline{z}) \\
df &= \frac{\partial f}{\partial x}\frac{1}{2}(dz + d\overline{z}) + \frac{\partial f}{\partial y}\frac{1}{2i}(dz - d\overline{z}) \\
&= \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right) dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right) d\overline{z} \\
&= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) d\overline{z} \\
&= \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \overline{z}} d\overline{z}
\end{aligned}$$

More generally, for X complex manifold of dimension n, let $z^1, \cdots, z^n$ be local holomorphic coordinates. We can define

$$\begin{aligned}
\partial f &:= \sum_j \frac{\partial f}{\partial z^j} dz^j \\
\overline{\partial} f &:= \sum_j \frac{\partial f}{\partial \overline{z}^j} d\overline{z}^j \\
\implies df &= \partial f + \overline{\partial} f
\end{aligned}$$

and we have

$$\partial : \Lambda^{p,q} \to \Lambda^{p+1,q} \qquad \overline{\partial} : \Lambda^{p,q} \to \Lambda^{p,q+1}$$

s.t.

$$\partial^2 = 0 \qquad \overline{\partial}^2 = 0$$

In particular,

$$\begin{aligned}
0 = d^2 &= (\partial + \overline{\partial})^2 \\
&= \partial^2 + \overline{\partial}^2 + \partial\overline{\partial} + \overline{\partial}\partial \\
&= 0 + 0 + \partial\overline{\partial} + \overline{\partial}\partial
\end{aligned}$$

Hence they anti-commute

$$\partial\overline{\partial} + \overline{\partial}\partial = 0$$

2.2.3 Curvature Form F of Holomorphic Line Bundle

Now we come back to

$$F := -i \sum_{k,j} \frac{\partial^2}{\partial z^j \partial \overline{z}^k} (\log(h)) dz^j \wedge d\overline{z}^k$$

We can write

$$\begin{aligned} F &= -i \sum_{j,k} \frac{\partial}{\partial z^j} \left(\frac{\partial}{\partial \overline{z}^k} \log(h) \right) dz^j \wedge d\overline{z}^k \\ &= -i \sum_j dz^j \frac{\partial}{\partial z^j} \left(\sum_k \frac{\partial}{\partial \overline{z}^k} \log(h) \right) \wedge d\overline{z}^k \\ &= -i \partial \overline{\partial} \log(h) \end{aligned}$$

Now F is readily seen to be closed.

$$\begin{aligned} dF &= -i(\partial + \overline{\partial}) \partial \overline{\partial} \log(h) \\ &= -i(\partial^2 \overline{\partial} + \overline{\partial} \partial \overline{\partial}) \log(h) \\ &= i \partial \overline{\partial} \log(h) = i \partial^2 \log(h) = 0 \end{aligned}$$

Thus F is closed. But in general F is not exact. It is tempting to argue F is exact by arguing the following

$$\begin{aligned} F &= -i \partial \overline{\partial} \log(h) = -i(\partial + \overline{\partial}) \overline{\partial} \log(h) \\ &= -id(\overline{\partial} \log(h)) \end{aligned}$$

But $\overline{\partial} \log(h)$ is not a well-defined form. Since

$$h \in \Gamma(X, L^{-1} \otimes \overline{L}^{-1})$$

This holds locally but not globally. The cohomology measures something global, but the curvature measures something local, really dependent on the metric. But the total amount of curvature is fixed.

Remark 2.3 (First Chern class independent of metric)**.** *More precisely, let h and h' be two metrics on L and let F, F' be two corresponding curvatures, i.e.*

$$F_{\overline{k}j} = -\partial_j \partial_{\overline{k}} \log(h) \qquad F'_{\overline{k}j} = -\partial_j \partial_{\overline{k}} \log(h')$$

However

$$\begin{aligned} F_{\overline{k}j} - F'_{\overline{k}j} &= -\partial_j \partial_{\overline{k}} \log(h) + \partial_j \partial_{\overline{k}} \log(h') \\ &= -\partial_j \partial_{\overline{k}} \log(\frac{h}{h'}) \end{aligned}$$

But $\frac{h}{h'}$ is strictly positive C^∞ function since

$$h \in L^{-1} \otimes \overline{L}^{-1}, \qquad h' \in L^{-1} \otimes \overline{L}^{-1} \implies \frac{h}{h'} \in \mathbb{1} \implies C^\infty \text{ function} > 0$$

Then say

$$\frac{h}{h'} = e^\phi \qquad \text{for certain } \phi \in C^\infty$$

Now

$$\begin{aligned} F_{\overline{k}j} - F'_{\overline{k}j} &= -\partial_j \partial_{\overline{k}} \phi \\ i(F_{\overline{k}j} - F'_{\overline{k}j}) dz^j \wedge d\overline{z}^k &= -i \partial_j \partial_{\overline{k}} \phi dz^j \wedge d\overline{z}^k \\ &= -i \partial \overline{\partial} \phi = -i(\partial + \overline{\partial}) \overline{\partial} \phi \\ &= -id(\overline{\partial} \phi) \qquad \text{exact form} \end{aligned}$$

Hence

$$[F]_{\text{dR}} = [F']_{\text{dR}}$$

Fact. Suppose we have Σ complex 1-dim submanifold of X. Then $L|_\Sigma$ is still a line bundle, and $h|_{L|_\Sigma}$ is a metric. Then F is a curvature form which restricts to Σ. Thus

$$\int_\Sigma F \qquad \text{is a intrinsic}$$

Let ψ be any meromorphic function of $L|_\Sigma$, which is not identically 0. Then we can prove the following

$$\frac{1}{2\pi}\int_\Sigma F = \#\text{ zeros of } \psi\ - \#\text{ poles of } \psi$$

□

3 Connections and Curvature of Vector Bundles

On an open set of $\mathbb{C}^n$, a vector-valued function $\varphi : \mathbb{C}^n \to \mathbb{C}^N$ is of the following form

$$\varphi(z) = \begin{pmatrix} \varphi^1(z) \\ \vdots \\ \varphi^N(z) \end{pmatrix} \qquad \text{here } N \text{ is the rank}$$

To deal with global notions, we again need some gluing rules. Let

$$X := \bigcup_{\mu} X_\mu \qquad \text{coordinate charts}$$

A vector bundle $E \to X$ is given by matrix-valued transition functions

$$\{t_{\mu\nu}{}^{\alpha}{}_{\beta}(z)\} \qquad \text{on } X_\mu \cap X_\nu$$

and we define gluing rule

$$\varphi \in \Gamma(X, E) \iff \varphi^\alpha_\mu(z_\mu) \text{ on } X_\mu \text{ satisfying} \qquad \varphi^\alpha_\mu(z_\mu) = t_{\mu\nu}{}^{\alpha}{}_{\beta}(z)\varphi^\beta_\nu(z_\nu) \qquad \text{on } X_\mu \cap X_\nu$$

Explicitly.

$$\begin{pmatrix} \varphi^1_\mu(z_\mu) \\ \vdots \\ \varphi^N_\mu(z_\mu) \end{pmatrix} = \begin{pmatrix} t_{\mu\nu}{}^{1}{}_{1} & \cdots & t_{\mu\nu}{}^{1}{}_{N} \\ \vdots & \cdots & \vdots \\ t_{\mu\nu}{}^{N}{}_{1} & \cdots & t_{\mu\nu}{}^{N}{}_{N} \end{pmatrix} \begin{pmatrix} \varphi^1_\nu(z_\nu) \\ \vdots \\ \varphi^N_\nu(z_\nu) \end{pmatrix} \qquad \text{on } X_\mu \cap X_\nu$$

□

We always assume transition functions are holomorphic, hence it is holomorphic vector bundle.

3.1 Covariant Derivatives of Sections of Vector Bundles

Let $E \to X$ be complex vector bundle. We want to equip E with metric $H_{\overline{\alpha}\beta}$. Take

$$\varphi = \varphi^\alpha_\mu(z_\mu) = \varphi^\alpha(z) \in \Gamma(X, E)$$

In the $\overline{\partial}$-direction

$$\nabla_{\overline{k}}\varphi^\alpha = \partial_{\overline{k}}\varphi^\alpha \in \Gamma(X, E \otimes \overline{\Lambda^{1,0}}) = \Gamma(X, E \otimes \Lambda^{0,1})$$

To construct ∇_j, we need the notion of a metric H on E. If $\varphi \in \Gamma(X, E)$

$$\varphi^\alpha \overline{\varphi^\beta} H_{\overline{\beta}\alpha} > 0 \qquad \varphi \neq 0$$

with $H_{\overline{\beta}\alpha}$ satisfying the condition that this expression is a scalar, i.e.

$$\varphi^\alpha_\mu(z_\mu)\overline{\varphi^\beta_\mu}(z_\mu)H_{\overline{\beta}\alpha}(z_\mu) = \varphi^\gamma_\nu(z_\nu)\overline{\varphi^\delta_\nu}(z_\nu)H_{\overline{\delta}\gamma}(z_\nu) \qquad \text{on } X_\mu \cap X_\nu$$

Definition 3.1. *$H = H_{\overline{\alpha}\beta}$ is metric on E if*

$$\overline{\varphi^\alpha} H_{\overline{\alpha}\beta}\varphi^\beta \qquad \textit{is a scalar}$$

Hence we've obtained a transformation law for $H_{\overline{\beta}\alpha}(z_\mu)$ and $H_{\overline{\delta}\gamma}(z_\nu)$. We can now generalize the formula

$$\nabla_j\varphi = h^{-1}\partial_j(h\varphi)$$

in the case of line bundle. Define the inverse of $H_{\overline{\beta}\alpha}$ by the following equation.

$$H^{\gamma\overline{\beta}}H_{\overline{\beta}\alpha} = \delta^\gamma_\alpha \qquad \text{identity matrix}$$

Similarly

$$H_{\overline{\beta}\alpha}H^{\alpha\overline{\lambda}} = \delta^{\overline{\lambda}}_{\overline{\beta}}$$

We now define the covariant derivative using a Key Formula.

Definition 3.2 (Chern Unitary Connection Covariant Derivative)**.**

$$\nabla_j\varphi^\alpha := H^{\alpha\overline{\gamma}}\partial_j(H_{\overline{\gamma}\beta}\varphi^\beta) \in \Gamma(X, E \otimes \Lambda^{1,0}) \tag{3}$$

One shall observe this generalizes (1)*.*

Components and Matrix Notation. Write $\varphi^\alpha = \begin{pmatrix} \varphi^1 \\ \vdots \\ \varphi^N \end{pmatrix}$ as vector bundle of rank N. Write

$$\psi_\beta = (\psi_1, \cdots, \psi_N)$$

Write

$$M^\alpha{}_\beta = \begin{pmatrix} M^1{}_1 & \cdots & M^1{}_N \\ \vdots & \cdots & \vdots \\ M^N{}_1 & \cdots & M^N{}_N \end{pmatrix}$$

Write matrix multiplication

$$(PQ)^\alpha{}_\beta = P^\alpha{}_\gamma Q^\gamma{}_\beta \qquad \text{where the first } \gamma \text{ is row index and the second } \gamma \text{ is column}$$

Similarly, we can multiply matrices with the type

$$K_{\overline{\lambda}\alpha} \qquad L_{\overline{\mu}\beta}$$

For example

$$H^{\alpha\overline{\gamma}} K_{\overline{\gamma}\alpha} H^{\alpha\overline{\mu}} K_{\overline{\mu}\beta}$$

□

Example 3.1 (Writing covariant derivative in Components and Matrix Notation)**.**

$$\nabla_j \varphi^\alpha = H^{\alpha\overline{\gamma}} \partial_j (H_{\overline{\gamma}\beta} \varphi^\beta)$$

In components

$$(\nabla_j \varphi)^\alpha = (H^{-1} \partial_j (H\varphi))^\alpha$$

and in matrix notation

$$\nabla_j \varphi = H^{-1} \partial_j (H\varphi)$$

3.2 Curvature of Vector Bundle

We want to know in which way they do not commute. Now we compute the commutators.

$$\begin{aligned}
[\nabla_{\overline{j}}, \nabla_{\overline{k}}]\varphi &= \partial_{\overline{j}}\partial_{\overline{k}}\varphi - \partial_{\overline{k}}\partial_{\overline{j}}\varphi = 0 \qquad \text{as standard formulas in flat spaces} \\
[\nabla_j, \nabla_k]\varphi &= \nabla_j (H^{-1}\partial_k (H\varphi)) - \nabla_k (H^{-1}\partial_j (H\varphi)) \\
&= H^{-1}(\partial_j (H(H^{-1}\partial_k (H\varphi))) - H^{-1}\partial_k (H(H^{-1}\partial_j (H\varphi)))) \\
&= H^{-1}\partial_j \partial_k (H\varphi) - H^{-1}\partial_k \partial_j (H\varphi) \\
&= H^{-1}(\partial_j\partial_k - \partial_k\partial_j)(H\varphi) = 0 \qquad \text{standard derivatives commute}
\end{aligned}$$

Here comes the most important one.

$$\begin{aligned}
[\nabla_j, \nabla_{\overline{k}}]\varphi^\alpha &= \nabla_j \nabla_{\overline{k}}\varphi - \nabla_{\overline{k}}(\nabla_j \varphi) \\
&= H^{-1}\partial_j (H\partial_{\overline{k}}\varphi) - \partial_{\overline{k}}(H^{-1}\partial_j (H\varphi)) \\
&= H^{-1}\partial_j (H\partial_{\overline{k}}\varphi) - \partial_{\overline{k}}(H^{-1}\left((\partial_j H)\varphi + H\partial_j \varphi\right)) \\
&= (H^{-1}\partial_j H)\partial_{\overline{k}}\varphi + H^{-1}H\partial_j\partial_{\overline{k}}\varphi - H^{-1}H\partial_{\overline{k}}\partial_j\varphi - \partial_{\overline{k}}(H^{-1}\partial_j H\varphi) \\
&= -\{\partial_{\overline{k}}(H^{-1}\partial_j H)\}\varphi^\alpha \\
&= F_{\overline{k}j}{}^\alpha{}_\beta \varphi^\beta \\
[\nabla_j, \nabla_{\overline{k}}]\varphi &= F_{\overline{k}j}\varphi \qquad \text{in matrix notation}
\end{aligned}$$

where

$$F_{\overline{k}j}{}^\alpha{}_\beta := -\partial_{\overline{k}}(H^{\alpha\overline{\gamma}}\partial_j H_{\overline{\gamma}\beta})$$

Definition 3.3 (Curvature of Vector Bundles)**.** *The curvature of E w.r.t. $H_{\overline{\alpha}\beta}$ is*

$$F_{\overline{k}j}{}^\alpha{}_\beta := -\partial_{\overline{k}}(H^{\alpha\overline{\gamma}}\partial_j H_{\overline{\gamma}\beta}) \tag{4}$$

Notice that in matrix notation

$$F_{\overline{k}j} = -\partial_{\overline{k}}(H^{-1}\partial_j H)$$

Also notice that F is a section of $\Lambda^{1,1} \otimes \mathrm{End}(E)$

Now we discuss Connection Form.

Definition 3.4 (Connection Form)**.** *Recall that*

$$\begin{aligned}\nabla_{\overline{k}}\varphi &= \partial_{\overline{k}}\varphi \\ \nabla_j\varphi &= H^{-1}\partial_j(H\varphi) \\ &= H^{-1}H\partial_j\varphi + H^{-1}(\partial_j H)\varphi = \partial_j\varphi + H^{-1}(\partial_j H)\varphi\end{aligned}$$

We define the connection form as

$$A := (H^{-1}\partial_j H)dz^j$$

i.e.

$$A = dz^j A^{\alpha}_{j\beta} + d\overline{z}^k A^{\alpha}_{\overline{k}\beta}$$

with

$$\begin{aligned}A^{\alpha}_{j\beta} &= (H^{-1}\partial_j H)^{\alpha}{}_{\beta} = H^{\alpha\overline{\gamma}}\partial_j H_{\overline{\gamma}\beta} \\ A^{\alpha}_{\overline{k}\beta} &= 0 \qquad \textit{no correction in } \overline{\partial}\textit{-direction}\end{aligned}$$

Hence

$$\begin{aligned}A &= dz^j A^{\alpha}_{j\beta} \qquad \textit{in component notations} \\ &= dz^j A_j \qquad A_j \textit{ is a matrix, } A_j = H^{-1}\partial_j H \textit{ matrix notation}\end{aligned}$$

Now we introduce the basic formula for the curvature.

Definition 3.5 (Curvature Form)**.**

$$F \equiv iF_{\overline{k}j}{}^{\alpha}{}_{\beta}dz^j \wedge d\overline{z}^k \in \Gamma(X, \Lambda^{1,1} \otimes \mathrm{End}(E))$$

But for simplicity we drop i since its cumbersome.

$$F := F_{\overline{k}j}{}^{\alpha}{}_{\beta}dz^j \wedge d\overline{z}^k$$

Lemma 3.1 (Basic formula for the curvature)**.**

$$F = dA + A \wedge A$$

The particular combination on the RHS transforms well even though A itself does not.

Proof. We compute the RHS.

$$\begin{aligned}dA &= d(\sum_j dz^j A_j) \qquad \text{later we drop summation in } j \\ &= (\overline{\partial} + \partial)(dz^j A_j) \\ &= d\overline{z}^k(\partial_{\overline{k}}A_j) \wedge dz^j + dz^k(\partial_k A_j) \wedge dz^j \\ &= -\partial_{\overline{k}}A_j dz^j \wedge d\overline{z}^k + dz^k(\partial_k A_j) \wedge dz^j \\ &= F_{\overline{k}j}dz^j \wedge d\overline{z}^k + (\partial_k A_j)dz^k \wedge dz^j\end{aligned}$$

Now

$$\partial_k A_j = \partial_k(H^{-1}\partial_j H) = (\partial_k(H^{-1}))\partial_j H + H^{-1}\partial_k\partial_j H$$

We claim that

$$\partial_k(H^{-1}) = -H^{-1}\partial_j H H^{-1} = -H^{-2}\partial_j H \tag{5}$$

To check the claim (5) we note

$$\begin{aligned}H^{-1}H &= 1 \\ \partial_k(H^{-1}H) &= 0 \\ \partial_k(H^{-1})H + H^{-1}\partial_k H &= 0 \\ \partial_k(H^{-1}) &= -H^{-1}\partial_k H H^{-1}\end{aligned}$$

Thus

$$\partial_k A_j = -H^{-1}(\partial_k H)H^{-1}\partial_j H + H^{-1}\partial_k\partial_j H$$
$$\partial_k A_j dz^k \wedge dz^j = -(H^{-1}\partial_k H)dz^k \wedge (H^{-1}\partial_j H)dz^j + H^{-1}(\partial_k\partial_j H)dz^k \wedge dz^j$$

But the last term is 0 due to anti-commute. Hence

$$\begin{aligned}\partial_k A_j dz^k \wedge dz^j &= -A_k dz^k \wedge A_j dz^j \\ &= -A \wedge A \qquad \text{in matrix notation}\end{aligned}$$

As a summary

$$\begin{aligned}dA &= F - A \wedge A \\ F &= dA + A \wedge A\end{aligned}$$

□

Theorem 3.1 (Second Bianchi Identity)**.**

$$dF + A \wedge F - F \wedge A = 0$$

Proof. We compute

$$\begin{aligned}dF &= d(dA + A \wedge A) \\ &= 0 + d(A \wedge A) \\ &= (dA) \wedge A + (-1)A \wedge dA \\ &= (dA + A \wedge A) \wedge A - A \wedge (dA + A \wedge A) \\ &= F \wedge A - A \wedge F\end{aligned}$$

□

Remark 3.1. *One can think of Second Bianchi Identity as*

$$d_A F = 0$$

where d_A is the extension of exterior differential.

A bit review

Remark 3.2 (Identities for the Curvature)**.**

$$\begin{aligned}\nabla_{\overline{k}}\varphi^\alpha &= \partial_{\overline{k}}\varphi^\alpha \\ \nabla_j\varphi^\alpha &= H^{\alpha\overline{\gamma}}\partial_j(H_{\overline{\gamma}\beta}\varphi^\beta) \\ &= H^{\alpha\overline{\gamma}}\left(\partial_j H_{\overline{\gamma}\beta}\varphi^\beta + H_{\overline{\gamma}\beta}\partial_j\varphi^\beta\right) \\ &= \delta^\alpha_\beta\partial_j\varphi^\beta + (H^{\alpha\overline{\gamma}}\partial_j H_{\overline{\gamma}\beta})\varphi^\beta \\ &= \partial_j\varphi^\alpha + (H^{\alpha\overline{\gamma}}\partial_j H_{\overline{\gamma}\beta})\varphi^\beta \\ \nabla_\ell\varphi^\alpha &= \partial_\ell\varphi^\alpha + A^\alpha_{\ell\beta}\varphi^\beta\end{aligned}$$

Provided

$$\begin{aligned}A^\alpha_{\overline{k}\beta} &= 0 \qquad \ell = \overline{k} \text{ in the } \overline{\partial}\text{-direction} \\ A^\alpha_{j\beta} &= H^{\alpha\overline{\gamma}}\partial_j H_{\overline{\gamma}\beta} \\ &= H^{-1}\partial_j H \qquad \ell = j \text{ in the } \partial\text{-direction}\end{aligned}$$

For Matrix connection

$$\begin{aligned}A &= A^\alpha_{\ell\beta} \\ &= dz^j A^\alpha_{j\beta} \qquad A^\alpha_{j\beta} := (H^{-1}\partial_j H)^\alpha_\beta\end{aligned}$$

Then we have

$$\begin{aligned}F_{\overline{k}j} &= -\partial_{\overline{k}}A_j \\ F &= dA + A \wedge A\end{aligned}$$

and the second Bianchi Identity

$$dF + A \wedge F - F \wedge A = 0$$

3.3 Induced Connections

To appreciated the second Bianchi Identity, we need to understand induced connections. What are these? Notice A vector bundle E gives rise to many other bundles.

Definition 3.6 (E_* Dual Bundle)**.** *The sections ψ_α of E_* can be paired with the sections φ^α of E to produce scalars. Given*

$$\varphi \in \Gamma(X, E), \qquad \psi \in \Gamma(X, E_*)$$

one define E_* by requiring an association

$$\psi_\alpha \varphi^\alpha \qquad \textit{gives a scalar}$$

Now for transition functions

$$\psi_{\mu\alpha}\varphi^\alpha_\mu = \psi_{\nu\beta}\varphi^\beta_\nu \qquad \textit{on } X_\mu \cap X_\nu$$

This is transition law for $\psi_{\mu\alpha}$ and $\psi_{\nu\beta}$. Thus given E vector bundle, we obtain E_* as dual vector bundle.

Definition 3.7 (Induced Connection on Dual Bundle)**.** *A connection on E, i.e., a way of differentiating sections of E, induces a connection on E_*. We obtain the induced connection by requiring the Leibniz Rule holds. Indeed, if Leibniz Rule holds,*

$$\partial_\ell(\psi_\alpha\varphi^\alpha) = (\nabla_\ell\psi_\alpha)\varphi^\alpha + \psi_\alpha(\nabla_\ell\varphi^\alpha) \tag{6}$$

Notice

1. *$\psi_\alpha\varphi^\alpha$ is scalar of ∂_ℓ on LHS is standard derivative.*
2. *$\nabla_\ell\varphi^\alpha$ is connection on E*
3. *Now we solve for $\nabla_\ell\psi_\alpha$.*

Explicitly.

$$\begin{aligned}
(\partial_\ell\psi_\alpha)\varphi^\alpha + \psi_\alpha(\partial_\ell\varphi^\alpha) &= (\nabla_\ell\psi_\alpha)\varphi^\alpha + \psi_\alpha(\partial_\ell\varphi^\alpha + A^\alpha_{\ell\beta}\varphi^\beta) \\
(\partial_\ell\psi_\alpha)\varphi^\alpha &= (\nabla_\ell\psi_\alpha)\varphi^\alpha + \psi_\alpha A^\alpha_{\ell\beta}\varphi^\beta \\
&= (\nabla_\ell\psi_\alpha)\varphi^\alpha + \psi_\beta A^\beta_{\ell\alpha}\varphi^\alpha \\
\partial_\ell\psi_\alpha &= \nabla_\ell\psi_\alpha + \psi_\beta A^\beta_{\ell\alpha} \\
\nabla_\ell\psi_\alpha &:= \partial_\ell\psi_\alpha - \psi_\beta A^\beta_{\ell\alpha} \qquad \text{connection on } E_*
\end{aligned}$$

Recall

$$\nabla_\ell\varphi^\alpha = \partial_\ell\varphi^\alpha + A^\alpha_{\ell\beta}\varphi^\beta \qquad \text{is connection on } E$$

Notice in matrix notation

$$\begin{aligned}
\nabla_\ell\psi &= \partial_\ell\psi - \psi A_\ell \\
\nabla_\ell\varphi &= \partial_\ell\varphi + A_\ell\varphi
\end{aligned}$$

We have both the sign difference, and that square matrix(connection form) multiplying on Left and Right differ. In particular

$$\begin{aligned}
\nabla_j\psi_\alpha &:= \partial_j\psi_\alpha - \psi_\beta(H^{\beta\overline{\gamma}}\partial_j H_{\overline{\gamma}\alpha}) \qquad \forall\ j = 1, \cdots, n \\
\nabla_{\overline{k}}\psi_\alpha &:= \partial_{\overline{k}}\psi_\alpha \qquad \forall\ k = 1, \cdots, n
\end{aligned}$$

□

Definition 3.8 (Product Vector Bundle)**.** *Now suppose we have 2 holomorphic vector bundles E and $\tilde{E}$. Then we can construct*

$$E \otimes \tilde{E} \qquad \textit{as product vector bundle}$$

with its transition functions as the product of the transition functions of E and $\tilde{E}$.

Definition 3.9 (Connection on Product Vector Bundle)**.** *In particular if we have connections ∇ and $\tilde{\nabla}$ on E and $\tilde{E}$, we again obtain a connection on the tensor product $E \otimes \tilde{E}$ via imposing Leibniz rule. More explicityly for*

$$\varphi \in \Gamma(X, E), \qquad \tilde{\varphi} \in \Gamma(X, \tilde{E})$$

we have

$$\varphi\tilde{\varphi} \in \Gamma(X, E \otimes \tilde{E})$$

and Leibniz rule requires

$$\nabla(\varphi\tilde{\varphi}) = (\nabla\varphi)\tilde{\varphi} + \varphi(\tilde{\nabla}\tilde{\varphi}) \tag{7}$$

Explicitly.

$$\begin{aligned}\nabla_\ell(\varphi\tilde\varphi)^{\alpha\tilde\alpha} &= (\nabla_\ell\varphi^\alpha)\tilde\varphi^{\tilde\alpha} + \varphi^\alpha(\nabla_\ell\tilde\varphi^{\tilde\alpha})\\ &= (\partial_\ell\varphi^\alpha + A^\alpha_{\ell\beta}\varphi^\beta)\tilde\varphi^{\tilde\alpha} + \varphi^\alpha(\partial_\ell\tilde\varphi^{\tilde\alpha} + \tilde A^{\tilde\alpha}_{\ell\tilde\beta}\tilde\varphi^{\tilde\beta})\\ &= \partial_\ell(\varphi\tilde\varphi)^{\alpha\tilde\alpha} + A^\alpha_{\ell\beta}\varphi^\beta\tilde\varphi^{\tilde\alpha} + \tilde A^{\tilde\alpha}_{\ell\tilde\beta}\varphi^\alpha\tilde\varphi^{\tilde\beta}\end{aligned}$$

The connection form on $E\otimes\tilde E$ is just

$$A_\ell + \tilde A_\ell$$

□

Definition 3.10 (Endomorphism $\mathrm{End}(E)$)**.** *Let E be a vector bundle, and let $\mathrm{End}(E)$ be the vector bundle of endomorphism*

$$T = T^\alpha_\beta \in \Gamma(X, \mathrm{End}(E))$$

i.e.,

$$\mathrm{End}(E) = E\otimes E_*$$

Definition 3.11 (Connection on $\mathrm{End}(E)$)**.** *If ∇ is connection o E and $\varphi\in\Gamma(X,E)$ with*

$$\nabla_\ell\varphi^\alpha = \partial_\ell\varphi^\alpha + A^\alpha_{\ell\beta}\varphi^\beta$$

then

$$\nabla_\ell T^\alpha_\beta := \partial_\ell T^\alpha_\beta + A^\alpha_{\ell\gamma}T^\gamma_\beta - T^\alpha_\gamma A^\gamma_{\ell\beta}$$

where

1. *$A^\alpha_{\ell\gamma}T^\gamma_\beta$ is the connection on E*
2. *and $T^\alpha_\gamma A^\gamma_{\ell\beta}$ is the connection on E_*.*

In matrix notation

$$\begin{aligned}\nabla_\ell T &= \partial_\ell T + A_\ell T - TA_\ell\\ &= \partial_\ell T + [A_\ell, T]\end{aligned}$$

We care because we want to differentiate the curvature, which is an endomorphism. Back to

$$F_{\bar kj}{}^\alpha{}_\beta \in \Gamma(X, \Lambda^{1,1}\otimes\mathrm{End}(E))$$

we want

$$\nabla_\ell F_{\bar kj} := \partial_\ell F_{\bar kj} + A_\ell F_{\bar kj} - F_{\bar kj}A_\ell$$

In the second Bianchi Identity

$$0 = dF + A\wedge F - F\wedge A$$

i.e., if

Definition 3.12 (Exterior Derivative d_A)**.** *we define d_A on $\Gamma(X,\Lambda^p\otimes\mathrm{End}(E))$ by*

$$\begin{aligned}T &= \frac{1}{p!}\sum T_{\ell_1,\cdots,\ell_p}{}^\alpha{}_\beta du^{\ell_1}\wedge\cdots\wedge du^{\ell_p} && p\text{-form}\\ d_AT &:= \frac{1}{p!}\sum du^m\nabla_m T_{\ell_1,\cdots,\ell_p}{}^\alpha{}_\beta\wedge du^{\ell_1}\wedge\cdots\wedge du^{\ell_p} && \text{defines the exterior derivative}\end{aligned}$$

By Second Bianchi Identity this yields

$$d_AF = 0$$

3.4 Chern-Weil Theory

Let $E\to X$ be a holomorphic vector bundle. We let $H_{\overline\alpha\beta}$ be a metric on E. Recall the curvature form

$$F_{\bar kj}{}^\alpha{}_\beta dz^j\wedge d\bar z^k$$

depends on the metric. But in fact the characteristic classes do not depend on the metric.

