[Giusti] Minimial Surfaces and Functions of Bounded Variation

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Contents

Chapter 1

Functions of Bounded Variation

1.1 Functions of Bounded Variation and Caccioppoli Sets

1.1.1 Definitions and Semicontinuity

Definition 1.1.1 (BV Functions). *Let $\Omega \subset \mathbb{R}^n$ be open set. $f \in L^1(\Omega)$.*

$$\int_\Omega |Df| := \sup\{\int_\Omega f \,\mathrm{div} g \, dx \mid g \in C_0^1(\Omega; \mathbb{R}^n), |g(x)| \leq 1\} \tag{1.1}$$

$f \in BV(\Omega)$ if $\int_\Omega |Df| < \infty$. $BV(\Omega)$ is space of $L^1(\Omega)$ functions of bounded variation in Ω.

Example 1.1.1. *If $f \in C^1(\Omega)$, $\int_\Omega |Df| = \int_\Omega |\nabla f| dx$ where $\nabla f \in C(\Omega; \mathbb{R}^n)$ is classical gradient. If $f \in W^{1,1}(\Omega)$, $\int_\Omega |Df| = \int_\Omega |\nabla f| dx$ where $\nabla f \in L^1(\Omega; \mathbb{R}^n)$ is weak gradient.*

Example 1.1.2. *We study $\varphi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \in \mathbb{R}^n \setminus E \end{cases}$ characteristic on E with C^2 boundary.*

- *If E is bounded, $\|\varphi_E\|_{L^1(\Omega)} = |E \cap \Omega| < \infty$ so $\varphi_E \in L^1(\Omega)$. But $\nabla \varphi_E$ distributional derivative is vector-valued Radon measure instead of $L^1(\Omega)$ function, hence $\varphi_E \notin W^{1,1}(\Omega)$. But on the other hand, we may compute $\int_\Omega |D\varphi_E|$. Let $g \in C_0^1(\Omega; \mathbb{R}^n)$ s.t. $|g| \leq 1$, so by Gauss-Green formula*

$$\int_\Omega \varphi_E \,\mathrm{div} g \, dx = \int_E \mathrm{div} g \, dx = \int_{\partial E} g \cdot \nu \, dH_{n-1} \leq H_{n-1}(\partial E \cap \Omega) \tag{1.2}$$

  for ν outer unit normal to ∂E. Taking supremum in g yields $\int_\Omega |D\varphi_E| < \infty$. Thus $W^{1,1}(\Omega) \subsetneq BV(\Omega)$.

- *We in fact prove $\int_\Omega |D\varphi_E| = H_{n-1}(\partial E \cap \Omega)$. Since E C^2 boundary, $v \in C^1(\partial E; \mathbb{R}^n)$ with $|\nu| = 1$. Since ∂E is closed in $\mathbb{R}^n$ and $\mathbb{R}^n$ is normal, we may apply Tietze Extension to extend ν to $N \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ with $|N| \leq 1$. By Urysohn's there exists $\eta \in C_0^\infty(\Omega)$ s.t. $|\eta| \leq 1$, so let $g = \eta N \in C_0^1(\Omega; \mathbb{R}^n)$*

$$\int_\Omega \varphi_E \,\mathrm{div} g \, dx = \int_E \mathrm{div} g \, dx = \int_{\partial E} \eta N \cdot \nu \, dH_{n-1} = \int_{\partial E} \eta \, dH_{n-1}$$

  Take supremum in g on LHS and in η on RHS yields (due to $H_{n-1} \llcorner \partial E$ is Radon measure on $\mathbb{R}^n$)

$$\int_\Omega |D\varphi_E| \geq \sup\{\int_{\partial E} \eta \, dH_{n-1} \mid \eta \in C_0^\infty(\Omega), |\eta| \leq 1\} = H_{n-1}(\partial E \cap \Omega) \tag{1.3}$$

  Hence (1.2) and (1.3) together gives, for E C^2 boundary

$$\int_\Omega |D\varphi_E| = H_{n-1} \llcorner \partial E(\Omega) := H_{n-1}(\partial E \cap \Omega) \tag{1.4}$$

  A side remark: (1.4) is true in fact for E with C^1 boundary.

Remark 1.1.1. *For $f \in BV(\Omega)$, the duality pairing $\langle Df, g \rangle := -\int_\Omega f \,\mathrm{div} g \, dx$ defines the distributional gradient $Df \in (C_0^1(\Omega; \mathbb{R}^n))'$ because $\int_\Omega |Df| = \sup_{g \in C_0^1(\Omega; \mathbb{R}^n)} \frac{|\langle Df, g \rangle|}{|g|} < \infty$. By Riesz, the bounded linear functional Df on $C_0^1(\Omega; \mathbb{R}^n)$ defines a vector-valued Radon measure Df on Ω with $\int_\Omega |Df|$ the total variation of Df on Ω. Since $|Df|$ is a Borel measure over Ω, one may measure $\int_A |Df|$ for $A \subset \Omega$ not necessarily open. In particular, if $f = \varphi_E$ for some E bounded and C^2 so that $\varphi_E \in BV(\Omega)$, since the two Borel measures $|D\varphi_E|$ and $H_{n-1} \llcorner \partial E$ agrees on all open sets as in (1.4), they agree on all Borel sets.*

Definition 1.1.2 (Perimeter & Caccioppoli Set)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and E a Borel set. The Perimeter of E in Ω is*

$$P(E,\Omega) := \int_\Omega |D\varphi_E| = \sup\{\int_E \operatorname{div} g\, dx \mid g \in C_0^1(\Omega;\mathbb{R}^n), |g| \leq 1\} \tag{1.5}$$

If $\Omega = \mathbb{R}^n$ write $P(E) := P(E,\mathbb{R}^n)$. The Borel set E is a Caccioppoli Set if it has locally finite perimeter, i.e., $P(E,\Omega) < \infty$ for each bounded open $\Omega \subset \mathbb{R}^n$.

Remark 1.1.2. *One has characterisations for Caccioppoli Sets E*

- *E is a Caccioppoli Set iff there exist vector-valued Radon measure ω over $\mathbb{R}^n$ s.t.*

  1. *ω has locally finite variation, i.e., for each bounded open $\Omega \subset \mathbb{R}^n$, $|\omega|(\Omega) < \infty$*
  2. *for all $g \in C_0^1(\Omega;\mathbb{R}^n)$ s.t. $|g| \leq 1$, one has $\int_E \operatorname{div} g\, dx = \int g \cdot d\omega$*

  Proof. $\Longrightarrow$ Since for each Ω bounded and open, $P(E,\Omega) = \int_\Omega |D\varphi_E| < \infty$ iff $\varphi_E \in BV(\Omega)$, $D\varphi_E$ defines a vector-valued Radon measure with locally finite variation over $\mathbb{R}^n$. Let $\omega = -D\varphi_E$, so for each fixed Ω,

  $$\int g \cdot d\omega = -\langle D\varphi_E, g\rangle = \int_\Omega \varphi_E \operatorname{div} g\, dx = \int_E \operatorname{div} g\, dx$$

  $\Longleftarrow$ Suppose such ω exists. Then for any $g \in C_0^1(\Omega;\mathbb{R}^n)$ s.t. $|g| \leq 1$

  $$\int_E \operatorname{div} g\, dx = \int g \cdot d\omega \leq |\omega|(\Omega) < \infty$$

  take supremum in g on LHS gives $P(E,\Omega) = \int_\Omega |D\varphi_E| \leq |\omega|(\Omega) < \infty$. □

  Definition 1.1.3 (Gauss-Green Measure $D\varphi_E$)**.** *For E Caccioppoli, the vector-valued Radon measure $D\varphi_E$ on $\mathbb{R}^n$ with locally finite variation that satisfies the above is called the Gauss-Green measure of E.*

- *For E any Borel Set, $\operatorname{supp} D\varphi_E \subset \partial E$ where*

  $$\operatorname{supp} D\varphi_E := \mathbb{R}^n \setminus \bigcup \left\{ A \text{ open} \mid \forall g \in C_0^1(A;\mathbb{R}^n),\, |g| \leq 1 \implies \int g \cdot D\varphi_E = 0 \right\}$$

  Proof. For any $x \notin \partial E$, there exists A open neighbor of x s.t. either $A \subset E$ or $A \subset E^c$. If $A \subset E^c$, $\varphi_E = 0$ on A, so for any $g \in C_0^1(A;\mathbb{R}^n)$, $|g| \leq 1$ one indeed has $\int g \cdot D\varphi_E = -\int \varphi_E \operatorname{div} g\, dx = 0$. If $A \subset E$, $\varphi_E = 1$ on A, so for such g, $\int g \cdot D\varphi_E = -\int_E \operatorname{div} g\, dx = -\int \operatorname{div} g\, dx = 0$ since g is compactly supported and one apply the divergence theorem. Thus for any $x \notin \partial E$, $x \notin \operatorname{supp} D\varphi_E$. □

- *E is a Caccioppoli Set iff the Gauss-Green formula holds in a generalized sense, i.e., for any $\Omega \subset \mathbb{R}^n$ open and bounded, and for any $g \in C_0^1(\Omega;\mathbb{R}^n)$ s.t. $|g| \leq 1$*

  $$\int_E \operatorname{div} g\, dx = -\int_{\partial E} g \cdot D\varphi_E \tag{1.6}$$

  Proof. $\Longrightarrow$ follows directly. $\Longleftarrow$ By the previous item, $\int_{\partial E} g \cdot D\varphi_E = \int g \cdot D\varphi_E$. Indeed, $\omega := -D\varphi_E$ has bounded variation on each open bounded Ω. Use the first item that characterises Caccioppoli set. □

- *Given Caccioppoli set E, one has useful identification of $\varphi_E \in BV$*

  Corollary 1.1.1. *For E Caccioppoli, and $\Omega \subset \mathbb{R}^n$ open. If either E or Ω is bounded, $\varphi_E \in BV(\Omega)$.*

  Proof. Since either E or Ω is bounded, $\|\varphi_E\|_{L^1(\Omega)} = |E \cap \Omega| < \infty$ hence $\varphi_E \in L^1(\Omega)$. Now one compute $\int_\Omega |D\varphi_E|$, and may proceed in 2 directions. If Ω itself is bounded, since E Caccioppoli gives locally finite perimeter, indeed $\int_\Omega |D\varphi_E| < \infty$. If on the other hand, E is bounded, for any $g \in C_0^1(\Omega;\mathbb{R}^n)$ s.t. $|g| \leq 1$, using (1.6)

  $$\int_\Omega \varphi_E \operatorname{div} g\, dx = \int_E \operatorname{div} g\, dx = -\int_{\partial E} g \cdot D\varphi_E$$

  ∂E is bounded and closed, hence compact. Then one may cover ∂E using sufficient large open ball B_R, and since E is Caccioppoli, $|D\varphi_E|$ defines locally finite variation positive measure

  $$-\int_{\partial E} g \cdot D\varphi_E \leq \int_{B_R \cap \Omega} |D\varphi_E| < \infty$$

  □

Theorem 1.1.1 (Semi-continuity). *Let $\Omega \subset \mathbb{R}^n$ open. $\{f_j\} \subset BV(\Omega)$ s.t. $f_j \to f$ in $L^1_{loc}(\Omega)$, then*

$$\int_\Omega |Df| \leq \liminf_{j\to\infty} \int_\Omega |Df_j| \tag{1.7}$$

Proof. For any $g \in C^1_0(\Omega; \mathbb{R}^n)$ s.t. $|g| \leq 1$

$$\int_\Omega f \operatorname{div} g \, dx = \lim_{j\to\infty} \int_\Omega f_j \operatorname{div} g \, dx \leq \liminf_{j\to\infty} \int_\Omega |Df_j|$$

take supremum in g on LHS. □

Remark 1.1.3. *The equality in (1.7) may not be achieved. Let $\Omega = (0, 2\pi)$ and $f_j(x) = \frac{1}{j}\sin(jx)$. Note $\int_0^{2\pi} |\frac{1}{j}\sin(jx)|dx \leq 2\pi\frac{1}{j} \to 0$ so $f_j \to 0$ in $L^1(0, 2\pi)$. But $f_j'(x) = \cos(jx)$ and $\int_0^{2\pi} |Df_j| = \int_0^{2\pi} |\cos(jx)|dx = 4$.*

Example 1.1.3 (Rational Balls). *One has some very interesting example set E, whose boundary $|\partial E| = H_{n-1}(\partial E) = \infty$ yet $P(E) = \int |D\varphi_E| < \infty$. In particular, this tells us $\int |D\varphi_E| = H_{n-1}(\partial E)$ is not true for non-smooth boundaries. We construct the rational balls $E := \bigcup_{i=0}^\infty B_i$ s.t.*

$$B_i := B_{\frac{1}{2^i}}(x_i) \text{ where } \{x_i\} \text{ truncates all rationals in } \mathbb{R}^n$$

one may first calculate the measure of E

$$|E| \leq \sum_i |B_i| = \sum_i \omega_n(\frac{1}{2^i})^n = \frac{\omega_n}{1-2^{-n}} < \infty$$

Then since $\overline{E} = \mathbb{R}^n$, we conclude $|\partial E| = H_{n-1}(\partial E) = \infty$. But on the other hand let $E_k = \bigcup_{i=0}^k B_i$ so $\varphi_{E_k} \to \varphi_E$ in $L^1(\mathbb{R}^n)$. Notice E_k has piecewise smooth boundary so $H_{n-1}(\partial E_k) = \int |D\varphi_{E_k}|$, and moreover

$$H_{n-1}(\partial E_k) \leq \sum_{i=0}^k H_{n-1}(\partial B_i) = \sum_{i=0}^k n\omega_n(\frac{1}{2^i})^{n-1} \leq n\omega_n \frac{1}{1-2^{1-n}} < \infty \qquad uniformly \text{ in } k$$

By semicontinuity

$$P(E) = \int |D\varphi_E| \leq \liminf_{k\to\infty} \int |D\varphi_{E_k}| = \liminf_{k\to\infty} H_{n-1}(\partial E_k) \leq n\omega_n \frac{1}{1-2^{1-n}} < \infty$$

So E is a Caccioppoli Set.

Proposition 1.1.1. *For $\Omega \subset \mathbb{R}^n$ open, $BV(\Omega)$ with norm $\|f\|_{BV} := \|f\|_{L^1} + \int_\Omega |Df|$ is a Banach Space.*

Proof. That $\|f\|_{BV}$ defines a norm follows from L^1 norm and homogeneity, subadditivity of total variation. To see $BV(\Omega)$ is complete, take Cauchy sequence $\{f_j\}$ in $BV(\Omega)$. Since $\{f_j\}$ is already Cauchy in $L^1(\Omega)$, there exists $f \in L^1(\Omega)$ s.t. $\|f - f_j\|_{L^1} \to 0$. Also, there exits N s.t. $\forall m, n \geq N$, $\int_\Omega |D(f_m - f_n)| \leq 1$, one has $\int_\Omega |Df_j| \leq \max_{1\leq i\leq N} \int_\Omega |Df_i| + 1$ uniformly bounded. Hence (1.7) semicontinuity gives $\int_\Omega |Df| < \infty$ so $f \in BV(\Omega)$. It suffices to show $\int_\Omega |D(f - f_j)| \to 0$. For any $\varepsilon > 0$, there exists N s.t. for any $j, k \geq N$, $\int_\Omega |D(f_j - f_k)| \leq \varepsilon$. Fix j, apply (1.7) semicontinuity to $\{f_j - f_k\}_k$ so $\int_\Omega |D(f_j - f)| \leq \liminf_{k\to\infty} \int_\Omega |D(f_j - f_k)| \leq \varepsilon$. Take ε to 0. □

Proposition 1.1.2. *Let $\Omega \subset \mathbb{R}^n$ open. $f, f_j \in BV(\Omega)$ s.t. $f_j \to f$ in $L^1_{loc}(\Omega)$ and $\int_\Omega |Df| = \lim_{j\to\infty} \int_\Omega |Df_j|$. Then for any $A \subset \Omega$ open, one has certain reverse direction to (1.7)*

$$\int_{\overline{A}\cap\Omega} |Df| \geq \limsup_{j\to 0} \int_{\overline{A}\cap\Omega} |Df_j|$$

in particular, if $\int_{\partial A\cap\Omega} |Df| = 0$, one has

$$\int_A |Df| = \lim_{j\to 0} \int_A |Df_j| \tag{1.8}$$

Proof. Let $B := \Omega \setminus \overline{A}$ so $B \subset \Omega$ open. By semicontinuity (1.7)

$$\int_A |Df| \leq \liminf_{j\to 0} \int_A |Df_j| \qquad \int_B |Df| \leq \liminf_{j\to 0} \int_B |Df_j|$$

one calculate

$$\begin{aligned}\int_{\overline{A}\cap\Omega} |Df| + \int_B |Df| &= \int_\Omega |Df| = \lim_{j\to\infty} \int_\Omega |Df_j| \\ &\geq \limsup_{j\to 0} \int_{\overline{A}\cap\Omega} |Df_j| + \liminf_{j\to\infty} \int_B |Df_j| \geq \limsup_{j\to 0} \int_{\overline{A}\cap\Omega} |Df_j| + \int_B |Df|\end{aligned}$$

since $f \in BV(\Omega)$, indeed $\int_B |Df| < \infty$ so one may cancel out. To see (1.8), one notice $A \subset \Omega$. □

1.1.2 Approximation by smooth functions

Definition 1.1.4. *$\eta(x)$ is mollifier if* $\begin{cases}\eta \in C_0^\infty(\mathbb{R}^n) \\ \operatorname{supp}\eta \subset B_1 \\ \int \eta\, dx = 1\end{cases}$ *If moreover,* $\begin{cases}\eta \geq 0 \\ \eta(x) = \mu(|x|)\end{cases}$ *η is positive symmetric.*

Standard example for such positive symmetric mollifier is $\eta = \frac{1}{\int \gamma dx}\gamma$ *where* $\gamma(x) := \begin{cases} 0 & |x| \geq 1 \\ \exp(\frac{1}{|x|^2-1}) & |x| < 1\end{cases}$

Definition 1.1.5. *Given a positive symmetric mollifier η, the rescaled mollifier $\eta_\varepsilon(x) := \frac{1}{\varepsilon^n}\eta(\frac{x}{\varepsilon})$ satisfies* $\operatorname{supp}\eta_\varepsilon \subset B_\varepsilon$*. Given $f \in L^1_{loc}(\Omega)$, define its mollification $f_\varepsilon := \eta_\varepsilon * f$*

$$f_\varepsilon(x) = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} \eta(\frac{x-y}{\varepsilon})f(y)\ dy = (-1)^n \int_{\mathbb{R}^n} \eta(z)f(x-\varepsilon z)\, dz = \int_{\mathbb{R}^n} \eta(z) f(x+\varepsilon z)\, dz \tag{1.9}$$

Lemma 1.1.1. *One has tools from mollification*

- *$f_\varepsilon \in C^\infty(\mathbb{R}^n)$, $f_\varepsilon \to f$ in $L^1_{loc}(\Omega)$. If $f \in L^1(\Omega)$, $f_\varepsilon \to f$ in $L^1(\Omega)$.*
- *If $A \leq f(x) \leq B$ for any $x \in \Omega$, then $A \leq f_\varepsilon(x) \leq B$ for any $x \in \Omega$.*
- *If $f, g \in L^1(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} f_\varepsilon g dx = \int_{\mathbb{R}^n} f g_\varepsilon dx$.*
- *If $f \in C^1(\mathbb{R}^n)$, then $(\frac{\partial}{\partial x_i} f)_\varepsilon = \frac{\partial}{\partial x_i}(f_\varepsilon)$ for $i = 1, \cdots, n$.*
- $\operatorname{supp} f := \overline{\{x \in \mathbb{R}^n \mid f \neq 0\}} \subset A$*, then* $\operatorname{supp} f_\varepsilon \subset A_\varepsilon := \{x \mid \operatorname{dist}(x, A) \leq \varepsilon\}$.

Proposition 1.1.3. *$\Omega \subset \mathbb{R}^n$ open, $f \in BV(\Omega)$. For $A \subset\subset \Omega$ open s.t. $\int_{\partial A} |Df| = 0$, one has*

$$\int_A |Df| = \lim_{\varepsilon\to 0} \int_A |Df_\varepsilon| dx \tag{1.10}$$

Proof. Since $f \in L^1(\Omega)$, $f_\varepsilon \to f$ in $L^1(\Omega)$, by semicontinuity (1.7), one has $\int_A |Df| \leq \liminf_{\varepsilon\to 0} \int_A |Df_\varepsilon|$. It suffices to prove $\int_A |Df| \geq \limsup_{\varepsilon\to 0} \int_A |Df_\varepsilon|$. For any $g \in C^1_0(A; \mathbb{R}^n)$ s.t. $|g| \leq 1$, using tools from mollification

$$\int_A f_\varepsilon \operatorname{div} g\, dx = \int_A f(\operatorname{div} g)_\varepsilon\, dx = \int_A f \operatorname{div}(g_\varepsilon)\, dx$$

$|g| \leq 1 \implies |g_\varepsilon| \leq 1$ and $\operatorname{supp} g \subset A \implies \operatorname{supp} g_\varepsilon \subset A_\varepsilon$. Hence taking supremum in g

$$\int_A |Df_\varepsilon| \leq \int_{A_\varepsilon} |Df|$$

Take lim sup on LHS and use continuity from above on RHS ($f \in BV(\Omega)$ defines a Radon measure $|Df|$)

$$\limsup_{\varepsilon\to 0} \int_A |Df_\varepsilon| \leq \lim \int_{A_\varepsilon} |Df| = \int_{\overline{A}} |Df|$$

Now by our assumption, RHS equals $\int_A |Df|$. □

Remark 1.1.4. *Note in (1.10) we require $A \subset\subset \Omega$ not because we need boundedness, but because we wish that A and A_ε do not touch $\partial\Omega$. And this problem is resolved for taking $\Omega = \mathbb{R}^n$, and indeed, one may do so for $A = A_\varepsilon = \mathbb{R}^n$ ($\partial A = \partial\mathbb{R}^n = \varnothing$). Now for any $f \in BV(\mathbb{R}^n)$, one has*

$$\int_{\mathbb{R}^n} |Df| = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} |Df_\varepsilon|\, dx \tag{1.11}$$

Indeed for E bounded Caccioppoli, $\varphi_E \in BV(\mathbb{R}^n)$ by Corollary 1.1.1, so (1.11) applies to φ_E.