Definition 3.13 (Characteristic Class)**.** *We can define for each $p \geq 1$ the object*

$$c_p(F) := \mathrm{Tr}(F \wedge \cdots \wedge F) \qquad p\text{-factors of } F$$

Since F is a $(1,1)$-form, valued in $\mathrm{End}(E)$*, this gives*

$$F \wedge \cdots \wedge F \qquad \text{wedge } p \text{ times gives a } (p,p)\text{-form valued in } \mathrm{End}(E)$$

Hence

$$\mathrm{Tr}(F \wedge \cdots \wedge F) \qquad \text{is a } (p,p)\text{-form}$$

Then Chern-Weil says

Theorem 3.2 (Chern-Weil)**.** *1. $c_p(F)$ is always a closed (p,p)-form, i.e.*

$$dc_p(F) = 0$$

2. $[c_p(F)]_{\mathrm{dR}}$, i.e., the equivalence class of $c_p(F)$ mod *exact forms is independent of the connection ∇, defines*

$$[c_p(F)] := \{pth \text{ Chern Class}\}$$

Proof. Apply the second Bianchi Identity. We begin by proving (1). For simplicity we prove for $p = 1$ and $p = 2$. For $p = 1$

$$c_1(F) = \mathrm{Tr}(F)$$

1. To see $c_1(F)$ is closed, we just differentiate by applying d

$$\begin{aligned} dc_1(F) &= d(\mathrm{Tr}(F)) = \mathrm{Tr}(dF) \\ &= \mathrm{Tr}(-A \wedge F + F \wedge A) \\ &= 0 \end{aligned}$$

   Since in general, given two square matrices M, N, we have $MN \neq NM$ but trace commutes $\mathrm{Tr}(MN) = \mathrm{Tr}(NM)$. This is due to

$$\begin{aligned} (MN)^\alpha_\beta &= M^\alpha_\gamma N^\gamma_\beta \\ (MN)^\alpha_\alpha &= M^\alpha_\gamma N^\gamma_\alpha \\ &= N^\gamma_\alpha M^\alpha_\gamma \\ &= (NM)^\gamma_\gamma \\ \mathrm{Tr}(MN) &= \mathrm{Tr}(NM) \end{aligned}$$

   What we're dealing with are forms. But this is indeed fine since F is 2 forms that commutes so

$$\begin{aligned} \mathrm{Tr}(A \wedge F) &= \mathrm{Tr}(F \wedge A) \\ dc_1(F) &= 0 \end{aligned}$$

2. To see independence of connection, given A and A' as two connections, and let F, F' be the two corresponding curvature form. The claim then is

$$c_1(F) - c_1(F') = d\{\text{ of Something}\}$$

   The key observation is to set

$$A := A' + B$$

   If A and A' are connections, this B is a 1-form, globally defined. Since

$$\nabla\varphi \qquad \text{is globally defined, and } \nabla\varphi = \partial\varphi + A\varphi$$

   as well as

$$\nabla'\varphi \qquad \text{is globally defined, and } \nabla'\varphi = \partial\varphi + A'\varphi$$

   Then subtracting

$$\nabla\varphi - \nabla'\varphi \qquad \text{is globally defined, and } (A - A')\varphi \text{ is globally defined}$$

   Let's introduce the following one-parameter family of connections

$$A_t := A' + tB$$

linking A' to A. Let

$$\begin{aligned} &F_t \qquad \text{be the curvature of } A_t \\ &F_t = dA_t + A_t \wedge A_t \end{aligned}$$

Next we write things as

$$\begin{aligned} c_1(F) - c_1(F') &= \mathrm{Tr}(F) - \mathrm{Tr}(F') \\ &= \int_0^1 \frac{d}{dt}(\mathrm{Tr}(F_t))\,dt \\ &= \int_0^1 \frac{d}{dt}\left(\mathrm{Tr}(dA_t + A_t \wedge A_t)\right)\,dt \\ &= \int_0^1 \mathrm{Tr}(d\dot{A}_t + \dot{A}_t \wedge A_t + A_t \wedge \dot{A}_t)\,dt \\ &= \int_0^1 \mathrm{Tr}(dB + B \wedge A_t + A_t \wedge B)\,dt \qquad \text{observe } \dot{A}_t = B \\ &= d\left(\int_0^1 \mathrm{Tr}(B)\,dt\right) \qquad \text{the latter cancel because } A \text{ and } B \text{ are one-forms} \end{aligned}$$

But since B is globally defined, this gives an exact form.

Now we prove the case for $p = 2$.

$$c_2(F) = \mathrm{Tr}(F \wedge F)$$

1. First, we show $c_2(F)$ is closed $(2,2)$-form.

$$\begin{aligned} dc_2(F) &= d\mathrm{Tr}(F \wedge F) = \mathrm{Tr}(dF \wedge F + F \wedge dF) \\ &= 2\mathrm{Tr}(dF \wedge F) = 2\mathrm{Tr}((-A \wedge F + F \wedge A) \wedge F) \qquad \text{one apply the Bianchi identity} \\ &= 2\mathrm{Tr}(-A \wedge F \wedge F + A \wedge F \wedge F) = 0 \end{aligned}$$

2. Next, we show that $[\,c_2(F)\,]_{\mathrm{dR}}$ is independent of the connection ∇. Once again, let ∇, ∇' be two connections, and set

$$A = A' + B \qquad \text{globally defined form}$$

One define

$$A_t := A' + tB$$

so

$$\begin{aligned} A_t &= A' \qquad t = 0 \\ A_t &= A \qquad t = 1 \end{aligned}$$

and define

$$F_t := dA_t + A_t \wedge A_t$$

Next we write

$$\begin{aligned} c_2(F) - c_2(F') &= \int_0^1 \frac{d}{dt} c_2(F_t)\,dt \\ &= \int_0^1 \mathrm{Tr}(\dot{F}_t \wedge F_t + F_t \wedge \dot{F}_t)\,dt \\ &= 2\int_0^1 \mathrm{Tr}(\dot{F}_t \wedge F_t)\,dt \end{aligned}$$

Noticing

$$\begin{aligned} \dot{F}_t &= d\dot{A}_t + \dot{A}_t \wedge A_t + A_t \wedge \dot{A}_t \\ &= dB + B \wedge A + A \wedge B \end{aligned}$$

One has

$$\begin{aligned} c_2(F) - c_2(F') &= 2\int_0^1 \mathrm{Tr}((dB + B \wedge A_t + A_t \wedge B) \wedge F_t)\,dt \\ &= 2\int_0^1 \mathrm{Tr}(dB \wedge F_t)\,dt + 2\int_0^1 \mathrm{Tr}((B \wedge A_t + A_t \wedge B) \wedge F_t)\,dt \end{aligned}$$

One would like to write the former as an exact differential.

$$\begin{aligned}\int_0^1 \mathrm{Tr}(dB \wedge F_t)\, dt &= d(\int_0^1 \mathrm{Tr}(B \wedge F_t)\, dt) + \int_0^1 \mathrm{Tr}(B \wedge dF_t)\, dt \\ &= d(\int_0^1 \mathrm{Tr}(B \wedge F_t)\, dt) + \int_0^1 \mathrm{Tr}(B \wedge (-A_t \wedge F_t) + B \wedge (F_t \wedge A_t))\, dt\end{aligned}$$

Using cancellation one obtain

$$c_2(F) - c_2(F') = 2d(\int_0^1 \mathrm{Tr}(B \wedge F_t)\, dt)$$

as an exact form.

□

4 Kähler Geometry

4.1 Introduction to Kähler Metric

Now we specialize to tangent bundle. Let

$$X = \bigcup_{\mu} X_\mu$$

be a complex manifold of dimension n.

Definition 4.1 (Tangent Bundle $T^{1,0}$). *We consider the following vector bundle with transition functions*

$$t_{\mu\nu}{}^{j}{}_{k}(z) := \left(\frac{\partial z_\mu^j}{\partial z_\nu^k}\right) \qquad \text{on } X_\mu \cap X_\nu \qquad \forall\ 1 \le j,\, k \le n$$

We define the tangent bundle $T^{1,0}$ where $\varphi \in \Gamma(X, T^{1,0})$ if any

$$\varphi_\mu^j(z_\mu) \qquad \text{on } X_\mu$$

satisfies the gluing rule

$$\varphi_\mu^j(z_\mu) = t_{\mu\nu}{}^{j}{}_{k}(z)\varphi_\nu^k(z_\nu)$$

On the vector bundle we pick a metric $H_{k\bar{j}}(z)$ on $T^{1,0}$, i.e, we want to let

$$|\varphi|^2 = H_{k\bar{j}}\varphi^k\overline{\varphi^j} \qquad \text{to be a scalar}$$

Now we have a Chern Unitary Connection on $T^{1,0}$.

$$\begin{aligned}
\nabla_{\bar{k}}\varphi^\ell &= \partial_{\bar{k}}\varphi^\ell \\
\nabla_j\varphi^\ell &= H^{\ell\overline{m}}\partial_j(H_{\overline{m}p}\varphi^p)
\end{aligned}$$

As we recall

$$\begin{aligned}
[\nabla_j, \nabla_{\bar{k}}]\varphi^\ell &= R_{\bar{k}j}{}^{\ell}{}_{p}\varphi^p \\
R_{\bar{k}j}{}^{\ell}{}_{p} &= -\partial_{\bar{k}}(H^{\ell\overline{m}}\partial_j H_{\overline{m}p}) \qquad \text{is the curvature}
\end{aligned}$$

Question: Why are we using the Chern Unitary Connection? The Chern Unitary connection is dictated by two conditions

1. We retain the complex structure via $\nabla_{\bar{k}}\varphi = \partial_{\bar{k}}\varphi$
2. and it is unitary by definition.

However in the case of tangent bundles, there is another natural connection ∇^{LC}, the Levi-Civita Connection, which is dictated by

1. unitarity
2. and by the fact that it is torsion 0.

To understand torsion free, for

$$\nabla_j\varphi^\ell = \partial_j\varphi^\ell + A_{jp}^\ell\varphi^p$$

We can define

Definition 4.2 (Torsion Tensor).

$$T_{jp}^\ell := A_{jp}^\ell - A_{pj}^\ell$$

Remark 4.1. *Notice this only makes sense for tangent bundles, since on a general bundle $E \to X$,*

$$\nabla_j\varphi^\alpha = \partial_j\varphi^\alpha + A_{j\beta}^\alpha\varphi^\beta$$

where j is base index and β is fiber index. Hence for general bundle it doesn't make sense to talk about torsion. In general, the most convenient connections are the ones that have torsion zero.

A natural question to ask is do these two connections lead to the same thing? These two conditions are quite different. Not surprisingly, the Chern Connection and the Levi-Civita Connection are different.

$$\nabla \neq \nabla^{LC}$$

Question: are there metrics $H_{\overline{k}j}$ for which

$$\nabla = \nabla^{LC}$$

This is non-trivial. We call these metrics Kähler.

Definition 4.3 (Kähler Metric)**.** *A metric $H_{\overline{k}j}$ on $T^{1,0}(X)$ is said to be Kähler if*

$$T^{\ell}_{jp} = 0 \qquad i.e. \qquad A^{\ell}_{jp} = A^{\ell}_{pj}$$

But this condition actually has a lot of remarkable properties for the manifold.

4.1.1 Global Implications of the Kähler Condition

For this we introduce

Definition 4.4 (Kähler Form)**.** *Given a metric $g_{\overline{k}j}$ on Tangent Bundle $T^{1,0}$, we create the $(1,1)$-form*

$$\omega = i g_{\overline{k}j} dz^j \wedge d\overline{z}^k$$

Lemma 4.1 (Characterisation of Kähler Metric)**.** *$g_{\overline{k}j}$ is Kähler iff*

$$d\omega = 0$$

i.e.

$$\partial_\ell g_{\overline{k}j} = \partial_j g_{\overline{k}\ell} \tag{8}$$

Proof. It suffices to just compute.

$$\begin{aligned} d\omega &= i d(g_{\overline{k}j} dz^j \wedge d\overline{z}^k) \\ &= i(dg_{\overline{k}j} \wedge dz^j \wedge d\overline{z}^k) \\ &= i\left(\frac{\partial}{\partial z^\ell} g_{\overline{k}j} dz^\ell + \frac{\partial}{\partial \overline{z}^\ell} g_{\overline{k}j} d\overline{z}^\ell\right) \wedge dz^j \wedge d\overline{z}^k \\ &= \frac{i}{2}\left((\frac{\partial}{\partial z^\ell} g_{\overline{k}j} - \frac{\partial}{\partial z^j} g_{\overline{k}\ell}) dz^\ell \wedge dz^j \wedge d\overline{z}^k + (\frac{\partial}{\partial \overline{z}^\ell} g_{\overline{k}j} - \frac{\partial}{\partial \overline{z}^k} g_{\overline{\ell}j}) d\overline{z}^\ell \wedge dz^j \wedge d\overline{z}^k \right) \end{aligned}$$

Hence $d\omega = 0$ implies both

$$\begin{aligned} \frac{\partial}{\partial z^\ell} g_{\overline{k}j} - \frac{\partial}{\partial z^j} g_{\overline{k}\ell} = 0 \\ \frac{\partial}{\partial \overline{z}^\ell} g_{\overline{k}j} - \frac{\partial}{\partial \overline{z}^k} g_{\overline{\ell}j} = 0 \end{aligned}$$

We observe now that these are exactly the same as

$$T^{\ell}_{jp} = 0$$

Indeed

$$A^{\ell}_{jp} = A^{\ell}_{pj} \implies g^{\ell\overline{m}} \partial_j g_{\overline{m}p} = g^{\ell\overline{m}} \partial_p g_{\overline{m}j}$$

□

4.1.2 Key Themes

Assume now that our manifold X is compact. The reason we do so is because we want to talk about the cohomology classes. Then a Kähler metric has an associated cohomology class. Recall

$$[\omega]_{\mathrm{dR}} := \{\text{equivalence class of } \omega \text{ modulo exact forms}\}$$

The Key theme in complex geometry is

1. Fix a Kähler class $[\omega]_{\mathrm{dR}}$.
2. Is there a representative metric in this given class with 'best' curvature properties?

3. Such a metric is called the canonical metric.

The answer: 'Best' will turn out to be metrics of constant scalar curvature and whether they exist is a deep and very hard question, which is the analogue in complex geometry of Einstein's Equation. Our program:

1. Understand better the curvature tensor of Kähler metrics (e.g. First Bianchi Identity, and properties of the Ricci Curvature tensor).

2. Constant scalar curvature condition becomes an explicit 2nd order non-linear PDE.

3. We can solve this PDE in some very important cases (e.g. Yau's solution of the Calabi conjecture).

4.2 Curvature Tensors of Kähler Metrics

Let X be a complex manifold, and $g_{\overline{k}j}$ be a Kähler metric.

Definition 4.5 (Curvature Tensor of Kähler Metric)**.** *Recall in general that the curvature of a metric $g_{\overline{k}j}$ is of the following form*

$$R_{\overline{k}j}{}^{\ell}{}_{p} := -\partial_{\overline{k}}(g^{\ell\overline{m}}\partial_j g_{\overline{m}p})$$

as special case of $F_{\overline{k}j}{}^{\alpha}{}_{\beta}$. Introduce lowering index

$$R_{\overline{k}j\overline{q}p} := g_{\overline{q}\ell}R_{\overline{k}j}{}^{\ell}{}_{p}$$

Definition 4.6 (Ricci Tensor)**.** *Ricci curvature is the contraction of the full curvature*

$$R_{\overline{k}j} := R_{\overline{k}j}{}^{\ell}{}_{\ell}$$

and the Ricci form

$$\mathrm{Ric}(\omega) := iR_{\overline{k}j}dz^j \wedge d\overline{z}^k$$

Lemma 4.2 (First Bianchi Identity)**.** *One has the First Bianchi Identity*

$$R_{\overline{k}j\overline{q}p} = R_{\overline{q}j\overline{k}p} = R_{\overline{q}p\overline{k}j} \tag{9}$$

i.e., we can permute 1, 3 indices and 2, 4 indices.

Proof. We compute

$$\begin{aligned} R_{\overline{k}j\overline{q}m} &= g_{\overline{q}\ell}(-\partial_{\overline{k}}(g^{\ell\overline{p}}\partial_j g_{\overline{p}m})) \\ &= -g_{\overline{q}\ell}\left(\partial_{\overline{k}}(g^{\ell\overline{p}})\partial_j g_{\overline{p}m} + g^{\ell\overline{p}}\partial_{\overline{k}}\partial_j g_{\overline{p}m}\right) \end{aligned}$$

Notice we have formula

$$\partial_{\overline{k}}(g^{\ell\overline{p}}) = -g^{\ell\overline{r}}(\partial_{\overline{k}}g_{\overline{r}s})g^{s\overline{p}}$$

In matrix notation this is

$$\partial_{\overline{k}}(G^{-1}) = -G^{-1}(\partial_{\overline{k}}G)G^{-1}$$

To see this we know

$$\begin{aligned} G^{-1}G &= I \\ \partial_{\overline{k}}(G^{-1})G + G^{-1}\partial_{\overline{k}}G &= 0 \\ \partial_{\overline{k}}(G^{-1}) &= -G^{-1}\partial_{\overline{k}}GG^{-1} \end{aligned}$$

Hence we use this formula and substitute to above.

$$\begin{aligned} R_{\overline{k}j\overline{q}m} &= g_{\overline{q}\ell}g^{\ell\overline{r}}(\partial_{\overline{k}}g_{\overline{r}s})g^{s\overline{p}}\partial_j g_{\overline{p}m} - g_{\overline{q}\ell}g^{\ell\overline{p}}\partial_{\overline{k}}\partial_j g_{\overline{p}m} \\ &= g_{\overline{q}\ell}g^{\ell\overline{r}}(\partial_{\overline{k}}g_{\overline{r}s})g^{s\overline{p}}\partial_j g_{\overline{p}m} - \partial_{\overline{k}}\partial_j g_{\overline{q}m} \qquad \text{since } g_{\overline{q}\ell}g^{\ell\overline{p}} = \delta^{\overline{p}}_{\overline{q}} \end{aligned}$$

Now to interchange indices, using that $g_{\overline{q}\ell}g^{\ell\overline{r}} = \delta^{\overline{r}}_{\overline{q}}$ one has

$$R_{\overline{k}j\overline{q}m} = (\partial_{\overline{k}}g_{\overline{q}s})g^{s\overline{p}}\partial_j g_{\overline{p}m} - \partial_{\overline{k}}\partial_j g_{\overline{q}m}$$

One may indeed interchange $\overline{k}$ and $\overline{q}$, and j and m using the Kähler property (8). □

Lemma 4.3 (Ricci: The fundamental identity in Kähler Geometry)**.** *Given Ricci curvature tensor $R_{\overline{k}j}$, one has the explicit form*

$$R_{\overline{k}j} = -\partial_j\partial_{\overline{k}}\log(\det(g_{\overline{q}p})) \tag{10}$$

Proof.

$$\begin{aligned} R_{\overline{k}j} &= R_{\overline{k}j}{}^{\ell}{}_{\ell} \\ &= -\partial_{\overline{k}}(g^{\ell\overline{p}}\partial_j g_{\overline{p}\ell}) \end{aligned}$$

It suffices to prove

$$g^{\ell\overline{p}}\partial_j g_{\overline{p}\ell} = \partial_j \log\det(g_{\overline{p}\ell})$$

Notice in matrix notation $G = (g_{\overline{p}\ell})$ this is

$$\operatorname{Tr}(G^{-1}\partial_j G) = \partial_j \log(\det(G))$$

Assume that G is diagonal, i.e.,

$$G = \begin{pmatrix} \lambda_1 & \cdots & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & \cdots & \lambda_n \end{pmatrix}$$

Then

$$\begin{aligned} \log(\det(G)) &= \log(\prod_{\ell} \lambda_\ell) \\ &= \sum_{\ell} \log(\lambda_\ell) \\ \partial_j(\log(\det(G))) &= \sum_{\ell} \partial_j \log(\lambda_\ell) = \sum_{\ell} \frac{\partial_j \lambda_\ell}{\lambda_\ell} \\ &= \operatorname{Tr}(G^{-1}\partial_j G) \end{aligned}$$

Hence we have the fundamental identity in Kähler Geometry

$$R_{\overline{k}j} = -\partial_{\overline{k}}\partial_j \log(\det(g_{\overline{p}\ell}))$$

□

Alternative Proof of (10) *using forms.* We claim the basic identity we need is the following: if

$$T = iT_{\overline{k}j}dz^j \wedge \overline{z}^k$$

Then

$$T \wedge \frac{\omega^{n-1}}{(n-1)!} = (\operatorname{Tr}(T))\frac{\omega^n}{n!} \tag{11}$$

where Trace denotes the contraction

$$\operatorname{Tr}(T) := g^{j\overline{k}}T_{\overline{k}j}$$

To check this, assume that both are diagonal, i.e.

$$T = i\sum_{\ell} T_{\overline{\ell}\ell}dz^\ell \wedge d\overline{z}^\ell$$

and

$$\omega = i\sum_{k} \omega_{\overline{k}k}dz^k \wedge d\overline{z}^k$$

Then up to some constant

$$\begin{aligned} T \wedge \omega^{n-1} &= (i\sum_{\ell} T_{\overline{\ell}\ell}dz^\ell \wedge d\overline{z}^\ell) \wedge (i\sum_{k_1} \omega_{\overline{k_1}k_1}dz^{k_1} \wedge d\overline{z}^{k_1}) \wedge \cdots \wedge (i\sum_{k_{n-1}} \omega_{\overline{k_{n-1}}k_{n-1}}dz^{k_{n-1}} \wedge d\overline{z}^{k_{n-1}}) \\ &= T_{\overline{\ell}\ell}(\prod_{p\neq\ell} g_{\overline{p}p})(dz^1 \wedge d\overline{z}^1 \wedge \cdots \wedge dz^n \wedge d\overline{z}^n) \\ &= \sum_{\ell}(g_{\overline{\ell}\ell}^{-1})T_{\overline{\ell}\ell}(\prod_{p} g_{\overline{p}p})(dz^1 \wedge d\overline{z}^1 \wedge \cdots \wedge dz^n \wedge d\overline{z}^n) = g^{m\overline{\ell}}T_{\overline{\ell}m}\omega^n \end{aligned}$$

We would like to use this formula to apply to Ricci curvature. Take the form

$$\omega = ig_{\overline{k}j}dz^j \wedge d\overline{z}^k$$

Then the variation writes

$$\begin{aligned}\delta\log(\omega^n) &= \frac{\delta(\omega^n)}{\omega^n} = \frac{\delta(\omega\wedge\cdots\wedge\omega)}{\omega^n} \\ &= \frac{1}{\omega^n}\left((\delta\omega\wedge\cdots\wedge\omega)+\cdots+(\omega\wedge\cdots\wedge\delta\omega)\right) \\ &= n\frac{\delta\omega\wedge\omega^{n-1}}{\omega^n} \\ &= n\frac{\mathrm{Tr}(\delta\omega)}{\omega^n}\frac{\omega^n}{n!}(n-1)! = \mathrm{Tr}(\delta\omega) = g^{j\overline{k}}\delta\omega_{\overline{k}j} = g^{q\overline{m}}\delta g_{\overline{m}q}\end{aligned}$$

where in the last line we used the trace identity. So in particular

$$\begin{aligned}\partial_j(\log(\omega^n)) &= g^{\overline{q}m}\partial_j(g_{\overline{m}q}) \\ R_{\overline{k}j} &= -\partial_{\overline{k}}(g^{q\overline{m}}\partial_j g_{\overline{m}q}) = -\partial_{\overline{k}}\partial_j\log(\omega^n)\end{aligned}$$

And we obtain in forms

$$R_{\overline{k}j} = -\partial_{\overline{k}}\partial_j(\log(\omega^n)) \tag{12}$$

□

Remark 4.2 (Geometric Consequence)**.** *Observe that $g_{\overline{p}\ell}$ is a metric on $T^{1,0}(X)$. This implies*

$$\det(g_{\overline{p}\ell})$$

is a metric on $\Lambda^n T^{1,0}(X)$ the maximum wedge power.

Definition 4.7. *If $V_1,\cdots,V_n$ are sections of $T^{1,0}(X)$, then*

$$V_1\wedge\cdots\wedge V_n$$

is a section of $\Lambda^n T^{1,0}(X)$.

Hence transition functions of $\Lambda^n T^{1,0}(X)$ correspond to $\det(g_{\overline{p}\ell})$. Notice $\Lambda^n T^{1,0}(X)$ is a Line Bundle.

In the following we reinterpret (10). Set

$$K_X^{-1} := \Lambda^n T^{1,0}(X)$$

Then

$$-\partial_{\overline{k}}\partial_j\log\det(g_{\overline{p}\ell}) = -\partial_{\overline{k}}\partial_\ell\log(\text{metric } h \text{ on } K_X^{-1}) = c_1(h)$$

Notice the RHS is the curvature of K_X^{-1}. Moreover assume that X is compact, then

$$[\,c_1(h)\,]_{\mathrm{dR}} = c_1(K_X^{-1})$$

Lemma 4.4 (Ricci Form)**.** *Given Ricci form* $\mathrm{Ric}(\omega)$

$$\begin{aligned}d(\mathrm{Ric}(\omega)) &= 0 \\ [\,\mathrm{Ric}(\omega)\,]_{\mathrm{dR}} &= c_1(K_X^{-1})\end{aligned}$$

Where K_X^{-1} is the maximum wedge powers of $T^{1,0}$. This is 'Anti-canonical Bundle'

Proof. To see $\mathrm{Ric}(\omega)$ is closed form, notice

$$\begin{aligned}d\mathrm{Ric}(\omega) &= -i(\partial+\overline{\partial})\partial\overline{\partial}(\log(\omega^n)) \\ &= -i(\partial^2\overline{\partial}+\overline{\partial}\partial\overline{\partial})(\log(\omega^n)) \\ &= -i\overline{\partial}\partial\overline{\partial}(\log(\omega^n)) \qquad \text{using } \partial^2=0 \\ &= i\partial\overline{\partial}^2(\log(\omega^n)) \qquad \text{using } \partial\overline{\partial}+\overline{\partial}\partial=0 \\ &= 0 \qquad \text{using } \overline{\partial}^2=0\end{aligned}$$

□

4.3 Calabi-Yau

Given X a compact, n-dim complex manifold.

Remark 4.3 (Calabi Conjecture)**.** *Given a tensor* $T_{\bar{k}j}$ $(1,1)$*-form, is there a Kähler metric* ω *for which*

$$\mathrm{Ric}(\omega) = T$$

i.e.

$$R_{\bar{k}j}(\omega) = T_{\bar{k}j}$$

where

$$T = iT_{\bar{k}j}dz^j \wedge d\bar{z}^k$$

Remark 4.4 (Necessary conditions on Calabi Conjectures)**.** *Clearly, a necessary condition is that* T *is closed* ($dT = 0$) *since*

$$\mathrm{Ric}(\omega) = i\partial\bar{\partial}\log(\det(g)) \implies d\mathrm{Ric}(\omega) = (\partial + \bar{\partial})i\partial\bar{\partial}\log(\det(g)) = 0$$

since $\partial^2 = \bar{\partial}^2 = 0$. *Also one needs*

$$[\,T\,]_{\mathrm{dR}} = [\,\mathrm{Ric}(\omega)\,]_{\mathrm{dR}} = c_1(K_X^{-1})$$

Hence, if there exists ω *s.t.*

$$\mathrm{Ric}(\omega) = T$$

then we must have

$$c_1(K_X^{-1}) = [\,\mathrm{Ric}(\omega)\,]_{\mathrm{dR}} = [\,T\,]_{\mathrm{dR}}$$

necessary condition for the solvability of the Einstein's equation

$$[\,T\,] = c_1(K_X^{-1})$$

Proof of Remark 4.4. Notice (12)

$$\mathrm{Ric}(\omega) = -i\partial\bar{\partial}\log(\omega^n)$$

But ω^n is an (n,n)-form, i.e., a section of

$$K_X \otimes \overline{K_X}$$

where K_X is the line bundle whose sections involve

$$f(z)dz^1 \wedge \cdots \wedge dz^n$$

So K_X is really the bundle of n-forms. This implies ω^n is a metric on K_X^{-1}. Why? A metric on a line bundle L is by definition, a strictly positive section of

$$L^{-1} \otimes \overline{L^{-1}}$$

Now let $L := K_X^{-1}$ we see that a metric on L is thus a positive section of $K_X \otimes \overline{K_X}$. Thus

$$\mathrm{Ric}(\omega) = -i\partial\bar{\partial}(\log(\omega^n))$$

is precisely the curvature of the bundle K_X^{-1}. Thus

$$[\,\mathrm{Ric}(\omega)\,]_{\mathrm{dR}} = c_1(K_X^{-1})$$

is independent of ω. □

Proposition 4.1 (Calabi Conjecture; S.T. Yau 1976)**.** *Given* T *satisfying* $dT = 0$ *and*

$$[\,T\,]_{\mathrm{dR}} = [\,\mathrm{Ric}(\omega)\,]_{\mathrm{dR}} = c_1(K_X^{-1})$$

Then in any Kähler class $[\,\omega_0\,]$, *there exists a unique* $\omega \in [\,\omega_0\,]$ *with*

$$\mathrm{Ric}(\omega) = T$$

Corollary 4.1. *In particular, suppose that we're on a manifold with* 0 *Chern class*

$$c_1(K_X^{-1}) = 0$$

Then in any Kähler class $[\,\omega_0\,]$, *there exists a unique Kähler metirc* $\omega \in [\,\omega_0\,]$ *with*

$$\mathrm{Ric}(\omega) = 0$$

Back then people didn't know whether metric with zero Ricci curvature exists. Then this is striking.

Remark 4.5. *Observe the equation*

$$\mathrm{Ric}(\omega) = 0$$

is the Euclidean analogue of Einstein's Equation in vacuum

$$R_{\bar{k}j} = 0$$

4.3.1 Reduction to PDE

How did Yau solve it? Reduction to partial differential equations.