(1.10) motivates our approximation of $f \in BV(\Omega)$ using smooth functions. Note approximation in BV norm should not be expected since the BV-closure of $C^\infty(\Omega)$ is $W^{1,1}(\Omega) \subsetneq BV(\Omega)$.

Theorem 1.1.2 (Approximation using C^∞)**.** *$\Omega \subset \mathbb{R}^n$ open, $f \in BV(\Omega)$. There exists $\{f_j\} \subset C^\infty(\Omega)$ s.t.*

$$\lim_{j\to\infty} \int_\Omega |f_j - f|\, dx = 0 \tag{1.12}$$

$$\lim_{j\to 0} \int_\Omega |Df_j| dx = \int_\Omega |Df| \tag{1.13}$$

Proof. Since $f \in BV(\Omega)$, $|Df|$ on Ω is finite measure, so $\forall\, \varepsilon > 0$, there exists $m \in \mathbb{N}$ s.t. $\int_{\Omega \setminus \Omega_0} |Df| < \varepsilon$ where

$$\Omega_k := \left\{ x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) > \frac{1}{m+k} \right\} \qquad k \geq 0 \tag{1.14}$$

Define sequence $\{A_i\}_{i\geq 1}$ s.t. $A_1 := \Omega_2$, $A_i := \Omega_{i+1} \setminus \overline{\Omega}_{i-1}$ for $i \geq 2$. Note A_i are open and $\Omega \subset \bigcup_{i\geq 1} A_i$. There exists smooth partition of unity $\{\phi_i\}$ subordinate to the cover $\{A_i\}$ s.t.

$$\phi_i \in C_0^\infty(A_i), \quad 0 \leq \phi_i \leq 1, \quad \sum_{i=1}^{\infty} \phi_i = 1$$

Note for any $x \in \Omega$, at most 2 of the A_i covers x, hence $\sum_i \phi_i$ is finite sum pointwise, thus $f = \sum_{i=1}^{\infty} f\phi_i$. One wish to construct certain mollification of f so that our desired approximation holds, and a common method is to mollify each $f\phi_i$ with ε_i chose for each $i \geq 1$ then sum them up. Each ε_i needs to satisfy (let $\Omega_{-1} := \varnothing$)

$$\mathrm{supp}(\eta_{\varepsilon_i} * (f\phi_i)) \subset \Omega_{i+2} \setminus \overline{\Omega}_{i-2} \tag{1.15}$$

$$\|\eta_{\varepsilon_i} * (f\phi_i) - f\phi_i\|_{L^1(\Omega)} < \varepsilon/2^i \tag{1.16}$$

$$\|\eta_{\varepsilon_i} * (fD\phi_i) - fD\phi_i\|_{L^1(\Omega)} < \varepsilon/2^i \tag{1.17}$$

and define $f_\varepsilon := \sum_{i=1}^{\infty} \eta_{\varepsilon_i} * (f\phi_i)$. Note $f_\varepsilon \in C^\infty(\Omega)$ since at each $x \in \Omega$, at most 4 supports from (1.15) covers x, hence finite sum of smooth functions gives smoothness. One immediately has from (1.16)

$$\int_\Omega |f_\varepsilon - f| dx \leq \sum_{i=1}^{\infty} \int_\Omega |\eta_{\varepsilon_i} * (f\phi_i) - f\phi_i|\, dx < \varepsilon$$

hence (1.12) holds. And by semicontinuity (1.7), one has $\int_\Omega |Df| \leq \liminf_{\varepsilon\to 0} \int_\Omega |Df_\varepsilon|$. It suffices to prove $\int_\Omega |Df| \geq \limsup_{\varepsilon \to 0} \int_\Omega |Df_\varepsilon|$. For any $g \in C_0^1(\Omega; \mathbb{R}^n)$ s.t. $|g| \leq 1$,

$$\int_\Omega f_\varepsilon \mathrm{div}\, g\, dx = \sum_{i=1}^{\infty} \int_\Omega \eta_{\varepsilon_i} * (f\phi_i) \mathrm{div}\, g\, dx = \sum_{i=1}^{\infty} \int_\Omega f\phi_i \mathrm{div}\, (\eta_{\varepsilon_i} * g)\, dx$$

notice

$$\mathrm{div}(\phi_i\, \eta_{\varepsilon_i} * g) = D\phi_i \cdot (\eta_{\varepsilon_i} * g) + \phi_i\, \mathrm{div}\, (\eta_{\varepsilon_i} * g)$$

hence

$$\begin{aligned}
\int_\Omega f_\varepsilon \mathrm{div}\, g\, dx &= \sum_{i=1}^{\infty} \int_\Omega f \left[\mathrm{div}(\phi_i\, \eta_{\varepsilon_i} * g) - D\phi_i \cdot (\eta_{\varepsilon_i} * g)\right] dx \\
&= \int_\Omega f \mathrm{div}(\phi_1\, \eta_{\varepsilon_1} * g)\, dx + \sum_{i=2}^{\infty} \int_\Omega f \mathrm{div}(\phi_i\, \eta_{\varepsilon_i} * g)\, dx - \sum_{i=1}^{\infty} \int_\Omega f D\phi_i \cdot (\eta_{\varepsilon_i} * g)\, dx \\
&= \int_\Omega f \mathrm{div}(\phi_1\, \eta_{\varepsilon_1} * g)\, dx + \sum_{i=2}^{\infty} \int_\Omega f \mathrm{div}(\phi_i\, \eta_{\varepsilon_i} * g)\, dx - \sum_{i=1}^{\infty} \int_\Omega \eta_{\varepsilon_i} * (fD\phi_i) \cdot g\, dx
\end{aligned}$$

notice the pointwise finite sum implies

$$\sum_{i=1}^{\infty} \phi_i = 1 \implies \sum_{i=1}^{\infty} D\phi_i = 0$$

hence one may add back the sum of gradients

$$\int_\Omega f_\varepsilon \mathrm{div}\, g\, dx = \int_\Omega f \mathrm{div}(\phi_1\, \eta_{\varepsilon_1} * g)\, dx + \sum_{i=2}^{\infty} \int_\Omega f \mathrm{div}(\phi_i\, \eta_{\varepsilon_i} * g)\, dx - \sum_{i=1}^{\infty} \int_\Omega [\eta_{\varepsilon_i} * (fD\phi_i) - fD\phi_i] \cdot g\, dx$$

now by direct estimate, (1.15) and (1.17) respectively

$$\int_\Omega f \mathrm{div}(\phi_1\, \eta_{\varepsilon_1} * g)\, dx \leq \int_\Omega |Df|$$

$$\sum_{i=2}^{\infty} \int_\Omega f \mathrm{div}(\phi_i\, \eta_{\varepsilon_i} * g)\, dx \leq 3 \int_{\Omega\setminus\Omega_0} |Df| < 3\varepsilon$$

$$\sum_{i=1}^{\infty} \int_\Omega [\eta_{\varepsilon_i} * (fD\phi_i) - fD\phi_i] \cdot g\, dx < \varepsilon$$

Hence taking supremum in g on LHS gives

$$\int_\Omega |Df_\varepsilon| \leq \int_\Omega |Df| + 4\varepsilon \qquad \implies \limsup_{\varepsilon\to 0} \int_\Omega |Df_\varepsilon| \leq \int_\Omega |Df|$$

and (1.13) immediately follows. □

Remark 1.1.5 (Boundary Behavior of Smooth Approximation)**.** *$\Omega \subset \mathbb{R}^n$ open, $f \in BV(\Omega)$. For every $\varepsilon > 0$, $N > 0$ and $x_0 \in \partial\Omega$, let f_ε be as above*

$$\lim_{\rho \to 0} \frac{1}{\rho^N} \int_{B_\rho(x_0) \cap \Omega} |f_\varepsilon - f| \, dx = 0 \tag{1.18}$$

Proof. For $\varepsilon > 0$, choose $m \in \mathbb{N}$, Ω_k as in (1.14) and f_ε as in Theorem 1.1.2. One wish to determine i_0 w.r.t. ρ so that for any $x \in B_\rho(x_0) \cap \Omega$, one may write

$$f_\varepsilon(x) - f(x) = \sum_{i=1}^{\infty} (\eta_{\varepsilon_i} * (f\phi_i) - f\phi_i) = \sum_{i=i_0}^{\infty} (\eta_{\varepsilon_i} * (f\phi_i) - f\phi_i)$$

Making use of (1.15), one needs i_0 to be the smallest integer i s.t. $\partial B_\rho(x_0) \cap \Omega$ touches supp$\eta_{\varepsilon_i} * (f\phi_i)$, i.e.

$$\frac{1}{m + i_0 + 2} \leq \rho \leq \frac{1}{m + i_0 + 1} \implies i_0 = \lceil \frac{1}{\rho} \rceil - m - 2$$

thus via (1.16), for some constant C independent of ρ

$$\int_{B_\rho(x) \cap \Omega} |f_\varepsilon - f| \, dx \leq \sum_{i=i_0}^{\infty} \|\eta_{\varepsilon_i} * (f\phi_i) - f\phi_i\|_{L^1(\Omega)} \leq C\, 2^{-i_0} = C\, 2^{-\frac{1}{\rho}}$$

where $2^{-\frac{1}{\rho}}$ goes to 0 exponentially fast. Hence multiplying both sides by $\frac{1}{\rho^N}$ and sending $\rho \to 0$ gives (1.18). □

1.1.3 Compactness Theorem and Existence of Minimizing Caccioppoli sets

One shall recall the GNS type Sobolev Embedding and Rellich Theorem from Sobolev Spaces.

Lemma 1.1.2 (Sobolev Embedding)**.** *$\Omega \subset \mathbb{R}^n$ bounded open. $\partial\Omega$ Lipschitz continuous. $1 \leq p \leq n$. Then*

$$W^{1,p}(\Omega) \subset L^q(\Omega) \qquad \forall\, 1 \leq q \leq \frac{np}{n-p} \tag{1.19}$$

i.e., for any such $1 \leq q \leq \frac{np}{n-p}$, there exists $C = C(n,\, p,\, q,\, \Omega)$ s.t.

$$\|f\|_{L^q} \leq C\, \|f\|_{W^{1,p}} \tag{1.20}$$

Lemma 1.1.3 (Rellich-Kondrachov)**.** *$\Omega \subset \mathbb{R}^n$ bounded open. $\partial\Omega$ Lipschitz continuous. $1 \leq p < n$. Then*

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega) \qquad \forall\, 1 \leq q < \frac{np}{n-p} \tag{1.21}$$

i.e., each uniformly bounded sequence $\{f_j\}$ in $W^{1,p}(\Omega)$ norm has a convergent subsequence $\{f_{j_k}\}$ in $L^q(\Omega)$ norm for each $q \in [1,\, \frac{np}{n-p})$.

Using above lemmas, one may show for the corresponding BV Embedding and a Compactness Theorem.

Theorem 1.1.3 (GNS-type BV Embedding)**.** *$\Omega \subset \mathbb{R}^n$ bounded open. $\partial\Omega$ Lipschitz continuous. Then*

$$BV(\Omega) \subset L^p(\Omega) \qquad \forall\, 1 \leq p \leq \frac{n}{n-1} \tag{1.22}$$

i.e., for any such $1 \leq p \leq \frac{n}{n-1}$, there exists $C = C(n,\, p,\, \Omega)$ s.t.

$$\|f\|_{L^p} \leq C\, \|f\|_{BV} \tag{1.23}$$

Proof. For any $f \in BV(\Omega)$, by smooth approximation Theorem 1.1.2, choose $\{f_j\} \subset C^\infty(\Omega)$ s.t. $\|f_j - f\|_{L^1} \to 0$ and $\int_\Omega |Df| = \lim_{j \to 0} |Df_j|$. Then there exists M large enough s.t. $\|f_j\|_{BV} \leq M$ uniformly. Since $C^\infty(\Omega) \subset W^{1,1}(\Omega)$, by Sobolev Embedding (1.19), for any $1 \leq p \leq \frac{n}{n-1}$, there exists $C = C(n,\, p,\, \Omega)$ s.t.

$$\|f_j\|_{L^p} \leq C \left(\|f_j\|_{L^1} + \|Df_j\|_{L^1} \right) \leq CM$$

uniformly in j. If $p = 1$, by definition of BV norm there's nothing to prove. For $1 < p \leq \frac{n}{n-1}$, the uniform boundedness of f_j in L^p implies, from reflexivity of L^p and Banach Alaoglu, a weakly convergent subsequence in L^p. Still denoting f_j, ones has $f_0 \in L^p$ s.t. $f_j \rightharpoonup f_0$ in L^p. Since Ω is bounded, by Hölder, a priori one knows $f_j,\, f_0 \in L^1(\Omega)$, and for any $g \in (L^1(\Omega))^* = L^\infty(\Omega)$ (so $g^{\frac{p-1}{p}} \in L^{p'}(\Omega)$)

$$|\int_\Omega (f_j - f_0)\, g|\, dx = |\int_\Omega (f_j - f_0) g^{\frac{p-1}{p}}\, g^{\frac{1}{p}}|\, dx \leq |\int_\Omega (f_j - f_0) g^{\frac{p-1}{p}}\, dx| \left\| g^{\frac{1}{p}} \right\|_{L^\infty(\Omega)} \to 0$$

hence one has $f_j \rightharpoonup f_0$ in L^1. But since we already know $f_j \to f$ in L^1, by uniqueness of L^1 strong limit, $f_0 = f$. Finally, by lower semicontinuity of weak convergence,

$$\|f\|_{L^p} \leq \liminf_{j\to 0} \|f_j\|_{L^p} \leq C \liminf_{j\to 0} \left(\|f_j\|_{L^1} + \|Df_j\|_{L^1}\right) = C\,\|f\|_{BV}$$

□

Theorem 1.1.4 (Compactness). *$\Omega \subset \mathbb{R}^n$ bounded open. $\partial\Omega$ Lipschitz continuous. Then*

$$BV(\Omega) \subset\subset L^p(\Omega) \qquad \forall\, 1 \leq p < \frac{n}{n-1} \tag{1.24}$$

i.e., each uniformly bounded sequence $\{f_j\}$ in $BV(\Omega)$ norm has a convergent subsequence $\{f_{j_k}\}$ in $L^p(\Omega)$ norm for each $p \in [1, \frac{n}{n-1})$. Moreover, the limiting function $f \in BV(\Omega)$.

Proof. Let $\{f_j\} \subset BV(\Omega)$ uniformly bounded by $\|f_j\|_{BV(\Omega)} \leq M$. By smooth approximation Theorem 1.1.2, $\forall\, j$, choose $\tilde{f}_j \in C^\infty(\Omega)$ s.t.

$$\int_\Omega |f_j - \tilde{f}_j| < \frac{1}{j}, \qquad \int_\Omega |D\tilde{f}_j| dx \leq M + 2$$

Now since $\{\tilde{f}_j\} \subset C^\infty(\Omega)$ is uniformly bounded in $W^{1,1}(\Omega)$ norm, by Rellich (1.21), there exists convergent subsequence, still denoting $\tilde{f}_j$, in L^p for any $1 \leq p < \frac{n}{n-1}$. Fix any such p, let $f \in L^p(\Omega)$ s.t. $\left\|\tilde{f}_j - f\right\|_{L^p} \to 0$. Note Ω is bounded, hence Hölder inequality gives convergence in L^1 (p' Hölder conjugate w.r.t p)

$$\int_\Omega |f - \tilde{f}_j|\, dx \leq \left(\int_\Omega |f - \tilde{f}_j|^p\, dx\right)^{\frac{1}{p}} |\Omega|^{\frac{1}{p'}} \to 0$$

and then one may apply semicontinuity (1.7) which gives

$$\int_\Omega |Df| \leq \liminf_{j\to\infty} |D\tilde{f}_j| dx \leq M + 2 < \infty$$

to conclude $f \in BV(\Omega)$. It suffices to show $\|f_j - f\|_{L^p} \to 0$. But by Minkowski

$$\|f_j - f\|_{L^p} \leq \left\|f_j - \tilde{f}_j\right\|_{L^p} + \left\|\tilde{f}_j - f\right\|_{L^p}$$

where the former term convergence due to BV Embedding (1.22) and DCT

$$|f_j - \tilde{f}_j|^p \leq |f_j|^p + |\tilde{f}_j|^p \in L^1(\Omega) \implies \left\|f_j - \tilde{f}_j\right\|_{L^p} \to 0$$

and the latter term converges by Rellich (1.21)

□

Theorem 1.1.5 (Existence of Minimizing Caccioppoli Set). *Let $\Omega \subset \mathbb{R}^n$ be bounded open, and let L be a Caccioppoli Set. Then there exists a Caccioppoli set E s.t. $E = L$ outside Ω and*

$$\int |D\varphi_E| = \inf\{\int |D\varphi_F| \mid F = L \textit{ outside } \Omega\}$$

i.e., $\exists$ E Caccioppoli s.t. $E = L$ outside Ω and

$$\int |D\varphi_E| \leq \int |D\varphi_F| \tag{1.25}$$

for any $F \subset \mathbb{R}^n$ Borel s.t. $F = L$ outside Ω.

Proof. One wish to use compactness that extracts a convergent subsequence in L^1. But notice we have no information about regularity of $\partial\Omega$, hence we first take $R > 0$ large s.t. $\Omega \subset\subset B_R(0)$ ball of radius R and we work with B_R. Take a minimizing sequence of sets $\{E_j\}$ s.t. $E_j = L$ outside Ω for any j and

$$\lim_{j\to\infty} \int_{B_R} |D\varphi_{E_j}| = \inf\{\int_{B_R} |D\varphi_F| \mid F = L \text{ outside } \Omega\} \tag{1.26}$$

notice L itself agrees with L outside Ω and since L is a Caccioppoli set, on B_R bounded open, $\int_{B_R} |D\varphi_L| < \infty$. Hence the RHS of (1.26) $< \infty$. Now we may take M large enough so $\int_{B_R} |D\varphi_{E_j}| < M$ uniformly bounded.

And since B_R are bounded, $\varphi_{E_j} \in L^1(B_R)$ for any j, and in particular, $\left\|\varphi_{E_j}\right\|_{L^1(B_R)} \leq |B_R| < \infty$ uniformly, so $\{\varphi_{E_j}\} \subset BV(B_R)$ is uniformly bounded in BV norm. B_R has smooth boundary, so Theorem 1.1.4 gives a convergent subsequence $\varphi_{E_j} \to f$ in $L^1(B_R)$. Again passing to subsequence, $\varphi_{E_j} \to f$ pointwise a.e., but φ_{E_j} are characteristic functions, so $f = \varphi_E$ agrees with characteristic function of some Borel set E a.e. Indeed $E = E_j = L$ outside Ω. And since $\varphi_{E_j} \to \varphi_E$ in $L^1(B_R)$, by semicontinuity (1.7), $\int_{B_R} |D\varphi_E| \leq \lim_{j\to\infty} \int_{B_R} |D\varphi_{E_j}|$

$$\int_{B_R} |D\varphi_E| = \inf\{\int_{B_R} |D\varphi_F| \mid F = L \text{ outside } \Omega\}$$

Finally we recover estimate on $\mathbb{R}^n$ from B_R. For any $F \subset \mathbb{R}^n$ Borel s.t. $F = L$ outside Ω

$$\begin{aligned}\int |D\varphi_E| &= \int_{B_R} |D\varphi_E| + \int_{B_R^c} |D\varphi_E| = \int_{B_R} |D\varphi_E| + \int_{B_R^c} |D\varphi_L| \\ &\leq \int_{B_R} |D\varphi_F| + \int_{B_R^c} |D\varphi_L| = \int_{B_R} |D\varphi_F| + \int_{B_R^c} |D\varphi_F| = \int |D\varphi_F|\end{aligned}$$

Since one may take $F = L$ to be the Caccioppoli Set, E is a Caccioppoli Set. □

Remark 1.1.6. *One has information for the minimizing set E from Theorem 1.1.5.*

- *L determines boundary values for E. Since $D\varphi_E$ is supported within ∂E, or more particularly, imagine E smooth so $\int_\Omega |D\varphi_E| = H_{n-1}(\partial E \cap \Omega)$ really measures the surface area of ∂E within Ω, then (1.25) indicates that '∂E within Ω' minimizes the surface area for all 'sets **within** Ω that has boundary $\partial L \cap \partial\Omega$'.*
- *Imagine $\partial L \cap \partial\Omega$ fixed, then it determines a surface spanning $\partial L \cap \partial\Omega$. But now curve the portion $\Omega \cap L$ towards Ω, it serves as obstacle forcing '∂E within Ω' away from the minimal surface spanned by $\partial L \cap \partial\Omega$.*

1.1.4 Coarea formula and Smooth Approximation of Caccioppolis sets

One shall recall Coarea formula for Lipschitz functions

Lemma 1.1.4 (Coarea Formula)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz for $n \geq m$. Then for any $A \subset \mathbb{R}^n$ Borel*

$$\int_A \sqrt{\det(Df^* Df)}(x)\, dx = \int_{\mathbb{R}^m} H_{n-m}\left(A \cap f^{-1}(y)\right) dy \tag{1.27}$$

With the Classical Coarea formula, one may prove for BV functions.

Theorem 1.1.6 (Coarea Formula)**.** *$\Omega \subset \mathbb{R}^n$ open. $f \in BV(\Omega)$. Denote $F_t := \{x \in \Omega \mid f(x) < t\}$, then*

$$\int_\Omega |Df| = \int_{-\infty}^{\infty} \left(\int_\Omega |D\varphi_{F_t}|\right) dt \tag{1.28}$$

Proof. $\leq$. First let $f \geq 0$. $\forall\, x \in \Omega$, $f(x) = \int_0^\infty \varphi_{F_t^c}\, dt = \int_0^\infty (1 - \varphi_{F_t})\, dt$, so $\forall\, g \in C_0^1(\Omega; \mathbb{R}^n)$ s.t. $|g| \leq 1$

$$\int_\Omega f \operatorname{div} g\, dx = \int_\Omega \left(\int_0^\infty (1 - \varphi_{F_t})\, dt\right) \operatorname{div} g\, dx = \int_0^\infty \left(\int_\Omega \operatorname{div} g\, dx - \int_\Omega \varphi_{F_t} \operatorname{div} g\, dx\right) dt$$

By Fubini, and then note compact support of g

$$= -\int_0^\infty \int_\Omega \varphi_{F_t} \operatorname{div} g\, dx\, dt \leq \int_0^\infty \int_\Omega |D\varphi_{F_t}|\, dt$$

Then let $f \leq 0$. $\forall\, x \in \Omega$, $f(x) = -\int_{-\infty}^0 \varphi_{F_t}\, dt$, so $\forall\, g \in C_0^1(\Omega; \mathbb{R}^n)$ s.t. $|g| \leq 1$

$$\int_\Omega f \operatorname{div} g\, dx = -\int_\Omega \left(\int_{-\infty}^0 \varphi_{F_t}\, dt\right) \operatorname{div} g\, dx = -\int_{-\infty}^0 \left(\int_\Omega \varphi_{F_t} \operatorname{div} g\, dx\right) dt \leq \int_{-\infty}^0 \int_\Omega |D\varphi_{F_t}|\, dt$$

Hence for any $f \in BV(\Omega)$, write $f = f^+ - f^-$ for $f^+, f^- \geq 0$, so

$$\int_\Omega f \operatorname{div} g\, dx \leq \int_\Omega \left(f^+ - f^-\right) \operatorname{div} g\, dx \leq \int_{-\infty}^\infty \int_\Omega |D\varphi_{F_t}|\, dt$$

taking supremum in g gives $\int_\Omega |Df| \leq \int_{-\infty}^\infty \int_\Omega |D\varphi_{F_t}|\, dt$.
$\geq$. One first show (1.28) for $f \in C(\Omega)$ continuous piecewise linear function. Let $\Omega = \bigcup_{i=1}^N \Omega_i$ for Ω_i disjoint,

open where $f(x) = \langle a_i, x\rangle + b_i$ for $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, $x \in \Omega_i$. Then $\int_\Omega |Df| = \sum_{i=1}^N |a_i||\Omega_i|$. On the other hand, F_t now has piecewise smooth boundary, so

$$\int_{\Omega_i} |D\varphi_{F_t}| = H_{n-1}(\partial F_t \cap \Omega_i) = H_{n-1}\{x \in \Omega_i \mid f(x) = t\} = H_{n-1}\{x \in \Omega_i \mid \langle a_i, x\rangle + b_i = t\}$$

Hence integrating w.r.t. t and by change of coordinates

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{\Omega_i} |D\varphi_{F_t}|\, dt &= \int_{-\infty}^{\infty} H_{n-1}\{x \in \Omega_i \mid \langle a_i, x\rangle + b_i = t\}\, dt \\ &= \int_{-\infty}^{\infty} |a_i| H_{n-1}\left\{x \in \Omega_i \mid \frac{\langle a_i, x\rangle}{|a_i|} + \frac{b_i}{|a_i|} = \frac{t}{|a_i|}\right\} d\left(\frac{t}{|a_i|}\right) \\ &= |a_i| \int_{-\infty}^{\infty} H_{n-1}\left(\Omega_i \cap \left\{\frac{\langle a_i, x\rangle}{|a_i|} + \frac{b_i}{|a_i|} = t\right\}\right) dt\end{aligned}$$

using Classical Coarea formula (1.27) with m = 1

$$= |a_i| \int_{\Omega_i} 1\, dx = |a_i||\Omega_i|$$

hence for $f \in C(\Omega)$ piecewise linear, (1.28) holds

$$\int_\Omega |Df| = \sum_{i=1}^N |a_i||\Omega_i| = \sum_{i=1}^N \int_{-\infty}^{\infty}\int_{\Omega_i} |D\varphi_{F_t}|\, dt = \int_{-\infty}^{\infty}\int_{\Omega} |D\varphi_{F_t}|\, dt$$

Now take any $f \in C^\infty(\Omega)$, approximate using sequence of $\{f_j\} \subset C(\Omega)$ continuous piecewise linear functions in $W^{1,1}(\Omega)$ norm. In particular, one has

$$\|f - f_j\|_{L^1(\Omega)} \to 0, \qquad \|Df\|_{L^1(\Omega)} = \lim_{j\to 0} \|Df_j\|_{L^1(\Omega)} \tag{1.29}$$

where the latter follows from $\|Df - Df_j\|_{L^1(\Omega)} \to 0$ and DCT. Denoting $F_{j,t} := \{x \in \Omega \mid f_j(x) < t\}$, one has

$$|f(x) - f_j(x)| = \int_{-\infty}^{\infty} |\varphi_{F_t}(x) - \varphi_{F_{j,t}}(x)|\, dt \implies \|f - f_j\|_{L^1(\Omega)} = \int_{-\infty}^{\infty}\int_\Omega |\varphi_{F_t}(x) - \varphi_{F_{j,t}}(x)|\, dx\, dt \to 0$$

hence there exists a subsequence $\varphi_{F_{j,t}} \to \varphi_{F_t}$ in $L^1(\Omega)$ a.e. t. Since (1.28) holds for each f_j,

$$\int_\Omega |Df| = \lim_{j\to 0}\int_\Omega |Df_j| = \lim_{j\to 0}\int_{-\infty}^{\infty}\int_\Omega |D\varphi_{F_{j,t}}|\, dt$$

one apply Fatou w.r.t. t

$$\geq \int_{-\infty}^{\infty}\left(\liminf_{j\to 0}\int_\Omega |D\varphi_{F_{j,t}}|\right) dt$$

then apply semicontinuity (1.7) for BV function

$$\geq \int_{-\infty}^{\infty}\int_\Omega |D\varphi_{F_t}|\, dt$$

and we conclude (1.28) for $f \in C^\infty(\Omega)$. But notice, we've really only used (1.29) in the above argument. Hence for any $f \in BV(\Omega)$, by Theorem 1.1.2, one may choose $\{f_j\} \subset C^\infty(\Omega)$ s.t. (1.29) holds. Then run the argument again, we conclude (1.28) for $f \in BV(\Omega)$.

□

To show for smooth approximation of sets, one needs Sard's lemma for smooth boundary construction.

Lemma 1.1.5 (Sard's Lemma). *$f : \mathbb{R}^n \to \mathbb{R}^m$ C^k where $k \geq \max\{n - m + 1, 1\}$. Let*

$$X := \{x \in \mathbb{R}^n \mid Jf(x) := \begin{bmatrix}\nabla f_1 \\ \cdots \\ \nabla f_m\end{bmatrix}(x) \text{ has rank } < m\}$$

denote the set of critical points of f. Then the image $f(X)$ has Lebesgue measure 0 in $\mathbb{R}^m$. In particular, if $m = 1$, then given C^k map $f : \mathbb{R}^n \to \mathbb{R}$ for $k \geq n$, one has

$$\partial\{x \in \mathbb{R}^n \mid f(x) < t\} = \{x \in \mathbb{R}^n \mid f(x) = t\} \quad C^k \text{ boundary for a.e. } t \in \mathbb{R} \tag{1.30}$$

Theorem 1.1.7 (Smooth approximation of Caccioppoli Set)**.** *For $E \subset \mathbb{R}^n$ bounded Caccioppoli set, there exists E_j sets with C^∞ boundary s.t.*

$$\int |\varphi_{E_j} - \varphi_E| dx \to 0 \qquad \int |D\varphi_E| = \lim_{j\to 0} \int |D\varphi_{E_j}| \tag{1.31}$$

Proof. Let η_ε be positive symmetric mollifier. For E Caccioppoli, one look at the mollification $(\varphi_E)_\varepsilon = \eta_\varepsilon * \varphi_E$. Since $(\varphi_E)_\varepsilon$ smooth and compactly supported, indeed $(\varphi_E)_\varepsilon \in BV(\mathbb{R}^n)$. Observe $0 \le (\varphi_E)_\varepsilon \le 1$ as inherited from φ_E, and denoting the set $E_{\varepsilon,t} := \{x \in \mathbb{R}^n \mid (\varphi_E)_\varepsilon(x) < t\}$, one has, by Coarea formula (1.28)

$$\int |D(\varphi_E)_\varepsilon| = \int_0^1 \left(\int |D\, \varphi_{E_{\varepsilon,t}}| \right) dt \tag{1.32}$$

But since E is bounded Caccioppoli, Corollary 1.1.1 gives $\varphi_E \in BV(\mathbb{R}^n)$. One may thus apply global mollification approximation (1.11)

$$\int |D\varphi_E| = \lim_{\varepsilon\to 0} \int |D(\varphi_E)_\varepsilon| = \lim_{\varepsilon\to 0} \int_0^1 \left(\int |D\, \varphi_{E_{\varepsilon,t}}| \right) dt$$

One now aims for the following claim. One wish to show for any $0 < t < 1$,

$$\int |\varphi_{E^c_{\varepsilon,t}} - \varphi_E|\, dx \le \frac{1}{\min\{1-t,\, t\}} \int |\,(\varphi_E)_\varepsilon - \varphi_E|\, dx \tag{1.33}$$

To do so, observe

$$\begin{aligned} (\varphi_E)_\varepsilon - \varphi_E \ge t \qquad & \textit{on } E^c_{\varepsilon,t} \setminus E \\ \varphi_E - (\varphi_E)_\varepsilon \ge 1 - t \qquad & \textit{on } E \setminus E^c_{\varepsilon,t} \end{aligned}$$

Hence

$$\begin{aligned} \int |(\varphi_E)_\varepsilon - \varphi_E|\, dx &= \int_{E^c_{\varepsilon,t}\setminus E} |(\varphi_E)_\varepsilon - \varphi_E|\, dx + \int_{E\setminus E^c_{\varepsilon,t}} |(\varphi_E)_\varepsilon - \varphi_E|\, dx \\ &\ge t\, |E^c_{\varepsilon,t} \setminus E| + (1-t)\, |E \setminus E^c_{\varepsilon,t}| \ge \min\{1-t,\, t\} \int |\varphi_{E^c_{\varepsilon,t}} - \varphi_E|\, dx \end{aligned}$$

which gives (1.33). By mollification, since $\varphi_E \in L^1(\mathbb{R}^n) \subset BV(\mathbb{R}^n)$, $\|(\varphi_E)_\varepsilon - \varphi_E\|_{L^1} \to 0$, hence RHS of (1.33) converges to 0 as $\varepsilon \to 0$ for each t, implying $\left\|\varphi_{E^c_{\varepsilon,t}} - \varphi_E\right\|_{L^1} \to 0$ for each t. But since E bounded, $E^c_{\varepsilon,t} = \{x \mid (\varphi_E)_\varepsilon \ge t\}$ is also bounded for any $0 < t < 1$. And because $\partial E^c_{\varepsilon,t} = \{x \mid (\varphi_E)_\varepsilon = t\}$ is smooth, from example 1.1.2, one has $\varphi_{E^c_{\varepsilon,t}} \in BV(\mathbb{R}^n)$. Hence for $0 < t < 1$, one has semicontinuity (1.7)

$$\liminf_{\varepsilon\to 0} \int |D\varphi_{E^c_{\varepsilon,t}}| \ge \int |D\varphi_E|$$

But because $\text{supp} D\varphi_{E^c_{\varepsilon,t}} \subset \partial E^c_{\varepsilon,t}$, under total variation, one has $\int |D\varphi_{E^c_{\varepsilon,t}}| = \int |D\varphi_{E_{\varepsilon,t}}|$. So

$$\int |D\varphi_E| = \lim_{\varepsilon\to 0} \int |D(\varphi_E)_\varepsilon| = \lim_{\varepsilon\to 0} \int_0^1 \left(\int |D\, \varphi_{E_{\varepsilon,t}}| \right) dt$$

By Fatou w.r.t. t

$$\ge \int_0^1 \left(\liminf_{\varepsilon\to 0} \int |D\varphi_{E_{\varepsilon,t}}| \right) dt = \int_0^1 \left(\liminf_{\varepsilon\to 0} \int |D\varphi_{E^c_{\varepsilon,t}}| \right) dt \ge \int |D\varphi_E|$$

now combining $\begin{cases} \liminf_{\varepsilon\to 0} \int |D\varphi_{E^c_{\varepsilon,t}}| \ge \int |D\varphi_E| \\ \int_0^1 \left(\liminf_{\varepsilon\to 0} \int |D\varphi_{E^c_{\varepsilon,t}}| \right) dt = \int |D\varphi_E| \end{cases}$ one must have for a.e. $0 < t < 1$

$$\liminf_{\varepsilon\to 0} \int |D\varphi_{E^c_{\varepsilon,t}}| = \int |D\varphi_E|$$

Now one is ready to apply Sard's lemma (1.30) to the set $\partial E^c_{\varepsilon,t} = \{x \in \mathbb{R}^n \mid (\varphi_E)_\varepsilon = t\}$, resulting in smooth boundary of $\partial E^c_{\varepsilon,t}$ for a.e. $0 < t < 1$. Take one such t. we have obtained

$$\begin{cases} \partial E^c_{\varepsilon,t} \quad \textit{smooth} \\ \left\|\varphi_{E^c_{\varepsilon,t}} - \varphi_E\right\|_{L^1} \to 0 \\ \liminf_{\varepsilon\to 0} \int |D\varphi_{E^c_{\varepsilon,t}}| = \int |D\varphi_E| \end{cases}$$

Take subsequence ε_j s.t. $\varepsilon_j \to 0$ as $j \to \infty$ and $\int |D\varphi_E| = \lim_{j\to 0} \int |D\varphi^c_{E_{\varepsilon_j},t}|$. Define $E_j := E^c_{\varepsilon_j,t}$. ∎

Remark 1.1.7. *Notice E_j bounded and smooth ensures $\varphi_{E_j} \subset BV(\mathbb{R}^n)$, and E bounded Caccioppoli ensures $\varphi_E \in BV(\mathbb{R}^n)$. Hence one may apply (1.8), so that for any $A \subset \mathbb{R}^n$ open*

$$\int_A |D\varphi_E| = \lim_{j\to 0} \int_A |D\varphi_{E_j}|$$

1.1.5 Isoperimetric Inequality

One shall first recall from Sobolev Space the GNS inequality as the tool from (1.19) and Poincaré Lemma

Lemma 1.1.6 (GNS Inequality)**.** *$1 \le p < n$. Then there exists $C = C(n, p)$ s.t.*

$$\|f\|_{L^{\frac{np}{n-p}}(\mathbb{R}^n)} \le C \|Df\|_{L^p(\mathbb{R}^n)} \qquad \forall f \in C_0^1(\mathbb{R}^n) \tag{1.34}$$

Lemma 1.1.7 (Poincaré)**.** *$\Omega \subset \mathbb{R}^n$ open, bounded, connected. $\partial\Omega$ Lipschitz continuous. $1 \le p \le \infty$. There there exists $C = C(n, p, \Omega)$ s.t.*

$$\left\| f - \fint_\Omega f\, dy \right\|_{L^p(\Omega)} \le C \|Df\|_{L^p(\Omega)} \qquad \forall f \in W^{1,p}(\Omega) \tag{1.35}$$

Corollary 1.1.2. *There exists $C_1 = C_1(n)$ and $C_2 = C_2(n)$ s.t.*

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le C_1 \|Df\|_{L^1(\mathbb{R}^n)} \qquad \forall f \in C_0^\infty(\mathbb{R}^n) \tag{1.36}$$

$$\|f - f_\rho\|_{L^{\frac{n}{n-1}}(B_\rho)} \le C_2 \|Df\|_{L^1(B_\rho)} \qquad \forall f \in C^\infty(B_\rho) \tag{1.37}$$

where $f_\rho := \fint_{B_\rho} f\, dy = \frac{1}{|B_\rho|}\int_{B_\rho} f\, dy$.

Proof. Apply (1.34) with $p = 1$ yields (1.36). Apply (1.19) with $\Omega = B_\rho$, $p = 1$ and $q = \frac{n}{n-1}$ gives

$$\|f - f_\rho\|_{L^{\frac{n}{n-1}}(B_\rho)} \le C\|f - f_\rho\|_{W^{1,1}(B_\rho)} = C\left(\|f - f_\rho\|_{L^1(B_\rho)} + \|Df\|_{L^1(B_\rho)}\right) \le C_2\|Df\|_{L^1(B_\rho)}$$

where the last inequality uses (1.35). □

One immediately has Sobolev Inequalities for BV function.

Theorem 1.1.8 (Sobolev for BV)**.** *There exists $C_1 = C_1(n)$ and $C_2 = C_2(n)$ s.t.*

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le C_1 \int |Df| \qquad \forall f \in BV(\mathbb{R}^n) \text{ and } \operatorname{supp} f \text{ compact} \tag{1.38}$$

$$\|f - f_\rho\|_{L^{\frac{n}{n-1}}(B_\rho)} \le C_2 \int_{B_\rho} |Df| \qquad \forall f \in BV(B_\rho) \tag{1.39}$$

where $f_\rho := \fint_{B_\rho} f\, dy = \frac{1}{|B_\rho|}\int_{B_\rho} f\, dy$.

Proof. One mimic the proof in (1.23). For $f \in BV(\mathbb{R}^n)$ with $\operatorname{supp} f$ compact, by smooth approximation Theorem 1.1.2, there exists $\{f_j\} \subset C_0^\infty(\mathbb{R}^n)$ with uniform compact support s.t. $\|f_j - f\|_{L^1(\mathbb{R}^n)} \to 0$ and $\int |Df| = \lim_{j\to\infty} \int |Df_j| dx$. Now Df_j is uniformly bounded in L^1 on $\mathbb{R}^n$, say by M. So one has from (1.36), $\|f_j\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le C_1\|Df_j\|_{L^1(\mathbb{R}^n)} \le C_1 M$ uniformly bounded. Since $L^{\frac{n}{n-1}}$ is Reflexive, a uniformly bounded sequence in $L^{\frac{n}{n-1}}$ has a weakly convergent subsequence by Banach Alaoglu, say $f_j \rightharpoonup f_0$ in $L^{\frac{n}{n-1}}$. But with uniform compact support for f_j and f_0, one has $f_j \rightharpoonup f_0$ in L^1 by Hölder. Since we already know $f_j \to f$ in L^1, $f_0 = f$. Now by lower semicontinuity of weak convergence

$$\left(\int |f|^{\frac{n}{n-1}}\, dx\right)^{\frac{n-1}{n}} \le \lim_{j\to\infty}\left(\int |f_j|^{\frac{n}{n-1}}\, dx\right)^{\frac{n-1}{n}} \le C_1 \lim_{j\to\infty}\|Df_j\|_{L^1(\mathbb{R}^n)} = C_1\int |Df|$$

thus we've proved (1.38). For $f \in BV(B_\rho)$, by smooth approximation Theorem 1.1.2, there exists $\{f_j\} \subset C^\infty(B_\rho)$ s.t. $\|f_j - f\|_{L^1(B_\rho)} \to 0$ and $\int_{B_\rho}|Df| = \lim_{j\to\infty}\int_{B_\rho}|Df_j|dx$, so $\|Df_j\|_{L^1(B_\rho)}$ is uniformly bounded, and by (1.37), $\{f_j - (f_j)_\rho\}$ is uniformly bounded in $L^{\frac{n}{n-1}}(B_\rho)$. Hence there exists weakly convergent subsequence $f_j - (f_j)_\rho \rightharpoonup f_0$ in $L^{\frac{n}{n-1}}(B_\rho)$, thus since B_ρ bounded, $f_j - (f_j)_\rho \rightharpoonup f_0$ weakly in $L^1(B_\rho)$ via Hölder. But $f_j - (f_j)_\rho \to f - f_\rho$ in L^1, so $f - f_\rho = f_0$. Again by the lower semicontinuity one has (1.39)

$$\left(\int_{B_\rho} |f - f_\rho|^{\frac{n}{n-1}}\, dx\right)^{\frac{n-1}{n}} \le \lim_{j\to\infty}\left(\int_{B_\rho} |f_j - (f_j)_\rho|^{\frac{n}{n-1}}\, dx\right)^{\frac{n-1}{n}} \le C_2 \lim_{j\to\infty}\|Df_j\|_{L^1(B_\rho)} = C_2\int_{B_\rho}|Df|$$

□

Theorem 1.1.9 (Isoperimetric Inequality)**.** *For $E \subset \mathbb{R}^n$ bounded Caccioppoli, there exists $C_1 = C_1(n)$ and $C_2 = C_2(n)$ s.t. for any open ball $B_\rho \subset \mathbb{R}^n$ with radius ρ*

$$|E|^{\frac{n-1}{n}} \leq C_1 \int |D\varphi_E| = C_1 P(E) \tag{1.40}$$

$$\min\{|E \cap B_\rho|, |E^c \cap B_\rho|\}^{\frac{n-1}{n}} \leq C_2 \int_{B_\rho} |D\varphi_E| = C_2 P(E, B_\rho) \tag{1.41}$$

Proof. Since E bounded Caccioppoli, $\varphi_E \in BV(\mathbb{R}^n)$ and $\text{supp}\varphi_E = \overline{E}$ is compact, one apply (1.38) and so (1.40) holds. Now let $f = \varphi_E$, then $f_\rho = \frac{1}{|B_\rho|} \int_{B_\rho} \varphi_E = \frac{|E \cap B_\rho|}{|B_\rho|}$, so

$$\begin{aligned}
\int_{B_\rho} |f - f_\rho|^{\frac{n}{n-1}}\, dx &= \int_{B_\rho \cap E} |1 - f_\rho|^{\frac{n}{n-1}}\, dx + \int_{B_\rho \cap E^c} |f_\rho|^{\frac{n}{n-1}}\, dx \\
&= |B_\rho \cap E| \left(\frac{|E^c \cap B_\rho|}{|B_\rho|}\right)^{\frac{n}{n-1}} + |B_\rho \cap E^c| \left(\frac{|E \cap B_\rho|}{|B_\rho|}\right)^{\frac{n}{n-1}} \\
&\geq \min\{|B_\rho \cap E|, |B_\rho \cap E^c|\} \left(\left(1 - \frac{|E \cap B_\rho|}{|B_\rho|}\right)^{\frac{n}{n-1}} + \left(\frac{|E \cap B_\rho|}{|B_\rho|}\right)^{\frac{n}{n-1}}\right)
\end{aligned}$$

Hence taking $\frac{n-1}{n}$ power gives

$$\left(\int_{B_\rho} |f - f_\rho|^{\frac{n}{n-1}}\, dx\right)^{\frac{n-1}{n}} \geq \min\{|B_\rho \cap E|, |B_\rho \cap E^c|\}^{\frac{n-1}{n}} \left(\left(1 - \frac{|E \cap B_\rho|}{|B_\rho|}\right)^{\frac{n}{n-1}} + \left(\frac{|E \cap B_\rho|}{|B_\rho|}\right)^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}$$

Notice for any $\theta \geq 1$ and $a, b \geq 0$, one has elementary inequality $(a+b)^\theta \leq 2^\theta \left(a^\theta + b^\theta\right)$. Letting $\theta = \frac{n}{n-1}$, $a = 1 - \frac{|E \cap B_\rho|}{|B_\rho|}$ and $b = \frac{|E \cap B_\rho|}{|B_\rho|}$, so

$$\left(\left(1 - \frac{|E \cap B_\rho|}{|B_\rho|}\right)^{\frac{n}{n-1}} + \left(\frac{|E \cap B_\rho|}{|B_\rho|}\right)^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \geq \left(2^{\frac{-n}{n-1}} \cdot 1\right)^{\frac{n-1}{n}} = \frac{1}{2}$$

independent of size of B_ρ. Hence apply (1.39) one has (1.41).