Lemma 4.5 ($\partial\overline{\partial}$-lemma)**.** *Let* $[\,\omega_0\,]$ *be a Kähler class. Then*

$$\omega \in [\,\omega_0\,] \iff \omega = \omega_0 + i\partial\overline{\partial}\varphi, \quad \omega_0 + i\partial\overline{\partial}\varphi > 0, \quad \varphi \textit{ is unique up to an additive constant}$$

Up to an additive constant

$$\omega \iff \varphi \quad \textit{(scalar function) Kähler potential}$$

We can try solving for φ in the following way. We want to solve

$$\begin{aligned}
\mathrm{Ric}(\omega) &= T \\
\mathrm{Ric}(\omega) - \mathrm{Ric}(\omega_0) &= T - \mathrm{Ric}(\omega_0) \\
-i\partial\overline{\partial}\log(\omega^n) + i\partial\overline{\partial}\log(\omega_0^n) &= T - \mathrm{Ric}(\omega_0) \qquad \text{in the same class } c_1(K_X^{-1}) \\
-i\partial\overline{\partial}\log(\frac{\omega^n}{\omega_0^n}) &= i\partial\overline{\partial}f \qquad \text{for some } f \text{ well-defined up to constant, using } \partial\overline{\partial}\text{-Lemma 4.5} \\
-\log(\frac{\omega^n}{\omega_0^n}) &= f \qquad \text{since } \frac{\omega^n}{\omega_0^n} \text{ is a scalar function} \\
\frac{\omega^n}{\omega_0^n} &= e^{-f} \\
\omega^n &= \omega_0^n e^{-f} \\
(\omega_0 + i\partial\overline{\partial}\varphi)^n &= \omega_0^n e^{-f} \qquad \text{using } \partial\overline{\partial}\text{-Lemma 4.5 once again, since we assume } \omega \in [\,\omega_0\,]
\end{aligned}$$

This is the well-known Monge-Ampère Equation. Notice $i\partial\overline{\partial}\varphi$ essentially involves the Hessian of φ. In coordinates, the equation is

$$\det((g_0)_{\overline{k}j} + \partial_j\partial_{\overline{k}}\varphi) = (\det(g_0)_{\overline{k}j})e^f$$

This is very nonlinear second order equation. This is solved by Yau, a big achievement in 1976. Idea is to use the Method of Continuity.

4.3.2 Method of Continuity

Imagine we have a space of equations. Imagine somewhere a point in the space, which is a equation that we want to solve. The key idea is to look at some other equation in the space such that we know how to solve. We want to connect these two equations via a path. We need requirements

1. Suppose at any point on the path that we can solve, we can solve for nearby equations. We hope to go all the way to the equation we want to solve.
2. But there is danger that the neighborhood for the equation we can solve becomes smaller and smaller and we cannot reach beyond. We need to guarantee that we do not get stuck. In order to show we do not get stuck, we need the idea of 'a priori estimate', the key in Partial Differential Equations. We need to prove the a priori estimate.

In this history one have all a priori estimates but for one. Yau gave the estimate and won the fields medal.

4.4 Solving Monge-Ampère Equation using Method of Continuity

Fix X complex compact manifold and ω_0 Kähler form and $f(z)$ scalar function. We want to solve

$$\begin{aligned}
(\omega_0 + i\partial\overline{\partial}\varphi)^n &= \omega_0^n e^{f(z)} \qquad (13) \\
\omega_0 + i\partial\overline{\partial}\varphi &> 0
\end{aligned}$$

Recall that

$$T(z) - \mathrm{Ric}(\omega_0) = i\partial\overline{\partial}f \qquad \text{for some } f \text{ only determined up to an additive constant}$$

4.4.1 Path of Equations

Set

$$\omega = \omega_0 + i\partial\overline{\partial}\varphi$$
$$\omega > 0 \iff \omega \quad \text{is a metric}$$

The equations that we know how to solve is

$$(\omega_0 + i\partial\overline{\partial}\varphi)^n = \omega_0^n$$

The solution is just $\varphi = 0$.

Remark 4.6 (Necessary condition)**.** *Observe a necessary condition for the existence of solutions to*

$$(\omega_0 + i\partial\overline{\partial}\varphi)^n = \omega_0^n e^f$$

is that

$$\int_X \omega_0^n e^f = \int_X \omega_0^n \tag{14}$$

Proof. Indeed the equation implies

$$\begin{aligned}\int_X \omega_0^n e^f &= \int_X (\omega_0 + i\partial\overline{\partial}\varphi)^n \\ &= \int_X \omega_0^n + C(\omega_0^{n-1} i\partial\overline{\partial}\varphi) + C(\omega_0^{n-2}(i\partial\overline{\partial}\varphi)^2) + \cdots + C(i\partial\overline{\partial}\varphi)^n\end{aligned}$$

We claim that

$$\int_X \omega_0^{n-1} i\partial\overline{\partial}\varphi = 0$$

Indeed, using Integration by parts

$$\begin{aligned}\int_X \omega_0^{n-1} i\partial\overline{\partial}\varphi &= \int_X \partial(\omega_0^{n-1} i\overline{\partial}\varphi) \qquad \text{since } \partial\omega_0^{n-1} = (n-1)(\partial\omega_0)\omega_0^{n-2} \text{ but } \partial\omega_0 = 0 \text{ since it is Kähler} \\ &= \int_X \partial\overline{\partial}(\omega_0^{n-1} i\varphi) \\ &= \int_X (\partial + \overline{\partial})\overline{\partial}(\omega_0^{n-1} i\varphi) \qquad \text{since } \overline{\partial}^2 = 0 \\ &= \int_X d(\overline{\partial}(\omega_0^{n-1} i\varphi)) = \int_{\partial X} \overline{\partial}(\omega_0^{n-1} i\varphi) = 0 \qquad \text{using } \partial X = \varnothing\end{aligned}$$

□

Now we make a choice of a path of equations linking what we want to solve, i.e., (13)

$$(\omega_0 + i\partial\overline{\partial}\varphi)^n = \omega_0^n e^f$$

to the equation that we know how to solve

$$(\omega_0 + i\partial\overline{\partial}\varphi)^n = \omega_0^n$$

The candidate is

$$(\omega_0 + i\partial\overline{\partial}\varphi)^n = \omega_0^n e^{tf + c_t}, \qquad \omega_0 + i\partial\overline{\partial}\varphi > 0 \qquad \forall\, 0 \le t \le 1 \tag{15}$$

We need to verify the necessary condition (14) as well, so we need

$$\begin{cases} (\omega_0 + i\partial\overline{\partial}\varphi)^n = \omega_0^n e^{tf + c_t} \\ \int_X \omega_0^n e^{tf + c_t} = \int_X \omega_0^n \end{cases} \qquad \forall\, 0 \le t \le 1 \tag{16}$$

Proof that the path (16) *is reasonable with proper choice of* c_t*.*

$$\begin{aligned} e^{c_t}\int_X \omega_0^n e^{tf} &= \int_X \omega_0^n \\ e^{c_t} &= \frac{\int_X \omega_0^n}{\int_X \omega_0^n e^{tf}} \\ c_t &= \log\left(\frac{\int \omega_0^n}{\int_X \omega_0^n e^{tf}}\right)\end{aligned}$$

Clearly at $t = 0$ one has $c_t = 0$, while at $t = 1$

$$c_t = \log(\frac{\int \omega_0^n}{\int_X \omega_0^n e^f}) = \log(1) = 0$$

Thus with choice of c_t

$$c_t = \log(\frac{\int \omega_0^n}{\int_X \omega_0^n e^{tf}}) \tag{17}$$

we can choose the path (15). □

We notice

$$t = 0 \implies \varphi = 0 \qquad \text{with } c_0 = 0$$
$$t = 1 \implies (\omega_0 + i\partial\overline{\partial}\varphi)^n = \omega_0^n \qquad \text{with } c_1 = 0$$

Remark 4.7. *If we can solve the equation (15) for some value t_0, then we can solve it for t close enough to t_0.*

We formally define the interval

$$I := \{t \in [0, 1] \mid (15) \text{ admits a solution}\} \tag{18}$$

It is clear that $0 \in I$. By method of continuity, if we're further able to show I is both closed and open, then by connectedness $I = [0, 1]$ the whole interval, and thus our equation (13) is solvable.

4.4.2 Open interval: Implicit Function Theorem

Proof that interval I (18) is open. For any $t \in I$, we want to show $(t - \delta, t + \delta) \subset I$ for some $\delta > 0$ sufficiently small. From this we consider the map

$$(t, \varphi) \mapsto \mathcal{F}(t, \varphi) := \frac{(\omega_0 + i\partial\overline{\partial}\varphi)^n}{\omega_0^n} - e^{f_t}$$

where

$$f_t := tf + c_t$$

and we apply the Implicit Function Theorem to be discussed in Lemma 4.6. □

Let's recall the implicit function theorem from Calculus.

Remark 4.8 (Implicit Function Theorem)**.** *Suppose we want to solve an equation of the type*

$$F(t, x) = 0$$

and we have one solution

$$F(t_0, x_0) = 0$$

Indeed for proper F, around the fixed t_0 one has a solution in the small neighborhood. But the danger is when the graph F is vertical at t_0 so on one side one has no solution, but on the other side there are two. But this is equivalent to say

$$\frac{\partial F}{\partial x}(t_0, x_0) = 0$$

So it suffices to require

$$\frac{\partial F}{\partial x}(t_0, x_0) \neq 0$$

Lemma 4.6 (Implicit Function Theorem on Banach Spaces)**.** *Let B_1 and B_2 be Banach Spaces, and consider a map*

$$\mathcal{F} : \mathbb{R} \times B_1 \to B_2 \qquad (t, x) \mapsto \mathcal{F}(t, x)$$

and we assume

$$\mathcal{F}(t_0, x_0) = 0$$

We assume also the following that $\mathcal{F} \in C^1$, and

$$\frac{\partial \mathcal{F}}{\partial x}(t_0, x) \qquad \text{is invertible with bounded inverse as a mapping } B_1 \to B_2$$

Then there exists an interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ with the property that there exists a neighborhood V of x_0 s.t. the following is true: for any $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, there exists a unique $x \in V$ that satisfies

$$\mathcal{F}(t, x) = 0$$

In our case, what is the function $\mathcal{F}$ that we want? We want

$$(t,\varphi) \xrightarrow{\mathcal{F}} \mathcal{F}(t,\varphi) := \frac{(\omega_0 + i\partial\bar{\partial}\varphi)^n}{\omega_0^n} - e^{f_t} \tag{19}$$

where

$$f_t := tf + c_t$$

We also need to specify what B_1 and B_2 are. A naive choice would be

$$B_1 = C^2$$
$$B_2 = C^0$$

But these naive choices do not work since they're hard to manipulate. The good choices are in fact, for fixed $\alpha \in (0,1)$, and choose

$$B_1 := C^{2,\alpha}(X)$$
$$B_2 := C^{0,\alpha}(X)$$

where $C^{k,\alpha}(X)$ are Hölder Spaces defined as follows

Definition 4.8 (Hölder Spaces on $\mathbb{R}^{2n}$)**.** *Consider $\Omega \Subset \mathbb{R}^{2n}$.*

$$C^{0,\alpha}(\Omega) := \{\psi \text{ functions on } \Omega \ \mid \sup|\psi(x)| + \sup_{x\neq y}\frac{|\psi(x)-\psi(y)|}{|x-y|^\alpha} < \infty\}$$

and for $k \in \mathbb{N}$

$$C^{0,\alpha}(\Omega) := \{\psi \text{ functions on } \Omega \ \mid D^\beta\psi \in C^{0,\alpha} \text{ for all } \beta \text{ with } |\beta| \leq k\}$$

What about on complex manifold X?

Definition 4.9 ($C^{k,\alpha}(X)$)**.** *For φ functions on X, it corresponds to $\psi_\mu(z_\mu)$ on $\Phi_\mu(X_\mu)$ so we require $\psi_\mu \in C^{k,\alpha}$ for all μ.*

But in fact, to address subsequent uniqueness, we choose

$$B_1 := \{\varphi \in C^{2,\alpha}(X) \mid \int_X \omega_0^n\varphi = 0\} \tag{20}$$
$$B_2 := \{\psi \in C^{0,\alpha}(X) \mid \int_X \omega_0^n\psi = 0\} \tag{21}$$

Lemma 4.7. *With B_1 and B_2 as in (20) and (21), and $\mathcal{F}$ as in (19), the map*

$$\frac{\partial\mathcal{F}}{\partial\varphi}(t_0,\varphi) : B_1 \to B_2 \qquad \text{is invertible with bounded inverse}$$

Remark 4.9. *How to understand $\frac{\partial F}{\partial\varphi}$?*

1. *For function of one variable $f : \mathbb{R} \to \mathbb{R}$, differentiability is*

$$\Delta f = \delta f + o(\delta f)$$
$$f'(x) = \frac{\delta f}{\delta x}$$

2. *For functions of several variables*

$$f : \mathbb{R}^n \to \mathbb{R}^d$$

Consider a change

$$\delta x : h \to \delta f$$

Then the derivative in the direction h is

$$\frac{\delta f}{\delta x} \qquad h \mapsto \frac{\delta f}{\delta x}$$

which is a linear map from $\mathbb{R}^n \to \mathbb{R}^d$.

3. For function on Banach Space

$$\mathcal{F} : B_1 \to B_2$$

Given a direction $h \in B_1$ let δf be the increase corresponding to $(\delta x)h$ the derivative is

$$h \mapsto \frac{\delta f}{\delta x}$$

Proof. We begin by computing

$$\begin{aligned}
\mathcal{F} &= \frac{(\omega_0 + i\partial\overline{\partial}\varphi)^n}{\omega_0^n} - e^{f_t} \\
\delta\mathcal{F} &= \frac{\delta(\omega_0 + i\partial\overline{\partial}\varphi)^n}{\omega_0^n} \qquad \text{since } f_t \text{ does not depend on } \varphi \\
&= n\frac{\delta(\omega_0 + i\partial\overline{\partial}\varphi) \wedge (\omega_0 + i\partial\overline{\partial}\varphi)^{n-1}}{\omega_0^n} \\
&= n\frac{i\partial\overline{\partial}(\delta\varphi) \wedge (\omega_0 + i\partial\overline{\partial}\varphi)^{n-1}}{\omega_0^n} \qquad \text{since } \delta\omega_0 = 0 \text{ and that } \delta(\partial\overline{\partial}\varphi) = \partial\overline{\partial}(\delta\varphi)
\end{aligned}$$

One recall the identities (11)

$$T \wedge \frac{\omega^{n-1}}{(n-1)!} = (\mathrm{Tr}(T))\frac{\omega^n}{n!} \qquad \mathrm{Tr}(T) := g^{j\overline{k}}T_{\overline{k}j}$$

Here for the choice with

$$\begin{aligned}
\omega &= \omega_0 + i\partial\overline{\partial}\varphi \\
\omega &= ig_{\overline{k}j}dz^j \wedge d\overline{z}^k
\end{aligned}$$

One may take

$$\begin{aligned}
T &= i\partial_j\partial_{\overline{k}}(\delta\varphi) \qquad \text{or in matrix notation } T = i\partial\overline{\partial}(\delta\varphi) \\
\mathrm{Tr}(T) &= g^{j\overline{k}}\partial_j\partial_{\overline{k}}(\delta\varphi) = \Delta_\omega(\delta\varphi)
\end{aligned}$$

Thus

$$\delta\mathcal{F} = (\mathrm{Tr}(T))\frac{\omega^n}{\omega_0^n} = \Delta_\omega(\delta\varphi)\frac{\omega^n}{\omega_0^n}$$

Let $h \in B_1$ be a given direction, and let

$$\delta\varphi = (\delta x)h$$

Consider

$$h \mapsto \frac{\delta\mathcal{F}}{\delta x} = \frac{1}{\delta x}\Delta_\omega((\delta x)h)\frac{\omega^n}{\omega_0^n} = (\Delta_\omega h)\frac{\omega^n}{\omega_0^n}$$

This is the formula. We first address Invertibility of $\frac{\partial\mathcal{F}}{\partial\varphi}$. For our

$$B_1 \overset{\delta\mathcal{F}}{\to} B_2$$

$\frac{\partial\mathcal{F}}{\partial\varphi}$ is invertible means that, setting $h = \delta\varphi$

$$\forall\ \psi \in B_2, \qquad \text{there exists unique } h \in B_1 \qquad s.t. \qquad (\Delta_\omega h)\frac{\omega^n}{\omega_0^n} = \psi$$

The equation can thus be rewritten as

$$(\Delta_\omega h)\frac{\omega^n}{\omega_0^n} = \psi \iff \Delta_\omega h = \psi\frac{\omega_0^n}{\omega^n}$$

We need to be able to solve for arbitrary $\psi \in B_2$. if Here we need a famous theorem regarding solvability of Laplacians.

Theorem 4.1. *Let (X, ω) be a compact Kähler manifold, ω Kähler form. Consider the equation*

$$\Delta_\omega h = \Psi$$

Then

1. *The equation admits a solution iff* $\int_X \omega^n \Psi = 0$

2. *The solution is unique up to an additive constant.*

3. *If* $\Psi \in C^{0,\alpha}$ *then* $h \in C^{2,\alpha}$. *Moreover, if* $\Psi \in C^\infty$, *then* $h \in C^\infty$.

In our case, the condition

$$0 = \int_X \omega^n \Psi = \int_X \omega^n \psi \frac{\omega_0^n}{\omega^n} = \int_X \psi \omega_0^n \qquad \forall\, \psi \in B_2$$

Hence by definition of B_2, the equation indeed admits a solution. □

4.4.3 Closed Interval: A priori Estimates

Now the hardest step is to show I (18) is closed.

$$I := \{t \in [0,1] \mid (15) \text{ admits a solution } \varphi \in B_1\}$$

i.e., we have to show that if $\{t_j\} \subset I$ and $t_j \to T$, then $T \in I$. In particular

1. $t_j \in I$ means that there exists $\varphi_j \in B_1$ satisfying

$$(\omega_0 + i\partial\overline{\partial}\varphi_j)^n = \omega_0^n e^{f_{t_j}}$$

2. $T \in I$ means that there exists $\varphi_T \in B_1$ satisfying

$$(\omega_0 + i\partial\overline{\partial}\varphi_T)^n = \omega_0^n e^{f_T}$$

Suppose φ_{t_j} converges in B_1, then

$$\varphi_T = \lim_{j\to\infty} \varphi_{t_j} \qquad \text{satisfies (15) at } T$$

In general, think about: if sequence of equations converge, does its solution converge? In general, no! The question is: if $t_j \to T$, do φ_{t_j} converge in B_1? The key observation in the theory of PDEs is that a weaker statement suffices!

$$\text{If } t_j \to T, \text{ is there a subsequence of } \{\varphi_{t_j}\} \text{ which converges in } B_1?$$

We do have tools to show that the sequence has a convergent subsequence. And the convergence of a subsequence can be achieved if we can prove some estimates. We have the model theorem for the existence of a convergent subsequence.

Theorem 4.2 (Arzela-Ascoli)**.** *Let* $\{f_j\}$ *be a sequence of functions on compact* $\Omega \Subset \mathbb{R}^n$. *Assume the following*

1. $\{f_j\}$ *is uniformly bounded for all* j, $|f_j| \le C$.

2. *the sequence* $\{f_j\}$ *is equi-continuous in the following sense: for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *positive so that* $|x-y| < \delta$ *implies*

$$|h_j(x) - h_j(y)| < \varepsilon \qquad \forall\, j$$

Then $\{h_j\}$ *admits a uniformly convergent subsequence.*

Remark 4.10. *Now how do we prove a sequence of functions is equi-continuous? An example of a sequence* $\{h_j\}$ *which is bounded and equi-continuous is a sequence that is*

1. *bounded*

2. *and satisfies*

$$|\nabla h_j| \le C \qquad \forall\, j$$

Indeed by the Mean Value Theorem

$$|h_j(x) - h_j(y)| \le \sup |\nabla h_j||x-y| \le C|x-y| \qquad \forall\, j$$

Hence we have the equi-continuity statement.

Now we go back to our problem. How do we convert the Arzela-Ascoli Theorem that we want to our case? Rather, we need to show why this is good enough and we prove the estimates. In our case, we want a subsequence converging in $B_1 \subset C^{2,\alpha}$. Simplify $\varphi_j := \varphi_{t_j}$. We claim the following:

Lemma 4.8. *If we can show that*

1. $\|\varphi_j\| \leq C$
2. $\|\Delta_{\omega_0}\varphi_j\| \leq C$
3. *Only two third order derivatives are uniformly bounded* $\|\nabla_j \nabla_{\bar{k}} \nabla_\ell \varphi\| \leq C$ *and* $\left\|\nabla_{\bar{j}} \nabla_k \nabla_{\bar{\ell}} \varphi\right\| \leq C$ *suffices*

Then we shall have convergence of a subsequence in $C^{2,\alpha}$.

The key in these estimates is the uniformity w.r.t. t_j. We shall prove the following a priori estimates. For φ solution to (15) with (16), i.e., for $F = tf + c_t$ for any $0 \leq t \leq 1$

$$\begin{aligned} (\omega_0 + i\partial\overline{\partial}\varphi)^n &= \omega_0^n e^F \\ \omega_0 + i\partial\overline{\partial}\varphi &> 0 \\ \int_X \omega_0^n \varphi &= 0 \\ \int_X \omega_0^n e^F &= \int_X \omega_0^n \end{aligned}$$

Lemma 4.9 (Estimate (a))**.** *There exists* $C_0 = C_0(X, \omega_0, \|F\|_{C^0})$ *s.t.*

$$\|\varphi\|_{C^0} \leq C_0 \tag{22}$$

Lemma 4.10 (Estimate (b))**.** *There exists* $C_2 = C_2(X, \omega_0, \|F\|_{C^0}, \inf_X \Delta F)$ *so that*

$$\|\Delta\varphi\|_{C^0} \leq C_2 \tag{23}$$

Notice that ω_0 *is a given metric, while*

$$\omega = \omega_0 + i\partial\overline{\partial}\varphi \qquad \textit{is another metric}$$

Here Δ *is the Laplacian w.r.t.* ω_0 *metric where*

$$\Delta F := (g_0)^{j\bar{k}} \partial_j \partial_{\bar{k}} F$$

Lemma 4.11 (Estimate (c))**.** *There exists* $C_3 = C_3(X, \omega_0, \|F\|_{C^0}, \|\nabla F\|_{C^0}, \left\|\nabla_j \nabla_{\bar{k}} F\right\|_{C^0}, \left\|\nabla_j \nabla_{\bar{k}} \nabla_\ell F\right\|_{C^0})$ *s.t.*

$$\left\|\nabla_j \nabla_{\bar{k}} \nabla_\ell \varphi\right\|_{C^0} \leq C_3 \tag{24}$$

here, e.g.,

$$\left\|\nabla_j \nabla_{\bar{k}} \nabla_\ell \varphi\right\|_{C^0}^2 = \sup_X \left\{ g_0^{j\overline{m}} g^{r\bar{k}} g_0^{\ell\bar{s}} \nabla_j \nabla_{\bar{k}} \nabla_\ell \varphi \overline{\nabla_m \nabla_{\bar{r}} \nabla_s \varphi} \right\}$$

and ∇ *is the connection w.r.t. the reference metric* ω_0.

Remark 4.11. *Notice here not all derivatives occur here in* (24). *Notice not all* 3*rd order derivatives of* φ *appear. The derivatives* $\nabla_j \nabla_k \nabla_\ell \varphi$ *and* $\nabla_{\bar{j}} \nabla_{\bar{k}} \nabla_{\bar{\ell}} \varphi$ *are missing.*

With a priori estimates (22), (23), (24) we're able to prove the following theorem.

Theorem 4.3 (Yau 1978)**.** *Consider the Monge-Ampère Equation.*

$$\begin{aligned} (\omega_0 + i\partial\overline{\partial}\varphi)^n &= \omega_0^n e^f \\ \omega_0 + i\partial\overline{\partial}\varphi &> 0 \\ \int_X \omega_0^n \varphi &= 0 \\ \int_X \omega_0^n e^f &= \int_X \omega_0^n \end{aligned}$$

Then for any finite integer $k \geq 3$, *and any* $0 < \alpha < 1$, *if* $f \in C^k$, *then there exists a unique solution* φ *to the above equation and* $\varphi \in C^{k+1,\alpha}(X)$.

Remark 4.12 (Heuristics). *$\varphi \in C^{k+1,\alpha}$ for $0<\alpha<1$ essentially means that φ is as close as possible to being of class C^{k+2}, i.e., if RHS of class C^k, then the solution φ is very close to being of class C^{k+2}. Informally, the solution gains 2 derivatives.*

Corollary 4.2. *If $f \in C^\infty(X)$, then there exists a unique $\varphi \in C^\infty(X)$.*

Proof. By Uniqueness in the previous theorem 4.3, the solution one obtains for k and $k+1$ must coincide. $\square$

We return to the method of continuity with a more precise setup. We return to the path of equations (15)

$$\begin{cases} (\omega_0 + i\partial\bar{\partial}\varphi)^n = \omega_0^n e^{tf+c_t} \quad \forall\, 0 \le t \le 1, \\ \quad \omega_0 + i\partial\bar{\partial}\varphi > 0, \\ \qquad \int_X \omega_0^n \varphi = 0, \\ \int_X \omega_0^n e^{tf+c_t} = \int_X \omega_0^n \quad \forall\, 0 \le t \le 1. \end{cases} \tag{$*_t$}$$

We introduced

$$I := \{t \in [0,1] \mid \ (*_t) \text{ admits a solution } \varphi \in B_1 \text{ where } B_1 := C^{k+1,\alpha}(X) \cap \{\int_X \omega_0^n \varphi = 0\}\} \tag{25}$$

similarly choose

$$B_2 := C^{k-1,\alpha}(X) \cap \{\int_X \omega_0^n \varphi = 0\}$$

By considering the map

$$\mathcal{F}(t,\varphi) : [0,1] \times B_1 \to B_2 \qquad \mathcal{F}(t,\varphi) := \frac{(\omega_0 + i\partial\bar{\partial}\varphi)^n}{\omega_0^n} - e^{tf+c_t}$$

We already know by the implicit function theorem that I is open. The Key remaining step is I is closed, i.e.,

$$t_j \to T,\ t_j \in I \implies T \in I$$

Let φ_j be the solution to path $(*_t)$ at $t = t_j$ which exists due to $t_j \in I$. Question: Does $\{\varphi_j\}$ have a convergent subsequence, and if so, in what norm?

$$(\omega_0 + i\partial\bar{\partial}\varphi_j)^n = \omega_0^n e^{f_{t_j}} \overset{t_j \to T}{\to} \omega_0^n e^{f_T}$$

But we have no idea for the LHS. We cannot conclude anything about the limit unless we know that

$$\varphi_j \overset{C^2}{\to} \varphi_T$$

for some function φ_T. We shall show that, using the a priori estimates (22), (23), (24), there exists a subsequence of $\{\varphi_j\}$ which converges in C^2, and that the limit

$$\varphi_T \in C^{k_1,\alpha}(X)$$

and satisfies the limiting equation

$$(\omega_0 + i\partial\bar{\partial}\varphi_T)^n = \omega_0^n e^{Tf+c_T}$$

The answer will be YES if $\{\varphi_j\}$ has a convergent subsequence converging to some $\varphi_T \in C^{k+1,\alpha}$ in a norm which is stronger than C^2.

Compactness and Weak Compactness

Theorem 4.4 (Compactness in $\mathbb{R}^n$; Bolzano-Weierstrass). *If we have a sequence $\{\varphi_j\} \subset \mathbb{R}^n$ with $|\varphi_j| \le C$, then there exists convergent subsequence $\{\varphi_{j_k}\}$ s.t.*

$$\varphi_{j_k} \to \varphi \qquad k \to \infty$$

Notice such compactness theorems cannot hold in infinite dimensions. Here is a simple example that one can easily see.

Example 4.1. *Let H be an infinite dimensional Hilbert Space. Let $\{e_j\}$ be an O.N.B. for H so $\|e_j\| = 1$ and $\langle e_j, e_k\rangle = 0$ for any $j \neq k$. Then $\{e_j\}$ does not admit any convergent subsequence. This is simple as one can look at the distance between any two of them. One can compute for any $j \neq k$*

$$\begin{aligned}\|e_j - e_k\|^2 &= \langle e_j - e_k, e_j - e_k\rangle \\ &= \langle e_j, e_j\rangle - \langle e_k, e_j\rangle - \langle e_j, e_k\rangle + \langle e_k, e_k\rangle \\ &= 2 \\ \|e_j - e_k\| &= \sqrt{2}\end{aligned}$$

Since in ∞-dimensions Compactness does not hold, we formulate something known as the Weak-Compactness. There are in fact 3 main such notions, which are useful in different contexts, which are useful in different contexts.

Theorem 4.5 (Banach Alaoglu Theorem)**.** *If $\mathcal{B}$ is a Banach space which is countable and reflexive, then any sequence $\{\varphi_j\} \subset \mathcal{B}$ which is bounded admits a Weak*-convergent subsequence, i.e., there exists a limiting function $\varphi_\infty \in \mathcal{B}$ and there exists $\{\varphi_{j_k}\}$ s.t. for any $\ell \in \mathcal{B}^*$ the space of bounded linear functionals on $\mathcal{B}$,*

$$\langle \ell, \varphi_{j_k}\rangle \to \langle \ell, \varphi_\infty\rangle$$

This convergence is very weak.

Theorem 4.6 (Rellich Compactness)**.** *Suppose that, fix $s < t$, and assume we have a sequence of functions $\{\varphi_j\} \subset H_{(t)}(X)$ where X is a compact manifold s.t. $\|\varphi_j\|_{(t)} \leq C$. Then there exists a subsequence $\{\varphi_{j_k}\}$ which converges in $H_{(s)}(X)$. In general $H_{(s)}(X) \supsetneq H_{(t)}(X)$.*

The above only converges for weaker norm $\|\cdot\|_{(s)} \leq \|\cdot\|_{(t)}$.

Theorem 4.7 (Weak Compactness for Hölder Spaces)**.** *Fix $0 < \alpha < \beta < 1$. Then any sequence $\{\varphi_j\} \subset C^{k,\beta}(X)$ satisfying*

$$\|\varphi_j\|_{C^{k,\beta}} \leq C$$

admits a convergent subsequence w.r.t. the norm $\|\cdot\|_{C^{k,\alpha}}$.