□

1.2 Traces of BV Function

1.2.1 preliminary lemmas

Lemma 1.2.1 (Lebesgue Differentiation)**.** *$f \in L^1(\mathbb{R}^n)$. Then for a.e. $x \in \mathbb{R}^n$*

$$\lim_{\rho \to 0} \frac{1}{\rho^n} \int_{B_\rho} |f(x+y) - f(x)| \, dy = 0 \tag{1.42}$$

One need Zorn's lemma for a Covering argument.

Lemma 1.2.2 (Zorn's Lemma)**.** *One needs a few definitions to make sense of Zorn's lemma.*

- *A set P is partially ordered by $\leq$ if*
  1. *$\leq$ is reflexive: $x \leq x$ for any $x \in P$*
  2. *$\leq$ is anti-symmetric: $x \leq y$ and $y \leq x$ implies $x = y$*
  3. *$\leq$ is transitive: $x \leq y$ and $y \leq z$ implies $x \leq z$*

  Note not all elements in P are required to be comparable. If a subset $S \subset P$ that inherits the partial order $\leq$ has every pair of elements comparable, S is called totally ordered.

- *An element $m \in P$ with partial order $\leq$ is maximal if there does not exist $s \in P$ s.t. $s \neq m$ and $m \leq s$. Note 'maximal' here does not need m to be comparable with all other elements in P.*

- *Given subset $S \subset P$ that inherits the partial order $\leq$. An element $u \in P$ is an upper bound of S if for any $s \in S$, $s \leq u$.*

Zorn's Lemma claims: Given a nonempty partially order set $(P, \leq)$. If every nonempty subset $S \subset P$ that inherits the order $\leq$ and is totally bounded has an upper bound $u \in P$, then P contains at least one maximal element m with order $\leq$.

Lemma 1.2.3 (Covering Lemma)**.** *$A \subset \mathbb{R}^n$. $\rho : A \to (0,1)$. Then there exists countable set $\{x_i\} \subset A$ s.t.*

$$B_{\rho(x_i)}(x_i) \cap B_{\rho(x_j)}(x_j) = \varnothing \quad for\ i \neq j \tag{1.43}$$

$$A \subset \bigcup_{i=1}^{\infty} B_{3\rho(x_i)}(x_i) \tag{1.44}$$

Proof. For $k \geq 1$, let $A_k := \{x \in A \mid \frac{1}{2^k} \leq \rho(x) < \frac{1}{2^{k-1}}\}$. One wish to define a sequence of sets L_k for each k. If $A_k = \varnothing$, let $L_k := \varnothing$. WLOG, assume $A_1 \neq \varnothing$. Let $\mathcal{L}_1 := \{L \subset A_1 \mid \forall\, x,\, y \in L,\, x \neq y,\, B_{\rho(x)}(x) \cap B_{\rho(y)}(y) = \varnothing\}$. For nonempty A_1, $\mathcal{L}_1$ is indeed nonempty because both the empty set and singletons are elements of $\mathcal{L}_1$. Now order $\mathcal{L}_1$ with inclusion. For any subcollection of $\mathcal{L}_1$ totally ordered with inclusion, indeed their union is element of $\mathcal{L}_1$ and is upper bounded. Hence $\mathcal{L}_1$ contains a maximal element via Zorn's lemma, call it L_1. Now assume for $L_1, \cdots, L_k$, one obtain L_{k+1} via taking the maximal element of the following collection ordered with inclusion

$$\mathcal{L}_{k+1} := \{L \subset A_{k+1} \mid \forall\, x,\, y \in L_1 \cup L_2 \cup \cdots \cup L_k \cup L,\, x \neq y,\, B_{\rho(x)}(x) \cap B_{\rho(y)}(y) = \varnothing\}$$

Notice $\varnothing \in \mathcal{L}_{k+1}$ is always true so Zorn's lemma applies. L_{k+1} could be empty even if A_{k+1} is nonempty. Moreover, for each L_k, for any $M \subset \mathbb{R}^n$ compact, $M \cap L_k$ must contain finitely many elements otherwise $\{B_{\rho(x)}(x)\}_{x \in M \cap L_k}$ as open cover of $M \cap \overline{L}_k$ does not have finite subcover, contradicting compactness of $M \cap \overline{L}_k$. Hence let M truncate collections of balls $\{\overline{B}_j\}$ with radius $j \in \mathbb{N}$, so each $\overline{B}_j \cap L_k$ is finite for any j. Thus pass j to ∞, L_k is countable. So $L := \bigcup_{k=1}^{\infty} L_k$ is countable set satisfying (1.43). To see (1.44), take any $z \in A = \bigcup_{k=1}^{\infty} A_k$. There must exist k s.t. $z \in A_k$. Now since L_k is maximal element of $\mathcal{L}_k$, $L_k \cup \{z\} \notin \mathcal{L}_k$. Hence there must exist $x \in L_1 \cup \cdots \cup L_k$ s.t. $x \neq z$ and $B_{\rho(x)}(x) \cap B_{\rho(z)}(z) \neq \varnothing$. Note by definition of A_k, $\frac{1}{2^k} \leq \rho(z) < \frac{1}{2^{k-1}}$, and by definition of $L_1 \cup \cdots \cup L_k$, $\frac{1}{2^k} \leq \rho(x) < 1$. Hence $\frac{1}{2}\rho(z) < \rho(x)$. But the balls $B_{\rho(x)}(x) \cap B_{\rho(z)}(z) \neq \varnothing$, so $z \in B_{3\rho(x)}(x)$. □

Using the covering lemma, one obtains a boundary differentiation lemma analogous to Lemma 1.2.1.

- $B_r(x) := \{z \in \mathbb{R}^n \mid |x - z| < r\}$ ball with center x radius r in $\mathbb{R}^n$
- $\mathscr{B}_\rho(y) := \{t \in \mathbb{R}^{n-1} \mid |y - t| < \rho\}$ ball with center y radius ρ in $\mathbb{R}^{n-1}$
- Let $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_n > 0\}$, $y \in \mathbb{R}^{n-1} = \partial\mathbb{R}^n_+$, $\rho > 0$. Upper cylinder with center y radius and height ρ

$$C^+_\rho(y) := \{(z,t) \in \mathbb{R}^{n-1} \times (0,\infty) \mid |y - z| < \rho,\, 0 < t < \rho\} = \mathscr{B}_\rho(y) \times (0,\, \rho)$$

Lemma 1.2.4. *μ positive Radon measure on $\mathbb{R}^n_+$ with $\mu(\mathbb{R}^n_+) < \infty$. Then for H_{n-1}-a.e. $y \in \mathbb{R}^{n-1} = \partial\mathbb{R}^n_+$,*

$$\lim_{\rho\to 0} \frac{1}{\rho^{n-1}} \mu(C^+_\rho(y)) = 0 \tag{1.45}$$

Proof. It suffices to show $\forall\, k > 0$, $A_k := \{y \in \mathbb{R}^{n-1} \mid \limsup_{\rho\to 0} \frac{1}{\rho^{n-1}}\mu(C^+_\rho(y)) > \frac{1}{k}\}$ is of H_{n-1} measure zero. Given $\varepsilon > 0$. Note for any $y \in A_k$, there exists $\rho_y < \varepsilon$ s.t.

$$\frac{1}{\rho_y^{n-1}}\mu(C^+_{\rho_y}(y)) > \frac{1}{2k} \iff \rho_y^{n-1} < 2k\,\mu(C^+_{\rho_y}(y))$$

Choose $\{y_j\} \subset A_k$ as in Lemma 1.2.3 with $\rho(y_j) = \rho_{y_j}$ so that $\mathscr{B}_{\rho_{y_j}}(y_j)$ are disjoint and $A_k \subset \bigcup_{j=1}^\infty \mathscr{B}_{3\rho_{y_j}}(y_j)$.

$$H_{n-1}(A_k) \le \sum_{j=1}^\infty H_{n-1}(\mathscr{B}_{3\rho_{y_j}}(y_j)) = \omega_{n-1}\sum_{j=1}^\infty (3\rho_{y_j})^{n-1} < \omega_{n-1}3^{n-1}2k\sum_{j=1}^\infty \mu(C^+_{\rho_{y_j}}(y_j))$$

But $C^+_{\rho_{y_j}}(y_j) = \mathscr{B}_{\rho_{y_j}}(y_j) \times (0,\, \rho_{y_j})$ are disjoint, and since $\rho_{y_j} < \varepsilon$ uniformly in j

$$H_{n-1}(A_k) \le \omega_{n-1}3^{n-1}2k\,\mu\{x \in \mathbb{R}^n_+ \mid 0 < x_n < \varepsilon\}$$

for any $\varepsilon > 0$. But $\mu(\mathbb{R}^n_+) < \infty$, so $\mu\{x \in \mathbb{R}^n_+ \mid 0 < x_n < \varepsilon\} \to 0$ as $\varepsilon \to 0$, hence $H_{n-1}(A_k) = 0\ \ \forall\, k > 0$. □

1.2.2 Existence and Property of Trace on $\mathbf{C}_R$

One first work with upper cylinder $C^+_R := C^+_R(0) = \mathscr{B}_R \times (0, R)$. Also denote $C_R := \mathscr{B}_R \times (-R,\, R)$.

Theorem 1.2.1 (Construction of Trace)**.** *$f \in BV(C^+_R)$. There exists $f^+ \in L^1(\mathscr{B}_R)$ s.t. for H_{n-1}-a.e. $y \in \mathscr{B}_R$*

$$\lim_{\rho\to 0}\frac{1}{\rho^n}\int_{C^+_\rho(y)} |f(z) - f^+(y)|\,dz = 0 \tag{1.46}$$

and for any $g \in C^1_0(C_R;\mathbb{R}^n)$, one has

$$\int_{C^+_R} f\,\mathrm{div} g\,dx = -\int_{C^+_R}\langle g,\, Df\rangle - \int_{\mathscr{B}_R} f^+\, g_n\, dH_{n-1} \tag{1.47}$$

Definition 1.2.1 (Trace of BV Function)**.** *$f \in BV(C^+_R)$. $f^+ \in L^1(\mathscr{B}_R)$ in Theorem 1.2.1 is trace of f on $\mathscr{B}_R$. Indeed (1.46) implies for H_{n-1}-a.e. $y \in \mathscr{B}_R$*

$$f^+(y) = \lim_{\rho\to 0}\frac{1}{|C^+_\rho(y)|}\int_{C^+_\rho(y)} f(z)\,dz \tag{1.48}$$

Proof. First suppose $f \in C^\infty(C^+_R)$. Then for any $0 < \varepsilon < R$, define $f^\varepsilon : \mathscr{B}_R \to \mathbb{R}$ as $f^\varepsilon(y) := f(y,\, \varepsilon)$. Hence denoting $Q_{\varepsilon',\varepsilon} := \mathscr{B}_R \times (\varepsilon',\, \varepsilon)$ for $0 \le \varepsilon' < \varepsilon \le R$, one has from FTC

$$\int_{\mathscr{B}_R} |f^\varepsilon(y) - f^{\varepsilon'}(y)|\,dH_{n-1}(y) \le \int_{\mathscr{B}_R}\int_{\varepsilon'}^{\varepsilon} |D_n f(y,t)|\,dt\,dH_{n-1}(y) = \int_{Q_{\varepsilon',\varepsilon}} |D_n f|\,dx \tag{1.49}$$

Since f smooth, RHS Cauchy in ε gives LHS Cauchy in ε, thus $\exists\, f^+ \in L^1(\mathscr{B}_R)$ s.t. $\|f^\varepsilon - f^+\|_{L^1(\mathscr{B}_R)} \to 0$. Take any $g \in C^1_0(C_R;\mathbb{R}^n)$, Since f smooth, for any $0 < \varepsilon < R$, and let $\nu = (\nu^1, \cdots, \nu^n)$ denote unit normal w.r.t. $\mathscr{B}_R \times \{x_n = \varepsilon\}$ and pointing downwards to $\mathbb{R}^{n-1}$, i.e., $\nu = (0, \cdots, 0, -1)$

$$\begin{aligned}
\int_{Q_{\varepsilon,R}} f\,\mathrm{div} g\,dx &= -\int_{Q_{\varepsilon,R}}\langle g,\, Df\rangle + \int_{\mathscr{B}_R\times\{x_n=\varepsilon\}} f(y,\, \varepsilon)\, g(y,\, \varepsilon)\cdot\nu\, dH_{n-1}(y) \\
&= -\int_{Q_{\varepsilon,R}}\langle g,\, Df\rangle - \int_{\mathscr{B}_R\times\{x_n=\varepsilon\}} f(y,\, \varepsilon)\, g_n(y,\, \varepsilon)\, dH_{n-1}(y) \\
&= -\int_{Q_{\varepsilon,R}}\langle g,\, Df\rangle - \int_{\mathscr{B}_R} f^\varepsilon(y)\, g^\varepsilon_n(y)\, dH_{n-1}(y)
\end{aligned}$$

letting $\varepsilon \to 0$, one obtain (1.47) for f smooth. To see for (1.46), for any $y \in \mathscr{B}_R$ and $0 < \rho < R$ s.t. $C^+_\rho(y) \subset C^+_R$

$$\begin{aligned}
\int_{C^+_\rho(y)} |f(z) - f^+(y)|\,dz &= \int_{\mathscr{B}_\rho(y)}\int_0^\rho |f(\eta,\, t) - f^+(y)|\,dt\,dH_{n-1}(\eta) \\
&\le \int_{\mathscr{B}_\rho(y)}\int_0^\rho |f(\eta,\, t) - f^+(\eta)|\,dt\,dH_{n-1}(\eta) + \int_{\mathscr{B}_\rho(y)}\int_0^\rho |f^+(\eta) - f^+(y)|\,dt\,dH_{n-1}(\eta) \\
&= \int_{\mathscr{B}_\rho(y)}\int_0^\rho |f(\eta,\, t) - f^+(\eta)|\,dt\,dH_{n-1}(\eta) + \rho\int_{\mathscr{B}_\rho(y)} |f^+(\eta) - f^+(y)|\,dH_{n-1}(\eta)
\end{aligned}$$

notice upon multiplying by ρ^{-n}, the second term goes to 0 for H_{n-1}-a.e. y due to Lebesgue Differentiation 1.2.1. For the first term, use Fubini and mimic (1.49)

$$\begin{aligned}\int_{\mathscr{B}_\rho(y)}\int_0^\rho |f(\eta,\,t)-f^+(\eta)|\,dt\,dH_{n-1}(\eta) &= \int_0^\rho\int_{\mathscr{B}_\rho(y)} |f^t(\eta)-f^+(\eta)|\,dH_{n-1}(\eta)\,dt\\ &\le \int_0^\rho\int_{\mathscr{B}_\rho(y)}\int_0^t |D_nf(\eta,\,\xi)|\,d\xi\,dH_{n-1}(\eta)\,dt\\ &\le \int_0^\rho\int_{Q_{0,t}(y)} |Df|\,dx\,dt \le \rho\int_{C_\rho^+(y)}|Df|\end{aligned}$$

now multiplying by ρ^{-n} and notice $|Df|$ is Radon measure on C_R^+ that is finite, one may use (1.45) with $\mu = |Df|$. Hence for H_{n-1}-a.e. $y \in \mathscr{B}_R$

$$\frac{1}{\rho^n}\int_{C_\rho^+(y)}|f(z)-f^+(y)|\,dz \le \frac{1}{\rho^{n-1}}\int_{C_\rho^+(y)}|Df| + \frac{1}{\rho^{n-1}}\int_{\mathscr{B}_\rho(y)}|f^+(\eta)-f^+(y)|\,dH_{n-1}(\eta)\to 0$$

and one concludes (1.46) for f smooth. In general for $f \in BV(C_R^+)$, approximate using $\{f_j\} \subset C^\infty(C_R^+)$ via Theorem 1.1.2. Recall remark (1.18), for any j, given n and H_{n-1}-a.e. $y \in \mathscr{B}_R$

$$\lim_{\rho\to 0}\frac{1}{\rho^n}\int_{C_\rho^+(y)}|f(z)-f_j(z)|\,dz = 0$$

Hence combining with f_j satisfying (1.46)

$$\frac{1}{\rho^n}\int_{C_\rho^+(y)}|f(z)-f_j^+(y)|\,dz \le \frac{1}{\rho^n}\int_{C_\rho^+(y)}|f(z)-f_j(z)|\,dz + \frac{1}{\rho^n}\int_{C_\rho^+(y)}|f_j(z)-f_j^+(y)|\,dz \to 0$$

for any j. Thus by uniqueness of L^1 limit, all traces f_j^+ coincide H_{n-1}-a.e. $y \in \mathscr{B}_R$. So define $f^+ := f_j^+$ for any such trace. One has (1.46) for $f \in BV(C_R^+)$. Finally, since $\|f - f_j\|_{L^1(C_R^+)} \to 0$ and $\int_{C_R^+}|Df_j| \to \int_{C_R^+}|Df|$, one wish to deduce (1.47) from

$$\int_{C_R^+} f_j\,\mathrm{div}g\,dx = -\int_{C_R^+}\langle g,\,Df_j\rangle - \int_{\mathscr{B}_R} f_j^+\,g_n\,dH_{n-1}$$

The first term converges due to $\|f - f_j\|_{L^1(C_R^+)} \to 0$ and the last term does not need to converge as $f^+ = f_j^+$ for any j. For the second term, note $\int_{C_R^+}|Df_j| \to \int_{C_R^+}|Df|$ convergence ensures uniform boundedness of $\int_{C_R^+}|Df_j|$. By Banach Alaoglu, the closed unit ball in norm is compact in the weak* topology. Hence identifying $\int_{C_R^+}|Df|$ as norm, there exists subsequence s.t. $Df_j \overset{*}{\rightharpoonup} Df$. But the vague topology convergence $\int_{C_R^+}\langle g,\,Df_j\rangle \to \int_{C_R^+}\langle g,\,Df\rangle$ is essentially the weak* topology convergence. Hence we're done.

□

Proposition 1.2.1 (Approximation in BV implies Approximation in Trace). $f \in BV(C_R^+)$. *If* $\{f_j\} \subset BV(C_R^+)$ *s.t.* $f_j \to f$ *in* $L^1(C_R^+)$ *and*

$$\lim_{j\to\infty}\int_{C_R^+}|Df_j| = \int_{C_R^+}|Df| \tag{1.50}$$

then

$$\lim_{j\to\infty}\int_{\mathscr{B}_R}|f_j^+ - f^+|\,dH_{n-1}(y) = 0 \tag{1.51}$$

Proof. For any $0 < \beta < R$, consider $Q_{0,\beta} := \mathscr{B}_R \times (0,\,\beta)$. Define $f_\beta : \mathscr{B}_R \to \mathbb{R}$ s.t. $f_\beta(y) := \frac{1}{\beta}\int_0^\beta f(y,t)\,dt$ for any $f \in BV(C_R^+)$. Then for a.e. β

$$\begin{aligned}\int_{\mathscr{B}_R}|f^+(y)-f_\beta(y)|\,dH_{n-1}(y) &= \int_{\mathscr{B}_R}|f^+(y)-\frac{1}{\beta}\int_0^\beta f(y,\,t)\,dt|\,dH_{n-1}(y)\\ &= \frac{1}{\beta}\int_0^\beta\int_{\mathscr{B}_R}|f^+(y)-f(y,\,t)|\,dH_{n-1}(y)\,dt \le \frac{1}{\beta}\int_0^\beta\int_{Q_{0,t}}|Df|\,dx\,dt \le \int_{Q_{0,\beta}}|Df|\,dx\end{aligned} \tag{1.52}$$

where the last line uses (1.49), initially shown for smooth f. To make sense of (1.49) for $f \in BV(C_R^+)$, one precisely needs smooth approximation from Theorem 1.1.2 where $\|f_\varepsilon \to f\|_{L^1(C_R^+)}$ implies for a.e. t

$$\int_{\mathscr{B}_R}|f_\varepsilon^+(y)-f_\varepsilon(y,\,t)|\,dH_{n-1}(y) \;\to\; \int_{\mathscr{B}_R}|f^+(y)-f(y,\,t)|\,dH_{n-1}(y)$$

and $\int_{C_R^+}|Df_\varepsilon| \to \int_{C_R^+}|Df|$ implies via (1.8) ($\int_{\mathscr{B}_R\times\{t\}}|Df| = 0$ for a.e. t otherwise uncountably many disjoint summing up contradicts $f \in BV(C_R^+)$) that $\int_{Q_{0,t}}|Df_\varepsilon| \to \int_{Q_{0,t}}|Df|$. Hence for $\{f_j\} \subset BV(C_R^+)$ as assumed

$$\int_{\mathscr{B}_R}|f_j^+ - f^+|\,dH_{n-1}(y) \le \int_{\mathscr{B}_R}|f_j^+ - (f_j)_\beta|\,dH_{n-1}(y) + \int_{\mathscr{B}_R}|(f_j)_\beta - f_\beta|\,dH_{n-1}(y) + \int_{\mathscr{B}_R}|f_\beta - f^+|\,dH_{n-1}(y)$$

using (1.52)

$$\le \int_{Q_{0,\,\beta}}|Df_j| + \int_{\mathscr{B}_R}|(f_j)_\beta - f_\beta|\,dH_{n-1}(y) + \int_{Q_{0,\,\beta}}|Df|$$

the middle term writes, using $\|f_j - f\|_{L^1(C_R^+)} \to 0$

$$\int_{\mathscr{B}_R}|(f_j)_\beta - f_\beta|\,dH_{n-1}(y) = \frac{1}{\beta}\int_0^\beta\int_{\mathscr{B}_R}|f_j(y,\,t) - f(y,\,t)|\,dH_{n-1}(y)\,dt = \frac{1}{\beta}\int_{C_R^+}|f_j - f|\,dx \to 0$$

Thus, since for a.e. β, $\int_{Q_{0,\,\beta}}|Df_j| \to \int_{Q_{0,\,\beta}}|Df|$, one has

$$\limsup_{j\to\infty}\int_{\mathscr{B}_R}|f_j^+ - f^+|\,dH_{n-1}(y) \le 2\int_{Q_{0,\,\beta}}|Df|$$

for a.e. β. Thus using $f \in BV(C_R^+)$ so $\int_{Q_{0,\,\beta}}|Df| \to 0$ as $\beta \to 0$, one arrives at (1.51). □

Note for $C_R^- := \mathscr{B}_R \times (-R, 0)$, one may similarly define $f^- \in L^1(\mathscr{B}_R)$ as trace for the function $f \in BV(C_R^-)$ via Theorem 1.2.1.