This norm $\|\cdot\|_{C^{k,\alpha}}$ is weaker than the norm $\|\cdot\|_{C^{k,\beta}}$. We want to apply this weak compactness for Hölder Spaces. Recall that we want $\{\varphi_j\}$ solution of

$$(\omega_0 + i\partial\overline{\partial}\varphi_j)^n = \omega_0^n e^{t_j f_{t_j} + c_{t_j}}$$

to have a subsequence converging to some function φ_T. Thus we want to show that there exists $\beta > \alpha$ s.t.

$$\|\varphi_j\|_{C^{k+1,\beta}} \leq C \qquad \forall\ j \tag{26}$$

Hence by Weak Compactness of Hölder Spaces, there would exist a subsequence φ_{j_k} converging in this weaker norm $\|\cdot\|_{C^{k+1,\alpha}}$ and hence its limit φ_T is in $C^{k+1,\alpha}$ as well. If $k \geq 2$, this allows us to take limits in the equation and that's what we want.

To conclude Closed Interval If we can prove the Estimates (22), (23) and (24) then one can prove (26)

$$\|\varphi_j\|_{C^{k+1,\beta}} \leq C \qquad \forall\ j$$

Observe that on the RHS, if say $k = 5$, we would need to obtain C^β bounds for all derivatives of φ up to order $k + 1 = 6$. On the LHS, the following are missing

1. Bounds for the gradient $\nabla_j\varphi$.
2. Bounds for the mixed Derivative $\nabla_j\nabla_{\overline{k}}\varphi$, $\nabla_{\overline{j}}\nabla_{\overline{k}}\varphi$ and $\nabla_j\nabla_k\varphi$.
3. Bounds for $\nabla_j\nabla_k\nabla_\ell\varphi$, $\nabla_{\overline{j}}\nabla_{\overline{k}}\nabla_{\overline{\ell}}\varphi$.
4. Bounds for all the derivatives of order ≥ 4.

Remark 4.13. *The General Theory for Elliptic PDE is the theory which will allow us to obtain all the missing derivatives from the ones listed in (22), (23) and (24). This is the product of years of research.*

Proof that I (25) is closed using A priori Estimates (22), (23), (24). Assume that there exists a sequence $t_j \in I$, $t_j \to T$ i.e., there exists $\varphi_j \in C^{k+1,\alpha}$ solution of the equation

$$(\omega_0 + i\partial\overline{\partial}\varphi_j)^n = \omega_0^n e^{t_j f + c_{t_j}}$$

and $t_j \to T$.

1. We claim that there exists a subsequence of φ_j (still denoted φ_j for simplicity) s.t. φ_j converges to some φ_T *in the norm* $C^{k+1,\alpha}$. But this implies then $(*_t)$ is solvable at $t = T$, i.e., $T \in I$. Hence I is closed.

2. Next, we show how the claim follows from the a priori estimates. To do so we introduce a second claim: the a priori estimates imply that for some $\alpha < \beta < 1$, there exists a constant independent of j such that

$$\|\varphi_j\|_{C^{k+1,\beta}} \leq C$$

   How does claim 1 follow from claim 2? We use Weak Compactness Theorem 4.7. Hence the key to prove is the Claim 2.

3. We're left with showing the A priori Estimates (a), (b), and (c) imply that

$$\|\varphi_j\|_{C^{k+1,\beta}} \leq C$$

   Luckily here we can apply general PDE Theory. This follows from the following observations (for simplicity we write φ instead of φ_j)

   (a) Our first estimate is

$$\Delta_{\omega_0}\varphi \leq C_2 \implies \|\partial_{\overline{p}}\partial_q\varphi\| \leq C_4$$

   This is because $\omega_0 + i\partial\overline{\partial}\varphi > 0$ positive-definite i.e.

$$(g_0)_{\overline{k}j} + \partial_j\partial_{\overline{k}}\varphi > 0$$

   and a norm for a positive-definite matrix $M_{\overline{p}q}$ is $\text{Tr}(M)$. Indeed, a positive definite(Hermitian) matrix $M_{\overline{p}q}$ can always be diagonalized, i.e.

$$M = UDU^* \qquad \text{unitary} \circ \text{diagonal} \circ \text{unitary}$$

   Thus

$$\text{Tr}(M) = \text{Tr}(D) = \sum_{i=1}^{n} \lambda_i \qquad \lambda_i \text{ eigenvalues of } D$$
$$|M_{\overline{p}q}| \leq C\text{Tr}(D) = C\text{Tr}(M)$$

   Thus it suffices to control the trace for a positive-definite matrix to control its norm. Now we take, assuming $(\omega_0)_{\overline{p}q} = \delta_{pq}$

$$M = \omega_0 + i\partial\overline{\partial}\varphi \implies |\delta_{pq} + \partial_{\overline{p}}\partial_q\varphi| \leq C(n + \Delta_{\omega_0}\varphi)$$

   with observation

$$\text{Tr}(\omega_0 + i\partial\overline{\partial}\varphi) = g_0^{j\overline{k}}((g_0)_{\overline{k}j} + \partial_j\partial_{\overline{k}}\varphi) = n + \Delta_0\varphi$$

   $\text{Tr}(\omega_0 + i\partial\overline{\partial}\varphi)$ is a norm for $\omega_0 + i\partial\overline{\partial}\varphi$, i.e. $n + \Delta\varphi$ is a norm for $\omega_0 + i\partial\overline{\partial}\varphi$. Hence any entry of $\omega_0 + i\partial\overline{\partial}\varphi$ is bounded by $n + \Delta\varphi$. Thus

$$\left\|(g_0)_{\overline{k}j} + \partial_j\partial_{\overline{k}}\varphi\right\|_\infty \leq n + \Delta\varphi$$

   Hence

$$\partial_j\partial_{\overline{k}}\varphi \leq n + \Delta\varphi \leq C_4$$

   (b) Set $\omega = \omega_0 + i\partial\overline{\partial}\varphi$ we have then

$$\omega \leq C_4\omega_0$$

   a control on unknown metric ω using our reference metric ω_0. Our next estimate is: There exists $C_5 > 0$ with

$$\omega \geq C_5\omega_0$$

   We claim the reverse estimate is also true. How to show this? We claim it suffices to show that for any eigenvalue λ_p of ω, we have

$$\lambda_p \geq C_5$$

It suffices to compare entries on the diagonal. By normalization, we assume eigenvalues of the reference metric are all 1. At this step we use the Monge-Ampère Equation.

$$\begin{aligned}
(\omega_0 + i\partial\overline{\partial}\varphi)^n &= \omega_0^n e^{tf+c_t} \\
\omega^n &= w_0^n e^F \qquad F := tf + c_t \\
\det(\omega^n) &= \det(\omega_0^n)e^F \\
\lambda_1 \cdots \lambda_n &= 1 \cdot e^F \\
\lambda_p \prod_{j\neq p} \lambda_j &= e^F \qquad \forall\ p \\
\lambda_p C_4^{n-1} &\geq e^F \\
\lambda_p &\geq C_4^{-(n-1)} e^{\min F} \qquad \min F = \min\{tf + c_t \mid 0 \leq t \leq 1\} > 0 \\
\lambda_p &\geq C_5
\end{aligned}$$

Thus the net outcome is $\partial_j\partial_{\overline{k}}\varphi$ bounded iff metrics ω_0 and $\omega_0 + i\partial\overline{\partial}\varphi$ are equivalent, i.e.

$$C_5\omega_0 \leq \omega_0 + i\partial\overline{\partial}\varphi \leq C_4\omega_0$$

where we used a lower bound on e^F.

(c) Next we claim the following: ω is a metric with Lipschitz coefficients. The coefficients of ω are

$$(g_0)_{\overline{p}q} + \partial_p\partial_{\overline{q}}\varphi$$

the gradients of the coefficients of ω are bounded. For example, suppose we differentiate

$$\partial_\ell\omega_{\overline{p}q} = \partial_\ell(g_0)_{\overline{p}q} + \partial_\ell\partial_p\partial_{\overline{q}}\varphi \qquad \text{where } \partial_\ell(g_0)_{\overline{p}q} \text{ is a fixed smooth matrix}$$

Then we conclude this is bounded using Estimate (c) (24). Hence $\omega_{\overline{p}q}$ is Lipschitz. Thus for any $0 < \beta < 1$, $\omega_{\overline{p}q} \in C^\beta$.

(d) The punchline is, now we differentiate the Monge-Ampère Equation

$$\begin{aligned}
\det(\omega_{\overline{p}q}) &= \det((\omega_0)_{\overline{p}q})e^F \\
\log\det(\omega_{\overline{p}q}) &= \log(\det((\omega_0)_{\overline{p}q})) + F \\
\partial_\ell(\log(\det(\omega_{\overline{p}q}))) &= \partial_\ell(\log(\det((\omega_0)_{\overline{p}q}))) + \partial_\ell F \\
g^{q\overline{p}}\partial_\ell g_{\overline{p}q} &= g^{q\overline{p}}\left(\partial_\ell((g_0)_{\overline{p}q}) + \partial_{\overline{q}}\partial_p\partial_\ell\varphi\right) = \partial_\ell(\log(\det((\omega_0)_{\overline{p}q}))) + \partial_\ell F \\
(g^{q\overline{p}}\partial_{\overline{q}}\partial_p)\partial_\ell\varphi &= -g^{q\overline{p}}\partial_\ell(g_0)_{\overline{p}q} + \partial_\ell(\log(\det((\omega_0)_{\overline{p}q}))) + \partial_\ell F
\end{aligned}$$

Thus $\partial_\ell\varphi$ satisfies a Laplace Equation whose coefficients are $g^{q\overline{p}}$. But $g_{\overline{p}q}$ are Lipschitz, and by Monge-Ampère Equation, $\det(g) \geq C$. Hence $g^{q\overline{p}}$ are also Lipschitz. In particular for any $\beta < 1$, $g^{p\overline{q}}$ is of class C^β. Let's use a theorem from Elliptic Regularity.

Theorem 4.8 (Elliptic Regularity). *If $g^{q\overline{p}}\partial_{\overline{q}}\partial_p\psi \in C^\beta$, $g^{q\overline{p}} \in C^\beta$, and $g^{q\overline{p}} \geq C_5$. Then $\psi \in C^{2,\beta}$.*

We apply Theorem 4.8 to our case, to get

$$\partial_\ell\varphi \in C^{2,\beta}$$

But this is true for any ℓ, thus $\varphi \in C^{3,\beta}$. We can differentiate again to get

$$\begin{aligned}
g^{q\overline{p}}\partial_{\overline{p}}\partial_q(\partial_m\partial_\ell\varphi) - g^{q\overline{r}}\partial_m g_{\overline{r}s}g^{s\overline{p}}\partial_{\overline{p}}\partial_q(\partial_\ell\varphi) &= \partial_m\{\cdots\} \\
g^{q\overline{p}}\partial_{\overline{p}}\partial_q(\partial_m\partial_\ell\varphi) &= \{C^\beta\} + \partial_m\{\cdots\} \in C^\beta
\end{aligned}$$

By the same Elliptic Regularity 4.8 we get

$$\partial_m\partial_\ell\varphi \in C^{2,\beta}$$

But thus $\varphi \in C^{4,\beta}$. Continuing we get

$$\varphi \in C^{k+1,\beta}$$

□

4.5 A Priori Estimates

4.5.1 a Priori Estimate: A

We denote

$$\begin{cases} \omega = \omega_0 + i\partial\overline{\partial}\varphi > 0 \\ \omega^n = \omega_0^n e^F \qquad F := tf + c_t \qquad 0 \le t \le 1 \end{cases}$$

The following is Yau's major contribution.

Theorem 4.9 (Estimate (a) (22))**.** *Let φ satisfy $(*_t)$ equations. Then there exists A_0 depending only on X, ω_0 and $\sup_X e^F$ s.t.*

$$\|\varphi\|_{C^0} \le A_0$$

Proof. Recall the algebraic identity:

$$\alpha^n - \beta^n = (\alpha - \beta)(\alpha^{n-1} + \alpha^{n-2}\beta + \alpha^{n-3}\beta^2 + \cdots + \beta^{n-1})$$

Since the multiplication of $(1,1)$-forms is commutative, the same identity holds for α, β given by $(1,1)$-forms. Thus

$$\begin{aligned} \omega^n - \omega_0^n &= (\omega - \omega_0)\left(\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1}\right) \\ &= i\partial\overline{\partial}\varphi(\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1}) \end{aligned}$$

From the Monge-Ampère Equations, we get

$$\begin{aligned} \omega_0^n e^F - \omega_0^n &= i\partial\overline{\partial}\varphi(\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1}) \\ (e^F - 1)\omega_0^n &= i\partial\overline{\partial}\varphi(\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1}) \\ \int_X \varphi(e^F - 1)\omega_0^n &= \int_X \varphi i\partial\overline{\partial}\varphi(\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1}) \\ &= \int_X \varphi i d\overline{\partial}\varphi\left(\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1}\right) \qquad \text{using } d = \partial + \overline{\partial} \text{ and } \overline{\partial}^2 = 0 \\ &= \int_X d\left(\varphi i\overline{\partial}\varphi(\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1})\right) \qquad \text{by Stokes Theorem this term integrates to 0} \\ &- \int_X (d\varphi) \wedge i\overline{\partial}\varphi(\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1}) \qquad \text{it suffices to deal with } \partial\varphi \wedge i\overline{\partial}\varphi \\ &- \int_X \varphi i\overline{\partial}\varphi d(\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1}) \qquad d\omega^{n-1} = (n-1)d\omega \wedge \omega^{n-2} = 0 \text{ since } \omega \text{ is Kähler} \end{aligned}$$

In the second term, notice that

$$\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1}$$

is a $(n-1, n-1)$-form. Furthermore,

$$(d\varphi) \wedge i\overline{\partial}\varphi = \partial\varphi \wedge i\overline{\partial}\varphi + \overline{\partial}\varphi \wedge i\overline{\partial}\varphi = \partial\varphi \wedge i\overline{\partial}\varphi$$

Hence the second term reduces to

$$(\partial\varphi \wedge i\overline{\partial}\varphi)(\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1})$$

Thus we find

$$\int_X \varphi(e^F - 1)\omega_0^n = -\int_X i\partial\varphi \wedge \overline{\partial}\varphi(\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1})$$

This implies an L^2 gradient estimate for φ! Indeed this follows from some simple observations

1. Consider $i\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_0^{n-1}$ and recall the identity for $T = iT_{\overline{k}j} dz^j \wedge d\overline{z}^k$

$$T \wedge \frac{\omega_0^{n-1}}{(n-1)!} = (g_0^{j\overline{k}} T_{\overline{k}j})\frac{\omega_0^n}{n!}$$

   Now apply this to

$$T = i\partial\varphi \wedge \overline{\partial}\varphi = i\sum \partial_j\varphi\partial_{\overline{k}}\varphi dz^j \wedge d\overline{z}^k$$

Then

$$i\partial\varphi \wedge \overline{\partial}\varphi \wedge \frac{\omega_0^{n-1}}{(n-1)!} = (g_0^{j\overline{k}}\partial_j\varphi\partial_{\overline{k}}\varphi)\frac{\omega_0^n}{n!}$$
$$= \|\partial\varphi\|^2_{\omega_0}\frac{\omega_0^n}{n!}$$
$$\int_X i\partial\varphi \wedge \overline{\partial}\varphi \wedge \frac{\omega_0^{n-1}}{(n-1)!} = \int_X \|\partial\varphi\|^2_{\omega_0}\frac{\omega_0^n}{n!}$$

On the RHS this is the L^2 norm of the gradient of φ.

2. All the terms on the RHS are positive, i.e.

$$\int_X i\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega^{n-1-p}\omega_0^p \geq 0 \qquad \forall\ p$$

In general we prove positivity of the following term

$$\int_X i\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega^p \wedge \omega_0^q$$

For our purposes, we say that a form of the type (k,k) is positive if it is a linear combination with positive coefficients of terms of the following type

$$ie_1 \wedge \overline{e}_1 \wedge (ie_2 \wedge \overline{e}_2) \wedge \cdots \wedge (ie_k \wedge \overline{e}_k)$$

Then

(a) If Φ and Ψ are positive forms of type (k,k) and (ℓ,ℓ) then $\Phi \wedge \Psi$ is a positive form of the type $(k+\ell, k+\ell)$.
(b) ω and ω_0 are both positive (positive hermitian forms can be diagonalized)
(c) $i\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega^p \wedge \omega_0^q$ is always positive
(d) All positive forms of type (n,n) must be proportional to ω_0^n with a positive coefficient.

3. We take for granted that we do have the eigenvalue inequality

$$\lambda_{\omega_0}\|\varphi\|_{L^2} \leq \left(\int_X |\overline{\partial}\varphi|^2\right)^{\frac{1}{2}} \qquad \forall\ \varphi \text{ s.t. } \int_X \varphi\omega_0^n = 0$$

4. Thus we can conclude

$$\int_X |\partial\varphi|^2_{\omega_0}\frac{\omega_0^n}{n!} \leq \int_X i\partial\varphi \wedge \overline{\partial}\varphi\frac{1}{(n-1)!}\left(\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots \omega_0^{n-1}\right)$$
$$= -\frac{1}{(n-1)!}\int_X \varphi(e^F - 1)\omega_0^n$$
$$0 \leq \int_X |\partial\varphi|^2_{\omega_0}\omega_0^n \leq n\int_X |\varphi||e^F - 1|\omega_0^n$$
$$\int_X |\partial\varphi|^2_{\omega_0}\omega_0^n \leq A\int_X |\varphi|\omega_0^n \qquad A := \sup_X(e^F + 1)$$

Furthermore

$$\int_X |\varphi|\omega_0^n \leq \left(\int_X |\varphi|^2\omega_0^n\right)^{\frac{1}{2}}$$

We use now the eigenvalue estimate for the Laplacian w.r.t. ω_0

$$\left(\int_X |\partial\varphi|^2_{\omega_0}\omega_0^n\right)^{\frac{1}{2}} \geq \lambda_{\omega_0}\left(\int_X |\varphi|^2\omega_0^n\right)^{\frac{1}{2}}$$

We find now

$$\lambda_{\omega_0}\|\varphi\|_{L^2}\left(\int_X |\partial\varphi|^2_{\omega_0}\omega_0^n\right)^{\frac{1}{2}} \leq A\|\varphi\|_{L^2}$$
$$\lambda_{\omega_0}\left(\int_X |\partial\varphi|^2_{\omega_0}\omega_0^n\right)^{\frac{1}{2}} \leq A$$
$$\|\varphi\|_{L^2} \leq \frac{A}{\lambda_{\omega_0}}$$

This gives the L^2 gradient bound of φ.

In fact we can keep improving. We shall try to estimate $\|\varphi\|_{L^p}$ for higher and higher p until we reach $p=\infty$ and obtain

$$\|\varphi\|_{L^2} \geq \|\varphi\|_{L^{p_1}} \geq \cdots \geq \|\varphi\|_{L^\infty} = \|\varphi\|_{C^0}$$

This can be done by adapting to the Monge-Ampère Equation the method known as the Moser Iteration. This method was designed for linear equation in divergence forms. Idea is we try to do the estimate for

$$\left\|\nabla(\varphi|\varphi|^{\frac{\alpha}{2}})\right\|_{L^2}^2 \qquad \forall\ \alpha \geq 0$$

Why we want to estimate $\varphi|\varphi|^{\frac{\alpha}{2}}$? This is because $\varphi^{1+\frac{\alpha}{2}}$ is difficult to deal with if $\varphi < 0$. Why $|\varphi|^{\frac{\alpha}{2}}$? Since $|\varphi| \geq 0$.

Lemma 4.12.

$$\frac{d}{dt}(t|t|^\alpha) = (\alpha+1)|t|^\alpha \qquad \forall\ t \in \mathbb{R}$$

Proof. The formula is obvious if $\alpha = 0$. So assume for $\alpha > 0$. Now the formula is true at $t=0$ since $\frac{d}{dt}(t|t|^\alpha) = 0$ which vanishes of order > 1 and

$$(\alpha+1)|t|^\alpha\big|_{t=0} = 0 \qquad \forall\ \alpha > 0$$

Thus we need to verify the formula for $t > 0$ and for $t < 0$. At $t > 0$, it is again obvious, since

$$t|t|^\alpha = t^{\alpha+1} \qquad |t|^\alpha = t^\alpha$$

At $t < 0$, set $t = -s$ with $s > 0$. Then

$$t|t|^\alpha = (-s)s^\alpha = -s^{\alpha+1} \implies \frac{d}{dt}(t|t|^\alpha) = -\frac{d}{ds}(-s^{\alpha+1}) = (\alpha+1)s^\alpha = (\alpha+1)|t|^\alpha$$

□

Next, we try to estimate

$$\left\|\nabla(\varphi|\varphi|^{\frac{\alpha}{2}})\right\|_{L^2(\omega_0)}^2$$

Previously we had

$$\begin{aligned}
\omega^n - \omega_0^n &= (\omega - \omega_0)\left(\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1}\right) \\
\omega_0^n e^F - \omega_0^n &= (i\partial\overline{\partial}\varphi)\left(\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1}\right) \\
(e^F - 1)\omega_0^n &= (i\partial\overline{\partial}\varphi)\left(\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1}\right) \\
\varphi|\varphi|^\alpha(e^F - 1)\omega_0^n &= \varphi|\varphi|^\alpha(i\partial\overline{\partial}\varphi)\left(\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1}\right) \qquad \text{we multiply both sides not by } \varphi\text{, but by } \varphi|\varphi|^\alpha \\
\int_X \varphi|\varphi|^\alpha(e^F - 1)\omega_0^n &= \int_X \varphi|\varphi|^\alpha(i\partial\overline{\partial}\varphi)\left(\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1}\right) \qquad d = \partial + \overline{\partial} \\
&= \int_X d(\varphi|\varphi|^\alpha i\partial\varphi\left(\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1}\right)) \qquad 0 \text{ by Stokes Theorem} \\
&\quad - \int_X d(\varphi|\varphi|^\alpha) i\overline{\partial}\varphi\left(\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1}\right) \qquad \text{reduces to } \int_X \partial(\varphi|\varphi|^\alpha) i\overline{\partial}\varphi\left(\omega^{n-1} + \cdots + \omega_0^{n-1}\right) \\
&\quad - \int_X \varphi|\varphi|^\alpha i\overline{\partial}\varphi d\left(\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1}\right) \qquad 0 \text{ since Kähler } d\omega^{n-1} = \cdots = d\omega_0^{n-1} = 0 \\
\int_X \varphi|\varphi|^\alpha(e^F - 1)\omega_0^n &= -\int_X \partial(\varphi|\varphi|^\alpha) i\overline{\partial}\varphi(\omega^{n-1} + \cdots + \omega_0^{n-1}) \qquad \text{we change of variables } \varphi \mapsto t \\
&= -(\alpha+1)\int_X |\varphi|^\alpha \partial\varphi \wedge i\overline{\partial}\varphi(\omega^{n-1} + \cdots + \omega_0^{n-1}) \qquad \forall\ \alpha \geq 0 \\
&= -(\alpha+1)\int_X \left(i|\varphi|^{\frac{\alpha}{2}}\partial\varphi \wedge |\varphi|^{\frac{\alpha}{2}}\overline{\partial}\varphi\right)(\omega^{n-1} + \cdots + \omega_0^{n-1}) \\
&= -(\alpha+1)\int_X \left(\frac{\partial(\varphi|\varphi|^{\frac{\alpha}{2}})}{(\frac{\alpha}{2}+1)} \wedge \frac{\overline{\partial}(\varphi|\varphi|^{\frac{\alpha}{2}})}{(\frac{\alpha}{2}+1)}\right)(\omega^{n-1} + \cdots + \omega_0^{n-1}) \qquad \text{using Lemma} \\
\int_X \varphi|\varphi|^\alpha(e^F - 1)\omega_0^n &= -\frac{\alpha+1}{(\frac{\alpha}{2}+1)^2}\int_X i\partial(\varphi|\varphi|^{\frac{\alpha}{2}}) \wedge \overline{\partial}(\varphi|\varphi|^{\frac{\alpha}{2}}) \wedge (\omega^{n-1} + \cdots + \omega_0^{n-1})
\end{aligned}$$

Now notice all the terms in the sum in the RHS have the same sign. Thus picking any piece

$$\frac{\alpha+1}{(\frac{\alpha}{2}+1)^2}\int_X i\partial(\varphi|\varphi|^{\frac{\alpha}{2}}) \wedge \overline{\partial}(\varphi|\varphi|^{\frac{\alpha}{2}}) \wedge \omega^{n-1} \leq \int_X |\varphi|^{\alpha+1}(e^F + 1)\omega_0^n$$

$$i\partial\psi \wedge \overline{\partial}\psi \wedge \omega_0^{n-1} = (g_0^{\overline{p}q}\partial_q\psi\partial_{\overline{p}}\psi)\frac{\omega_0^n}{n} = |\nabla\psi|_{\omega_0}^2\frac{\omega_0^n}{n} \qquad \text{making the observation}$$

$$\frac{\alpha+1}{(\frac{\alpha}{2}+1)^2}\left\|\nabla(\varphi|\varphi|^{\frac{\alpha}{2}})\right\|_{L^2(\omega_0)}^2 \leq \int_X |\varphi|^{\alpha+1}(e^F + 1)\omega_0^n$$

Notice this is the same argument as in the previous class with $\varphi \mapsto \varphi|\varphi|^{\frac{\alpha}{2}}$ for any $\alpha \geq 0$. Now to exploit this, we shall apply Sobolev Inequality on (X, ω_0). This inequality says that

Lemma 4.13 (Sobolev Inequality on (X, ω_0))**.**

$$\left(\int_X |u|^{2\beta}\omega_0^n\right)^{\frac{1}{\beta}} \leq C_{\omega_0}\left(\int_X |\nabla u|^2_{\omega_0}\omega_0^n + \int_X |u|^2\omega_0^n\right) \qquad \beta := \frac{n}{n-1} > 1 \tag{27}$$

Let's simplify our notation and say all integrals are w.r.t. ω_0. Since we're allowed to choose u, we choose

$$u = \varphi|\varphi|^{\frac{\alpha}{2}}$$

Let's do some preliminary calculations.

$$|u|^2 = (|\varphi||\varphi|^{\frac{\alpha}{2}})^2 = |\varphi|^{2+\alpha} = |\varphi|^p$$

where we set $p := \alpha + 2 \geq 2$ so that $p - 1 = \alpha + 1$ and

$$|u|^{2\beta} = |\varphi|^{p\beta}$$

Applying (27) we have

$$\begin{aligned}
\left(\int_X |\varphi|^{p\beta}\right)^{\frac{1}{\beta}} &\leq C\left(\int_X |\nabla(\varphi|\varphi|^{\frac{\alpha}{2}})|^2 + \int_X |\varphi|^p\right) \\
&\leq C\left(\frac{(\frac{\alpha}{2}+1)^2}{\alpha+1}\int_X |\varphi|^{p-1}(e^F+1) + \int_X |\varphi|^p\right) \\
&\leq C\left(p\int_X |\varphi|^{p-1} + \int_X |\varphi|^p\right)
\end{aligned}$$

Thus we have gained the control of

$$\|\varphi\|_{L^{p\beta}} \lesssim \|\varphi\|_{L^{p-1}} + \|\varphi\|_{L^p}$$

We want to iterate this game. Let's simplify the above inequality that we just obtained as follows. What's the idea?

1. On the Right Hand Side, first we control $\|\varphi\|_{L^{p-1}}$ by $\|\varphi\|_{L^p}$ by Hölder's

$$\begin{aligned}
\int_X |\varphi|^{p-1} &\leq \left(\int_X |\varphi|^p\right)^{\frac{p-1}{p}}\left(\int_X 1\right)^{\frac{1}{p}} \\
\|\varphi\|^{p-1}_{L^{p-1}} &\leq \|\varphi\|^{p-1}_{L^p} V_0^{\frac{1}{p}}
\end{aligned}$$

   We do have the same homogeneity and the powers add up to 1.

2. We can now write

$$p\int_X |\varphi|^{p-1} + \int_X |\varphi|^p \leq p\,\|\varphi\|^{p-1}_{L^p} V_0^{\frac{1}{p}} + \|\varphi\|^p_{L^p}$$

$$\|\varphi\|^p_{L^{p\beta}} \leq C\left\{p\,\|\varphi\|^{p-1}_{L^p} V_0^{\frac{1}{p}} + \|\varphi\|^p_{L^p}\right\} \tag{28}$$

   Thus we can control $\|\varphi\|_{L^{p\beta}}$ from the norm $\|\varphi\|_{L^p}$!

For the purpose of Moser iteration, we use the following corollary:

Corollary 4.3.

$$\max\{1, \|\varphi\|_{L^{p\beta}}\} \leq (Cp)^{\frac{1}{p}}\max(1, \|\varphi\|_p)$$

This inequality is easier to iterate.

Proof. We consider two cases.

1. First case, if $\|\varphi\|_{L^p} \leq 1$, then from the previous inequality (28), we see that

$$\begin{aligned}
\|\varphi\|^p_{L^{p\beta}} &\leq C\left(pV_0^{\frac{1}{p}} + 1\right) \leq Cp \qquad \forall\, p \geq 2 \\
\|\varphi\|_{L^{p\beta}} &\leq (Cp)^{\frac{1}{p}}
\end{aligned}$$

2. In the second case assume $\|\varphi\|_{L^p} > 1$, then on the RHS of (28)

$$\begin{aligned} p\,\|\varphi\|_{L^p}^{p-1} V_0^{\frac{1}{p}} + \|\varphi\|_{L^p}^{p} &\le p\,\|\varphi\|_{L^p} V_0^{\frac{1}{p}} + \|\varphi\|_{L^p}^{p} \\ &\le (Cp)\,\|\varphi\|_{L^p}^{p} \\ \|\varphi\|_{L^{p\beta}}^{p} &\le Cp\,\|\varphi\|_{L^p}^{p} \\ \|\varphi\|_{L^{p\beta}} &\le (Cp)^{\frac{1}{p}}\,\|\varphi\|_{L^p} \end{aligned}$$

□

Now we write the Corollary 4.3 in the log form

$$\log\max(1,\|\varphi\|_{L^{p\beta}}) \le \frac{1}{p}\log(Cp) + \log\max(1,\|\varphi\|_{L^p}) \tag{29}$$

Next we apply (29) with $p \mapsto p\beta^k$. We get

$$\begin{aligned} \log\max(1,\|\varphi\|_{L^{p\beta^k}}) &\le \frac{1}{p\beta^{k-1}}\log(Cp\beta^{k-1}) + \log\max(1,\|\varphi\|_{L^{p\beta^{k-1}}}) \\ &\le \frac{1}{p\beta^{k-1}}\log(Cp\beta^{k-1}) + \frac{1}{p\beta^{k-2}}\log(Cp\beta^{k-2}) + \log\max(1,\|\varphi\|_{L^{p\beta^{k-2}}}) \\ &\le \sum_{\ell=0}^{k-1}\frac{1}{p\beta^{\ell}}\log(Cp\beta^{\ell}) + \log\max(1,\|\varphi\|_{L^p}) \end{aligned}$$

Now we let $k \to \infty$. We get a geometric series on the RHS

$$\begin{aligned} \log\max(1,\|\varphi\|_{L^\infty}) &\le \sum_{\ell=0}^{\infty}\frac{1}{p\beta^{\ell}}\log(Cp\beta^{\ell}) + \log(\max(1,\|\varphi\|_{L^p})) \\ \log\max(1,\|\varphi\|_{L^\infty}) &\le C_p + \log(\max(1,\|\varphi\|_{L^p})) \end{aligned}$$

Here we're allowed to take any p we want. Take $p = 2$, we get

$$\log\max(1,\|\varphi\|_{L^\infty}) \le C_2 + \log\max(1,\|\varphi\|_{L^2})$$

Since we know that $\|\varphi\|_{L^2} \le C$, then

$$\begin{aligned} \log\max(1,\|\varphi\|_{\infty}) &\le C \\ \|\varphi\|_{\infty} &\le C \end{aligned}$$

□

4.5.2 a Priori Estimate: B

Theorem 4.10 (Estimate (b) (23))**.**

$$\Delta_0\varphi \le C$$

where

$$\Delta_0\varphi := g_0^{p\bar{q}}\partial_p\partial_{\bar{q}}\varphi \qquad \textit{the Laplacian w.r.t. } \omega_0$$

Proof. The strategy is: We try to apply the maximum principle, in showing that the quantity we want to estimate satisfies a Laplace Inequality, and look at the points where this quantity attains its maximum. There are two metric

$$\begin{aligned} \omega_0 &\implies \Delta_{\omega_0} = \Delta_0 \\ \omega = \omega_0 + i\partial\bar{\partial}\varphi &\implies \Delta_\omega = \Delta \end{aligned}$$

We shall use Δ instead of Δ_{ω_0} because we shall need to differentiate the Monge-Ampère Equation and we have seen it in the proof of openness and the use of the Implicit Function Theorem) that it is the Laplacian w.r.t. the unknown metric ω which appears. Thus we compute

$$\Delta(\Delta_0\varphi)$$

and hope to extract a differential inequality. Geometrically we're making use of the Ricci curvature. To make the Ricci appear, it is better to compute with the endomorphism. We shall try to compute

$$\Delta(\mathrm{Tr}(h))$$

What happens? The method is: We introduce the Endomorphism h by

$$h^p_q := g_0^{p\overline{m}} g_{\overline{m}q} = g_0^{-1} g$$
$$\omega_0 = (g_0)_{\overline{p}q} = i(g_0)_{\overline{k}j} dz^j \wedge d\overline{z}^k$$
$$\omega = (g)_{\overline{p}q} = i g_{\overline{k}j} dz^j \wedge d\overline{z}^k$$

Fixing ω_0 we have one-to-one correspondence between h and g. We shall prove the following key inequality: If φ satisfies the Monge-Ampère Equation

$$(\omega_0 + i\partial\overline{\partial}\varphi)^n = \omega_0^n e^F$$

then

$$\Delta \log(\mathrm{Tr}(h)) \geq -C\,\mathrm{Tr}(h^{-1}) \tag{30}$$

for C constant depending only on ω_0.

1. We begin with quoting the Maximum Principle.

   Proposition 4.2 (Maximum Principle)**.** *In calculus, at a local maximum x_0 of a function $f(x)$, we must have*

   $$f''(x_0) \leq 0$$

   In several variables, at a local maximum x_0 of a function $f(x)$, $x \in \mathbb{R}^n$, we must have

   $$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)(x_0) \leq 0$$

   Hence

   $$\Delta f(x_0) = \mathrm{Tr}(\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)(x_0)) \leq 0$$

   Now do we use the Maximum Principle?

   (a) We find an inequality satisfied by Δf.

   (b) We look at this inequality at a maximum point of Δf.

   (c) We hope that the inequality gives some useful information.

   In the case at hand, assume that we have succeeded in showing (30) for some constant C_1. According to the maximum principle, we look at the inequality at a maximum point. At the maximum point of $\mathrm{Tr}(h)$, we also have a maximum point for $\log(\mathrm{Tr}(h))$. This is simply because log is increasing. Thus

   $$0 \geq \Delta(\log \mathrm{Tr} h)(z_0) \geq -C_1 \mathrm{Tr}(h^{-1})$$

   But this doesn't seem useful. The RHS is negative anyway. However, by a slight modification, we get exactly what we want. For this, observe that

   $$\begin{aligned}\mathrm{Tr}(h^{-1}) &= \mathrm{Tr}(g^{-1} g_0) = g^{p\overline{q}} (g_0)_{\overline{q}p} \\ &= g^{p\overline{q}} (g_{\overline{q}p} - \partial_p \partial_{\overline{q}} \varphi) \\ &= n - \Delta\varphi\end{aligned}$$

   Let's now consider the expression

   $$\begin{aligned}\Delta(\log(\mathrm{Tr}(h)) - A\varphi) &= \Delta \log(\mathrm{Tr}(h)) - A\Delta\varphi \\ &= \Delta \log(\mathrm{Tr}(h)) + A\left(\mathrm{Tr}(h^{-1}) - n\right) \\ &\geq -C_1 \mathrm{Tr}(h^{-1}) + A\mathrm{Tr}(h^{-1}) - An\end{aligned}$$

   Take now

   $$A := 2C_1$$

   Then

   $$\Delta(\log(\mathrm{Tr}(h)) - A\varphi) \geq C_1 \mathrm{Tr}(h^{-1}) - C_3 \qquad C_3 := -An$$

   Now apply the maximum principle to this. What happens is the following. Let z_1 be a local maximum point of $\log(\mathrm{Tr}(h)) - A\varphi$. Then

   $$0 \geq \Delta\left(\log(\mathrm{Tr}(h)) - A\varphi\right)(z_1) \geq C_1 \mathrm{Tr}(h^{-1})(z_1) - C_3$$
   $$\mathrm{Tr}(h^{-1})(z_1) \leq C_4 \qquad C_4 := \frac{C_3}{C_1}$$

Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of h at z_1. Thus we have

$$\frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} \leq C_4$$

and then

$$\lambda_1, \cdots, \lambda_n \geq C_5 := \frac{1}{C_4}$$

But now from the Monge-Ampère Equation,

$$\omega^n = \omega_0^n e^F \iff \frac{\omega^n}{\omega_0^n} = e^F$$

Thus the product

$$\begin{aligned}
\prod_{\ell=1}^{n} \lambda_\ell &= e^F \\
C_5^{n-1}\lambda_r \leq \left(\prod_{\ell\neq r} \lambda_\ell\right) \lambda_r &= e^F \\
\lambda_r \leq C_6 := \frac{\max e^F}{C_5^{n-1}} &\qquad \forall\, 1 \leq r \leq n
\end{aligned}$$

Now at an arbitrary point $z \in X$, we can write

$$\begin{aligned}
\left(\log(\mathrm{Tr}(h)) - A\varphi\right)(z) &\leq \left(\log(\mathrm{Tr}(h)) - A\varphi\right)(z_1) \\
&\leq \left(\log(nC_6) - A\varphi\right)(z_1) \\
\left(\log(\mathrm{Tr}(h)) - A\varphi\right)(z) &\leq C_7 - A\varphi(z_1)
\end{aligned}$$

Now we can rewrite

$$\begin{aligned}
\log(\mathrm{Tr}(h(z))) &\leq C_7 + A(\varphi(z) - \varphi(z_1)) \\
&\leq C_7 + A\mathrm{osc}(\varphi) \\
\mathrm{Tr}(h(z)) &\leq C_8 e^{A\mathrm{osc}(\varphi)} \\
&\leq C_8 e^{2A\|\varphi\|_{C^0}} \\
&\leq C_9 \qquad \text{using Estimate (22)}
\end{aligned}$$

Now using

$$g_0^{p\overline{q}}((g_0)_{\overline{q}p} + \partial_p\partial_{\overline{q}}\varphi) = n + \Delta_0\varphi$$

We have

$$n + \Delta_0\varphi \leq C_9$$

2. In the second step we prove the key inequality (30). To derive this, we need to express the LHS in terms of curvatures ${R_{\overline{j}k}}^{\ell}{}_{m}$ of ω_0 since the Monge-Ampère Equation is an assignment of the Ricci curvature $R_{\overline{j}k}$ of ω. Observe that we have two metrics and hence two notions of curvatures. In the following we discuss relation between Curvatures

$$\begin{aligned}
\omega_0 &\implies {R_{\overline{j}k}}^{\ell}{}_{m} \implies R_{\overline{j}k} \\
\omega = \omega_0 + i\partial\overline{\partial}\varphi &\implies {\mathcal{R}_{\overline{j}k}}^{\ell}{}_{m} \implies \mathcal{R}_{\overline{j}k}
\end{aligned}$$

How are the curvatures related? The basic formula is

$${\mathcal{R}_{\overline{j}k}}^{\ell}{}_{m} - {R_{\overline{j}k}}^{\ell}{}_{m} = -\partial_{\overline{j}}(\mathcal{D}_k h h^{-1})^{\ell}_{m} \tag{31}$$

where $\mathcal{D}_j$ is the covariant derivative w.r.t. ω. Let's assume this basic formula (31) for the moment. We return to the calculation of $\Delta\log(\mathrm{Tr}(h))$. We begin by calculating $\Delta(\mathrm{Tr}(h))$. Write

$$\begin{aligned}
\Delta\mathrm{Tr}(h) &= g^{p\overline{q}}\partial_{\overline{q}}\partial_p(\mathrm{Tr}(h)) = g^{p\overline{q}}\partial_{\overline{q}}\mathcal{D}_p(\mathrm{Tr}(h)) \\
&= g^{p\overline{q}}\partial_{\overline{q}}\mathrm{Tr}(\mathcal{D}_p h) \\
&= g^{p\overline{q}}\partial_{\overline{q}}\mathrm{Tr}((\mathcal{D}_p h h^{-1})h) \\
&= g^{p\overline{q}}\left(\mathrm{Tr}(\partial_{\overline{q}}(\mathcal{D}_p h h^{-1})h) + \mathrm{Tr}(\mathcal{D}_p h h^{-1})\partial_{\overline{q}} h\right)
\end{aligned}$$

Now notice

$$\mathrm{Tr}(\partial_{\overline{q}}(\mathcal{D}_p hh^{-1})h) = \partial_{\overline{q}}(\mathcal{D}_p hh^{-1})^{\ell}_m \cdot h^m_\ell \overset{(31)}{=} -\left(\mathcal{R}_{\overline{q}p}{}^{\ell}{}_m - R_{\overline{q}p}{}^{\ell}{}_m\right) \cdot h^m_\ell$$

Hence

$$\Delta \mathrm{Tr}(h) = g^{p\overline{q}}\left(-\mathcal{R}_{\overline{q}p}{}^{\alpha}{}_\beta + R_{\overline{q}p}{}^{\alpha}{}_\beta\right) h^\beta_\alpha + g^{p\overline{q}}\mathrm{Tr}(\mathcal{D}_p hh^{-1})\partial_{\overline{q}} h$$

Now we simplify

$$\begin{aligned} g^{p\overline{q}}\mathcal{R}_{\overline{q}p}{}^{\alpha}{}_\beta h^\beta_\alpha &= \mathcal{R}^\alpha_\beta h^\beta_\alpha \\ &= \mathcal{R}^\alpha_\beta g_0^{\beta\overline{\gamma}} g_{\overline{\gamma}\alpha} \qquad \text{by definition of } h \\ &= \mathcal{R}_{\overline{\gamma}\beta} g_0^{\beta\overline{\gamma}} \qquad \text{lowering indices} \end{aligned}$$

On the other hand

$$\begin{aligned} \mathcal{R}_{\overline{\gamma}\beta} &= -\partial_\beta\partial_{\overline{\gamma}}\log(\det(\omega^n)) \\ &= -\partial_\beta\partial_{\overline{\gamma}}\log(\det(\frac{\omega^n}{\omega_0^n})) - \partial_\beta\partial_{\overline{\gamma}}\log(\det(\omega_0^n)) \qquad \text{Recall Monge-Ampère } \omega^n = w_0^n e^F \\ \mathcal{R}_{\overline{\gamma}\beta} &= -\partial_\beta\partial_{\overline{\gamma}}F + R_{\overline{\gamma}\beta} \\ \mathcal{R}_{\overline{\gamma}\beta} g_0^{\beta\overline{\gamma}} &= -g_0^{\beta\overline{\gamma}}\partial_\beta\partial_{\overline{\gamma}}F + g_0^{\beta\overline{\gamma}} R_{\overline{\gamma}\beta} \\ &= -\Delta_0 F + R \qquad \text{where } R \text{ is scalar curvature of } \omega_0 \end{aligned}$$

Thus we find

$$\Delta(\mathrm{Tr}(h)) = \Delta_0 F - R + g^{p\overline{q}} R_{\overline{q}p}{}^{\alpha}{}_\beta h^\beta_\alpha + g^{p\overline{q}}\mathrm{Tr}(\mathcal{D}_p hh^{-1})\partial_{\overline{q}} h \tag{32}$$

Next we compute (exploiting the strict positivity of some quantity)

$$\begin{aligned} \Delta\log(\mathrm{Tr}(h)) &= g^{p\overline{q}}\partial_{\overline{q}}\partial_p \log(\mathrm{Tr}(h)) \\ &= g^{p\overline{q}}\partial_{\overline{q}}\left(\frac{\partial_p(\mathrm{Tr}(h))}{\mathrm{Tr}(h)}\right) \\ &= g^{p\overline{q}}\left(\frac{\partial_{\overline{q}}\partial_p(\mathrm{Tr}(h))}{\mathrm{Tr}(h)} - \frac{\partial_p(\mathrm{Tr}(h))\partial_{\overline{q}}(\mathrm{Tr}(h))}{(\mathrm{Tr}(h))^2}\right) \\ &= \frac{\Delta(\mathrm{Tr}(h))}{\mathrm{Tr}(h)} - \frac{g^{p\overline{q}}\partial_p(\mathrm{Tr}(h))\partial_{\overline{q}}(\mathrm{Tr}(h))}{(\mathrm{Tr}(h))^2} \end{aligned}$$

We use the previous formula so

$$\Delta\log(\mathrm{Tr}(h)) = \frac{\Delta_0 F - R + g^{p\overline{q}} R_{\overline{q}p}{}^{\alpha}{}_\beta h^\beta_\alpha}{\mathrm{Tr}(h)} + \frac{g^{p\overline{q}}\mathrm{Tr}(\mathcal{D}_p hh^{-1})\partial_{\overline{q}} h}{\mathrm{Tr}(h)} - \frac{g^{p\overline{q}}\partial_p(\mathrm{Tr}(h))\partial_{\overline{q}}(\mathrm{Tr}(h))}{(\mathrm{Tr}(h))^2}$$

We do estimates on each term

(a) Estimating the terms on the RHS

$$\begin{aligned} \frac{\Delta_0 F - R}{\mathrm{Tr}(h)} &\geq -C_1\frac{1}{\mathrm{Tr}(h)} \\ \frac{n}{\lambda_1 + \cdots + \lambda_n} = \frac{n}{\mathrm{Tr}(h)} &\leq \mathrm{Tr}(h^{-1}) = \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} \\ \implies \frac{\Delta_0 F - R}{\mathrm{Tr}(h)} &\geq -C_2(\mathrm{Tr}(h^{-1})) \end{aligned}$$

(b) The next term that we estimate is

$$g^{p\overline{q}} R_{\overline{q}p}{}^{\alpha}{}_\beta h^\beta_\alpha$$

Recall that

$$\begin{aligned} h &= g_0^{-1} g \\ g &= g_0 h \\ g^{-1} &= h^{-1} g_0^{-1} \\ g^{p\overline{q}} &= (h^{-1})^p_r (g_0)^{r\overline{q}} \end{aligned}$$

Now

$$g^{p\overline{q}} R_{\overline{q}p}{}^{\alpha}{}_{\beta} h^{\beta}_{\alpha} = (h^{-1})^{p}_{r} R_{p}^{r\alpha}{}_{\beta} h^{\beta}_{\alpha}$$
$$|g^{p\overline{q}} R_{\overline{q}p}{}^{\alpha}{}_{\beta} h^{\beta}_{\alpha}| \leq (\mathrm{Tr}(h^{-1}))|R_m(\omega_0)|\mathrm{Tr}(h)$$
$$|\frac{g^{p\overline{q}} R_{\overline{q}p}{}^{\alpha}{}_{\beta} h^{\beta}_{\alpha}}{\mathrm{Tr}(h)}| \leq C_{\omega_0}\mathrm{Tr}(h^{-1}) \qquad \text{as desired}$$

(c) Aubin-Yau Inequality. (purely algebraic)

$$\frac{g^{p\overline{q}}\mathrm{Tr}(\mathcal{D}_p h h^{-1})\partial_{\overline{q}} h}{\mathrm{Tr}(h)} - \frac{g^{p\overline{q}}\partial_p(\mathrm{Tr}(h))\partial_{\overline{q}}(\mathrm{Tr}(h))}{(\mathrm{Tr}(h))^2} \geq 0 \tag{33}$$

If so then indeed our desired inequality (30) holds.

Lemma 4.14 (Formulas Relating the covariant derivatives with respect to ω and ω_0)**.** *Notations*

$$\omega_0 \implies \nabla_\ell$$
$$\omega = \omega_0 + i\partial\overline{\partial}\varphi \implies \mathcal{D}_\ell$$

Then for any vector field V^m and differential form W_m we have

$$\mathcal{D}_j V^m - \nabla_j V^m = (\mathcal{D}_j h h^{-1})^m_p V^p \qquad h := g_0^{-1} g,\ h^{\ell}_m = (g_0)^{\ell\overline{r}} g_{\overline{r}m}$$
$$\mathcal{D}_j W_m - \nabla_j W_m = -W_p(\mathcal{D}_j h h^{-1})^p_m$$

Proof. To see this, write (by the Chern Unitary Connection)

$$\begin{aligned}
\nabla_j V &= g_0^{-1}\partial_j(g_0 V) = hg^{-1}\partial_j(gh^{-1}V) \\
&= hg^{-1}\mathcal{D}_j(gh^{-1}V) \qquad g_{\overline{s}m}(h^{-1})^{m}{}_{r}V^r \qquad \text{section of an antiholomorphic bundle} \\
&= hg^{-1}g\mathcal{D}_j(h^{-1}V) \\
&= h\left(\mathcal{D}_j(h^{-1}V)\right) \\
&= h\left(-h^{-1}(\mathcal{D}_j h)h^{-1}V + h^{-1}\mathcal{D}_j V\right) \\
&= -(\mathcal{D}_j h)h^{-1}V + \mathcal{D}_j V
\end{aligned}$$

□

Now we want to get relation between curvatures. Apply $\partial_{\overline{k}}$ to both sides

$$\begin{aligned}
\partial_{\overline{k}}\nabla_j V^m &= \partial_{\overline{k}}(\mathcal{D}_j V^m) - \partial_{\overline{k}}((\mathcal{D}_j h h^{-1})V^m) \\
\nabla_j(\partial_{\overline{k}} V^m) - R_{\overline{k}j}{}^{m}{}_{p}V^p &= \mathcal{D}_j(\partial_{\overline{k}}V^m) - \mathcal{R}_{\overline{k}j}{}^{m}{}_{p}V^p - \partial_{\overline{k}}((\mathcal{D}_j h h^{-1})V^m) \\
\nabla_j(\partial_{\overline{k}} V^m) - \mathcal{D}_j(\partial_{\overline{k}}V^m) &= R_{\overline{k}j}{}^{m}{}_{p}V^p - \mathcal{R}_{\overline{k}j}{}^{m}{}_{p}V^p - \partial_{\overline{k}}((\mathcal{D}_j h h^{-1})V^m) \\
-(\mathcal{D}_j h)h^{-1}(\partial_{\overline{k}}V^m) &= R_{\overline{k}j}{}^{m}{}_{p}V^p - \mathcal{R}_{\overline{k}j}{}^{m}{}_{p}V^p - \partial_{\overline{k}}(\mathcal{D}_j h h^{-1})V^m - (\mathcal{D}_j h h^{-1})\partial_{\overline{k}}V^m \qquad \text{using Lemma 4.14} \\
R_{\overline{k}j}{}^{m}{}_{p}V^p - \mathcal{R}_{\overline{k}j}{}^{m}{}_{p}V^p &= \partial_{\overline{k}}(\mathcal{D}_j h h^{-1})V^m \\
R_{\overline{k}j}{}^{m}{}_{p} - \mathcal{R}_{\overline{k}j}{}^{m}{}_{p} &= \partial_{\overline{k}}(\mathcal{D}_j h h^{-1})^m_p
\end{aligned}$$

Then next thing on the checklist is to prove the Aubin-Yau Inequality (33).

Proof of Aubin-Yau (33)*.* We begin by making the first term on the LHS more explicit. Recall

$$\begin{aligned}
\mathcal{D}_j V^m - \nabla_j V^m &= (\mathcal{D}_j h h^{-1})^m_p V^p \\
\mathcal{D}_j(g^{m\overline{r}}) - \nabla_j(g^{m\overline{r}}) &= (\mathcal{D}_j h h^{-1})^m_p g^{p\overline{r}} \\
-\nabla_j(g^{m\overline{r}}) &= (\mathcal{D}_j h h^{-1})^m_p g^{p\overline{r}} \\
-g_{\overline{r}\lambda}\nabla_j(g^{m\overline{r}}) &= (\mathcal{D}_j h h^{-1})^m_p g^{p\overline{r}} g_{\overline{r}\lambda} \\
&= (\mathcal{D}_j h h^{-1})^m_\lambda \\
-g_{\overline{r}\lambda}\left(-g^{m\overline{s}}(\nabla_j g_{\overline{s}\ell})g^{\ell\overline{r}}\right) &= (\mathcal{D}_j h h^{-1})^m_\lambda \\
g^{m\overline{s}}\nabla_j g_{\overline{s}\lambda} &= (\mathcal{D}_j h h^{-1})^m_\lambda \\
(\mathcal{D}_j h h^{-1})^m_\lambda &= g^{m\overline{s}}\nabla_j \varphi_{\overline{s}\lambda}
\end{aligned}$$

Next we calculate

$$\begin{aligned}\partial_{\overline{k}} h_m^\lambda &= \nabla_{\overline{k}}((g_0)^{\lambda\overline{r}} g_{\overline{r}m}) \\ &= g_0^{\lambda\overline{r}} \nabla_{\overline{k}} g_{\overline{r}m} \\ &= g_0^{\lambda\overline{r}} \nabla_{\overline{k}} \varphi_{\overline{r}m}\end{aligned}$$

Altogether we have

$$\begin{aligned}g^{j\overline{k}} \mathrm{Tr}((\mathcal{D}_j h) h^{-1} \partial_{\overline{k}} h) &= g^{j\overline{k}} (g^{m\overline{s}} \nabla_j \varphi_{\overline{s}\lambda}) g_0^{\lambda\overline{r}} \nabla_{\overline{k}} \varphi_{r\overline{m}} \\ &= g^{j\overline{k}} g^{m\overline{s}} g_0^{\lambda\overline{r}} \nabla_j \varphi_{\overline{s}\lambda} \nabla_{\overline{k}} \varphi_{\overline{r}m} \qquad (34)\end{aligned}$$

We work now at an arbitrary point z_0. We claim that in suitable normal coordinates, we can assume that at z_0,

$$\begin{aligned}(g_0)_{\overline{p}q} &= \delta_{\overline{p}q}, \qquad \nabla_j = \partial_j \\ \varphi_{\overline{p}q} &= (\delta_{\overline{p}q}) \varphi_{\overline{p}p} \qquad \text{simultaneous diagonalization}\end{aligned}$$

Thus at diagonal, $j = k$, $m = s$ and $\lambda = r$

$$\begin{aligned}g^{j\overline{k}} \mathrm{Tr}((\mathcal{D}_j h h^{-1}) \partial_{\overline{k}} h) &= \delta^{j\overline{k}} (1 + \varphi_{\overline{k}k})^{-1} \delta^{m\overline{s}} (1 + \varphi_{\overline{m}m})^{-1} \delta^{\lambda\overline{r}} \partial_k \varphi_{\overline{s}r} \partial_{\overline{k}} \varphi_{\overline{r}s} \\ &= \sum_{k,m,r} \frac{1}{(1 + \varphi_{\overline{k}k})(1 + \varphi_{\overline{m}m})} |\partial_k \varphi_{\overline{m}r}|^2 \qquad (35)\end{aligned}$$

We need to compare this expression to

$$\begin{aligned}g^{j\overline{k}} \partial_j (\mathrm{Tr}(h)) \partial_{\overline{k}} (\mathrm{Tr}(h)) &= \sum \frac{1}{1 + \varphi_{\overline{k}k}} \partial_k (\sum (1 + \varphi_{\overline{m}m})) \partial_{\overline{k}} (\sum (1 + \varphi_{\overline{\ell}\ell})) \\ &= \sum_k \frac{1}{1 + \varphi_{\overline{k}k}} \sum_m \partial_k \varphi_{\overline{m}m} \sum_\ell \partial_{\overline{k}} \varphi_{\overline{\ell}\ell}\end{aligned}$$

We apply the Cauchy-Schwarz Inequality so that

$$\begin{aligned}g^{j\overline{k}} \partial_j (\mathrm{Tr}(h)) \partial_{\overline{k}} (\mathrm{Tr}(h)) &= \sum_{m\ell} (\sum_k \frac{1}{1 + \varphi_{\overline{k}k}} \partial_k \varphi_{\overline{m}m} \partial_{\overline{k}} \varphi_{\overline{\ell}\ell}) = \sum_{m,\ell,k} \left(\frac{\partial_k \varphi_{\overline{m}m}}{(1 + \varphi_{\overline{k}k})^{\frac{1}{2}}} \cdot \frac{\partial_{\overline{k}} \varphi_{\overline{\ell}\ell}}{(1 + \varphi_{\overline{k}k})^{\frac{1}{2}}} \right) \\ &\leq \sum_{m\ell} \left(\sum_k \frac{|\partial_k \varphi_{\overline{m}m}|^2}{1 + \varphi_{\overline{k}k}} \right)^{\frac{1}{2}} \left(\sum_k \frac{|\partial_{\overline{k}} \varphi_{\overline{\ell}\ell}|^2}{1 + \varphi_{\overline{k}k}} \right)^{\frac{1}{2}} \\ &\leq |\sum_m \left(\sum_k \frac{|\partial_k \varphi_{\overline{m}m}|^2}{1 + \varphi_{\overline{k}k}} \right)^{\frac{1}{2}}|^2 \\ &= |\sum_m (1 + \varphi_{\overline{m}m})^{\frac{1}{2}} \left(\sum_k \frac{|\partial_k \varphi_{\overline{m}m}|^2}{(1 + \varphi_{\overline{m}m})(1 + \varphi_{\overline{k}k})} \right)^{\frac{1}{2}}|^2 \qquad \text{make the trace of } h \text{ appear} \\ &\leq \left(\sum_m (1 + \varphi_{\overline{m}m}) \right) \sum_{k,m} \frac{|\partial_k \varphi_{\overline{m}m}|^2}{(1 + \varphi_{\overline{m}m})(1 + \varphi_{\overline{k}k})} \qquad \text{Cauchy Schwarz}\end{aligned}$$

$$\frac{g^{j\overline{k}} \partial_j (\mathrm{Tr}(h)) \partial_{\overline{k}} (\mathrm{Tr}(h))}{\mathrm{Tr}(h)} \leq \sum_{k,m} \frac{|\partial_k \varphi_{\overline{m}m}|^2}{(1 + \varphi_{\overline{m}m})(1 + \varphi_{\overline{k}k})}$$

But this is less terms than RHS of (35). □

□

4.5.3 a Priori Estimate: C

Proof of (24). Recall

$$\begin{aligned}\omega_0 &\implies \nabla \implies R_{\bar{j}k}{}^{\ell}{}_{m} \implies R_{\bar{j}k} \\ \omega = \omega_0 + i\partial\overline{\partial}\varphi &\implies \mathcal{D} \implies \mathcal{R}_{\bar{j}k}{}^{\ell}{}_{m} \implies \mathcal{R}_{\bar{j}k}\end{aligned}$$

We shall prove: Let

$$S_0 := g_0^{p\overline{q}} g^{m\overline{r}} g_0^{\lambda\overline{r}} \nabla_p \varphi_{\overline{\gamma}\lambda} \nabla_{\overline{q}} \varphi_{\overline{r}m} \sim |\nabla\overline{\nabla}\nabla\varphi|^2_{\omega_0}$$

Then

$$S_0 \leq C$$

1. Observe that $S_0 \sim S$ with
$$S = g^{p\overline{q}} g^{m\overline{r}} g^{\lambda\overline{r}} \nabla_p \varphi_{\overline{\gamma}\lambda} \nabla_{\overline{q}} \varphi_{\overline{r}m}$$
since we know that
$$c\omega_0 \leq \omega \leq C\omega_0$$
by the estimate for $\Delta_0 h$ C_2 estimate.