Proposition 1.2.2 (Extension Property for BV)**.** *For $f_1 \in BV(C_R^+)$ and $f_2 \in BV(C_R^-)$, let $f^+,\, f^- \in L^1(\mathscr{B}_R)$ be their trace respectively. Then for $f : C_R = \mathscr{B}_R \times (-R,\,R) \to \mathbb{R}$ defined as $f := \begin{cases} f_1 & in\ C_R^+ \\ f_2 & in\ C_R^- \end{cases}$, one has $f \in BV(C_R)$ and*

$$\int_{\mathscr{B}_R}|f^+ - f^-|\,dH_{n-1}(y) = \int_{\mathscr{B}_R}|Df| \tag{1.53}$$

Proof. Note from (1.47) applied to f_1 and f_2 respectively, one has for any $g \in C_0^1(C_R;\mathbb{R}^n)$

$$\int_{C_R^+} f_1\,\mathrm{div} g\,dx = -\int_{C_R^+}\langle g,\,Df_1\rangle - \int_{\mathscr{B}_R} f^+\,g_n\,dH_{n-1}$$
$$\int_{C_R^-} f_2\,\mathrm{div} g\,dx = -\int_{C_R^-}\langle g,\,Df_2\rangle + \int_{\mathscr{B}_R} f^-\,g_n\,dH_{n-1}$$

Notice on C_R^-, while deriving (1.47) for smooth f, one take unit normal $\nu = (0,\cdots,0,1)$ pointing upwards to $\mathbb{R}^{n-1}$. Hence the last term involving g_n has opposite signs. One take sum of the above to obtain

$$\int_{C_R} f\,\mathrm{div} g\,dx = -\int_{C_R^+}\langle g,\,Df_1\rangle - \int_{C_R^-}\langle g,\,Df_2\rangle - \int_{\mathscr{B}_R}(f^+ - f^-)\,g_n\,dH_{n-1} \tag{1.54}$$

Now if require $|g| \le 1$, one has

$$|\int_{C_R} f\,\mathrm{div} g\,dx| \le \int_{C_R^+}|Df_1| + \int_{C_R^-}|Df_2| + \int_{\mathscr{B}_R}|f^+|\,dH_{n-1} + \int_{\mathscr{B}_R}|f^-|\,dH_{n-1} < \infty$$

Hence $f \in BV(C_R)$. But on the other hand, by definition of distributional gradient Df

$$\int_{C_R} f\,\mathrm{div} g\,dx = -\int_{C_R}\langle g,\,Df\rangle = -\int_{C_R^+}\langle g,\,Df\rangle - \int_{C_R^-}\langle g,\,Df\rangle - \int_{\mathscr{B}_R}\langle g,\,Df\rangle$$

Notice f coincides with f_1 and f_2 respectively on C_R^+ and C_R^-, hence

$$\int_{C_R} f\,\mathrm{div} g\,dx = -\int_{C_R^+}\langle g,\,Df_1\rangle - \int_{C_R^-}\langle g,\,Df_2\rangle - \int_{\mathscr{B}_R}\langle g,\,Df\rangle \tag{1.55}$$

Now combining (1.54) and (1.55) gives

$$\int_{\mathscr{B}_R}(f^+ - f^-)\,g_n\,dH_{n-1} = \int_{\mathscr{B}_R}\langle g,\,Df\rangle$$

so

$$\int_{\mathscr{B}_R}|Df| = \sup_{\substack{g\in C_0^1(C_R;\mathbb{R}^n)\\|g|\le 1}}|\int_{\mathscr{B}_R}\langle g,\,Df\rangle| = \sup_{\substack{g\in C_0^1(C_R;\mathbb{R}^n)\\|g|\le 1}}|\int_{\mathscr{B}_R}(f^+ - f^-)\,g_n\,dH_{n-1}| = \int_{\mathscr{B}_R}|f^+ - f^-|\,dH_{n-1}$$

where the last equality holds by Riesz Representation. Hence we're done with (1.53). □

1.2.3 Trace on Lipschitz Domains

One has systematic tools to reduce a Domain to C_R. Let $\Omega \subset \mathbb{R}^n$ open with $\partial\Omega$ Lipschitz.

- Since $\partial\Omega$ Lipschitz, for any $x_0 \in \partial\Omega$, there exists a neighborhood around x_0 s.t. the intersection of $\partial\Omega$ and the neighborhood is locally the graph of a Lipschitz function. Due to topology in $\mathbb{R}^n$, one is in fact free to choose the neighborhood as simple geometric objects. Via translation, one may first put $x_0 = 0$ as the origin, then rotate $\partial\Omega$ so that one may choose a cylinder $C(R) = \mathscr{B}_R \times (-\frac{R}{2}, \frac{R}{2})$ with $\mathscr{B}_R$ radius $R > 0$ and height $\frac{R}{2}$, as well as a local Lipschitz function $w : \mathscr{B}_R \subset \mathbb{R}^{n-1} \to (-\frac{R}{2}, \frac{R}{2})$ where the local boundary and interior writes

$$\partial\Omega \cap C(R) = \{(y, t) \in C(R) = \mathscr{B}_R \times (-\frac{R}{2}, \frac{R}{2}) \mid t = w(y)\} \tag{1.56}$$

$$\Omega \cap C(R) = \{(y, t) \in C(R) \mid t > w(y)\} \tag{1.57}$$

- One may further flatten out the local boundary by introducing the variables

$$(y, \tau) = (y, t - w(y)) \in C_R^+ = \mathscr{B}_R \times (0, R)$$

hence for $f \in BV(\Omega \cap C(R))$, one may further define for $g \in BV(C_R^+)$ via

$$g(y, \tau) := f(y, w(y) + \tau) = f(y, t) \tag{1.58}$$

- Apply Theorem 1.2.1 to $g \in BV(C_R^+)$, there exists trace $g^+ \in L^1(\mathscr{B}_R)$. One define $f^+ \in L^1(\partial\Omega \cap C(R))$ for $f \in BV(\Omega \cap C(R))$ as the trace on local Lipschitz boundary via

$$f^+(y, w(y)) := g^+(y) \tag{1.59}$$

Theorem 1.2.2 (Construction of Trace)**.** *$\Omega \subset \mathbb{R}^n$ open and bounded with $\partial\Omega$ Lipschitz. $f \in BV(\Omega)$. Then there exists trace $\varphi \in L^1(\partial\Omega)$ s.t. for H_{n-1}-a.e. $x \in \partial\Omega$*

$$\lim_{\rho\to 0} \frac{1}{\rho^n} \int_{B_\rho(x)\cap\Omega} |f(z) - \varphi(x)|\, dz = 0 \tag{1.60}$$

And for any $g \in C_0^1(\mathbb{R}^n; \mathbb{R}^n)$ one has, denoting ν outer unit normal w.r.t. $\partial\Omega$

$$\int_\Omega f \,\mathrm{div} g \, dx = -\int_\Omega \langle g, Df\rangle + \int_{\partial\Omega} \varphi \langle g, \nu\rangle \, dH_{n-1} \tag{1.61}$$

Proof. For $\Omega \subset \mathbb{R}^n$ bounded, $\partial\Omega$ is compact. Hence consider open cover $\{C_x(R)\}_{x\in\partial\Omega}$ where $C_x(R)$ is the cylinder s.t. upon translation and rotation, (1.56) and (1.57) holds for x positioned at the origin. There exists finite subcover $\{C_{x_i}(R_i)\}_{i=1}^N$. Given $f \in BV(\Omega)$, upon defining local trace $f_i^+ \in L^1(\partial\Omega \cap C_{x_i}(R_i))$ for each $f|_{C_{x_i}(R_i)}$ as in (1.59), one observe that on their overlaps they must agree H_{n-1}-a.e. due to uniqueness of L^1 limit. Hence $\varphi(x) := f_i^+(x)$ for i s.t. $x \in C_{x_i}(R_i)$ is a well-defined $L^1(\partial\Omega)$ function. Note for any $x \in \partial\Omega$, and for i s.t. $x \in C_{x_i}(R_i)$, there exists $\rho < \frac{R_i}{2}$ s.t. $B_\rho(x) \subset C_{x_i}(R_i)$. Hence (1.60) follows directly from (1.46) as a local behavior. To derive (1.61), one needs partition of unity. Denote $\Gamma_i := C_{x_i}(R_i)$ for $i \geq 1$ and $\Gamma_0 \subset\subset \Omega$ chosen s.t. $\overline{\Omega} \subset \bigcup_{i=0}^N \Gamma_i$ is open cover. One may choose a smooth partition of unity subordinate to $\{\Gamma_i\}_0^N$ s.t.

$$0 \leq \phi_i \leq 1, \qquad \mathrm{supp}\phi_i \subset \Gamma_i, \qquad \sum_{i=0}^N \phi_i = 1 \text{ in } \overline{\Omega}$$

Hence $f = \sum_{i=0}^N f\phi_i$ in Ω and $\varphi = \sum_{i=1}^N \varphi\phi_i$ on $\partial\Omega$ since $\Gamma_0 \subset\subset \Omega$. By definition of distributional derivative $D(f\phi_0) \in D'$ and that $\mathrm{supp} f\phi_0 \subset \Gamma_0 \subset\subset \Omega$, for any $g \in C_0^1(\mathbb{R}^n; \mathbb{R}^n)$

$$\int_\Omega f\,\phi_0 \,\mathrm{div} g\, dx = \int f\,\phi_0 \,\mathrm{div} g\, dx = -\int \langle g, D(f\phi_0)\rangle = -\int_\Omega \langle g, D(f\phi_0)\rangle \tag{1.62}$$

while for $i = 1, \cdots, N$, one apply flattening boundary and then (1.47) on each $C_{R_i}^+$ to obtain

$$\int_\Omega f\phi_i \,\mathrm{div} g\, dx = -\int_\Omega \langle g, D(f\phi_i)\rangle + \int_{\partial\Omega} \varphi\phi_i \langle g, \nu\rangle \, dH_{n-1} \tag{1.63}$$

Hence summing up (1.62) and (1.63) gives (1.61). □

Proposition 1.2.3 (Approximation in BV implies Approximation in Trace). *$\Omega \subset \mathbb{R}^n$ open and bounded, $\partial\Omega$ Lipschitz. $f \in BV(\Omega)$. If $\{f_j\} \subset BV(\Omega)$ s.t. $f_j \to f$ in $L^1(\Omega)$ and*

$$\lim_{j\to\infty} \int_\Omega |Df_j| = \int_\Omega |Df| \tag{1.64}$$

then, letting φ_j be trace for f_j and φ trace for f

$$\lim_{j\to\infty} \int_{\partial\Omega} |\varphi_j - \varphi| \, dH_{n-1} = 0 \tag{1.65}$$

Remark 1.2.1. *Let $\Omega \subset \mathbb{R}^n$ open and bounded, $\partial\Omega$ Lipschitz. $f \in BV(\Omega)$.*

- *By smooth approximation Theorem 1.1.2, there exists $\{f_j\} \subset C^\infty(\Omega)$ s.t. $\|f_j - f\|_{L^1(\Omega)} \to 0$ and $\lim\limits_{j\to 0} \int_\Omega |Df_j| \, dx = \int_\Omega |Df|$. As in Proposition 1.2.1, or essentially (1.18), letting φ_j be trace for f_j and φ trace for f, one has $\varphi_j = \varphi$ for any j.*
- *Let $A \subset\subset \Omega$ open with ∂A Lipschitz. Then $f|_A \in BV(A)$ and $f|_{\Omega\setminus\overline{A}}$, hence denote f_A^-, $f_A^+ \in L^1(\partial A)$ as their trace respectively.*

  1. *One has immediately via differentiation (1.60) that for H_{n-1}-a.e. $x \in \partial A$*

  $$\lim_{\rho\to 0} \frac{1}{\rho^n} \int_{B_\rho(x)\cap A} |f(z) - f_A^-(x)| \, dz = 0 \qquad \lim_{\rho\to 0} \frac{1}{\rho^n} \int_{B_\rho(x)\cap(\Omega\setminus\overline{A})} |f(z) - f_A^+(x)| \, dz = 0 \tag{1.66}$$

  2. *Via Extension property for BV Proposition 1.2.2, denoting ν as outer unit normal w.r.t. ∂A, one has important characterisation for the measures $|Df|$ and Df on ∂A*

  $$\int_{\partial A} |Df| = \int_{\partial A} |f_A^+ - f_A^-| \, dH_{n-1}(y) \tag{1.67}$$

  $$\int_{\partial A} Df = \int_{\partial A} \left(f_A^+ - f_A^-\right) \nu \, dH_{n-1}(y) \tag{1.68}$$

  In particular, let $\Omega = B_R$ and $A = B_\rho$ for $\rho < R$, and denote f_ρ^-, $f_\rho^+ \in L^1(\partial B_\rho)$ as trace for $f|_{B_\rho}$ and $f|_{B_R\setminus\overline{B}_\rho}$ respectively. One has, for some N_1, $N_2 \subset \mathbb{R}$ set measure 0

  $$\lim_{\substack{t\to\rho^- \\ t\notin N_1}} \int_{\partial B_1} |f(tx) - f_\rho^-(\rho x)| \, dH_{n-1}(x) = 0 \qquad \lim_{\substack{t\to\rho^+ \\ t\notin N_2}} \int_{\partial B_1} |f(tx) - f_\rho^+(\rho x)| \, dH_{n-1}(x) = 0 \tag{1.69}$$

  Proof. It suffices to prove for f_ρ^-. Notice, by a change of variables, for any $\frac{\rho}{2} < t < \rho$

  $$\begin{aligned}\int_{\partial B_1} |f(tx) - f_\rho^-(\rho x)| \, dH_{n-1}(x) &= \frac{1}{\rho^n} \int_{\partial B_\rho} |f(\frac{t}{\rho}x) - f_\rho^-(x)| \, dH_{n-1}(x) \\ &\le \frac{1}{\rho^n} \frac{1}{(\rho-t)^n} \int_{\partial B_\rho} \int_{B_{2(\rho-t)}(x)\cap B_\rho} |f(z) - f_\rho^-(x)| \, dz \, H_{n-1}(x)\end{aligned}$$

  where the last inequality holds for a.e. t. Denote the set that it fails by N_1. Now since $f \in L^1(B_R)$, one may apply DCT and use the inner part of (1.66)

  $$\begin{aligned}\limsup_{\substack{t\to\rho^- \\ t\notin N_1}} \int_{\partial B_1} |f(tx) - f_\rho^-(\rho x)| \, dH_{n-1}(x) &\le \limsup_{\substack{t\to\rho^- \\ t\notin N_1}} \frac{1}{\rho^n} \int_{\partial B_\rho} \frac{1}{(\rho-t)^n} \int_{B_{2(\rho-t)}(x)\cap B_\rho} |f(z) - f_\rho^-(x)| \, dz \, H_{n-1}(x) \\ &\le \frac{1}{\rho^n} \int_{\partial B_\rho} \left(\lim_{\substack{t\to\rho^- \\ t\notin N_1}} \frac{1}{(\rho-t)^n} \int_{B_{2(\rho-t)}(x)\cap B_\rho} |f(z) - f_\rho^-(x)| \, dz \right) H_{n-1}(x) \\ &= 0\end{aligned}$$

  □

  Also, since $f \in BV(\Omega)$, $|Df|$ is of finite measure. Due to countable additivity of measure for $|Df|$, for a.e. ρ, one has $\int_{\partial B_\rho} |Df| = 0$, hence

  $$f_\rho^+(x) = f(x) = f_\rho^-(x) \qquad \text{for } H_{n-1} - \text{a.e. } x \in \partial B_\rho \text{ for a.e. } \rho \tag{1.70}$$

- *Let $A \subset \Omega$ open with ∂A Lipschitz, and $f \in BV(A)$. One may extend f to Ω by $F := \begin{cases} f & in\ A \\ 0 & in\ \Omega \setminus A \end{cases}$ hence denoting F_A^-, $F_A^+ \in L^1(\partial A)$ as trace for $F|_A$, $F|_{\Omega\setminus\overline{A}}$, one has $F_A^- = f_A^-$ as trace of f on ∂A, and $F_A^+ = 0$.*

  1. *from (1.67)*
$$\int_\Omega |DF| - \int_A |Df| = \int_{\Omega\cap\partial A} |DF| = \int_{\Omega\cap\partial A} |f_A^-|\, dH_{n-1} \tag{1.71}$$
  2. *from (1.68), denoting ν as inner unit normal w.r.t. ∂A*
$$\int_\Omega DF - \int_A Df = \int_{\Omega\cap\partial A} DF = \int_{\Omega\cap\partial A} f_A^-\, \nu\, dH_{n-1} \tag{1.72}$$

In particular, one may further compute 3 perimeters for subsets of Caccioppoli set w.r.t. some ball. Let $\Omega = B_R$ and $A = B_\rho$ for $\rho < R$, and $f = \varphi_E$ for $E \subset \mathbb{R}^n$ Caccioppoli. Then $F = \varphi_{E\cap B_\rho}$. Due to (1.70), for a.e. ρ, $\varphi_E = \varphi_{E,\rho}^-$ for H_{n-1}-a.e. $x \in \partial B_\rho$. Note $\partial B_\rho \cap B_R = \partial B_\rho$, so

  1. *from (1.71)*
$$P(E\cap B_\rho,\, B_R) = P(E,\, B_\rho) + H_{n-1}(E\cap\partial B_\rho) \qquad for\ a.e.\ \rho\ s.t.\ (1.70)\ holds \tag{1.73}$$
  2. *similarily, from (1.72), denoting ν as inner unit normal w.r.t. ∂B_ρ*
$$\int_{B_R} D\varphi_{E\cap B_\rho} = \int_{B_\rho} D\varphi_E + \int_{\partial B_\rho} \varphi_E\, \nu\, dH_{n-1} \qquad for\ a.e.\ \rho\ s.t.\ (1.70)\ holds \tag{1.74}$$

Now let $A = B_R \setminus \overline{B}_\rho$, then $F = \varphi_{E\cap(B_R\setminus\overline{B}_\rho)}$, so for a.e. ρ, $\varphi_E = \varphi_{E,\rho}^+$ for H_{n-1}-a.e. $x \in \partial B_\rho$

$$P(E\setminus\overline{B}_\rho,\, B_R) = P(E,\, B_R\setminus\overline{B}_\rho) + H_{n-1}(E\cap\partial B_\rho) \qquad for\ a.e.\ \rho\ s.t.\ (1.70)\ holds \tag{1.75}$$

Furthermore for A as above, $B_R\setminus(E\cap(B_R\setminus\overline{B}_\rho)) = (B_R\setminus E)\cap(B_R\setminus\overline{B}_\rho)$, then using that mutual disjoint sets share same perimeter

$$P((B_R\setminus E)\cap(B_R\setminus\overline{B}_\rho),\, B_R) = P(E\cap(B_R\setminus\overline{B}_\rho),\, B_R) = P(E\setminus\overline{B}_\rho,\, B_R)$$

one has, again by mutual disjoint sets sharing same perimeter

$$\begin{aligned} P(E\cup\overline{B}_\rho,\, B_R) &= P(B_R\setminus(E\cup\overline{B}_\rho),\, B_R) = P((B_R\setminus E)\cap(B_R\setminus\overline{B}_\rho),\, B_R) = P(E\setminus\overline{B}_\rho,\, B_R) \\ &= P(E,\, B_R\setminus\overline{B}_\rho) + H_{n-1}(E\cap\partial B_\rho) \qquad for\ a.e.\ \rho\ s.t.\ (1.70)\ holds \end{aligned} \tag{1.76}$$

Hence one may measure perimeter of subsets for E in big ball using perimeter of E in small balls and the boundary quantity $H_{n-1}(E\cap\partial B_\rho)$ via (1.73), (1.75) and (1.76).