2. Key observation for our proof:
$$S = |\mathcal{D}hh^{-1}|^2_\omega$$

3. We claim that
$$\Delta_\omega S \geq -C_1 S - C_2 \tag{36}$$

4. We claim that for A large enough, we have
$$\Delta_\omega(S + A\text{Tr}(h)) \geq C_3 S - C_4 \tag{37}$$

Now assuming these, we have the desired estimate. Let z_1 be a point where $S + A\text{Tr}(h)$ attains its maximum. Then
$$0 \geq \Delta_\omega(S + A\text{Tr}(h))(z_1) \geq C_3 S(z_1) - C_4$$
$$S(z_1) \leq \frac{C_4}{c_3} = C_5$$

At any point z, we have
$$\begin{aligned} S(z) + A\text{Tr}(h(z)) &\leq (S + A\text{Tr}(h))(z_1) \\ &\leq C_5 + A \max \text{Tr}(h) \\ &\leq C_6 \qquad \text{by the } C_2 \text{ estimate (23)} \\ S(z) &\leq C_6 \qquad \forall\ z \in X \end{aligned}$$

Now we prove the claims.

Proof of (37). The key thing is to compute
$$\Delta_\omega(\text{Tr}(h)) \overset{(32)}{=} \Delta_0 F - R + h^{-1}{}^p_\ell R_p^{\ell}{}^r{}_s h^s_r + g^{p\overline{q}} \text{Tr}(\mathcal{D}_p h h^{-1} \partial_{\overline{q}} h)$$

Since we already know that h and h^{-1} are bounded by the C^2 estimate (23).
$$\begin{aligned} \Delta_\omega(\text{Tr}(h)) &\geq -C_7 - C_8 + g^{p\overline{q}} \text{Tr}(\mathcal{D}_p h h^{-1} \partial_{\overline{q}} h) \\ &\geq -C_9 + g^{p\overline{q}} \text{Tr}(\mathcal{D}_p h h^{-1} \partial_{\overline{q}} h) \\ &\overset{(34)}{=} -C_9 + g^{p\overline{q}} g^{m\overline{\gamma}} g_0^{\lambda\overline{r}} \nabla_p \varphi_{\overline{\gamma}\lambda} \nabla_{\overline{q}} \varphi_{\overline{r}m} \\ &\geq C_{10} S \qquad \text{since } g_0^{\lambda\overline{r}} \sim g^{\lambda\overline{r}} \end{aligned}$$

Thus
$$\begin{aligned} \Delta_\omega(S + A\text{Tr}(h)) &\geq -C_1 S - C_2 + A(-C_9 + C_{10} S) \\ &\geq (-C_1 + AC_{10}) S - C_2 - AC_9 \\ &\geq C_{12} S - C_{13} \end{aligned}$$

□

Thus we need only to show
$$\Delta_\omega S \geq -C_1 S - C_2$$

Proof of (36). Recall the notations $S = |\mathcal{D}hh^{-1}|^2$ and $h = g_0^{-1}g$. We rewrite

$$\begin{aligned}
S &= \langle \mathcal{D}hh^{-1}, \mathcal{D}hh^{-1}\rangle \\
\Delta_\omega S &= g^{p\overline{q}}\mathcal{D}_p\mathcal{D}_{\overline{q}}\langle \mathcal{D}hh^{-1}, \mathcal{D}hh^{-1}\rangle \\
&= g^{p\overline{q}}\langle \mathcal{D}_p\mathcal{D}_{\overline{q}}(\mathcal{D}hh^{-1}), \mathcal{D}hh^{-1}\rangle \\
&\quad + g^{p\overline{q}}\langle \mathcal{D}hh^{-1}, \mathcal{D}_{\overline{p}}\mathcal{D}_q(\mathcal{D}hh^{-1})\rangle \\
&\quad + g^{p\overline{q}}\langle \mathcal{D}_{\overline{q}}(\mathcal{D}hh^{-1}), \mathcal{D}_{\overline{p}}(\mathcal{D}hh^{-1})\rangle \\
&\quad + g^{p\overline{q}}\langle \mathcal{D}_p(\mathcal{D}hh^{-1}), \mathcal{D}_q(\mathcal{D}hh^{-1})\rangle \\
g^{p\overline{q}}\langle \mathcal{D}_{\overline{q}}(\mathcal{D}hh^{-1}), \mathcal{D}_{\overline{p}}(\mathcal{D}hh^{-1})\rangle &= |\overline{\mathcal{D}}(\mathcal{D}hh^{-1})|^2 \geq 0 \\
g^{p\overline{q}}\langle \mathcal{D}_p(\mathcal{D}hh^{-1}), \mathcal{D}_q(\mathcal{D}hh^{-1})\rangle &= |\mathcal{D}(\mathcal{D}hh^{-1})|^2 \geq 0 \\
\Delta_\omega S &\geq \langle g^{p\overline{q}}\mathcal{D}_p\mathcal{D}_{\overline{q}}(\mathcal{D}hh^{-1}), \mathcal{D}hh^{-1}\rangle + \langle \mathcal{D}hh^{-1}, g^{q\overline{p}}\mathcal{D}_{\overline{p}}\mathcal{D}_q(\mathcal{D}hh^{-1})\rangle
\end{aligned}$$

Let's work out the first term.

$$\begin{aligned}
g^{p\overline{q}}\mathcal{D}_p(\mathcal{D}_{\overline{q}}(\mathcal{D}_\ell hh^{-1}))^\alpha_\beta &= g^{p\overline{q}}\mathcal{D}_p(-R_{\overline{q}\ell}{}^\alpha{}_\beta + \mathcal{R}_{\overline{q}\ell}{}^\alpha{}_\beta) \\
&= -g^{p\overline{q}}\mathcal{D}_p R_{\overline{q}\ell}{}^\alpha{}_\beta + \mathcal{D}_p(g^{p\overline{q}}\mathcal{R}_{\overline{q}\ell}{}^\alpha{}_\beta)
\end{aligned}$$

We need to work a bit harder to get Ricci tensor. For this we need the second Bianchi identity. If ω is Kähler, then

$$\mathcal{D}_p(\mathcal{R}_{\overline{q}\ell}{}^\alpha{}_\beta) = \mathcal{D}_\ell\mathcal{R}_{\overline{q}p}{}^\alpha{}_\beta$$

This is only true for Kähler. In fact we can compare to the second Bianchi Identity which is valid for any connection. If we have a connection $\mathcal{D}$ on a vector bundle E, then

$$d_{\mathcal{D}}F = 0$$

i.e.

$$\mathcal{D}^*_\ell F_{\overline{q}p}{}^\alpha{}_\beta - \mathcal{D}^*_p F_{\overline{q}\ell}{}^\alpha{}_\beta = 0$$

where $\mathcal{D}^*$ is the covariant derivative in $\mathrm{End}(E)$. But in our case $\mathcal{D}$ is the covariant derivative on all the indices ℓ, p, α, β. With this second Bianchi Identity, we can write

$$\begin{aligned}
\mathcal{D}_p(g^{p\overline{q}}\mathcal{R}_{\overline{q}\ell}{}^\alpha{}_\beta) &= g^{p\overline{q}}\mathcal{D}_p\mathcal{R}_{\overline{q}\ell}{}^\alpha{}_\beta \\
&= g^{p\overline{q}}\mathcal{D}_\ell\mathcal{R}_{\overline{q}p}{}^\alpha{}_\beta \qquad \text{Second Bianchi} \\
&= \mathcal{D}_\ell(g^{p\overline{q}}\mathcal{R}_{\overline{q}p}{}^\alpha{}_\beta) \\
&= \mathcal{D}_\ell\mathcal{R}^\alpha_\beta
\end{aligned} \tag{38}$$

Now recall our Monge-Ampére Equation is precisely designed so that

$$\mathcal{R}_{\overline{\gamma}\beta} = T_{\overline{\gamma}\beta}$$

where T is a given $(1,1)$-tensor. Hence

$$\begin{aligned}
\mathcal{D}_\ell\mathcal{R}^\alpha_\beta &= \mathcal{D}_\ell(g^{\alpha\overline{\gamma}}\mathcal{R}_{\overline{\gamma}\beta}) = \mathcal{D}_\ell(g^{\alpha\overline{\gamma}}T_{\overline{\gamma}\beta}) \\
&= g^{\alpha\overline{\gamma}}\mathcal{D}_\ell T_{\overline{\gamma}\beta}
\end{aligned}$$

Furthermore, we're dealing with

$$|\mathcal{D}_\ell\mathcal{R}^\alpha_\beta| \leq C_3S^{\frac{1}{2}} + C_4 \qquad \text{Recall that } \mathcal{D}_pW - \nabla_pW = (\mathcal{D}_phh^{-1})W$$

How about the other term? The other term is of the same size.

$$|g^{p\overline{q}}\mathcal{D}_pR_{\overline{q}\ell}{}^\alpha{}_\beta| \leq C_3S^{\frac{1}{2}} + C_4 \qquad \text{use } \mathcal{D}_p - \nabla_p \sim \mathcal{D}_phh^{-1}$$

Thus by Cauchy-Schwarz

$$|\langle g^{p\overline{q}}\mathcal{D}_p(\mathcal{D}_{\overline{q}}(\mathcal{D}_\ell hh^{-1})), \mathcal{D}hh^{-1}\rangle| \leq C_5S + C_6$$

It remains to estimate the second term

$$\langle \mathcal{D}hh^{-1}, g^{p\overline{q}}\mathcal{D}_{\overline{p}}\mathcal{D}_q(\mathcal{D}hh^{-1})\rangle$$

To exploit the formula linking curvatures, we need to change the order, i.e., we permute $\mathcal{D}_{\overline{p}}$ and $\mathcal{D}_q$. Thus

$$\begin{aligned}
g^{q\overline{p}}\mathcal{D}_{\overline{p}}\mathcal{D}_q(\mathcal{D}_\ell hh^{-1})^\alpha_\beta &= g^{q\overline{p}}\mathcal{D}_q\mathcal{D}_{\overline{p}}(\mathcal{D}_\ell hh^{-1})^\alpha_\beta \\
&+ g^{q\overline{p}}\left(-\mathcal{R}_{\overline{p}q}{}^m{}_\ell(\mathcal{D}_m hh^{-1})^\alpha_\beta + \mathcal{R}_{\overline{p}q}{}^\alpha{}_\gamma(\mathcal{D}_\ell hh^{-1})^\gamma_\beta - \mathcal{R}_{\overline{p}q}{}^\gamma{}_\beta(\mathcal{D}_\ell hh^{-1})^\alpha_\gamma\right) \\
&= g^{q\overline{p}}\mathcal{D}_q\mathcal{D}_{\overline{p}}(\mathcal{D}_\ell hh^{-1})^\alpha_\beta - \mathcal{R}^m_\ell(\mathcal{D}_m hh^{-1})^\alpha_\beta + \mathcal{R}^\alpha_\gamma(\mathcal{D}_\ell hh^{-1})^\gamma_\beta - \mathcal{R}^\gamma_\beta(\mathcal{D}_\ell hh^{-1})^\alpha_\gamma \\
|g^{p\overline{q}}\mathcal{D}_{\overline{p}}\mathcal{D}_q(\mathcal{D}hh^{-1})| &\leq C_7 S^{\frac{1}{2}} + C_8 S^{\frac{1}{2}} + C_9 \\
&\leq C_{10}S^{\frac{1}{2}} + C_{11} \\
|\langle \mathcal{D}hh^{-1}, g^{p\overline{q}}\mathcal{D}_{\overline{p}}\mathcal{D}_q(\mathcal{D}hh^{-1})\rangle| &\leq C_{12}S + C_{13}
\end{aligned}$$

□

Finally we want to prove the second Bianchi identity for Kähler metric.

Proof of (38). We already know

$$\mathcal{D}^*_p(\mathcal{R}_{\overline{q}\ell}{}^{\overline{\alpha}}{}_\beta) = \mathcal{D}^*_\ell \mathcal{R}_{\overline{q}p}{}^\alpha{}_\beta$$

On the LHS and RHS this is

$$\begin{aligned}
\mathcal{D}^*_p(\mathcal{R}_{\overline{q}\ell}{}^{\overline{\alpha}}{}_\beta) &= \mathcal{D}_p\mathcal{R}_{\overline{q}\ell}{}^\alpha{}_\beta + A^m_p{}_\ell\mathcal{R}_{\overline{q}m}{}^\alpha{}_\beta \\
\mathcal{D}^*_\ell\mathcal{R}_{\overline{q}p}{}^\alpha{}_\beta &= \mathcal{D}_\ell\mathcal{R}_{\overline{q}p}{}^\alpha{}_\beta + A^m_\ell{}_p\mathcal{R}_{\overline{q}m}{}^\alpha{}_\beta
\end{aligned}$$

Thus the difference is

$$0 = (A^m_{p\ell} - A^m_{\ell p})\mathcal{R}_{\overline{q}m}{}^\alpha{}_\beta$$

for Kähler Metric. □

This concludes the proof of the whole Calabi-Yau conjecture. □

A Mid Term

We setup our discussion.

1. Let X be a compact n-dim complex manifold.
2. We write
$$X = \bigcup_{\mu} X_\mu$$
where $\{X_\mu\}$ is family of open charts as a covering of X.
3. Consider one-to-one and onto map to an open set $\Phi_\mu(X_\mu)$ in $\mathbb{C}^n$
$$\Phi_\mu : X_\mu \to \Phi_\mu(X_\mu) \subset \mathbb{C}^n \qquad z \mapsto z_\mu := (z_\mu^1, \cdots, z_\mu^n)$$
s.t. the transition functions are holomorphic
$$\Phi_\nu \circ \Phi_\mu^{-1} : \Phi_\mu(X_\mu \cap X_\nu) \subset \mathbb{C}^n \to \Phi_\nu(X_\mu \cap X_\nu) \subset \mathbb{C}^n \qquad z_\mu \mapsto z_\nu$$
with invertible Jacobian matrix
$$(\frac{\partial z_\mu^j}{\partial z_\nu^k})_{1 \le j,\, k \le n}$$
4. We define a function f on an open set $\Omega \subset X$ to be holomorphic if $f \circ \Phi_\mu^{-1}$ is a holomorphic function on $\Phi_\mu(X_\mu \cap \Omega) \subset \mathbb{C}^n$ for any μ.

A.1 Holomorphic Line Bundle

We define holomorphic line bundle and sections on the line bundle

1. Let a holomorphic line bundle $L \to X$ be specified by its transition functions $t_{\mu\nu}(z)$
$$L \leftrightarrow \{t_{\mu\nu}\}$$
that are holomorphic
   (a) Invertible $t_{\mu\nu} \neq 0$ on $X_\mu \cap X_\nu$
   (b) and satisfies cocycle condition
$$t_{\mu\nu}(z) t_{\nu\rho}(z) = t_{\mu\rho}(z) \qquad \text{on } X_\mu \cap X_\nu \cap X_\rho$$
2. A section $\varphi \in \Gamma(X, L)$ is defined by a collection of function $\varphi_\mu(z_\mu)$ defined on $\Phi_\mu(X_\mu)$
$$\varphi \in \Gamma(X, L) \leftrightarrow \{\varphi_\mu(z_\mu)\}$$
which satisfies the gluing condition
$$\varphi_\mu(z_\mu) = t_{\mu\nu}(z)\varphi_\nu(z_\nu) \qquad \forall\ z \in X_\mu \cap X_\nu$$
For convenience we drop μ for $\varphi_\mu(z_\mu)$ and write $\varphi(z)$.

Problem A.1. *What is a metric h on L?*

Answer A.1. *A metric h on L is a section of $L^{-1} \otimes \overline{L}^{-1}$ satisfying*
$$h(z) > 0 \qquad \forall\ z$$
The transition functions of L^{-1} are $t_{\mu\nu}(z)^{-1}$ and those for $\overline{L}^{-1}$ are $\overline{t_{\mu\nu}(z)}^{-1}$. Hence gluing condition satisfies
$$h_\mu(z_\mu) = t_{\mu\nu}(z)^{-1}\overline{t_{\mu\nu}(z)}^{-1} h_\nu(z_\nu)$$
$$h_\mu(z_\mu) = |t_{\mu\nu}(z)|^{-2} h_\nu(z_\nu) > 0 \qquad \forall\ z \in X_\mu \cap X_\nu$$

Problem A.2. *Fix a metric h and let ∇ be the corresponding Chern unitary connection on L. What are the explicit formulas for $\nabla_j\varphi$ and $\nabla_{\overline{k}}\varphi$?*

Answer A.2. *Define the Chern unitary connection(covariant derivative) on L in the $\overline{\partial}$-drection*

$$\nabla_{\overline{k}}\varphi := \partial_{\overline{k}}\varphi \in \Gamma(X, L\otimes\Lambda^{0,1}) \qquad \forall\ \varphi\in\Gamma(X,L)$$

Define in the ∂-direction

$$\begin{aligned}\nabla_j\varphi &:= h^{-1}\partial_j(h\varphi) \in \Gamma(X, L\otimes\Lambda^{1,0}) \qquad \forall\ \varphi\in\Gamma(X,L)\\ &= \partial_j\varphi + (\partial_j(\log(h)))\varphi\end{aligned}$$

Problem A.3. *Show that the commutator of ∇_j and $\nabla_{\overline{k}}$ is of the form*

$$[\nabla_j, \nabla_{\overline{k}}]\varphi = F_{\overline{k}j}\varphi \qquad \forall\ \varphi\in\Gamma(X,L)$$

and determine explicitly $F_{\overline{k}j}$.

Answer A.3.

$$\begin{aligned}[\nabla_j, \nabla_{\overline{k}}]\varphi &= \nabla_j\nabla_{\overline{k}}\varphi - \nabla_{\overline{k}}\nabla_j\varphi\\ &= h^{-1}\partial_j(h(\partial_{\overline{k}}\varphi)) - \partial_{\overline{k}}(h^{-1}\partial_j(h\varphi))\\ &= h^{-1}\left((\partial_j h)(\partial_{\overline{k}}\varphi) + h\partial_j\partial_{\overline{k}}\varphi\right) - \partial_{\overline{k}}\left(h^{-1}(\partial_j h)\varphi + \partial_j\varphi\right)\\ &= h^{-1}(\partial_j h)\partial_{\overline{k}}\varphi - \partial_j\partial_{\overline{k}}\varphi - \partial_{\overline{k}}(h^{-1}\partial_j h)\varphi - h^{-1}\partial_j h\partial_{\overline{k}}\varphi - \partial_{\overline{k}}\partial_j\varphi\\ &= h^{-1}(\partial_j h)\partial_{\overline{k}}\varphi - (\partial_{\overline{k}}(h^{-1}\partial_j h)\varphi + h^{-1}\partial_j h\partial_{\overline{k}}\varphi)\\ &= -\partial_{\overline{k}}(h^{-1}\partial_j h)\varphi\\ &= -(\partial_j\partial_{\overline{k}}(\log(h)))\varphi\end{aligned}$$

Hence

$$F_{\overline{k}j} := -(\partial_j\partial_{\overline{k}}(\log(h)))$$

Problem A.4. *Let the curvature form F be define*

$$F := \sum_{j,k=1}^{n} F_{\overline{k}j}dz^j\wedge d\overline{z}^k = -\sum_{j,k=1}^{n}\partial_j\partial_{\overline{k}}(\log(h))\varphi$$

Then F is a closed form, and its de Rham cohomology class $[\,F\,]$ is independent of the choice of the metric h.

Answer A.4. *We can write*

$$\begin{aligned}F &= -\sum_{j,k}\frac{\partial}{\partial z^j}\left(\frac{\partial}{\partial\overline{z}^k}\log(h)\right)dz^j\wedge d\overline{z}^k\\ &= -\sum_j dz^j\frac{\partial}{\partial z^j}\left(\sum_k\frac{\partial}{\partial\overline{z}^k}\log(h)\right)\wedge d\overline{z}^k\\ &= -\partial\overline{\partial}\log(h)\end{aligned}$$

Now F is readily seen to be closed.

$$\begin{aligned}dF &= -(\partial+\overline{\partial})\partial\overline{\partial}\log(h)\\ &= -(\partial^2\overline{\partial} + \overline{\partial}\partial\overline{\partial})\log(h)\\ &= \partial\overline{\partial}\log(h) = \partial^2\log(h) = 0\end{aligned}$$

Before moving on, we recall the definition of de Rham cohomology. Let F be a p-form which is closed, i.e., $dF = 0$. Then

$$[F]_{\mathrm{dR}} := F/\{\text{exact forms } d\psi \text{ where } \psi\in\Lambda^{p-1}\}$$

is defined as the de Rham cohomology class of F. Now we show this object is independent of the metric h. Let h and h' be two metrics on L and let F, F' be two corresponding curvatures, i.e.

$$F_{\overline{k}j} = -\partial_j\partial_{\overline{k}}\log(h) \qquad F'_{\overline{k}j} = -\partial_j\partial_{\overline{k}}\log(h')$$

However

$$\begin{aligned}F_{\overline{k}j} - F'_{\overline{k}j} &= -\partial_j\partial_{\overline{k}}\log(h) + \partial_j\partial_{\overline{k}}\log(h')\\ &= -\partial_j\partial_{\overline{k}}\log(\frac{h}{h'})\end{aligned}$$

But $\frac{h}{h'}$ is strictly positive C^∞ function since

$$h \in L^{-1} \otimes \overline{L}^{-1}, \qquad h' \in L^{-1} \otimes \overline{L}^{-1} \implies \frac{h}{h'} \in \mathbb{1} \implies C^\infty \text{ function} > 0$$

Then due to positivity, say

$$\frac{h}{h'} = e^\phi \qquad \text{for certain } \phi \in C^\infty$$

Now

$$\begin{aligned} F_{\overline{k}j} - F'_{\overline{k}j} &= -\partial_j \partial_{\overline{k}} \phi \\ (F_{\overline{k}j} - F'_{\overline{k}j}) dz^j \wedge d\overline{z}^k &= -\partial_j \partial_{\overline{k}} \phi dz^j \wedge d\overline{z}^k \\ &= -\partial \overline{\partial} \phi = -(\partial + \overline{\partial}) \overline{\partial} \phi \\ &= -d(\overline{\partial}\phi) \qquad \text{exact form} \end{aligned}$$

Hence due to quotient and $\overline{\partial}\phi$ as 1-form, we conclude

$$[F]_{\mathrm{dR}} = [F']_{\mathrm{dR}}$$

A.2 Holomorphic Vector Bundle

Set $r \in \mathbb{Z}^+$.

1. Define holomorphic transition function ${t_{\mu\nu}}^\alpha{}_\beta(z)$ for $1 \le \alpha, \beta \le r$ on $X_\mu \cap X_\nu$ s.t.
   (a) ${t_{\mu\nu}}^\alpha{}_\beta(z) = \delta^\alpha_\beta$ if $\mu = \nu$
   (b) and satisfies the cocycle condition

   $${t_{\mu\nu}}^\alpha{}_\beta(z) {t_{\nu\rho}}^\beta{}_\gamma(z) = {t_{\mu\rho}}^\alpha{}_\gamma(z) \qquad \forall\ z \in X_\mu \cap X_\nu \cap X_\rho$$

   Notice a necessary condition for co-cycle is $(t_{\mu\nu}) = (t_{\nu\mu})^{-1}$ as inverse matrices.
2. Define the holomorphic vector bundle $E \to X$ by its space of sections $\varphi \in \Gamma(X, E)$, whose elements are characterized by vector-valued functions $\varphi^\alpha_\mu(z_\mu)$ on $\Phi_\mu(X_\mu)$ for any $1 \le \alpha \le r$ s.t. the gluing rule holds

   $$\varphi^\alpha_\mu(z_\mu) = {t_{\mu\nu}}^\alpha{}_\beta(z) \varphi^\beta_\nu(z_\nu) \qquad \forall\ z \in X_\mu \cap X_\nu$$

Problem A.5. *Let H be metric on E defined on each $\Phi_\mu(X_\mu)$ s.t. it is a positive-definite Hermitian matrix $(H_\mu)_{\overline{\beta}\alpha}(z_\mu)$ and for any $\varphi \in \Gamma(X, E)$*

$$\overline{\varphi^\beta_\mu} (H_\mu)_{\overline{\beta}\alpha} \varphi^\alpha_\mu$$

transforms like a scalar, i.e., is invariant under $\mu \mapsto \nu$. Define the Chern Unitary Connection ∇ on E w.r.t. H. Give explicit formulas for ∇, both in terms of indices, and in terms of matrices.

Answer A.5. *Define ∇ s.t. in the $\overline{\partial}$-direction*

$$\nabla_{\overline{k}} \varphi^\alpha = \partial_{\overline{k}} \varphi^\alpha \in \Gamma(X, E \otimes \Lambda^{0,1}) \qquad \forall\ 1 \le \alpha \le r, \qquad \forall\ \varphi \in \Gamma(X, E)$$

Define in ∂-direction

$$\nabla_j \varphi^\alpha := H^{\alpha\overline{\gamma}} \partial_j (H_{\overline{\gamma}\beta} \varphi^\beta) \in \Gamma(X, E \otimes \Lambda^{1,0}) \qquad \forall\ 1 \le \alpha \le r, \qquad \forall\ \varphi \in \Gamma(X, E)$$

Using Einstein summation convention, in components

$$(\nabla_j \varphi)^\alpha = (H^{-1} \partial_j (H\varphi))^\alpha$$

and in matrix notation

$$\nabla_j \varphi = H^{-1} \partial_j (H\varphi)$$

Problem A.6. *Show that the commutator of ∇_j and $\nabla_{\overline{k}}$ is of the form*

$$[\nabla_j, \nabla_{\overline{k}}] \varphi^\alpha = {F_{\overline{k}j}}^\alpha{}_\beta \varphi^\beta \qquad \forall\ \varphi \in \Gamma(X, E)$$

and give explicit expressions for ${F_{\overline{k}j}}^\alpha{}_\beta$ both in terms of indices and in terms of matrices.

Answer A.6. *We compute*

$$\begin{aligned}
[\nabla_j, \nabla_{\overline{k}}]\varphi^\alpha &= \nabla_j \nabla_{\overline{k}} \varphi - \nabla_{\overline{k}}(\nabla_j \varphi) \\
&= H^{-1}\partial_j(H\partial_{\overline{k}}\varphi) - \partial_{\overline{k}}(H^{-1}\partial_j(H\varphi)) \\
&= H^{-1}\partial_j(H\partial_{\overline{k}}\varphi) - \partial_{\overline{k}}(H^{-1}\left((\partial_j H)\varphi + H\partial_j\varphi\right)) \\
&= (H^{-1}\partial_j H)\partial_{\overline{k}}\varphi + H^{-1}H\partial_j\partial_{\overline{k}}\varphi - H^{-1}H\partial_{\overline{k}}\partial_j\varphi - \partial_{\overline{k}}(H^{-1}\partial_j H\varphi) \\
&= -\{\partial_{\overline{k}}(H^{-1}\partial_j H)\}\varphi^\alpha \\
&=: F_{\overline{k}j}{}^{\alpha}{}_{\beta}\varphi^\beta \qquad \textit{in components} \\
[\nabla_j, \nabla_{\overline{k}}]\varphi &= F_{\overline{k}j}\varphi \qquad \textit{in matrix notation}
\end{aligned}$$

where we define $F \in \Gamma(X, \Lambda^{1,1} \otimes \mathrm{End}(E))$ as

$$\begin{aligned}
F_{\overline{k}j}{}^{\alpha}{}_{\beta} &:= -\partial_{\overline{k}}(H^{\alpha\overline{\gamma}}\partial_j H_{\overline{\gamma}\beta}) \qquad \textit{in components} \\
F_{\overline{k}j} &:= -\partial_{\overline{k}}(H^{-1}\partial_j H) \qquad \textit{in matrix notation}
\end{aligned}$$

Problem A.7. *If j is an unbarred index, set*

$$A^\alpha_{j\beta} := H^{\alpha\overline{\gamma}}\partial_j H_{\overline{\gamma}\beta}$$

where $H^{\alpha\overline{\gamma}}$ is the inverse matrix of $H_{\overline{\gamma}\beta}$, i.e.

$$H^{\alpha\overline{\gamma}}H_{\overline{\gamma}\beta} = \delta^\alpha_\beta$$

Otherwise set $A^\alpha_{j\beta} = 0$. We also define the connection matrix

$$A_j := A^\alpha_{j\beta}$$

and the connection form

$$A := \sum_{j=1}^{n} A_j dz^j$$

Show that the connection ∇ can be expressed as

$$\nabla\varphi = d\varphi + A\varphi$$

Answer A.7. *For any $1 \le \alpha \le r$*

$$\begin{aligned}
\nabla_{\overline{k}}\varphi^\alpha &= \partial_{\overline{k}}\varphi^\alpha \\
\nabla_j\varphi^\alpha &= H^{\alpha\overline{\gamma}}\partial_j(H_{\overline{\gamma}\beta}\varphi^\beta) \\
&= H^{\alpha\overline{\gamma}}\left(\partial_j(H_{\overline{\gamma}\beta})\varphi^\beta + H_{\overline{\gamma}\beta}\partial_j\varphi^\beta\right) \\
&= (H^{\alpha\overline{\gamma}}\partial_j H_{\overline{\gamma}\beta})\varphi^\beta + \delta^\alpha_\beta\partial_j\varphi^\beta \\
&= (H^{\alpha\overline{\gamma}}\partial_j H_{\overline{\gamma}\beta})\varphi^\beta + \partial_j\varphi^\alpha \\
\implies \nabla_\ell\varphi^\alpha &= \partial_\ell\varphi^\alpha + A^\alpha_{\ell\beta}\varphi^\beta \qquad \forall\ \ell = j, \overline{k} \qquad \ell, k = 1, \cdots, n \\
\nabla\varphi &= d\varphi + A\varphi \qquad \textit{in matrix notation}
\end{aligned}$$

Given that we've defined

$$\begin{aligned}
A^\alpha_{\overline{k}\beta} &= 0 \qquad \ell = \overline{k} \textit{ in the } \overline{\partial}\textit{-direction} \\
A^\alpha_{j\beta} &= H^{\alpha\overline{\gamma}}\partial_j H_{\overline{\gamma}\beta} \\
&= H^{-1}\partial_j H \qquad \ell = j \textit{ in the } \partial\textit{-direction}
\end{aligned}$$

Problem A.8. *Let the curvature form*

$$F := \sum_{j,k=1}^{n} F_{\overline{k}j} dz^j \wedge d\overline{z}^k$$

Show that

$$F = dA + A \wedge A$$

Deduce the second Bianchi Identity

$$dF + A \wedge F - F \wedge A = 0$$

Answer A.8. *1. We compute the RHS.*

$$\begin{aligned}
dA &= d(\sum_j dz^j A_j) \qquad \textit{later we drop summation in } j \\
&= (\overline{\partial} + \partial)(dz^j A_j) \\
&= d\overline{z}^k(\partial_{\overline{k}} A_j) \wedge dz^j + dz^k(\partial_k A_j) \wedge dz^j \\
&= -\partial_{\overline{k}} A_j dz^j \wedge d\overline{z}^k + dz^k(\partial_k A_j) \wedge dz^j \\
&= F_{\overline{k}j} dz^j \wedge d\overline{z}^k + (\partial_k A_j) dz^k \wedge dz^j
\end{aligned}$$

Now

$$\partial_k A_j = \partial_k(H^{-1}\partial_j H) = (\partial_k(H^{-1}))\partial_j H + H^{-1}\partial_k\partial_j H$$

Notice

$$\partial_k(H^{-1}) = -H^{-1}\partial_j H H^{-1}$$

To check the claim we note

$$\begin{aligned}
H^{-1}H &= 1 \\
\partial_k(H^{-1}H) &= 0 \\
\partial_k(H^{-1})H + H^{-1}\partial_k H &= 0 \\
\partial_k(H^{-1}) &= -H^{-1}\partial_k H H^{-1}
\end{aligned}$$

Thus

$$\begin{aligned}
\partial_k A_j &= -H^{-1}(\partial_k H)H^{-1}\partial_j H + H^{-1}\partial_k\partial_j H \\
\partial_k A_j dz^k \wedge dz^j &= -(H^{-1}\partial_k H)dz^k \wedge (H^{-1}\partial_j H)dz^j + H^{-1}(\partial_k\partial_j H)dz^k \wedge dz^j
\end{aligned}$$

But the last term is 0 *due to anti-commute. Hence*

$$\begin{aligned}
\partial_k A_j dz^k \wedge dz^j &= -A_k dz^k \wedge A_j dz^j \\
&= -A \wedge A \qquad \textit{in matrix notation}
\end{aligned}$$

As a summary

$$\begin{aligned}
dA &= F - A \wedge A \\
F &= dA + A \wedge A
\end{aligned}$$

2. Now we deduce the second Bianchi Identity. We compute

$$\begin{aligned}
dF &= d(dA + A \wedge A) \\
&= 0 + d(A \wedge A) \\
&= (dA) \wedge A + (-1)A \wedge dA \\
&= (dA + A \wedge A) \wedge A - A \wedge (dA + A \wedge A) \\
&= F \wedge A - A \wedge F
\end{aligned}$$

A.3 Dual Bundle

We consider the dual bundle E_* defined by requirements that its sections are given on $\Phi_\mu(z_\mu)$ by vector-valued functions $(\psi_\mu)_\alpha(z_\mu)$ for $1 \leq \alpha \leq r$ that satisfies the condition that

$$(\psi_\mu)_\alpha(z_\mu)\varphi_\mu^\alpha(z_\mu)$$

is a scalar, i.e., it is invariant under $\mu \mapsto \nu$.

Problem A.9. *Show that there exists a unique connection on E_*, denoted by ∇ for simplicity, which satisfies Leibniz's rule*

$$\partial_\ell(\psi_\alpha\varphi^\alpha) = (\nabla_\ell\psi)_\alpha\varphi^\alpha + \psi_\alpha(\nabla_\ell\varphi)^\alpha \qquad \forall\ \psi \in \Gamma(X, E_*) \qquad \forall\ \varphi \in \Gamma(X, E)$$

and ∇ is given by

$$\begin{aligned}
\nabla_\ell\psi_\alpha &= \partial_\ell\psi_\alpha - \psi_\beta A_{\ell\alpha}^\beta \qquad \textit{in components} \\
\nabla\psi &= d\psi - \psi A \qquad \textit{in matrix notation}
\end{aligned}$$

Answer A.9. *Assume there exists ∇ s.t. Leibniz rule holds, then ∇ necessarily satisfies*

$$\begin{aligned}(\partial_\ell\psi_\alpha)\varphi^\alpha + \psi_\alpha(\partial_\ell\varphi^\alpha) &= (\nabla_\ell\psi_\alpha)\varphi^\alpha + \psi_\alpha(\partial_\ell\varphi^\alpha + A^\alpha_{\ell\beta}\varphi^\beta)\\ (\partial_\ell\psi_\alpha)\varphi^\alpha &= (\nabla_\ell\psi_\alpha)\varphi^\alpha + \psi_\alpha A^\alpha_{\ell\beta}\varphi^\beta\\ &= (\nabla_\ell\psi_\alpha)\varphi^\alpha + \psi_\beta A^\beta_{\ell\alpha}\varphi^\alpha \qquad \textit{relabelling}\\ \partial_\ell\psi_\alpha &= \nabla_\ell\psi_\alpha + \psi_\beta A^\beta_{\ell\alpha}\\ \nabla_\ell\psi_\alpha &:= \partial_\ell\psi_\alpha - \psi_\beta A^\beta_{\ell\alpha} \qquad \textit{as connection on } E_*\end{aligned}$$

In particular

$$\begin{aligned}\nabla_j\psi_\alpha &:= \partial_j\psi_\alpha - \psi_\beta(H^{\beta\overline{\gamma}}\partial_j H_{\overline{\gamma}\alpha}) \qquad \forall\ j = 1,\cdots,n\\ \nabla_{\overline{k}}\psi_\alpha &:= \partial_{\overline{k}}\psi_\alpha \qquad \forall\ k = 1,\cdots,n\end{aligned}$$

Hence

$$\begin{aligned}\nabla_\ell\psi &= \partial_\ell\psi - \psi A_\ell \qquad \textit{in components}\\ \nabla\psi &= d\psi - \psi A \qquad \textit{in matrix notation}\end{aligned}$$

Recalling A vanishes in the ∂-direction.

Problem A.10. *Show that the commutator of ∇_j and $\nabla_{\overline{k}}$ on E_* are given by*

$$\begin{aligned}[\nabla_j, \nabla_{\overline{k}}]\psi_\alpha &= -\psi_\beta F_{\overline{k}j}{}^\beta{}_\alpha \qquad \textit{in components}\\ [\nabla_j, \nabla_{\overline{k}}]\psi &= -\psi F_{\overline{k}j} \qquad \textit{in matrix notation}\end{aligned}$$

where $F_{\overline{k}j}{}^\alpha{}_\beta$ is the curvature of E.

Answer A.10. *We compute*

$$\begin{aligned}[\nabla_j, \nabla_{\overline{k}}]\psi_\alpha &= \nabla_j\nabla_{\overline{k}}\psi_\alpha - \nabla_{\overline{k}}\nabla_j\psi_\alpha\\ &= \partial_j\partial_{\overline{k}}\psi_\alpha - \partial_{\overline{k}}\psi_\beta(H^{\beta\overline{\gamma}}\partial_j H_{\overline{\gamma}\alpha}) - \partial_{\overline{k}}\partial_j\psi_\alpha + (\partial_{\overline{k}}\psi_\beta)(H^{\beta\overline{\gamma}}\partial_j H_{\overline{\gamma}\alpha}) + \psi_\beta\partial_{\overline{k}}(H^{\beta\overline{\gamma}}\partial_j H_{\overline{\gamma}\alpha})\\ &= -\psi_\beta(-\partial_{\overline{k}}(H^{\beta\overline{\gamma}}\partial_j H_{\overline{\gamma}\alpha}))\\ &= -\psi_\beta F_{\overline{k}j}{}^\beta{}_\alpha \qquad \textit{in components}\\ [\nabla_j, \nabla_{\overline{k}}]\psi &= -\psi F_{\overline{k}j} \qquad \textit{in matrix notation}\end{aligned}$$

Problem A.11. *Check that the notion of metric H on E can be given by a simple equivalent definition in terms of E_*: a metric H on E is a section of the bundle $E_*\otimes\overline{E_*}$ satisfying the condition that the Hermitian form on $\Gamma(X,E)$ defined by*

$$\varphi \in \Gamma(X,E) \mapsto \overline{\varphi}H\varphi \qquad \textit{is positive definite} \tag{39}$$

Answer A.11. *Assume $H \in \Gamma(X, E_*\otimes\overline{E_*})$ s.t. (39) holds. Then by gluing rule, for $t_{\mu\nu}{}^\alpha{}_\beta$ the transition functions on $E \to X$*

$$\begin{aligned}(H_\mu)_{\overline{\beta}\alpha} &= (t_{\mu\nu}{}^{-1})^\alpha{}_\delta(\overline{t_{\mu\nu}}^{-1})^\beta{}_\gamma(H_\nu)_{\overline{\gamma}\delta}\\ \overline{t_{\mu\nu}}^\beta{}_\gamma(H_\mu)_{\overline{\beta}\alpha}t_{\mu\nu}{}^\alpha{}_\delta &= (H_\nu)_{\overline{\gamma}\delta} \qquad \textit{using positive-definiteness of } t_{\mu\nu}{}^\alpha{}_\beta\\ \overline{t_{\mu\nu}}^\beta{}_\gamma\overline{\varphi_\nu^\gamma}(z_\nu)(H_\mu)_{\overline{\beta}\alpha}t_{\mu\nu}{}^\alpha{}_\delta\varphi_\nu^\delta(z_\nu) &= \overline{\varphi_\nu^\gamma}(z_\nu)(H_\nu)_{\overline{\gamma}\delta}\varphi_\nu^\delta(z_\nu)\\ \overline{\varphi_\mu^\beta}(z_\mu)(H_\mu)_{\overline{\beta}\alpha}\varphi_\mu^\alpha(z_\mu) &= \overline{\varphi_\nu^\gamma}(z_\nu)(H_\nu)_{\overline{\gamma}\delta}\varphi_\nu^\delta(z_\nu) \qquad \textit{on } X_\mu\cap X_\nu\end{aligned}$$

Hence

$$\overline{\varphi^\beta}H_{\overline{\beta}\alpha}\varphi^\alpha \qquad \textit{transforms as a scalar} \qquad \forall\ \varphi\in\Gamma(X,E)$$

On the other hand, (39) is important to go backwards, as we need to take inverse of $\overline{\varphi_\nu^\gamma}(z_\nu)(H_\nu)_{\overline{\gamma}\delta}\varphi_\nu^\delta(z_\nu)$ on both sides, which only makes sense when $\overline{\varphi}H\varphi$ is positive definite. Then conclude using characterization of sections in $\Gamma(X, E_*\otimes\overline{E_*})$ using transition functions of the form $t_{\mu\nu}^{-1}\otimes\overline{t_{\mu\nu}^{-1}}$.

A.4 Kähler Geometry

We now apply the set-up to case of $E = T^{1,0}X$, whose sections are given in local coordinates as

$$\varphi = \sum_{p=1}^{n} \varphi^p(z) \frac{\partial}{\partial z^p}$$

1. A metric on $T^{1,0}(X)$ is denoted by $g_{\overline{p}q}(z)$
2. The curvature of its Chern Unitary Connection is denoted as $R_{\overline{k}j}{}^{m}{}_{q}$

$$R_{\overline{k}j}{}^{m}{}_{q} := -\partial_{\overline{k}}(g^{m\overline{p}}\partial_j g_{\overline{p}q})$$

3. Given a metric $g_{\overline{p}q}$ we define its symplectic form ω as

$$\omega := i \sum_{p,q=1}^{n} g_{\overline{p}q} dz^q \wedge d\overline{z}^p$$

Problem A.12. *Define the notion of Kähler metric.*

Answer A.12. *For connection form A defined via*

$$\nabla_j \varphi^\alpha = \partial_j \varphi^\alpha + A^\alpha_{j\beta}\varphi^\beta$$

In particular

$$\begin{aligned} A^\alpha_{j\beta} &= (H^{-1}\partial_j H)^\alpha{}_\beta = H^{\alpha\overline{\gamma}}\partial_j H_{\overline{\gamma}\beta} \\ A^\alpha_{\overline{k}\beta} &= 0 \qquad \textit{no correction in } \overline{\partial}\textit{-direction} \end{aligned}$$

we define the torsion-tensor as

$$T^\ell_{jp} := A^\ell_{jp} - A^\ell_{pj}$$

Now a metric $H_{\overline{k}j}$ on $T^{1,0}(X)$ is said to be Kähler if

$$T^\ell_{jp} = 0 \qquad i.e. \qquad A^\ell_{jp} = A^\ell_{pj}$$

In particular, we introduce an important characterisation of Kähler metric. $g_{\overline{k}j}$ is Kähler iff

$$d\omega = 0$$

i.e.

$$\partial_\ell g_{\overline{k}j} = \partial_j g_{\overline{k}\ell} \tag{40}$$

To see this, it suffices to just compute.

$$\begin{aligned} d\omega &= id(g_{\overline{k}j}dz^j \wedge d\overline{z}^k) \\ &= i(dg_{\overline{k}j} \wedge dz^j \wedge d\overline{z}^k) \\ &= i\left(\frac{\partial}{\partial z^\ell} g_{\overline{k}j} dz^\ell + \frac{\partial}{\partial \overline{z}^\ell} g_{\overline{k}j} d\overline{z}^\ell\right) \wedge dz^j \wedge d\overline{z}^k \\ &= \frac{i}{2}\left((\frac{\partial}{\partial z^\ell} g_{\overline{k}j} - \frac{\partial}{\partial z^j} g_{\overline{k}\ell}) dz^\ell \wedge dz^j \wedge d\overline{z}^k + (\frac{\partial}{\partial \overline{z}^\ell} g_{\overline{k}j} - \frac{\partial}{\partial \overline{z}^k} g_{\overline{\ell}j}) d\overline{z}^\ell \wedge dz^j \wedge d\overline{z}^k \right) \end{aligned}$$

Hence $d\omega = 0$ implies both

$$\begin{aligned} \frac{\partial}{\partial z^\ell} g_{\overline{k}j} - \frac{\partial}{\partial z^j} g_{\overline{k}\ell} = 0 \\ \frac{\partial}{\partial \overline{z}^\ell} g_{\overline{k}j} - \frac{\partial}{\partial \overline{z}^k} g_{\overline{\ell}j} = 0 \end{aligned}$$

We observe now that these are exactly the same as

$$T^\ell_{jp} = 0$$

So indeed

$$A^\ell_{jp} = A^\ell_{pj} \implies g^{\ell\overline{m}}\partial_j g_{\overline{m}p} = g^{\ell\overline{m}}\partial_p g_{\overline{m}j}$$

Problem A.13 (First Bianchi Identity)**.** *Assume the metric $g_{\overline{p}q}$ is Kähler, and set*

$$R_{\overline{k}j\overline{m}q} := g_{\overline{m}p} R_{\overline{k}j}{}^{p}{}_{q}$$

Establish the identities

$$R_{\overline{k}j\overline{m}q} = R_{\overline{m}j\overline{k}q} = R_{\overline{m}q\overline{k}j}$$

Answer A.13. *We compute*

$$\begin{aligned} R_{\overline{k}j\overline{m}q} &= g_{\overline{m}\ell}(-\partial_{\overline{k}}(g^{\ell\overline{p}}\partial_j g_{\overline{p}q})) \\ &= -g_{\overline{m}\ell}\left(\partial_{\overline{k}}(g^{\ell\overline{p}})\partial_j g_{\overline{p}q} + g^{\ell\overline{p}}\partial_{\overline{k}}\partial_j g_{\overline{p}q}\right) \end{aligned}$$

Notice we have formula

$$\partial_{\overline{k}}(g^{\ell\overline{p}}) = -g^{\ell\overline{r}}(\partial_{\overline{k}} g_{\overline{r}s}) g^{s\overline{p}}$$

In matrix notation this is

$$\partial_{\overline{k}}(G^{-1}) = -G^{-1}(\partial_{\overline{k}} G)G^{-1}$$

To see this we know

$$\begin{aligned} G^{-1}G &= I \\ \partial_{\overline{k}}(G^{-1})G + G^{-1}\partial_{\overline{k}} G &= 0 \\ \partial_{\overline{k}}(G^{-1}) &= -G^{-1}\partial_{\overline{k}} G G^{-1} \end{aligned}$$

Hence we use this formula and substitute to above.

$$\begin{aligned} R_{\overline{k}j\overline{m}q} &= g_{\overline{m}\ell} g^{\ell\overline{r}}(\partial_{\overline{k}} g_{\overline{r}s}) g^{s\overline{p}}\partial_j g_{\overline{p}q} - g_{\overline{m}\ell} g^{\ell\overline{p}}\partial_{\overline{k}}\partial_j g_{\overline{p}q} \\ &= g_{\overline{m}\ell} g^{\ell\overline{r}}(\partial_{\overline{k}} g_{\overline{r}s}) g^{s\overline{p}}\partial_j g_{\overline{p}q} - \partial_{\overline{k}}\partial_j g_{\overline{m}q} \qquad \textit{since } g_{\overline{m}\ell} g^{\ell\overline{p}} = \delta^{\overline{p}}_{\overline{m}} \end{aligned}$$

Now to interchange indices, using again that $g_{\overline{m}\ell} g^{\ell\overline{r}} = \delta^{\overline{r}}_{\overline{m}}$ one has

$$R_{\overline{k}j\overline{m}q} = (\partial_{\overline{k}} g_{\overline{m}s}) g^{s\overline{p}}\partial_j g_{\overline{p}q} - \partial_{\overline{k}}\partial_j g_{\overline{m}q}$$

One may indeed interchange $\overline{k}$ and $\overline{m}$, and j and q using the Kähler property (40) hence

$$\begin{aligned} R_{\overline{k}j\overline{m}q} &= (\partial_{\overline{k}} g_{\overline{m}s}) g^{s\overline{p}}\partial_j g_{\overline{p}q} - \partial_{\overline{k}}\partial_j g_{\overline{m}q} \\ &= (\partial_{\overline{m}} g_{\overline{k}s}) g^{s\overline{p}}\partial_j g_{\overline{p}q} - \partial_{\overline{m}}\partial_j g_{\overline{k}q} = R_{\overline{m}j\overline{k}q} \\ &= (\partial_{\overline{m}} g_{\overline{k}s}) g^{s\overline{p}}\partial_q g_{\overline{p}j} - \partial_{\overline{m}}\partial_q g_{\overline{k}j} = R_{\overline{m}q\overline{k}j} \end{aligned}$$

Problem A.14. *In what follows we need the following simple algebraic identity. Let*

$$M = i\sum_{p,q=1}^{n} M_{\overline{p}q} dz^q \wedge d\overline{z}^p$$

be a Hermitian $(1,1)$-form, i.e., satisfying

$$\overline{M}_{\overline{p}q} = M_{\overline{q}p}$$

Then

$$M \wedge \frac{\omega^{n-1}}{(n-1)!} = \mathrm{Tr}(M)\frac{\omega^n}{n!} \tag{41}$$

where

$$\mathrm{Tr}(M) := g^{q\overline{p}} M_{\overline{p}q}$$

Prove this identity.

Answer A.14. *To check this, assume that both are diagonal, i.e.*

$$M = i\sum_{\ell} M_{\overline{\ell}\ell} dz^\ell \wedge d\overline{z}^\ell$$

and

$$\omega = i\sum_{k} g_{\overline{k}k} dz^k \wedge d\overline{z}^k$$

This uses the fact that a Hermitian metric ω and a Hermitian form M can always be simultaneously diagonalized. Then

$$\begin{aligned}
M \wedge \frac{\omega^{n-1}}{(n-1)!} &= (i\sum_{\ell} M_{\bar{\ell}\ell} dz^{\ell} \wedge d\bar{z}^{\ell}) \wedge (i\sum_{k_1} g_{\overline{k_1}k_1} dz^{k_1} \wedge d\bar{z}^{k_1}) \wedge \cdots \wedge (i\sum_{k_{n-1}} g_{\overline{k_{n-1}}k_{n-1}} dz^{k_{n-1}} \wedge d\bar{z}^{k_{n-1}}) \\
&= i^n \sum_{\ell} M_{\bar{\ell}\ell} (\prod_{p\neq\ell} g_{\bar{p}p})(dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n) \\
&= i^n \sum_{\ell} (g^{-1}_{\bar{\ell}\ell}) M_{\bar{\ell}\ell} (\prod_{p} g_{\bar{p}p})(dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n) \\
&= g^{m\bar{\ell}} M_{\bar{\ell}m} \frac{\omega^n}{n!} = \mathrm{Tr}(M)\frac{\omega^n}{n!}
\end{aligned}$$

Problem A.15. *Letting $M_{\bar{p}q} = \delta g_{\bar{p}q}$ where $\delta g_{\bar{p}q}$ is an arbitrary Hermitian variation of the metric. Deduce that*

$$\delta \log(\omega^n) = \mathrm{Tr}(\delta g) = g^{q\bar{p}} \delta g_{\bar{p}q}$$

and hence for any partial derivative ∂_ℓ

$$\partial_\ell \log(\omega^n) = g^{q\bar{p}} \partial_\ell g_{\bar{p}q}$$

Answer A.15. *The variation writes*

$$\begin{aligned}
\delta \log(\omega^n) = \frac{\delta(\omega^n)}{\omega^n} &= \frac{\delta(\omega \wedge \cdots \wedge \omega)}{\omega^n} \\
&= \frac{1}{\omega^n}((\delta\omega \wedge \cdots \wedge \omega) + \cdots + (\omega \wedge \cdots \wedge \delta\omega)) \\
&= n\frac{\delta\omega \wedge \omega^{n-1}}{\omega^n} \\
&\overset{(41)}{=} n\frac{\mathrm{Tr}(\delta\omega)}{\omega^n}\frac{\omega^n}{n!}(n-1)! = \mathrm{Tr}(\delta\omega) = g^{j\bar{k}}\delta\omega_{\bar{k}j} = g^{q\bar{p}}\delta g_{\bar{p}q}
\end{aligned}$$

and hence for any partial derivative

$$\partial_\ell(\log(\omega^n)) = g^{\bar{q}m} \partial_\ell(g_{\overline{m}q})$$

Problem A.16. *Define the Ricci curvature of the metric $g_{\bar{k}j}$ by*

$$R_{\bar{k}j} = R_{\bar{k}j}{}^{q}{}_{q}$$

Show that

$$R_{\bar{k}j} = -\partial_{\bar{k}}\partial_j \log \omega^n$$

Answer A.16. *Note*

$$\begin{aligned}
R_{\bar{k}j} &= R_{\bar{k}j}{}^{q}{}_{q} \\
&= -\partial_{\bar{k}}(g^{q\bar{p}}\partial_j g_{\bar{p}q})
\end{aligned}$$

Notice from the previous problem, we derived

$$\partial_j \log(\omega^n) = g^{q\bar{p}} \partial_j g_{\bar{p}q}$$

Hence plugging in we obtain

$$R_{\bar{k}j} = -\partial_{\bar{k}}\partial_j \log(\omega^n)$$

A.5 Anti-Canonical Bundle and Calabi Conjecture

Let the canonical bundle K_X be the bundle $\Lambda^{n,0}$ of $(n,0)$-forms, and let the anti-canonical bundle be the bundle K_X^{-1}. Observe that ω^n is the positive section of $K_X \otimes \overline{K_X}$, and hence ω^n can be viewed as a metric on K_X^{-1}.

Problem A.17. *Explain why the Ricci curvature $R_{\bar{k}j}$ can then be interpreted as the curvature of K_X^{-1} with respect to the metric ω^n.*

Answer A.17. *Since ω^n is an (n,n)-form, it is a section of*

$$K_X \otimes \overline{K_X}$$

where K_X is the bundle of n-forms, i.e. the line bundle whose sections involve

$$f(z)dz^1 \wedge \cdots \wedge dz^n$$

This implies ω^n is a metric on K_X^{-1}. Since a metric on a line bundle L is by definition, a strictly positive section of

$$L^{-1} \otimes \overline{L^{-1}}$$

Now let $L := K_X^{-1}$ we see that a metric on L is thus a positive section of $K_X \otimes \overline{K_X}$. Thus

$$R_{\overline{k}j} = -\partial_{\overline{k}}\partial_j(\log(\omega^n))$$

is precisely the curvature of the bundle K_X^{-1} due to definition of curvature for holomorphic line bundles.

Problem A.18. *Let the Ricci form* $\mathrm{Ric}(\omega)$ *be defined by*

$$\mathrm{Ric}(\omega) := \sum_{\overline{k},j}^{n} R_{\overline{k}j} dz^j \wedge d\overline{z}^k = -\partial\overline{\partial}(\log(\omega^n))$$

Explain why $\mathrm{Ric}(\omega)$ *is a closed form, and why we have*

$$[i\mathrm{Ric}(\omega)] = c_1(K_X^{-1})$$

Observe that the right-hand side is independent of the Kähler metric ω.

Answer A.18. 1. *To see* $\mathrm{Ric}(\omega)$ *is closed form, notice*

$$\begin{aligned} d\mathrm{Ric}(\omega) &= -(\partial + \overline{\partial})\partial\overline{\partial}(\log(\omega^n)) \\ &= -(\partial^2\overline{\partial} + \overline{\partial}\partial\overline{\partial})(\log(\omega^n)) \\ &= -\overline{\partial}\partial\overline{\partial}(\log(\omega^n)) \qquad \textit{using } \partial^2 = 0 \\ &= \partial\overline{\partial}^2(\log(\omega^n)) \qquad \textit{using } \partial\overline{\partial} + \overline{\partial}\partial = 0 \\ &= 0 \qquad \textit{using } \overline{\partial}^2 = 0 \end{aligned}$$

2. *Recall the definition for de Rham cohomology*

$$[\mathrm{Ric}(\omega)]_{\mathrm{dR}} := \mathrm{Ric}(\omega)/\{\textit{exact forms } d\psi \textit{ where } \psi \in \Lambda^1\}$$

Assume two metrics ω_1^n and ω_2^n gives rise to two Ricci Forms

$$i\mathrm{Ric}(\omega_1) = -i\partial\overline{\partial}(\log(\omega_1^n)) \qquad i\mathrm{Ric}(\omega_2) = -i\partial\overline{\partial}(\log(\omega_2^n))$$

Notice

$$\begin{aligned} R^1_{\overline{k}j} - R^2_{\overline{k}j} &= -\partial_{\overline{k}}\partial_j(\log(\omega_1^n)) + \partial_{\overline{k}}\partial_j(\log(\omega_2^n)) \\ &= -\partial_{\overline{k}}\partial_j(\log(\frac{\omega_1^n}{\omega_2^n})) \end{aligned}$$

But $\frac{\omega_1^n}{\omega_2^n}$ is a strictly positive C^∞ function since

$$\omega_1^n \in K_X \otimes \overline{K_X}, \qquad \omega_2^n \in K_X \otimes \overline{K_X}, \qquad \frac{\omega_1^n}{\omega_2^n} \in \mathbb{1}$$

Thus

$$\frac{\omega_1^n}{\omega_2^n} = e^\phi \qquad \textit{for certain } \phi \in C^\infty \textit{ and } \phi > 0$$

Now

$$\begin{aligned} R^1_{\overline{k}j} - R^2_{\overline{k}j} &= -\partial_{\overline{k}}\partial_j\phi \\ i(R^1_{\overline{k}j} - R^2_{\overline{k}j})dz^j \wedge d\overline{z}^k &= -i\partial_{\overline{k}}\partial_j\phi dz^j \wedge d\overline{z}^k \\ &= -i\partial\overline{\partial}\phi = -i(\partial + \overline{\partial})\overline{\partial}\phi \\ &= -id(\overline{\partial}\phi) \qquad \textit{exact form} \end{aligned}$$

Hence

$$[\,i\mathrm{Ric}(\omega_1)\,] = [\,i\mathrm{Ric}(\omega_2)\,]$$

is independent of the metric ω^n. *Thus in particular*

$$[\,i\mathrm{Ric}(\omega)\,] = [\,c_1(\mathrm{Ric}(\omega))\,]_{\mathrm{dR}} = c_1(K_X^{-1})$$

Problem A.19. *Let* $M_{\overline{k}j}$ *be a given Hermitian matrix, and let*

$$M := \sum_{\overline{k}j} M_{\overline{k}j} dz^j \wedge d\overline{z}^k$$

We consider the following equation for a Kähler metric ω

$$\mathrm{Ric}(\omega) = M$$

Find necessary conditions on M *for the existence of solutions.*

Answer A.19. 1. *One necessary condition for existence of solution* ω *is that* M *is closed, i.e.* $dM = 0$. *This is because*

$$d\mathrm{Ric}(\omega) = 0$$

is closed

2. *Another necessary condition is*

$$[\,M\,]_{\mathrm{dR}} = c_1(K_X^{-1})$$

since we necessarily have

$$[\,M\,]_{\mathrm{dR}} = [\,\mathrm{Ric}(\omega)\,]_{\mathrm{dR}} = c_1(K_X^{-1})$$

Problem A.20. *Formulate the Calabi Conjecture.*

Answer A.20. *Given* M *satisfying* $dM = 0$ *and*

$$[\,M\,]_{\mathrm{dR}} = [\,\mathrm{Ric}(\omega)\,]_{\mathrm{dR}} = c_1(K_X^{-1})$$

Then in any Kähler class $[\,\omega_0\,]$, *there exists a unique* $\omega \in [\,\omega_0\,]$ *with*

$$\mathrm{Ric}(\omega) = M$$

B Final

B.1 Chern Connection and Curvature w.r.t. two metrics

Let X be a complex manifold of dimension n. Let ω and $\hat{\omega}$ be two Hermitian metrics on X. In local coordinates, they write

$$\hat{\omega} := i \sum_{j,\overline{k}=1}^{n} \hat{g}_{\overline{k}j} dz^j \wedge d\overline{z}^k$$
$$\omega := i \sum_{j,\overline{k}=1}^{n} g_{\overline{k}j} dz^j \wedge d\overline{z}^k$$

Let $\hat{\nabla}$ and ∇ denote their Chern unitary connections respectively w.r.t. $\hat{\omega}$ and ω. Let $\hat{R}_{\overline{k}j}{}^{p}{}_{q}$ and $R_{\overline{k}j}{}^{p}{}_{q}$ be the corresponding curvature tensors.