1.2.4 Converse to Trace Construction

Theorem 1.2.3 (Converse to Trace Construction)**.** *Let $\varphi \in L^1(\mathscr{B}_R)$ for $R > 0$ and compactly supported. For any $\varepsilon > 0$, there exists $f \in W^{1,1}(C_R^+)$ s.t. φ is trace of f and*

$$\int_{C_R^+} |f|\, dx \le \varepsilon \int_{\mathscr{B}_R} |\varphi|\, dH_{n-1} \tag{1.77}$$
$$\int_{C_R^+} |Df|\, dx \le (1+\varepsilon) \int_{\mathscr{B}_R} |\varphi|\, dH_{n-1} \tag{1.78}$$

Proof. There exists $\{\varphi_j\} \subset C^\infty(\mathscr{B}_R)$ s.t. $\|\varphi_j - \varphi\|_{L^1(\mathscr{B}_R)} \to 0$ with $\varphi_0 = 0$, $\|\varphi_j\|_{L^1(\mathscr{B}_R)} \le 2\,\|\varphi\|_{L^1(\mathscr{B}_R)}$ and

$$\int_{\mathscr{B}_R} |\varphi_j - \varphi_{j+1}|\, dH_{n-1} \le 2^{-j-1}\left(1+\frac{\varepsilon}{2}\right)\int_{\mathscr{B}_R} |\varphi|\, dH_{n-1} \implies \sum_{j=0}^\infty \|\varphi_j - \varphi\|_{L^1(\mathscr{B}_R)} \le \left(1+\frac{\varepsilon}{2}\right)\|\varphi\|_{L^1(\mathscr{B}_R)}$$

Now one may construct f with support on neighborhood of $\mathscr{B}_R$. Let $\{t_k\} \subset (0, R)$ be strictly decreasing sequence to 0. Define $f : C_R^+ \to \mathbb{R}$ s.t. for $x \in \mathscr{B}_R$, $t \in (0, R)$

$$f(x,\, t) := \begin{cases} 0 & if\ t > t_0 \\ \frac{t-t_{k+1}}{t_k-t_{k+1}}\varphi_k(x) + \frac{t_k-t}{t_k-t_{k+1}}\varphi_{k+1}(x) & if\ t_k \ge t > t_{k+1}\ for\ k \ge 0 \end{cases}$$

Hence one may calculate for any $t_k \geq t > t_{k+1}$ for $k \geq 0$

$$|D_i f| \leq |D_i \varphi_k(x)| + |D_i \varphi_{k+1}(x)| \quad 1 \leq i \leq n-1$$
$$|D_n f| \leq \frac{1}{t_k - t_{k+1}} |\varphi_k(x) - \varphi_{k+1}(x)|$$

Hence one calculate $\int_{C_R^+} |f|\, dx$ and $\int_{C_R^+} |Df|\, dx$ s.t.

$$\begin{aligned}
\int_{C_R^+} |f|\, dx &= \int_0^R \int_{\mathscr{B}_R} |f|\, dH_{n-1}(x)\, dt = \sum_{k=0}^{\infty} \int_{t_{k+1}}^{t_k} \int_{\mathscr{B}_R} |f|\, dH_{n-1}(x)\, dt \\
&\leq \sum_{k=0}^{\infty} \int_{t_{k+1}}^{t_k} \left(\|\varphi_k\|_{L^1(\mathscr{B}_R)} + \|\varphi_{k+1}\|_{L^1(\mathscr{B}_R)} \right) dt = \sum_{k=0}^{\infty} \left(\|\varphi_k\|_{L^1(\mathscr{B}_R)} + \|\varphi_{k+1}\|_{L^1(\mathscr{B}_R)} \right) (t_k - t_{k+1}) \\
&\leq 4\, \|\varphi\|_{L^1(\mathscr{B}_R)} \sum_{k=0}^{\infty} (t_k - t_{k+1}) = 4t_0\, \|\varphi\|_{L^1(\mathscr{B}_R)} \\
\int_{C_R^+} |Df|\, dx &= \sum_{k=0}^{\infty} \int_{t_{k+1}}^{t_k} \int_{\mathscr{B}_R} |Df|\, dH_{n-1}(x)\, dt \leq \sum_{k=0}^{\infty} \int_{t_{k+1}}^{t_k} \sum_{i=1}^{n} \int_{\mathscr{B}_R} |D_i f|\, dH_{n-1}(x)\, dt \\
&\leq \sum_{k=0}^{\infty} \int_{t_{k+1}}^{t_k} \left(\sum_{i=1}^{n-1} \left(\|D_i \varphi_k\|_{L^1(\mathscr{B}_R)} + \|D_i \varphi_{k+1}\|_{L^1(\mathscr{B}_R)} \right) + \frac{1}{t_k - t_{k+1}} \|\varphi_k - \varphi_{k+1}\|_{L^1(\mathscr{B}_R)} \right) dt \\
&\leq \sum_{k=0}^{\infty} \left(\left(\|D\varphi_k\|_{L^1(\mathscr{B}_R)} + \|D\varphi_{k+1}\|_{L^1(\mathscr{B}_R)} \right) (t_k - t_{k+1}) + \|\varphi_k - \varphi_{k+1}\|_{L^1(\mathscr{B}_R)} \right) \\
&\leq \sum_{k=0}^{\infty} \left(\|D\varphi_k\|_{L^1(\mathscr{B}_R)} + \|D\varphi_{k+1}\|_{L^1(\mathscr{B}_R)} \right) (t_k - t_{k+1}) + \left(1 + \frac{\varepsilon}{2}\right) \|\varphi\|_{L^1(\mathscr{B}_R)}
\end{aligned}$$

But one is left to choose t_k freely. Hence choose t_k s.t. $4t_0 < \varepsilon$ and for $k \geq 0$

$$(t_k - t_{k+1}) \leq \frac{\varepsilon\, \|\varphi\|_{L^1(\mathscr{B}_R)}}{1 + \|D\varphi_k\|_{L^1(\mathscr{B}_R)} + \|D\varphi_{k+1}\|_{L^1(\mathscr{B}_R)}} 2^{-k-2}$$

Hence one obtain (1.77) and (1.78), whence $f \in W^{1,1}(C_R^+)$. To see φ really is trace for f, denote $f_t(x) := f(x,t)$ and compute for $t_k \geq t > t_{k+1}$, following construction in Theorem 1.2.1 and DCT

$$\int_{\mathscr{B}_R} |f_t(x) - \varphi(x)|\, dH_{n-1}(x) \leq \int_{\mathscr{B}_R} \left|\frac{t - t_{k+1}}{t_k - t_{k+1}} \varphi_k(x) - \varphi(x)\right| dH_{n-1}(x) + \int_{\mathscr{B}_R} \left|\frac{t_k - t}{t_k - t_{k+1}} \varphi_{k+1}(x) - \varphi(x)\right| dH_{n-1}(x) \overset{k\to\infty}{\to} 0$$

Hence by uniqueness of L^1 limits, φ is indeed trace for f.

□

Theorem 1.2.4 (Converse to Trace Construction)**.** *$\Omega \subset \mathbb{R}^n$ open bounded, $\partial\Omega$ Lipschitz. $\varphi \in L^1(\partial\Omega)$. Then for any $\varepsilon > 0$, there exists $f \in W^{1,1}(\Omega)$ s.t. φ is trace of f and*

$$\int_\Omega |f|\, dx \leq \varepsilon \int_{\partial\Omega} |\varphi|\, dH_{n-1} \tag{1.79}$$
$$\int_\Omega |Df|\, dx \leq A \int_{\partial\Omega} |\varphi|\, dH_{n-1} \tag{1.80}$$

for $A = A(\partial\Omega)$ but independent of f, φ, ε. If moreover $\partial\Omega$ is C^1, one may choose $A = (1+\varepsilon)$. Also, f may be taken to be supported on arbitrary small neighborhood of $\partial\Omega$ by controlling t_0 via ε.

Chapter 2

Reduced Boundary

2.1 Construction and Properties

As a preliminary, one finds substitution for general Borel sets so that their measure theoretic boundary and topological boundary agree. We work with sets satisfying Lemma 2.1.1 from later on.

Lemma 2.1.1. *Let $E \subset \mathbb{R}^n$ Borel. Then there exists $\tilde{E}$ Borel s.t. $|\tilde{E}\Delta E| = 0$ differ by Lebesgue measure 0 and*

$$0 < |\tilde{E} \cap B_\rho(x)| < \omega_n \rho^n \quad \textit{for any } \rho > 0 \textit{ and } x \in \partial\tilde{E} \tag{2.1}$$

Proof. Define

$$E_0 := \{x \in \mathbb{R}^n \mid \textit{ there exists } \rho > 0 \textit{ s.t. } |E \cap B_\rho(x)| = 0\}$$
$$E_1 := \{x \in \mathbb{R}^n \mid \textit{ there exists } \rho > 0 \textit{ s.t. } |E \cap B_\rho(x)| = |B_\rho(x)| = \omega_n \rho^n\}$$

One see both E_0 and E_1 are open. For $x \in E_0$, take $\rho > 0$ s.t. $|E \cap B_\rho(x)| = 0$. Then for any $y \in B_\rho(x)$, let $\rho_0 := \rho - |x-y|$, so $B_{\rho_0}(y) \subset B_\rho(x)$ hence $|E \cap B_{\rho_0}(y)| = 0$. Due to existence of ρ_0, $y \in E_0$, i.e., the neighborhood $B_\rho(x) \subset E_0$. So E_0 open. For $x \in E_1$, there exists $\rho > 0$ s.t. $|E \cap B_\rho(x)| = |B_\rho(x)|$, i.e., $|B_\rho(x) \cap E^c| = 0$. Again, for any $y \in B_\rho(x)$, let $\rho_0 := \rho - |x-y|$, so $B_{\rho_0}(y) \subset B_\rho(x)$, thus $|B_{\rho_0}(y) \cap E^c| = 0$. Hence $y \in E_1$, we have $B_\rho(x) \subset E_1$, so E_1 is open. One may further show that $|E_0 \cap E| = 0$. Since for any $x \in E_0$, one may choose ρ_x s.t. $|E \cap B_{\rho_x}(x)| = 0$, and it indeed covers $E_0 \subset \bigcup_{x \in E_0} B_{\rho_x}(x)$, we may choose sequence $\{x_j\} \subset E_0$ as index for covering. One compute, due to $|B_{\rho_{x_j}}(x_j) \cap E| = 0$ for any j

$$|E_0 \cap E| \le |\bigcup_{j=1}^{\infty} B_{\rho_{x_j}}(x_j) \cap E| \le \sum_{j=1}^{\infty} |B_{\rho_{x_j}}(x_j) \cap E| = 0$$

Similarly, $|E_1 \setminus E| = 0$ by replacing E in above computation with E^c. Since E_0, E_1 open, $\tilde{E} := (E \cup E_1) \setminus E_0$ is Borel. And indeed one has $|\tilde{E}\Delta E| = 0$ via the following

$$|E \setminus \tilde{E}| = |E \cap ((E \cup E_1) \setminus E_0)^c| = |E \cap ((E \cup E_1)^c \cup E_0)| = |(E \cap E^c \cap E_1^c) \cup (E \cap E_0)| = |E_0 \cap E| = 0$$
$$|\tilde{E} \setminus E| = |(E \cup E_1) \cap E_0^c \cap E^c| = |(E \cap E_0^c \cap E^c) \cup (E_1 \cap E_0^c \cap E^c)| \le |E_1 \setminus E| = 0$$

Now for any $x \in \partial\tilde{E}$, since E_0, E_1 open, $x \notin E_0 \cup E_1$. Hence for any $\rho > 0$, (2.1) holds. □

2.1.1 Reduced Boundary and Uniform Density Estimate

Definition 2.1.1 (Reduced Boundary)**.** *Given $E \subset \mathbb{R}^n$ Caccioppoli. $x \in \partial^* E$ reduced boundary if*

$$\int_{B_\rho(x)} |D\varphi_E| > 0 \quad \textit{for any } \rho > 0 \tag{2.2}$$

and hence, defining

$$\nu_\rho(x) := \frac{\int_{B_\rho(x)} D\varphi_E}{\int_{B_\rho(x)} |D\varphi_E|} \quad \textit{for any } \rho > 0 \tag{2.3}$$

One require the limits $\lim_{\rho\to 0} \nu(x)$ exists and has length 1

$$\nu(x) := \lim_{\rho \to 0} \nu(x) = \lim_{\rho\to 0} \frac{\int_{B_\rho(x)} D\varphi_E}{\int_{B_\rho(x)} |D\varphi_E|} \tag{2.4}$$

$$|\nu(x)| = 1 \tag{2.5}$$

i.e.,

$$\partial^* E := \{x \in \partial E \mid (2.2) \text{ holds for any } \rho > 0, \text{ and the limiting object } (2.3) \text{ satisfies } (2.4) \text{ and } (2.5)\}$$

We call ν the (measure-theoretical) inner unit normal.

Recall the Lebesgue-Besicovitch differentiation.

Lemma 2.1.2 (Lebesgue-Besicovitch differentiation)**.** *μ_1, μ_2 Borel measures on $\mathbb{R}^n$, then*

$$D_{\mu_2}\mu_1 := \lim_{\rho\to 0} \frac{\mu_1(B_\rho(x))}{\mu_2(B_\rho(x))}$$

is defined μ_2-a.e. on $\mathbb{R}^n$, and $D_{\mu_2}\mu_1 \in L^1_{loc}(\mathbb{R}^n, \mu_2)$. If furthermore, $\mu_1 \ll \mu_2$, i.e., μ_1 is absolutely continuous w.r.t. μ_2 in the sense that $\mu_2(E) = 0$ implies $\mu_1(E)$ for any $E \subset \mathbb{R}^n$ Borel, then we write

$$\mu_1 = D_{\mu_2}\mu_1 \cdot \mu_2 \qquad \text{on all Borel sets}$$

Remark 2.1.1. *Note $D\varphi_E$ is indeed absolutely continuous w.r.t. $|D\varphi_E|$. Hence apply Lemma 2.1.2, one has*

$$\nu(x) := \lim_{\rho\to 0} \frac{\int_{B_\rho(x)} D\varphi_E}{\int_{B_\rho(x)} |D\varphi_E|} \text{ exists and } |\nu(x)| = 1 \qquad |D\varphi_E| - a.e.\ x \in \mathbb{R}^n \tag{2.6}$$

and the following measures agree

$$D\varphi_E = \nu|D\varphi_E| \qquad \text{on all Borel sets} \tag{2.7}$$

In particular, $D\varphi_E = \nu|D\varphi_E|$ on ∂E, and the set $\partial E \setminus \partial^* E$ has $|D\varphi_E|$-measure zero.

Example 2.1.1. *One has 2 examples. One for smooth boundary and one for Lipschitz.*

- *Let $E \subset \mathbb{R}^n$ be bounded, Caccioppoli with C^1 boundary ∂E. Then $\partial^* E = \partial E$.*

  Proof. Let $A = E$ and $f = \varphi_E$ in (1.68), one has via Extension property for $\varphi_E \in BV(\mathbb{R}^n)$ that (for this step, ∂E Lipschitz suffices)

  $$D\varphi_E = \nu dH_{n-1} \llcorner \partial E \qquad \text{on Borel sets} \tag{2.8}$$

  where ν denote the classical **inner unit normal w.r.t.** ∂E. And because supp$D\varphi_E \subset \partial E$, one writes for any $\rho > 0$

  $$\int_{B_\rho(x)} D\varphi_E = \int_{B_\rho(x)\cap\partial E} \nu\, dH_{n-1}$$

  while C^1 boundary ensure via (1.4) that

  $$\int_{B_\rho(x)} |D\varphi_E| = H_{n-1}(B_\rho(x) \cap \partial E)$$

  hence one has explicit formula for ν_ρ

  $$\nu_\rho(x) = \frac{\int_{B_\rho(x)\cap\partial E} \nu\, dH_{n-1}}{H_{n-1}(B_\rho(x) \cap \partial E)} \qquad \text{for any } x \in \partial E$$

  Since $\nu \in C(\partial E; \mathbb{R}^n)$, differentiation gives $\lim_{\rho\to 0} \nu_\rho(x) = \nu(x)$ for any $x \in \partial E$. Hence $|\nu| = 1$ as inherited. □

- *Let $E = (0,1) \times (0,1) \subset \mathbb{R}^2$. Notice except for the four corners, the boundaries are piecewise C^∞, hence these parts belong to $\partial^* E$. Now for any corner x, one may compute*

  $$|\nu(x)| = \lim_{\rho\to 0} \frac{|\int_{B_\rho(x)} D\varphi_E|}{\int_{B_\rho(x)} |D\varphi_E|} = \frac{1}{\sqrt{2}}$$

  Hence the four corners do not belong to $\partial^* E$.

One has Uniform Density estimates, which says bounded oscillation in normal directions at a given boundary point $x \in \partial E$ prevents densities of E and E^c from disappearing under blow-up limit. In particular, if $x \in \partial^* E$, it indeed satisfies our assumption, so uniform density estimate holds. For simplicity, let $0 \in \partial E$ via translation.

Theorem 2.1.1 (Uniform Density Estimates). *$E \subset \mathbb{R}^n$ be Caccioppoli and $0 \in \partial E$. If there exists $\rho_0 > 0$ and $q > 0$ constants s.t. for any $\rho < \rho_0$*

$$\int_{B_\rho} |D\varphi_E| > 0$$

$$|\nu_\rho(0)| = \left| \frac{\int_{B_\rho} D\varphi_E}{\int_{B_\rho} |D\varphi_E|} \right| \geq q > 0 \tag{2.9}$$

Then for any $\rho < \rho_0$, one has uniform estimates on the density

$$\frac{|E \cap B_\rho|}{\rho^n} \geq C_1(n, q) > 0 \tag{2.10}$$

$$\frac{|E^c \cap B_\rho|}{\rho^n} \geq C_2(n, q) > 0 \tag{2.11}$$

$$0 < C_3(n, q) \leq \frac{\int_{B_\rho} |D\varphi_E|}{\rho^{n-1}} \leq C_4(n, q) < \infty \tag{2.12}$$

for constants C_1, C_2, C_3, C_4 only relevant to n, q.

Proof. Since E Caccioppoli, $\varphi_E \in BV(B_{\rho_0})$. Denoting ν as inner unit normal w.r.t. ∂B_ρ one has via (1.74)

$$\int D\varphi_{E\cap B_\rho} = \int_{B_\rho} D\varphi_E + \int_{\partial B_\rho} \varphi_E \, \nu \, dH_{n-1} \qquad \textit{for a.e. } \rho < \rho_0$$

evaluate the vector-valued measure on some constant unit vector $e \in \mathbb{S}^{n-1}$ gives, for ρ s.t. (1.74) holds

$$0 = -\int \operatorname{div}(e) \, \varphi_{E\cap B_\rho} = \int \langle e, \, D\varphi_{E\cap B_\rho} \rangle = \int_{B_\rho} \langle e, \, D\varphi_E \rangle + \int_{\partial B_\rho} \varphi_E \, \nu \cdot e dH_{n-1}$$

Hence for any $e \in \mathbb{S}^{n-1}$ projection is less than the Hausdorff measure

$$\left| \int_{B_\rho} \langle e, \, D\varphi_E \rangle \right| = \left| \int_{\partial B_\rho} \varphi_E \, \nu \cdot e dH_{n-1} \right| \leq \int_{\partial B_\rho} \varphi_E \, dH_{n-1} = H_{n-1}(E \cap \partial B_\rho) \leq C\rho^{n-1}$$

taking supremum on LHS and using Riesz Representation yields

$$\left| \int_{B_\rho} D\varphi_E \right| \leq H_{n-1}(E \cap \partial B_\rho) \tag{2.13}$$

Using (2.13) and (2.9) further gives, bounding perimeter by the projection, hence the Hausdorff measure.

$$\int_{B_\rho} |D\varphi_E| \leq \frac{1}{q} \left| \int_{B_\rho} D\varphi_E \right| \leq C_4 \rho^{n-1} \quad \textit{for a.e. } \rho < \rho_0 \textit{ s.t. (1.74) holds}$$

Now using continuity from above of the measure $|D\varphi_E|$, we conclude the second part to (2.12) for all $\rho < \rho_0$. Now, using (1.73) and similar reasons as above, one has

$$\begin{aligned} P(E \cap B_\rho) &= P(E, B_\rho) + H_{n-1}(E \cap \partial B_\rho) \quad \textit{for a.e. } \rho < \rho_0 \\ &= \int_{B_\rho} |D\varphi_E| + \int_{\partial B_\rho} \varphi_E \, dH_{n-1} \leq \left(\frac{1}{q} + 1\right) \int_{\partial B_\rho} \varphi_E \, dH_{n-1} \end{aligned}$$

Since $E \cap B_\rho$ is bounded Caccioppoli, via isoperimetric inequality (1.40) and noting $P(E \cap B_\rho) = \int |D\varphi_{E\cap B_\rho}|$

$$|E \cap B_\rho|^{\frac{n-1}{n}} \leq \left(\frac{1}{q} + 1\right) C(n) \int_{\partial B_\rho} \varphi_E \, dH_{n-1} \tag{2.14}$$

for some $C(n)$ from (1.40). Notice by coarea formula, denoting $g(\rho) = |E \cap B_\rho|$

$$g(R) = |E \cap B_R| = \int_{B_R} \varphi_E \, dx = \int_0^R \int_{\partial B_\rho} \varphi_E \, dH_{n-1} \, d\rho \implies g'(\rho) = \int_{\partial B_\rho} \varphi_E \, dH_{n-1}$$

Hence (2.14) writes

$$g(\rho)^{\frac{n-1}{n}} \leq \left(\frac{1}{q} + 1\right) C(n) g'(\rho) \implies \rho \leq \left(\frac{1}{q} + 1\right) C(n) \, n \, g(\rho)^{\frac{1}{n}} \implies \left(\frac{1}{C(n) \, n \left(\frac{1}{q} + 1\right)} \right)^n \leq \frac{|E \cap B_\rho|}{\rho^n}$$

denoting $C_1 := \left(\frac{1}{C(n)n\left(\frac{1}{q}+1\right)}\right)^n$ and using continuity from below of the measure $|E \cap B_\rho|$ in ρ, one conclude (2.10) for every $\rho < \rho_0$. Note for E^c, $D\varphi_{E^c} = -D\varphi_E$ due to for any $g \in C_0^1(\mathbb{R}^n; \mathbb{R}^n)$

$$\int \langle g,\, D\varphi_{E^c}\rangle = -\int \varphi_{E^c} \operatorname{div}(g)\, dx = -\int (1-\varphi_E) \operatorname{div}(g)\, dx = \int \varphi_E \operatorname{div}(g)\, dx = -\int \langle g,\, D\varphi_E\rangle$$

whence $|D\varphi_E| = |D\varphi_{E^c}|$ and the above same argument runs with $C_2 = C_1$, resulting in (2.11). To see first part to (2.12), notice from (2.10) and (2.11), one has

$$C_1\rho^n \le \min\{|E\cap B_\rho|,\, |E^c \cap B_\rho|\} \implies C_1^{\frac{n-1}{n}} \rho^{n-1} \le \min\{|E\cap B_\rho|,\, |E^c \cap B_\rho|\}^{\frac{n-1}{n}}$$

Hence applying Poincaré inequality (1.41) one has, for some $\tilde{C}(n) > 0$

$$C_1^{\frac{n-1}{n}} \rho^{n-1} \le \tilde{C}(n) \int_{B_\rho} |D\varphi_E| \implies 0 < \frac{C_1^{\frac{n-1}{n}}}{\tilde{C}(n)} \le \frac{1}{\rho^{n-1}} \int_{B_\rho} |D\varphi_E|$$

define $C_3 := \frac{C_1^{\frac{n-1}{n}}}{\tilde{C}(n)}$ yields the first part of (2.12). ☐

2.1.2 Blow-up Limit

One define the tangent plane and half spaces for given $z \in \partial^* E$ (hence $\nu(z)$ is well-defined and $|\nu(z)| = 1$)

- Tanget Hyperplane to $\partial^* E$ at z is $T(z) := \{x \in \mathbb{R}^n \mid \langle \nu(z),\, x - z\rangle = 0\}$
- Half spaces to $\partial^* E$ at z on the same and opposite side with $\nu(z)$ are respectively

$$T^+(z) := \{x \in \mathbb{R}^n \mid \langle \nu(z),\, x - z\rangle > 0\}$$
$$T^-(z) := \{x \in \mathbb{R}^n \mid \langle \nu(z),\, x - z\rangle < 0\}$$

One may now show that the blowup limit of a point in reduced boundary actually converges to the half space on the same side as the outer normal. For simplicity, via translation and rotation, one assume $0 \in \partial^* E$, and the inner normal $\nu(0)$ is parallel to the x_1-axis that points towards $-\infty$. One wish to obtain the limit $T^+(0)$. But before the proof, one needs a De La Vallée Poussin Theorem to guarantee convergence in LHS of (2.17) given the L^1_{loc} convergence.

Lemma 2.1.3 (De La Vallée Poussin Theorem). *Given E_j sequence of Caccioppoli Sets in $\mathbb{R}^n$. Suppose $\varphi_{E_j} \to \varphi_E$ in $L^1_{loc}(\mathbb{R}^n)$ and that $\int_{\mathbb{R}^n} |D\varphi_{E_j}| \le M < \infty$ the total variation is uniformly bounded. Then up to a subsequence, the convergence holds in vague topology*

$$\int g \cdot D\varphi_{E_j} \to \int g \cdot D\varphi_E \quad \forall\ g \in C_0^1(\mathbb{R}^n)$$

and for a.e. ρ

$$\lim_{j\to\infty} \int_{B_\rho} D\varphi_{E_j} = \int_{B_\rho} D\varphi_E \tag{2.15}$$

Theorem 2.1.2 (Blow-up Limit of Reduced Boundary). *$E \subset \mathbb{R}^n$ Caccioppoli. $0 \in \partial^* E$ with $\nu(0) = (-1, 0, \cdots, 0)$. For any $t > 0$, define the set for blowup*

$$E_t := \{x \in \mathbb{R}^n \mid t\, x \in E\} \tag{2.16}$$

Then there exists a subsequence $t_j \to 0^+$ s.t. $E_j := E_{t_j} \to T^+ := T^+(0)$ in $L^1_{loc}(\mathbb{R}^n)$ sense. Moreover, for every open set $A \subset \mathbb{R}^n$ s.t. $H_{n-1}(\partial A \cap T(0)) = 0$ one has convergence in perimeter

$$\lim_{t_j \to 0} \int_A |D\varphi_{E_j}| = \int_A |D\varphi_{T^+}| = H_{n-1}(T(0) \cap A) \tag{2.17}$$

Proof. One wish to extract a convergent subsequence using compactness argument. First note in our setting, the targeting limit is $T^+ = \{x \in \mathbb{R}^n \mid x_1 < 0\}$. Fix $\rho > 0$. Now by change of variables, for any $g \in C_0^1(B_\rho; \mathbb{R}^n)$, write $\tilde{g}(x) := g(x/t)$

$$\begin{aligned}
\int_{B_\rho} \langle g,\, D\varphi_{E_t}\rangle &= -\int_{B_\rho} \operatorname{div}(g(x))\, \varphi_{E_t}(x)\, dx = -\int_{B_\rho} \operatorname{div}(\tilde{g}(tx))\, \varphi_E(tx)\, dx \\
&= -\int_{B_\rho} t \operatorname{div}(\tilde{g})(tx)\, \varphi_E(tx)\, dx = -\frac{1}{t^{n-1}} \int_{B_{t\rho}} \operatorname{div}(\tilde{g})(y)\, \varphi_E(y)\, dy \\
&= \frac{1}{t^{n-1}} \int_{B_{t\rho}} \langle \tilde{g},\, D\varphi_E\rangle \implies \int_{B_\rho} D\varphi_{E_t} = \frac{1}{t^{n-1}} \int_{B_{t\rho}} D\varphi_E
\end{aligned} \tag{2.18}$$

And by considering total variation, one has

$$\int_{B_\rho} |D\varphi_{E_t}| = \frac{1}{t^{n-1}} \int_{B_{t\rho}} |D\varphi_E| \tag{2.19}$$

With tools (2.18) and (2.19), one proceeds in two directions. First, making use of $0 \in \partial^* E$, in particular (2.4)

$$\lim_{t\to 0} \frac{1}{\int_{B_\rho} |D\varphi_{E_t}|} \begin{pmatrix} \int_{B_\rho} D_1\varphi_{E_t} \\ \int_{B_\rho} D_2\varphi_{E_t} \\ \vdots \\ \int_{B_\rho} D_n\varphi_{E_t} \end{pmatrix} = \lim_{t\to 0} \frac{\int_{B_\rho} D\varphi_{E_t}}{\int_{B_\rho} |D\varphi_{E_t}|} = \lim_{t\to 0} \frac{\int_{B_{t\rho}} D\varphi_E}{\int_{B_{t\rho}} |D\varphi_E|} = \nu(0) = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.20}$$

Second, one make an immediate observation that for each $\rho > 0$, $\{\varphi_{E_t}\}_t \subset BV(B_\rho)$ because E is Caccioppoli, and for each t, $B_{t\rho}$ is bounded, hence $\varphi_E \in BV(B_{t\rho})$ and RHS of (2.19) is bounded. An immediate consequence is that E_t are Caccioppoli Set for any t. Again, since $0 \in \partial^* E$, one has uniform density estimate. Applying second part of (2.12), together with (2.19) yields

$$\limsup_{t\to 0} \int_{B_\rho} |D\varphi_{E_t}| = \limsup_{t\to 0} \frac{1}{t^{n-1}} \int_{B_{t\rho}} |D\varphi_E| \le C < \infty \tag{2.21}$$

Hence the sequence of functions $\{\varphi_{E_t}\}$ is uniformly bounded in $BV(B_\rho)$ norm for each $\rho > 0$. Thus by compactness theorem 1.1.4, there exists a subsequence $\{\varphi_{E_j}\}$ where $E_j := E_{t_j}$ s.t. $\varphi_{E_j} \to f$ in $L^1_{loc}(\mathbb{R}^n)$ (by unique limit on each ball B_ρ) and that $f \in BV(\mathbb{R}^n)$. Since f is L^1 limit of characteristic functions, $f = \varphi_C$ for some Borel set $C \subset \mathbb{R}^n$. Since $\varphi_C \in BV(\mathbb{R}^n)$, indeed C is Caccioppoli. Moreover, by De La Vallée Poussin Theorem (2.15), for a.e. ρ s.t. $\int_{\partial B_\rho} |D\varphi_C| = 0$, one has approximation in vector-valued radon measure

$$\lim_{t_j\to 0} \int_{B_\rho} D\varphi_{E_j} = \int_{B_\rho} D\varphi_C \tag{2.22}$$

hence combining with (2.20) gives, for the x_1 direction

$$\lim_{t_j\to 0} \int_{B_\rho} |D\varphi_{E_j}| = -\lim_{t_j\to 0} \int_{B_\rho} D_1\varphi_{E_j} = -\int_{B_\rho} D_1\varphi_C$$

Now since $\varphi_{E_j} \to \varphi_C$ in $L^1_{loc}(\mathbb{R}^n)$, by semicontinuity 1.1.1

$$\int_{B_\rho} |D\varphi_C| \le \lim_{t_j\to 0} \int_{B_\rho} |D\varphi_{E_j}| = -\int_{B_\rho} D_1\varphi_C \tag{2.23}$$

but since any other $\int_{B_\rho} D_i\varphi_C = 0$ for $i \ge 2$ as in (2.20), the equality in (2.23) holds. Now by Lebesgue-Besicovitch Differentiation 2.1.2

$$D_1\varphi_C = \left(\lim_{t\to 0} \frac{\int_{B_\rho} D_1\varphi_C}{\int_{B_\rho} |D\varphi_C|} \right) |D\varphi_C| = -|D\varphi_C| \qquad \textit{on all Borel sets}$$

$$D\varphi_C = \left(\lim_{t\to 0} \frac{\int_{B_\rho} D\varphi_C}{\int_{B_\rho} |D\varphi_C|} \right) |D\varphi_C| = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} |D\varphi_C| \qquad \textit{on all Borel sets}$$

Hence $D_i\varphi_C = 0$ as Borel measure for $i \ge 2$. Therefore φ_C depends only on x_1 and $D_1\varphi_C < 0$ implies φ_C is non-increasing in x_1. Thus $C = \{x \in \mathbb{R}^n \mid x_1 < \lambda\}$ a.e. for some $\lambda \in \mathbb{R}$. One wish to determine λ. Suppose $\lambda < 0$, then we may construct ball $B_{|\lambda|}$ around 0 that does not intersect C, so using $\varphi_{E_j} \to \varphi_C$ in $L^1_{loc}(\mathbb{R}^n)$

$$\begin{aligned} 0 = |C \cap B_{|\lambda|}| &= \int_{B_{|\lambda|}} \varphi_C(x)\,dx = \lim_{t_j\to 0} \int_{B_{|\lambda|}} \varphi_{E_j}(x)\,dx \\ &= \lim_{t_j\to 0} \frac{1}{t_j^n} \int_{B_{|\lambda|}} \varphi_E(t_j x)\,d(t_j x) = \lim_{t_j\to 0} \frac{1}{t_j^n} \int_{B_{|\lambda| t_j}} \varphi_E(y)\,dy \\ &= \lim_{t_j\to 0} \frac{|E \cap B_{|\lambda| t_j}|}{t_j^n} \ge C_1 > 0 \end{aligned}$$

for some C_1 from (2.10), contradicting our assumption. If $\lambda > 0$, use

$$\begin{aligned} 0 = |C^c \cap B_{|\lambda|}| &= \int_{B_{|\lambda|}} \varphi_{C^c}(x)\, dx = \lim_{t_j \to 0} \int_{B_{|\lambda|}} \varphi_{E_j^c}(x)\, dx \\ &= \lim_{t_j \to 0} \frac{1}{t_j^n} \int_{B_{|\lambda| t_j}} \varphi_{E^c}(y)\, dy = \lim_{t_j \to 0} \frac{|E^c \cap B_{|\lambda| t_j}|}{t_j^n} \geq C_2 > 0 \end{aligned}$$

for some C_2 from (2.11). Hence $\lambda = 0$, and so $C = T^+ = \{x \in \mathbb{R}^n \mid x_1 < 0\}$ a.e. It remains to show for any open set $A \subset \mathbb{R}^n$ s.t. $H_{n-1}(\partial A \cap T(0)) = 0$, (2.17) holds. First note that, since T^+ has smooth boundary, one use remark 1.1.1 so that $|D\varphi_{T^+}| = H_{n-1} \llcorner \partial T^+ = H_{n-1} \llcorner T(0)$ as Borel measures. So if $H_{n-1}(\partial A \cap T(0)) = 0$ for some A open, in fact $\int_{\partial A} |D\varphi_{T^+}| = 0$. But this is condition for (1.8) where the equality in semicontinuity holds in subdomains. Hence apply (1.8), one directly arrives at (2.17).

□

Corollary 2.1.1 (Density Estimates on single side of Tangent Plane to Reduced Boundary)**.** *Let $E \subset \mathbb{R}^n$ Caccioppoli, and $0 \in \partial^* E$ with $\nu(0) = (-1, 0, \cdots, 0)$. Then the volumne density on single side vanishes*

$$\lim_{\rho \to 0} \frac{1}{\rho^n} |E \cap B_\rho \cap T^-| = 0 \tag{2.24}$$

$$\lim_{\rho \to 0} \frac{1}{\rho^n} |(B_\rho \setminus E) \cap T^+| = 0 \tag{2.25}$$

and for any $\rho, \varepsilon > 0$, denoting

$$S_{\rho,\,\varepsilon} := B_\rho \cap \{x \in \mathbb{R}^n \mid |\langle \nu(0), x \rangle| < \varepsilon \rho\} = B_\rho \cap \{x \in \mathbb{R}^n \mid |x_1| < \varepsilon \rho\}$$

the perimeter density takes up constant portion for any $\varepsilon > 0$

$$\lim_{\rho \to 0} \frac{1}{\rho^{n-1}} \int_{S_{\rho,\,\varepsilon}} |D\varphi_E| = \omega_{n-1} \tag{2.26}$$

where ω_{n-1} is volumne of $n-1$-dim unit ball.

Proof. Under definition (2.16), $T_\rho^+ = T^+$ and $T_\rho^- = T^-$ for any $\rho > 0$. By change of variables as in (2.18)

$$\begin{aligned} \frac{1}{\rho^n} |E \cap B_\rho \cap T^-| &= \frac{1}{\rho^n} \int_{B_\rho} \varphi_E(x) \varphi_{T^-}(x)\, dx = \int_{B_1} \varphi_E(\rho y)\, \varphi_{T^-}(\rho y)\, dy \\ &= \int_{B_1} \varphi_{E_\rho}(y)\, \varphi_{T_\rho^-}(y)\, dy = |E_\rho \cap B_1 \cap T^-| \\ \frac{1}{\rho^n} |(B_\rho \setminus E) \cap T^+| &= \frac{1}{\rho^n} \int_{B_\rho} \varphi_{E^c}(x)\, \varphi_{T^+}(x)\, dx = \int_{B_1} \varphi_{E^c}(\rho y)\, \varphi_{T^+}(\rho y)\, dy \\ &= \int_{B_1} \varphi_{E_\rho^c}(y)\, \varphi_{T_\rho^+}(y)\, dy = |(B_1 \setminus E_\rho) \cap T^+| \end{aligned}$$

But from Theorem 2.1.2, $E_\rho \to T^+$ in $L^1_{loc}(\mathbb{R}^n)$ up to a subsequence, hence

$$\lim_{\rho \to 0} \frac{1}{\rho^n} |E \cap B_\rho \cap T^-| = \lim_{\rho \to 0} |E_\rho \cap B_1 \cap T^-| = |T^+ \cap B_1 \cap T^-| = 0$$

$$\lim_{\rho \to 0} \frac{1}{\rho^n} |(B_\rho \setminus E) \cap T^+| = \lim_{\rho \to 0} |(B_1 \setminus E_\rho) \cap T^+| = |(B_1 \setminus T^+) \cap T^+| = 0$$

so (2.24) and (2.25) hold. Moreover, by the exact same procedure with $S_{\rho,\,\varepsilon}$ in place of B_ρ and $S_{1,\varepsilon}$ in place of B_1 as in (2.18), one has

$$\frac{1}{\rho^{n-1}} \int_{S_{\rho,\,\varepsilon}} |D\varphi_E| = \int_{S_{1,\,\varepsilon}} |D\varphi_{E_\rho}|$$

and since $S_{1,\varepsilon}$ is open set with $H_{n-1}(\partial S_{1,\,\varepsilon} \cap T) = 0$, apply (2.17) to conclude (2.26)

$$\lim_{\rho \to 0} \frac{1}{\rho^{n-1}} \int_{S_{\rho,\,\varepsilon}} |D\varphi_E| = \lim_{\rho \to 0} \int_{S_{1,\,\varepsilon}} |D\varphi_{E_\rho}| = H_{n-1}(T \cap S_{1,\,\varepsilon}) = \omega_{n-1}$$

□

The above Corollary 2.1.1 says for small enough balls $B_\rho(x)$, most of $E \cap B_\rho(x)$ lies in T^+, the same side w.r.t. inner normal ν; while most of $B_\rho(x) \setminus E$ lies in T^-, the outside. For small enough balls, the hyperplane T splits B_ρ into 2 parts which nearly corresponds to inner part E and outside part $\mathbb{R}^n \setminus E$.

2.2 Regularity of Reduced Boundary

2.2.1 Characterisation of $|D\varphi_E|$ using ∂E^*

The purpose of this section is to argue that for $E \subset \mathbb{R}^n$ Caccioppoli

- $\partial^* E$ is countable union of C^1 hypersurfaces up to set of $|D\varphi_E|$-measure zero.
- $\int_\Omega |D\varphi_E| = H_{n-1}(\partial^* E \cap \Omega)$ so $|D\varphi_E| = H_{n-1} \llcorner \partial^* E$ as Radon measures.
- $\partial^* E$ is dense in ∂E.

One shall first recall the precise definition for Hausdorff measure.

Definition 2.2.1. *Let $A \subset \mathbb{R}^n$, $0 \le k < \infty$ and $0 < \delta \le \infty$. We define the k-dim Hausdorff outer measure at step δ*

$$H_k^\delta(A) := \frac{\omega_k}{2^k} \inf \left\{ \sum_{j=1}^\infty \operatorname{diam}(S_j)^k \mid A \subset \bigcup_{j=1}^\infty S_j,\ \operatorname{diam}(S_j) < \delta \ \forall\, j \right\} \tag{2.27}$$

and consequently define

$$H_k(A) := \lim_{\delta \to 0} H_k^\delta(A) = \sup_{0<\delta\le\infty} H_k^\delta(A)$$

as k-dim Hausdorff measure. Here $\omega_k := \Gamma(\frac{1}{2})^k / \Gamma(\frac{k}{2}+1)$ for $k \ge 0$ is measure of unit ball in $\mathbb{R}^k$.

Lemma 2.2.1 (Ratio Estimate)**.** *$E \subset \mathbb{R}^n$ Caccioppoli. $B \subset \partial^* E$. Then*

$$H_{n-1}(B) \le 2 \cdot 3^{n-1} \int_B |D\varphi_E| \tag{2.28}$$

Proof. Since $|D\varphi_E|$ is Radon measure on $\mathbb{R}^n$, it can be approximated from the outside by open sets. Given B, for any $\eta > 0$, there exists A open s.t. $B \subset A$ and

$$\int_A |D\varphi_E| \le \int_B |D\varphi_E| + \eta \tag{2.29}$$

Moreover, for any $\varepsilon > 0$, apply (2.26) to arbitrary $x \in B$, there exists $0 < \rho(x) < \varepsilon$ s.t. using openness of A

$$B_{\rho(x)}(x) \subset A \qquad \textit{and} \qquad \int_{B_{\rho(x)}(x)} |D\varphi_E| \ge \frac{1}{2}\rho(x)^{n-1}\omega_{n-1} \tag{2.30}$$

One think about covering B using balls $\{B_{\rho(x)}(x)\}$ via lemma 1.2.3. So there exists $\{x_i\} \subset B$ s.t.

$$B \subset \bigcup_{i=1}^\infty B_{3\rho(x_i)}(x_i) \qquad \textit{and} \qquad B_{\rho(x_i)}(x_i) \cap B_{\rho(x_j)}(x_j) = \varnothing \ \textit{for}\ i \ne j$$

and (2.30) holds for each x_i. Hence one may bound, using $B_{\rho(x_i)}(x_i) \subset A$ and disjoint, and then (2.29)

$$\begin{aligned}\sum_{i=1}^\infty (3\rho(x_i))^{n-1} &\le \sum_{i=1}^\infty 3^{n-1} \frac{2}{\omega_{n-1}} \int_{B_{\rho(x_i)}(x_i)} |D\varphi_E| \le \frac{2\cdot 3^{n-1}}{\omega_{n-1}} \int_A |D\varphi_E| \\ &\le \frac{2\cdot 3^{n-1}}{\omega_{n-1}} \left(\int_B |D\varphi_E| + \eta \right)\end{aligned}$$

Hence recalling (2.27), since $B \subset \bigcup_{i=1}^\infty B_{3\rho(x_i)}(x_i)$ with $\rho(x_i) < \varepsilon$ universal bound in i

$$H_{n-1}^\varepsilon(B) \le \frac{\omega_{n-1}}{2^{n-1}} \inf \left\{ \sum_{i=1}^\infty (2\cdot 3\rho(x_i))^{n-1} \mid \rho(x_j) < \varepsilon \,\forall\, j \right\} \le 2\cdot 3^{n-1} \left(\int_B |D\varphi_E| + \eta \right)$$

take supremum in ε on LHS to obtain $H_{n-1}(B)$. Take $\eta \to 0$ to conclude (2.28). □

One shall be precise of our notion of C^1 hypersurface.

Definition 2.2.2. *Let Γ_{n-1} be collection of $H \subset \mathbb{R}^n$ s.t. there exists A open containing H and a C^1 function $f : A \to \mathbb{R}$ s.t.*

$$f(x) = 0 \quad \textit{and} \quad Df(x) \ne 0 \qquad \forall\, x \in \overline{H}$$

One needs a criterion to determine sets of Γ_{n-1}.

- Recall Whitney Extension Theorem

  Lemma 2.2.2. *Let $C \subset \mathbb{R}^n$ closed and $f : C \to \mathbb{R}$, $\nu : C \to \mathbb{R}^n$ be continuous. If for each compact $K \subset C$*

$$\sup\left\{\frac{|f(y) - f(x) - \nu(x)\cdot(y-x)|}{|x-y|} \mid 0 < |x-y| \le \delta,\ x, y \in K\right\} \to 0 \qquad as\ \delta \to 0$$

  Then there exists a global C^1 function $\overline{f} : \mathbb{R}^n \to \mathbb{R}$ s.t. $\overline{f} = f$ and $D\overline{f} = \nu$ on C.

- One hence obtain criterion

  Lemma 2.2.3. *Let $C \subset \mathbb{R}^n$ compact. If there exists $\nu : C \to \mathbb{R}^n$ continuous s.t. $\nu \neq 0$ and*

$$\lim_{|x-y|\to 0} \frac{\langle \nu(x),\, x-y\rangle}{|x-y|} = 0 \qquad uniformly\ for\ x, y \in C \tag{2.31}$$

  Then $C \in \Gamma_{n-1}$.