Problem B.1. *Write down the explicit formulas for $\hat{\nabla}$ and ∇ acting on vector fields V^m and differential forms W_p where m and p are unbarred directions.*

Answer B.1. *We denote j as unbarred direction and $\overline{k}$ as barred direction. For vector fields*

$$\begin{aligned}
\hat{\nabla}_j V^m &= \hat{g}^{-1}\partial_j(\hat{g}V^m) = \hat{g}^{m\bar{\ell}}\partial_j(\hat{g}_{\bar{\ell}p}V^p) \\
\hat{\nabla}_{\overline{k}} V^m &= \partial_{\overline{k}} V^m \\
\nabla_j V^m &= g^{-1}\partial_j(gV^m) = g^{m\bar{\ell}}\partial_j(g_{\bar{\ell}p}V^p) \\
\nabla_{\overline{k}} V^m &= \partial_{\overline{k}} V^m
\end{aligned}$$

For differential forms, we compute using Dual Bundle

$$\begin{aligned}
\hat{\nabla}_j W_p &= \partial_j W_p - W_q(\hat{g}^{q\bar{\ell}}\partial_j \hat{g}_{\bar{\ell}p}) \\
\hat{\nabla}_{\overline{k}} W_p &= \partial_{\overline{k}} W_p \\
\nabla_j W_p &= \partial_j W_p - W_q(g^{q\bar{\ell}}\partial_j g_{\bar{\ell}p}) \\
\nabla_{\overline{k}} W_p &= \partial_{\overline{k}} W_p
\end{aligned}$$

Problem B.2. *Let h be the relative endomorphism with respect to both metrics $\hat{\omega}$ and ω defined by*

$$h^p_q := \hat{g}^{p\overline{m}} g_{\overline{m}q}$$

Show that h is a positive endomorphism w.r.t. both metrics $\hat{\omega}$ and ω. Hence so is h^{-1}.

Answer B.2. *Denote $\langle \cdot, \cdot \rangle_{\hat{\omega}}$ and $\langle \cdot, \cdot \rangle_{\omega}$ as inner product w.r.t. to the two metrics. We first show that*

$$\langle V, V \rangle_{\omega} = \langle hV, V \rangle_{\hat{\omega}}$$

and it follows that h is positive w.r.t. $\hat{\omega}$. To do so, for any $V \neq 0$ vector field

$$\begin{aligned}
\langle V, V \rangle_{\omega} &= g_{\overline{k}j} V^j \overline{V^k} \\
&= \hat{g}_{\overline{k}p} \left(\hat{g}^{p\overline{m}} g_{\overline{m}j}\right) V^j \overline{V^k} \\
&= \hat{g}_{\overline{k}p} h^p{}_j V^j \overline{V^k} \\
&= \hat{g}_{\overline{k}p} (hV)^p \overline{V^k} = \langle hV, V \rangle_{\hat{\omega}}
\end{aligned}$$

Since LHS is a inner product and positive, then $h > 0$ w.r.t. $\hat{\omega}$. Similarly, h^{-1} is also positive w.r.t. $\hat{\omega}$. Interchanging roles of ω and $\hat{\omega}$ we obtain positivity of h and h^{-1} w.r.t. both metrics.

Problem B.3. *Show that the covariant derivatives ∇, $\hat{\nabla}$ on vector fields are related by the formula*

$$\nabla_j V^m - \hat{\nabla}_j V^m = (\nabla_j h h^{-1})^m{}_p V^p \tag{42}$$

and on differential forms are related by the formula

$$\nabla_j W_p - \hat{\nabla}_j W_p = -W_q(\nabla_j h h^{-1})^q{}_p \tag{43}$$

Answer B.3. *Notice*

$$h = \hat{g}^{-1}g \implies \hat{g} = gh^{-1}$$

Hence

$$\begin{aligned}
\hat{\nabla}_j V &= \hat{g}^{-1}\partial_j(\hat{g}V) = hg^{-1}\partial_j(gh^{-1}V) \\
&= hg^{-1}\nabla_j(gh^{-1}V) \qquad g_{\overline{s}m}(h^{-1})^m{}_r V^r \qquad \textit{section of an antiholomorphic bundle} \\
&= hg^{-1}g\nabla_j(h^{-1}V) \\
&= h\left(\nabla_j(h^{-1}V)\right) \\
&= h\left(-h^{-1}(\nabla_j h)h^{-1}V + h^{-1}\nabla_j V\right) \\
&= -(\nabla_j h)h^{-1}V + \nabla_j V
\end{aligned}$$

On the other hand

$$\begin{aligned}
\hat{\nabla}_j W &= \partial_j W - W(\hat{g}^{-1}\partial_j\hat{g}) \\
&= \partial_j W - W(hg^{-1}\partial_j(gh^{-1})) \\
&= \partial_j W - W(hg^{-1}(\partial_j gh^{-1} - gh^{-1}(\partial_j h)h^{-1})) \\
&= \partial_j W - W(g^{-1}\partial_j g - h^{-1}\partial_j h) \\
&= \partial_j W - W(g^{-1}\partial_j g) + W(h^{-1}\partial_j h) \\
&= \nabla_j W + W(\nabla_j hh^{-1})
\end{aligned}$$

Rearranging both yields (42) *and* (43).

Problem B.4. *Show that the curvature tensors of* ω *and* $\hat{\omega}$ *are related by*

$$R_{\overline{k}j}{}^m{}_p - \hat{R}_{\overline{k}j}{}^m{}_p = -\partial_{\overline{k}}(\nabla_j hh^{-1})^m_p \tag{44}$$

Answer B.4. *Apply* $\partial_{\overline{k}}$ *to both sides of* (42)

$$\begin{aligned}
\partial_{\overline{k}}\hat{\nabla}_j V^m &= \partial_{\overline{k}}(\nabla_j V^m) - \partial_{\overline{k}}((\nabla_j hh^{-1})V^m) \\
\hat{\nabla}_j(\partial_{\overline{k}}V^m) - \hat{R}_{\overline{k}j}{}^m{}_p V^p &= \nabla_j(\partial_{\overline{k}}V^m) - R_{\overline{k}j}{}^m{}_p V^p - \partial_{\overline{k}}((\nabla_j hh^{-1})V^m) \\
\hat{\nabla}_j(\partial_{\overline{k}}V^m) - \nabla_j(\partial_{\overline{k}}V^m) &= \hat{R}_{\overline{k}j}{}^m{}_p V^p - R_{\overline{k}j}{}^m{}_p V^p - \partial_{\overline{k}}((\nabla_j hh^{-1})V^m) \\
-(\nabla_j h)h^{-1}(\partial_{\overline{k}}V^m) &= \hat{R}_{\overline{k}j}{}^m{}_p V^p - R_{\overline{k}j}{}^m{}_p V^p - \partial_{\overline{k}}(\nabla_j hh^{-1})V^m - (\nabla_j hh^{-1})\partial_{\overline{k}}V^m \qquad \textit{using } (42) \\
\hat{R}_{\overline{k}j}{}^m{}_p V^p - R_{\overline{k}j}{}^m{}_p V^p &= \partial_{\overline{k}}(\nabla_j hh^{-1})V^m \\
\hat{R}_{\overline{k}j}{}^m{}_p - R_{\overline{k}j}{}^m{}_p &= \partial_{\overline{k}}(\nabla_j hh^{-1})^m_p \\
R_{\overline{k}j}{}^m{}_p - \hat{R}_{\overline{k}j}{}^m{}_p &= -\partial_{\overline{k}}(\nabla_j hh^{-1})^m_p
\end{aligned}$$

B.2 Kähler Geometry and Calabi Conjecture

Let X be a complex n-dim Kähler Manifold.

Problem B.5. *Define the first Chern Class* $c_1(X)$ *of the manifold* X.

Answer B.5. *For a general connection* ∇ *on the holomorphic tangent bundle* $T^{1,0}X$, $c_1(X)$ *is represented by:*

$$c_1(X) = [\mathrm{Tr}(R)]_{\mathrm{dR}} \in H^{1,1}(X, \mathbb{R}),$$

where R *is the curvature form of* ∇. *In our case*

$$[F]_{\mathrm{dR}} := F/\{\textit{exact forms } d\psi \textit{ where } \psi \in \Lambda^0\}$$

Problem B.6. *Let*

$$\omega = i\sum_{j,\overline{k}=1}^{n} g_{\overline{k}j}dz^j \wedge d\overline{z}^k$$

be any Kähler metric on X. *Show that its Ricci form*

$$\mathrm{Ric}(\omega) = i\sum_{j,\overline{k}} R_{j\overline{k}}\,dz^j \wedge d\overline{z}^k,$$

is a closed $(1,1)$*-form and its de Rham cohomology class is always* $c_1(X)$.

Answer B.6. *The Ricci curvature form associated to* ω *is:*

$$\mathrm{Ric}(\omega) = i \sum_{j,\overline{k}} R_{j\overline{k}}\, dz^j \wedge d\overline{z}^k,$$

where $R_{j\overline{k}} = -\partial_j \partial_{\overline{k}} \log\det(g)$. *We first show it is closed* $(1,1)$*-form. Indeed*

$$\begin{aligned}
d\mathrm{Ric}(\omega) &= -i(\partial + \overline{\partial})\partial\overline{\partial}(\log(\omega^n)) \\
&= -i(\partial^2\overline{\partial} + \overline{\partial}\partial\overline{\partial})(\log(\omega^n)) \\
&= -i\overline{\partial}\partial\overline{\partial}(\log(\omega^n)) \qquad \textit{using } \partial^2 = 0 \\
&= i\partial\overline{\partial}^2(\log(\omega^n)) \qquad \textit{using } \partial\overline{\partial} + \overline{\partial}\partial = 0 \\
&= 0 \qquad \textit{using } \overline{\partial}^2 = 0
\end{aligned}$$

Conclude by observing that Ricci form is defined as the trace of the curvature form. The first Chern class is thus the de Rham cohomology class

$$c_1(X) = [\mathrm{Ric}(\omega)]_{\mathrm{dR}} \in H^{1,1}(X, \mathbb{R}).$$

Problem B.7. *We consider the question of whether* X *admits a Kähler-Einstein metric of negative scalar-curvature, that is, whether* X *admits a Kähler metric* ω *which satisfies the following Euclidean analogue of Einstein's Equation*

$$\mathrm{Ric}(\omega) = -\omega$$

i.e.

$$R_{\overline{k}j} = -g_{\overline{k}j} \tag{45}$$

1. *Explain why a necessary condition for the existence of such a Kähler-Einstein metric is that* $c_1(X)$ *must be negative-definitive, in the sense that* $-c_1(X)$ *admits a representative which is a Kähler metric.*

2. *Furthermore, explain why a Kähler-Einstein metric* ω *satisfying* (45) *must be in the cohomology class* $-c_1(X)$.

Answer B.7.

1. *We first discuss necessity of* $-c_1(X)$ *being positive. The Ricci form* $Ric(\omega)$ *represents the first Chern class* $c_1(X) = [Ric(\omega)]_{\mathrm{dR}}$. *If* $\mathrm{Ric}(\omega) = -\omega$, *then*

$$c_1(X) = [-\omega]_{\mathrm{dR}} = -[\omega]_{\mathrm{dR}}$$

Since ω *is a Kähler metric,* $[\omega]_{\mathrm{dR}}$ *is a positive* $(1,1)$*-cohomology class. Thus,* $-c_1(X)$ *must be positive, i.e.,* $-c_1(X)$ *admits a Kähler metric representative. This makes* $c_1(X)$ *negative-definite.*

2. *Cohomology class of* ω. *From the equation* (45)*, take cohomology classes*

$$[\mathrm{Ric}(\omega)]_{\mathrm{dR}} = -[\omega]_{\mathrm{dR}}$$

By definition, $[\mathrm{Ric}(\omega)]_{\mathrm{dR}} = c_1(X)$. *Substituting*

$$c_1(X) = -[\omega]_{\mathrm{dR}} \implies [\omega]_{\mathrm{dR}} = -c_1(X).$$

Thus, ω *lies in the cohomology class* $-c_1(X)$.

Problem B.8. *Assume now* $c_1(X)$ *is negative definite, and let* $\hat{\omega} \in c_1(X)$ *be a Kähler metric. Since the de Rham cohomology class of* $\mathrm{Ric}(\hat{\omega}) + \hat{\omega}$ *is* 0*, by* $\partial\overline{\partial}$*-lemma, we can write*

$$\mathrm{Ric}(\hat{\omega}) + \hat{\omega} = i\partial\overline{\partial}F \tag{46}$$

for some function F *which is unique up to an additive constant. Show that if*

$$\omega = \hat{\omega} + i\partial\overline{\partial}\varphi > 0 \tag{47}$$

is a Kähler-Einstein metric, then after possibly shifting φ *by an additive constant,* φ *must satisfy Monge-Ampère Equation*

$$(\hat{\omega} + i\partial\overline{\partial}\varphi)^n = \hat{\omega}^n e^{\varphi + F} \tag{48}$$

Answer B.8. *We want to express* $\text{Ric}(\omega) - \text{Ric}(\hat{\omega})$ *in two different ways. Using* ω *satisfies* (45) *and* $\hat{\omega}$ *satisfies* (46) *we write*

$$\begin{aligned}
\text{Ric}(\omega) - \text{Ric}(\hat{\omega}) &= \hat{\omega} - \omega - i\partial\overline{\partial}F \\
-i\partial\overline{\partial}\log(\omega^n) + i\partial\overline{\partial}\log(\hat{\omega}^n) &= -i\partial\overline{\partial}(F+\varphi) \qquad \textit{using Ricci and (47)} \\
-i\partial\overline{\partial}(\log(\frac{\omega^n}{\hat{\omega}^n})) &= -i\partial\overline{\partial}(F+\varphi) \\
i\partial\overline{\partial}(\log(\frac{\omega^n}{\hat{\omega}^n})) &= i\partial\overline{\partial}(F+\varphi) \\
\log(\frac{\omega^n}{\hat{\omega}^n}) &= F+\varphi \qquad \textit{using } \frac{\omega^n}{\hat{\omega}^n} \textit{ is scalar function} \\
\frac{\omega^n}{\hat{\omega}^n} &= e^{F+\varphi} \\
\omega^n &= \hat{\omega}^n e^{F+\varphi} \\
(\hat{\omega} + i\partial\overline{\partial}\varphi)^n &= \hat{\omega}^n e^{F+\varphi} \qquad \textit{using (46) again}
\end{aligned}$$

B.3 C^0 Estimate

We consider the problem of a priori estimate for the Monge-Ampère Equation (48).

Problem B.9. *Show that any* C^2 *solution* φ *of the equation* (48) *must satisfy the following* C^0 *estimate*

$$\|\varphi\|_{C^0} \leq \|F\|_{C^0} \tag{49}$$

Answer B.9. *Proof.* First we consider z_0 where φ attains its maximum. Then at such point, the Hessian $(\partial_j \partial_{\overline{k}}\varphi)$ of φ must be a non-positive matrix. Hence at z_0

$$\omega = \hat{\omega} + i\partial\overline{\partial}\varphi \leq \hat{\omega}$$

But then

$$\begin{aligned}
\omega^n(z_0) &= (\hat{\omega}^n e^{\varphi+F})(z_0) \leq \hat{\omega}^n(z_0) \\
e^{\varphi(z_0)} &\leq e^{-F(z_0)} \\
\max_X \varphi &\leq \|F\|_{C^0}
\end{aligned}$$

On the other hand consider z_1 where φ attains its minimum. Then the Hessian $(\partial_j \partial_{\overline{k}}\varphi)$ of φ must be a non-negative matrix. Hence at z_1

$$\omega = \hat{\omega} + i\partial\overline{\partial}\varphi \geq \hat{\omega}$$

Then

$$\begin{aligned}
\omega^n(z_1) &= (\hat{\omega}^n e^{\varphi+F})(z_1) \geq \hat{\omega}^n(z_1) \\
e^{\varphi(z_1)} &\geq e^{-F(z_1)} \\
\min_X \varphi &\geq -\|F\|_{C^0}
\end{aligned}$$

Thus we conclude (49)

$$\|\varphi\|_{C^0} \leq \|F\|_{C^0}$$

□

B.4 C^2 Estimate

We still consider the Monge-Ampère Equation (48) or its equivalent formulation using Ricci curvature (45). Let h be the endomorphism defined by

$$h^p_q := \hat{g}^{p\overline{m}} g_{\overline{m}q}$$

Let the trace of h be defined as usual by

$$\text{Tr}(h) := h^p_p = \hat{g}^{p\overline{m}} g_{\overline{m}p}$$

Problem B.10. *Show that* $\text{Tr}(h)$ *satisfies the differential identity*

$$\Delta(\text{Tr}(h)) = (h^{-1})^p_\ell \hat{R}_p{}^{\ell}{}_s{}^r + \text{Tr}(h) + g^{p\overline{q}}\text{Tr}(\nabla_p h h^{-1} \partial_{\overline{q}} h) \tag{50}$$

where Δ *is the Laplacian on scalars with respect to* ω $\Delta = g^{p\overline{q}}\nabla_{\overline{q}}\nabla_p$.

Answer B.10. *We calculate*

$$\begin{aligned}\Delta(\mathrm{Tr}(h)) &= g^{p\overline{q}}\nabla_{\overline{q}}\nabla_p(\mathrm{Tr}(h)) = g^{p\overline{q}}\nabla_{\overline{q}}(\mathrm{Tr}(\nabla_p h))\\ &= g^{p\overline{q}}\nabla_{\overline{q}}(\mathrm{Tr}(\nabla_p h h^{-1})h)\\ &= g^{p\overline{q}}(\mathrm{Tr}(\nabla_{\overline{q}}(\nabla_p h h^{-1})h) + \mathrm{Tr}(\nabla_p h h^{-1})\partial_{\overline{q}}h)\end{aligned}$$

We compute the first term. Notice using (44)

$$\begin{aligned}g^{p\overline{q}}\mathrm{Tr}(\nabla_{\overline{q}}(\nabla_p h h^{-1})h) &= g^{p\overline{q}}(\hat{R}_{\overline{q}p}{}^{m}{}_{\ell} - R_{\overline{q}p}{}^{m}{}_{\ell})h^{\ell}_{m}\\ &= g^{p\overline{q}}\hat{R}_{\overline{q}p}{}^{m}{}_{\ell}h^{\ell}_{m} - g^{p\overline{q}}R_{\overline{q}p}{}^{m}{}_{\ell}h^{\ell}_{m}\end{aligned}$$

Using (45) *we know*

$$\begin{aligned}g^{p\overline{q}}R_{\overline{q}p}{}^{m}{}_{\ell}h^{\ell}_{m} &= R^{m}_{\ell}h^{\ell}_{m}\\ &= R^{m}_{\ell}\hat{g}^{\ell\overline{k}}g_{\overline{k}m}\\ &= R_{\overline{k}\ell}\hat{g}^{\ell\overline{k}}\\ &= -g_{\overline{k}\ell}\hat{g}^{\ell\overline{k}}\\ &= -\mathrm{Tr}(h)\end{aligned}$$

On the other hand

$$\begin{aligned}g^{p\overline{q}}\hat{R}_{\overline{q}p}{}^{m}{}_{\ell}h^{\ell}_{m} &= ((\hat{g}h)^{-1})^{p\overline{q}}\hat{R}_{\overline{q}p}{}^{m}{}_{\ell}h^{\ell}_{m}\\ &= (h^{-1}\circ\hat{g}^{-1})^{p\overline{q}}\hat{R}_{\overline{q}p}{}^{m}{}_{\ell}h^{\ell}_{m}\\ &= (h^{-1})^{p}_{\ell}\hat{g}^{\ell\overline{q}}\hat{R}_{\overline{q}p}{}^{m}{}_{\ell}h^{\ell}_{m}\\ &= (h^{-1})^{p}_{\ell}\hat{R}_{p}{}^{\ell r}{}_{s}h^{s}_{r}\end{aligned}$$

Collecting terms we obtain

$$\Delta(\mathrm{Tr}(h)) = (h^{-1})^{p}_{\ell}\hat{R}_{p}{}^{\ell r}{}_{s}h^{s}_{r} + \mathrm{Tr}(h) + g^{p\overline{q}}\mathrm{Tr}(\nabla_p h h^{-1})\partial_{\overline{q}}h$$

Problem B.11. *Establish the following general identity.*

$$\Delta(\log(\mathrm{Tr}(h))) = \frac{\Delta(\mathrm{Tr}(h))}{\mathrm{Tr}(h)} - \frac{g^{p\overline{q}}\partial_p\mathrm{Tr}(h)\partial_{\overline{q}}\mathrm{Tr}(h)}{(\mathrm{Tr}(h))^2} \tag{51}$$

Answer B.11.

$$\begin{aligned}\Delta\log(\mathrm{Tr}(h)) &= g^{p\overline{q}}\partial_{\overline{q}}\partial_p\log(\mathrm{Tr}(h))\\ &= g^{p\overline{q}}\partial_{\overline{q}}\left(\frac{\partial_p(\mathrm{Tr}(h))}{\mathrm{Tr}(h)}\right)\\ &= g^{p\overline{q}}\left(\frac{\partial_{\overline{q}}\partial_p(\mathrm{Tr}(h))}{\mathrm{Tr}(h)} - \frac{\partial_p(\mathrm{Tr}(h))\partial_{\overline{q}}(\mathrm{Tr}(h))}{(\mathrm{Tr}(h))^2}\right)\\ &= \frac{\Delta(\mathrm{Tr}(h))}{\mathrm{Tr}(h)} - \frac{g^{p\overline{q}}\partial_p(\mathrm{Tr}(h))\partial_{\overline{q}}(\mathrm{Tr}(h))}{(\mathrm{Tr}(h))^2}\end{aligned}$$

Problem B.12. *Deduce the following identity for the above Monge-Ampère Equation.*

$$\Delta(\log(\mathrm{Tr}(h))) = \frac{(h^{-1})^{p}_{\ell}\hat{R}_{p}{}^{\ell r}{}_{s}h^{s}_{r}}{\mathrm{Tr}(h)} + 1 + \left(\frac{g^{p\overline{q}}\mathrm{Tr}(\nabla_p h h^{-1}\partial_{\overline{q}}h)}{\mathrm{Tr}(h)} - \frac{g^{p\overline{q}}\partial_p(\mathrm{Tr}(h))\partial_{\overline{q}}(\mathrm{Tr}(h))}{(\mathrm{Tr}(h))^2}\right) \tag{52}$$

Answer B.12. *We put* (50) *and* (51) *together.*

$$\begin{aligned}\Delta(\log(\mathrm{Tr}(h))) &= \frac{\Delta(\mathrm{Tr}(h))}{\mathrm{Tr}(h)} - \frac{g^{p\overline{q}}\partial_p(\mathrm{Tr}(h))\partial_{\overline{q}}(\mathrm{Tr}(h))}{(\mathrm{Tr}(h))^2}\\ &= \frac{1}{\mathrm{Tr}(h)}\Big((h^{-1})^{p}_{\ell}\hat{R}_{p}{}^{\ell r}{}_{s}h^{s}_{r} + \mathrm{Tr}(h) + g^{p\overline{q}}\mathrm{Tr}(\nabla_p h h^{-1})\partial_{\overline{q}}h\Big) - \frac{g^{p\overline{q}}\partial_p(\mathrm{Tr}(h))\partial_{\overline{q}}(\mathrm{Tr}(h))}{(\mathrm{Tr}(h))^2}\\ &= \frac{(h^{-1})^{p}_{\ell}\hat{R}_{p}{}^{\ell r}{}_{s}h^{s}_{r}}{\mathrm{Tr}(h)} + 1 + \left(\frac{g^{p\overline{q}}\mathrm{Tr}(\nabla_p h h^{-1})\partial_{\overline{q}}h}{\mathrm{Tr}(h)} - \frac{g^{p\overline{q}}\partial_p(\mathrm{Tr}(h))\partial_{\overline{q}}(\mathrm{Tr}(h))}{(\mathrm{Tr}(h))^2}\right)\end{aligned}$$

Problem B.13. *Explain why* $\mathrm{Tr}(h)$ *can be viewed as a norm for* h*, and similarly* $\mathrm{Tr}(h^{-1})$ *as a norm for* h^{-1}*. And hence we obtain*

$$|(h^{-1})^p_\ell \hat{R}_p{}^{\ell}{}_s{}^r h^s_r| \leq C_{\hat{\omega}_0}(\mathrm{Tr}(h))(\mathrm{Tr}(h^{-1})) \tag{53}$$

for $C_{\hat{\omega}}$ *constant depending only on the curvature tensor of the reference metric* $\hat{\omega}$*.*

Answer B.13. *The trace* $\mathrm{Tr}(h) = \hat{g}^{p\overline{m}} g_{\overline{m}p}$ *measures the "size" of* h *as the sum of its eigenvalues. Similarly,* $\mathrm{Tr}(h^{-1}) = g^{p\overline{m}} \hat{g}_{\overline{m}p}$ *measures the size of* h^{-1}*. For positive-definite Hermitian endomorphisms, these traces act as* L^1*-norms on eigenvalues. Due to the given reference metric, one can choose* $C_{\hat{\omega}}$ *large to bound the components of* $\hat{R}$*. Now by the Cauchy-Schwarz inequality*

$$\left|(h^{-1})^p_\ell \hat{R}_p{}^{\ell}{}_s{}^r h^s_r\right| \leq C_{\hat{\omega}} \cdot \mathrm{Tr}(h) \cdot \mathrm{Tr}(h^{-1})$$

Problem B.14. *Deduce the following differential inequality using Aubin-Yau* (33)

$$\Delta \log(\mathrm{Tr}(h)) \geq -C_{\hat{\omega}} \mathrm{Tr}(h^{-1}) + 1 \tag{54}$$

Answer B.14. *Recall Aubin-Yau's Inequality* (33) *algebraic inequality.*

$$\frac{g^{p\overline{q}}\mathrm{Tr}(\nabla_p h h^{-1})\partial_{\overline{q}} h}{\mathrm{Tr}(h)} - \frac{g^{p\overline{q}}\partial_p(\mathrm{Tr}(h))\partial_{\overline{q}}(\mathrm{Tr}(h))}{(\mathrm{Tr}(h))^2} \geq 0$$

Hence plugging (53) *and* (33) *into* (52) *we obtain*

$$\begin{aligned}\Delta \log(\mathrm{Tr}(h)) &\geq -\frac{C_{\hat{\omega}}(\mathrm{Tr}(h))(\mathrm{Tr}(h^{-1}))}{\mathrm{Tr}(h)} + 1 + \left(\frac{g^{p\overline{q}}\mathrm{Tr}(\nabla_p h h^{-1})\partial_{\overline{q}} h}{\mathrm{Tr}(h)} - \frac{g^{p\overline{q}}\partial_p(\mathrm{Tr}(h))\partial_{\overline{q}}(\mathrm{Tr}(h))}{(\mathrm{Tr}(h))^2}\right) \\ &\geq -C_{\hat{\omega}}\mathrm{Tr}(h^{-1}) + 1 + 0 = -C_{\hat{\omega}}\mathrm{Tr}(h^{-1}) + 1\end{aligned}$$

Problem B.15. *Deduce that there exists a constant* K*, depending only on* $\hat{\omega}$ *and* $\|\varphi\|_{C^0}$*, so that at any point* $z \in X$*, we have*

$$\mathrm{Tr}(h(z)) \leq K$$

Answer B.15. *Let's now consider the expression for* A *to be chosen*

$$\begin{aligned}\Delta(\log(\mathrm{Tr}(h)) - A\varphi) &= \Delta \log(\mathrm{Tr}(h)) - A\Delta\varphi \\ &= \Delta \log(\mathrm{Tr}(h)) + A\left(\mathrm{Tr}(h^{-1}) - n\right) \\ &\geq -C_{\hat{\omega}}\mathrm{Tr}(h^{-1}) + 1 + A\mathrm{Tr}(h^{-1}) - An\end{aligned}$$

Take now

$$A := 2C_{\hat{\omega}}$$

Then

$$\Delta(\log(\mathrm{Tr}(h)) - A\varphi) \geq C_{\hat{\omega}}\mathrm{Tr}(h^{-1}) - C_3 \qquad C_3 := -An - 1$$

Now apply the maximum principle to this. Let z_1 *be a local maximum point of* $\log(\mathrm{Tr}(h)) - A\varphi$*. Then*

$$0 \geq \Delta\left(\log(\mathrm{Tr}(h)) - A\varphi\right)(z_1) \geq C_{\hat{\omega}}\mathrm{Tr}(h^{-1})(z_1) - C_3$$

$$\mathrm{Tr}(h^{-1})(z_1) \leq C_4 \qquad C_4 := \frac{C_3}{C_{\hat{\omega}}}$$

Let $\lambda_1, \cdots, \lambda_n$ *be the eigenvalues of* h *at* z_1*. Thus we have*

$$\frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} \leq C_4$$

and then

$$\lambda_1, \cdots, \lambda_n \geq C_5 := \frac{1}{C_4}$$

But now from the Monge-Ampère Equation (48)*,*

$$\omega^n = \hat{\omega}^n e^{\varphi + F} \iff \frac{\omega^n}{\hat{\omega}^n} = e^{\varphi + F}$$

Thus the product

$$\prod_{\ell=1}^{n} \lambda_\ell = e^{\varphi+F}$$

$$C_5^{n-1}\lambda_r \le \left(\prod_{\ell\neq r} \lambda_\ell\right) \lambda_r = e^{\varphi+F}$$

$$\lambda_r \le C_6 := \frac{\max e^{2F}}{C_5^{n-1}} \qquad \forall\ 1 \le r \le n$$

where in the last step we used C^0 *estimate* (49)*. Now at an arbitrary point* $z \in X$*, we can write*

$$\begin{aligned}
\left(\log(\mathrm{Tr}(h)) - A\varphi\right)(z) &\le \left(\log(\mathrm{Tr}(h)) - A\varphi\right)(z_1) \\
&\le \left(\log(nC_6) - A\varphi\right)(z_1) \\
\left(\log(\mathrm{Tr}(h)) - A\varphi\right)(z) &\le C_7 - A\varphi(z_1)
\end{aligned}$$

Now we can rewrite

$$\begin{aligned}
\log(\mathrm{Tr}(h(z))) &\le C_7 + A(\varphi(z) - \varphi(z_1)) \\
&\le C_7 + A osc(\varphi) \\
\mathrm{Tr}(h(z)) &\le C_8 e^{A osc(\varphi)} \\
&\le C_8 e^{2A\|\varphi\|_{C^0}} \\
&\le K \qquad \textit{using Estimate } (49)
\end{aligned}$$