  Proof. Apply Whitney Extension Theorem with C compact and function $= 0$. Then there exists C^1 function $f : \mathbb{R}^n \to \mathbb{R}$ s.t. $f = 0$ and $Df = \nu$ on C. Since $\nu \neq 0$, conclude $C \in \Gamma_{n-1}$. □

Before we prove the main theorem for this section, one needs 2 lemmas to reduce our problem

Lemma 2.2.4 (Egoroff Theorem)**.** *Let μ be measure on $\mathbb{R}^n$ and f_k, $f : \mathbb{R}^n \to \mathbb{R}^m$ μ-measurable. Let $A \subset \mathbb{R}^n$ μ-measurable with $\mu(A) < \infty$ and $f_k \to f$ μ-a.e. on A. Then for any $\varepsilon > 0$, there exists $B \subset A$ μ-measurable s.t. $\mu(A \setminus B) < \varepsilon$ and $f_k \to f$ uniformly on B.*

Lemma 2.2.5 (Lusin Theorem)**.** *Let μ be Borel regular measure on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ μ-measurable. Let $A \subset \mathbb{R}^n$ μ-measurable and $\mu(A) < \infty$. Then for any $\varepsilon > 0$, there exists $K \subset A$ compact s.t. $\mu(A \setminus K) < \varepsilon$ and $f|_K$ is continuous.*

Theorem 2.2.1 (Structure Theorem for Caccioppoli Set)**.** *If $E \subset \mathbb{R}^n$ is Caccioppoli, then*

- $$\partial^* E = \bigcup_{i=1}^{\infty} C_i \cup N \tag{2.32}$$

  where N is $|D\varphi_E|$-measure zero and $C_i \in \Gamma_{n-1}$ compact for all i.

- *For each Borel set $B \subset \partial^* E$,*

$$\int_B |D\varphi_E| = H_{n-1}(B) \tag{2.33}$$

  and moreover, for every open set $\Omega \subset \mathbb{R}^n$

$$P(E,\Omega) = \int_\Omega |D\varphi_E| = H_{n-1}(\partial^* E \cap \Omega) = H_{n-1}\llcorner\partial^* E(\Omega) \tag{2.34}$$

$$\int_\Omega D\varphi_E = \int_{\partial^* E\cap\Omega} \nu(x)\, dH_{n-1} = \nu(x) H_{n-1}\llcorner\partial^* E(\Omega) \tag{2.35}$$

- *Reduced Boundary is dense in ∂E*

$$\overline{\partial^* E} = \partial E \tag{2.36}$$

Proof. Given E Caccioppoli

- 1. Since E is not necessarily bounded (in particular, $|D\varphi_E|(\partial^* E)$ isn't necessarily finite), one make use of σ-finite of $|D\varphi_E|$ to partition $\mathbb{R}^n = \bigcup_{i=1}^\infty \Omega_i$ into open bounded disjoint domains so $|D\varphi_E|(\Omega_i) < \infty$. Then $\partial^* E = \bigcup_{i=1}^\infty (\partial^* E \cap \Omega_i)$. If (2.32) is proved for all bounded Caccioppoli sets, then $\partial^* E \cap \Omega_i = \bigcup_j C_{i,j} \cup N_i$ so $\partial^* E = \bigcup_i \bigcup_j C_{i,j} \cup N_i$ is still countable union with $|D\varphi_E|(\bigcup_i N_i) \le \sum_i |D\varphi_E|(N_i) = 0$. Hence it suffices to prove (2.32) for bounded Caccioppoli Sets.

  2. Given E bounded Caccioppoli, $\partial^* E$ is bounded so $|D\varphi_E|(\partial^* E) < \infty$. Let $|D\varphi_E|$ be our finite measure on the space $\partial^* E$, and denote for any $x \in \partial^* E$, the limits (2.24) and (2.25)

$$\lim_{\rho\to 0} \frac{1}{\rho^n} |E \cap B_\rho(x) \cap T^-(x)| = 0 \qquad \lim_{\rho\to 0} \frac{1}{\rho^n} |(B_\rho(x) \setminus E) \cap T^+(x)| = 0 \tag{2.37}$$

as our pointwise limiting function sequences. Then Egoroff theorem applies, saying for any $i > 0$, there exists $F_i \subset \partial^* E$ that is $|D\varphi_E|$-measurable s.t.

$$|D\varphi_E|(\partial^* E \setminus F_i) < \frac{1}{2i} \tag{2.38}$$

and (2.37) holds uniformly on F_i. Now to apply Lusin theorem, first notice one has $\nu(x) = \lim_{\rho \to 0} \frac{\int_{B_\rho(x)} D\varphi_E}{\int_{B_\rho(x)} |D\varphi_E|}$ defined on $\partial^* E$ as in (2.4). As Radon measure, $|D\varphi_E|$ is Borel-regular, and with $|D\varphi_E|(F_i) < \infty$, Lusin's theorem says there exists $C_i \subset F_i$ compact s.t.

$$|D\varphi_E|(F_i \setminus C_i) < \frac{1}{2i} \tag{2.39}$$

and $\nu|_{C_i}$ is continuous. Thus $\partial^* E = (\partial^* E \setminus (\bigcup_i C_i)) \cup \bigcup_i C_i$ where

$$|D\varphi_E|(\partial^* E \setminus (\bigcup_i C_i)) = \lim_{i \to \infty} \mu(\partial^* E \setminus C_i) = 0$$

so $N = \partial^* E \setminus (\bigcup_i C_i)$ is our $|D\varphi_E|$-measure zero set. It suffices to show $C_i \in \Gamma_{n-1}$.

3. Fix any such C_i. By Egoroff, convergence (2.37) happens on C_i uniformly, so for any $\varepsilon \in (0,1)$, there exists $\sigma \in (0,1)$ s.t. for any $\rho < 2\sigma$

$$|E \cap B_\rho(z) \cap T^-(z)| < \rho^n (\frac{1}{4}\omega_n 2^{-n})\varepsilon^n \qquad \textit{uniformly on } z \in C_i \tag{2.40}$$

$$\begin{aligned} |E \cap B_\rho(z) \cap T^+(z)| &= |B_\rho(z) \cap T^+(z)| - |(B_\rho(z) \setminus E) \cap T^+(z)| = \rho^n \omega_n \frac{1}{2} - |(B_\rho(z) \setminus E) \cap T^+(z)| \\ &> \rho^n \omega_n \frac{1}{2} - \rho^n(\frac{1}{4}\omega_n 2^{-n})\varepsilon^n = \frac{\omega_n \rho^n}{2}(1 - \frac{\varepsilon^n}{2^{n+1}}) \qquad \textit{uniformly on } z \in C_i \end{aligned} \tag{2.41}$$

Using (2.40) and (2.41), we wish to show for any $\varepsilon \in (0,1)$, there exists $\sigma > 0$ s.t. $\frac{|\langle \nu(x), x-y \rangle|}{|x-y|} < \varepsilon$ uniformly for any x, $y \in C_i$ s.t. $|x - y| < \sigma$. Since C_i are compact, ν is continuous on C_i by Lusin, and by definition for reduced boundary (2.5) that $|\nu(x)| = 1 \neq 0$, applying lemma 2.2.3 concludes that $C_i \in \Gamma_{n-1}$.

4. To show uniform convergence, first suppose there exists some $\varepsilon \in (0,1)$ s.t. for any $\sigma \in (0,1)$, there exists x, $y \in C_i$ s.t. $|x - y| < \sigma$ yet $\langle \nu(x), y - x \rangle < -\varepsilon |x - y|$. By definition of $T^-(x)$, this implies $y \in T^-(x)$ for such x, y. And indeed, for any $|z - y| < \varepsilon |x - y|$, one has

$$\langle \nu(x), z - x \rangle = \langle \nu(x), z - y \rangle + \langle \nu(x), y - x \rangle \le |z - y| - \varepsilon |x - y| < 0$$

hence $z \in T^-(x)$. Thus $B_{\varepsilon|x-y|}(y) \subset T^-(x)$. Moreover, that $|z - y| < \varepsilon |x - y|$ implies

$$|x - z| \le |x - y| + |y - z| < |x - y| + \varepsilon |x - y| < 2|x - y|$$

so $B_{\varepsilon|x-y|}(y) \subset B_{2|x-y|}(x)$. Hence we have

$$B_{\varepsilon|x-y|}(y) \subset T^-(x) \cap B_{2|x-y|}(x) \implies |B_{\varepsilon|x-y|}(y)| \le |T^-(x) \cap B_{2|x-y|}(x)| \tag{2.42}$$

Now, since we require $|x - y| < \sigma$, choose $\rho = 2|x - y| < 2\sigma$, one may apply (2.40) with $z = x$

$$|E \cap B_{2|x-y|}(x) \cap T^-(x)| < \frac{\omega_n \varepsilon^n}{4} |x - y|^n \tag{2.43}$$

and then use (2.41) with $\rho = \varepsilon |x - y| < \sigma$ and $z = y$

$$|E \cap B_{\varepsilon|x-y|}(y)| \ge |E \cap B_{\varepsilon|x-y|}(y) \cap T^+(y)| > \frac{\omega_n \varepsilon^n |x - y|^n}{2}(1 - \frac{\varepsilon^n}{2^{n+1}}) > \frac{\omega_n \varepsilon^n}{4} |x - y|^n \tag{2.44}$$

But now (2.43) and (2.44) together yields

$$|E \cap B_{2|x-y|}(x) \cap T^-(x)| < |E \cap B_{\varepsilon|x-y|}(y)| \implies |B_{2|x-y|}(x) \cap T^-(x)| < |B_{\varepsilon|x-y|}(y)|$$

contradicting the inclusion (2.42). Hence our assumption for existence of ε fails if we require that $\langle \nu(x), y - x \rangle < -\varepsilon |x - y|$.

5. Then, we suppose certain $\varepsilon \in (0,1)$ exists and for any $\sigma \in (0,1)$, there exists x, $y \in C_i$ s.t. $|x-y| < \sigma$ yet $\langle \nu(x), y-x \rangle > \varepsilon|x-y|$. One derive the similar inequalities for (2.40) and (2.41) but on $B_\rho(z) \setminus E$

$$|(B_\rho(z) \setminus E) \cap T^+(z)| < \rho^n (\frac{1}{4}\omega_n 2^{-n})\varepsilon^n \qquad uniformly\ on\ z \in C_i \tag{2.45}$$

$$\begin{aligned} |(B_\rho(z) \setminus E) \cap T^-(z)| &= |B_\rho(z) \cap T^-(z)| - |B_\rho(z) \cap E \cap T^-(z)| = \rho^n \omega_n \frac{1}{2} - |B_\rho(z) \cap E \cap T^-(z)| \\ &> \rho^n \omega_n \frac{1}{2} - \rho^n (\frac{1}{4}\omega_n 2^{-n})\varepsilon^n = \frac{\omega_n \rho^n}{2}(1 - \frac{\varepsilon^n}{2^{n+1}}) \qquad uniformly\ on\ z \in C_i \end{aligned} \tag{2.46}$$

By definition of $T^+(x)$, $y \in T^+(x)$, and indeed for any $|z-y| < \varepsilon|x-y|$

$$\langle \nu(x), z-x \rangle = \langle \nu(x), z-y \rangle + \langle \nu(x), y-x \rangle > -\varepsilon|x-y| + \varepsilon|x-y| > 0$$

hence $z \in T^+(x)$. Thus $B_{\varepsilon|x-y|}(y) \subset T^+(x)$. Moreover, that $|z-y| < \varepsilon|x-y|$ implies $B_{\varepsilon|x-y|}(y) \subset B_{2|x-y|}(x)$. Hence we have

$$B_{\varepsilon|x-y|}(y) \subset T^+(x) \cap B_{2|x-y|}(x) \implies |B_{\varepsilon|x-y|}(y)| \leq |T^+(x) \cap B_{2|x-y|}(x)| \tag{2.47}$$

Apply (2.45) with $z = x$ and $\rho = 2|x-y| < 2\sigma$ yields

$$|(B_{2|x-y|}(x) \setminus E) \cap T^+(x)| < \frac{\omega_n \varepsilon^n}{4}|x-y|^n \tag{2.48}$$

Then use (2.46) with $z = y$ and $\rho = \varepsilon|x-y| < \sigma$

$$|B_{\varepsilon|x-y|}(y) \setminus E| \geq |(B_{\varepsilon|x-y|}(y) \setminus E) \cap T^-(y)| > \frac{\omega_n \varepsilon^n}{4}|x-y|^n \tag{2.49}$$

together (2.48) and (2.49) yields

$$|T^+(x) \cap B_{2|x-y|}(x)| < |B_{\varepsilon|x-y|}(y)|$$

and we have contradiction.

6. We conclude that $\frac{\langle \nu(x), x-y \rangle}{|x-y|}$ converges to 0 uniformly for x, $y \in C_i$ for any i. Hence by lemma 2.2.3, $C_i \in \Gamma_{n-1}$ for any i. Thus we've proved (2.32).

- 1. For any $B \subset \partial^* E$, $B = (B \cap (\bigcup_i C_i)) \cup (B \setminus \bigcup_i C_i)$, but by continuity from above and the Ratio Estimate (2.28)

$$H_{n-1}(B \setminus \bigcup_i C_i) = \lim_{i \to \infty} H_{n-1}(B \cap C_i^c) \leq \lim_{i \to \infty} 2 \cdot 3^{n-1} \int_{B \setminus C_i} |D\varphi_E| \leq 2 \cdot 3^{n-1} \lim_{i \to \infty} \frac{1}{i} = 0$$

where (2.38) and (2.39) gives

$$\int_{B \setminus C_i} |D\varphi_E| \leq \int_{\partial^* E \setminus C_i} |D\varphi_E| \leq \int_{\partial^* E \setminus F_i} |D\varphi_E| + \int_{F_i \setminus C_i} |D\varphi_E| < \frac{1}{i}$$

Hence $H_{n-1}(B) = H_{n-1}(\bigcup_i (B \cap C_i))$. One may write

$$\bigcup_i (B \cap C_i) = (B \cap C_1) \cup \bigcup_{i=1}^{\infty} (B \cap C_{i+1}) \setminus (B \cap C_i) \qquad into\ disjoint\ union$$

Since subsets of $C_i \in \Gamma_{n-1}$ still belong to Γ_{n-1}, it suffices to prove (2.33) for $B \in \Gamma_{n-1}$.

2. Given $B \in \Gamma_{n-1}$, there exists open set $A \supset \overline{B}$ and $f : A \to \mathbb{R}$ that is C^1 s.t.

$$f(x) = 0 \qquad and \qquad Df(x) \neq 0 \qquad \forall x \in \overline{B}$$

Since f is C^1, up to taking subset, one may assume $Df(x) \neq 0$ on the open set A. Now $0 \in \mathbb{R}$ is a regular value for the map $f : A \to \mathbb{R}$, so taking its preimage

$$V := \{x \in A \mid f(x) = 0\}$$

one has from preimage theorem that V is C^1 regular hypersurface of dimension $n-1$.

3. From Hausdorff measure properties, $H_{n-1} \llcorner V$ has density for any $x \in B$
$$\lim_{\rho \to 0} \frac{H_{n-1} \llcorner V(B_\rho(x))}{\rho^{n-1}} = \omega_{n-1} \qquad for\ any\ x \in B$$
and because $x \in B \subset \partial^* E$, taking $\varepsilon = 1$ in density estimates for $|D\varphi_E|$ in (2.26) gives
$$\lim_{\rho \to 0} \frac{\int_{B_\rho(x)} |D\varphi_E|}{\rho^{n-1}} = \omega_{n-1} \qquad for\ any\ x \in B$$
hence $\lim_{\rho \to 0} \frac{H_{n-1} \llcorner V(B_\rho(x))}{\int_{B_\rho(x)} |D\varphi_E|} = 1$ for any $x \in B$. Now by Lebesgue-Besicovitch differentiation 2.1.2 and Ratio estimate (2.28) implying $H_{n-1} \ll |D\varphi_E|$ on $\partial^* E$, and that $B \subset V$
$$H_{n-1}(B) = H_{n-1} \llcorner V(B) = \lim_{\rho \to 0} \frac{H_{n-1} \llcorner V(B_\rho(x))}{\int_{B_\rho(x)} |D\varphi_E|} \cdot \int_B |D\varphi_E| = \int_B |D\varphi_E| \qquad \forall\ B \subset \partial^* E\ Borel$$
concluding (2.33).
4. From (2.33), take $B = \partial^* E \cap \Omega$ for any $\Omega \subset \mathbb{R}^n$ open gives
$$H_{n-1} \llcorner \partial^* E(\Omega) = H_{n-1}(\partial^* E \cap \Omega) = \int_{\partial^* E \cap \Omega} |D\varphi_E|$$
But according to Lebesgue-Besicovitch differentiation 2.1.2, in particular, remark 2.1.1, one has $|D\varphi_E|(\partial E \setminus \partial^* E) = 0$. Hence the above writes, upon using $\text{supp}(D\varphi_E) \subset \partial E$
$$H_{n-1} \llcorner \partial^* E(\Omega) = \int_{\partial E \cap \Omega} |D\varphi_E| = \int_\Omega |D\varphi_E| = P(E, \Omega)$$
concluding (2.34). (2.35) follows immediately from (2.7).

- To show density, one show that for any A open set intersecting ∂E, it must also intersect $\partial^* E$. In other words, if A open not does intersect $\partial^* E$, by (2.34)
$$0 = H_{n-1}(\partial^* E \cap A) = \int_A |D\varphi_E|$$
but since A is open, ∂E is closed, they do not intersect. Hence $\overline{\partial^* E} = \partial E$.

□

2.2.2 Lipschitz Regularity of Reduced Boundary

Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$ where $|\alpha| = (\sum_{i=1}^n \alpha_i^2)^{\frac{1}{2}} = 1$ is unit vector. Denote $D_\alpha = \sum_{i=1}^n \alpha_i D_i$.

Lemma 2.2.6 (Moving Ball Integration). *Let E Caccioppoli set in Ω (i.e., $E \subset\subset \Omega$ or $E \subset \Omega$ and part of ∂E agree with $\partial\Omega$) with $\Omega \subset \mathbb{R}^n$ open. Fix unit vector α. Let $z \in \Omega$, $\rho > 0$, and $\tau > 0$ s.t. for any $0 < t < \tau$, $B_\rho(z + t\alpha) \subset\subset \Omega$. Then*
$$|E \cap B_\rho(z + \tau\alpha)| - |E \cap B_\rho(z)| = \int_0^\tau \int_{B_\rho(z+t\alpha)} D_\alpha \varphi_E \, dt \tag{2.50}$$

Proof. One choose $g_k \subset C_0^\infty(\Omega)$ s.t. $0 \le g_k \le 1$, $g_k = 1$ on $B_{\rho - \frac{1}{k}}(z)$ and $\text{supp}(g_k) \subset B_\rho(z)$. Hence $g_k \to \varphi_{B_\rho(z)}$ in L^1. Moreover, for $B_\rho(z + \tau\alpha) \subset \Omega$, one has $g_k(x - \tau\alpha) \to \varphi_{B_\rho(z)}(x - \tau\alpha) = \varphi_{B_\rho(z+\tau\alpha)}(x)$ in L^1, so
$$|E \cap B_\rho(z + \tau\alpha)| = \int_E \varphi_{B_\rho(z+\tau\alpha)} \, dx = \lim_{k \to \infty} \int_E g_k(x - \tau\alpha) \, dx, \qquad |E \cap B_\rho(z)| = \lim_{k \to \infty} \int_E g_k(x) \, dx$$
But due to $g_k \in C_0^\infty(\Omega)$ and that $\text{supp}(g_k(x - t\alpha)) \subset B_\rho(z + t\alpha) \subset\subset \Omega$ for any $0 < t < \tau$
$$\begin{aligned}\int_E g_k(x - \tau\alpha) - g_k(x) \, dx &= -\int_E \int_0^\tau \alpha \cdot \nabla g_k(x - t\alpha) \, dt \, dx = -\int_\Omega \int_0^\tau \varphi_E \, \alpha \cdot \nabla g_k(x - t\alpha) \, dt \, dx \\ &= -\int_0^\tau \int_\Omega \varphi_E \, \alpha \cdot \nabla g_k(x - t\alpha) \, dx \, dt = \int_0^\tau \int_\Omega g_k(x - t\alpha) \, \alpha \cdot D\varphi_E \, dt\end{aligned}$$
Now take limit on both sides to arrive at
$$|E \cap B_\rho(z + \tau\alpha)| - |E \cap B_\rho(z)| = \int_0^\tau \int_\Omega \varphi_{B_\rho(z+t\alpha)}(x) D_\alpha \varphi_E \, dt = \int_0^\tau \int_{B_\rho(z+t\alpha)} D_\alpha \varphi_E \, dt$$

□

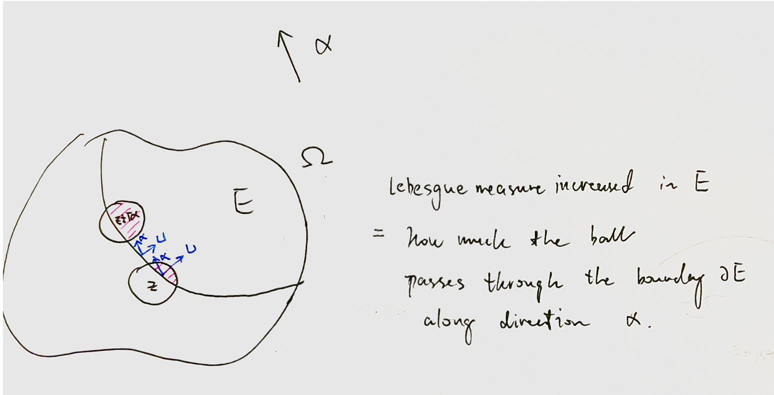

Figure 2.1: Moving Ball Integration measure increase Lemma 2.2.6

Lemma 2.2.7 (Boundary points Can't Escape E along α with uniformly positive normal projection)**.** *Let E Caccioppoli set in Ω with $\Omega \subset \mathbb{R}^n$ open. Suppose there exists $\alpha \in \mathbb{R}^n$ unit vector and a lower bound $p > 0$ s.t.*

$$\nu(x) \cdot \alpha = \lim_{\rho \to 0} \frac{\int_{B_\rho(x)} D_\alpha \varphi_E}{\int_{B_\rho(x)} |D\varphi_E|} \geq p > 0 \tag{2.51}$$

for $|D\varphi_E|$-a.e. $x \in \Omega$ (notice $\nu(x)$ exists $|D\varphi_E|$-a.e. x due to (2.6)). Let $z \in \partial E \cap \Omega$.

- *For any $k > 0$ s.t. the line segment $[z, z + k\alpha] \subset \Omega$, then $z + k\alpha \in \mathring{E}$, i.e interior of E.*
- *For any $k < 0$ s.t. the line segment $[z + k\alpha, z] \subset \Omega$, then $z + k\alpha \in (\Omega \setminus E)^\circ$, i.e. interior of $\Omega \setminus E$.*

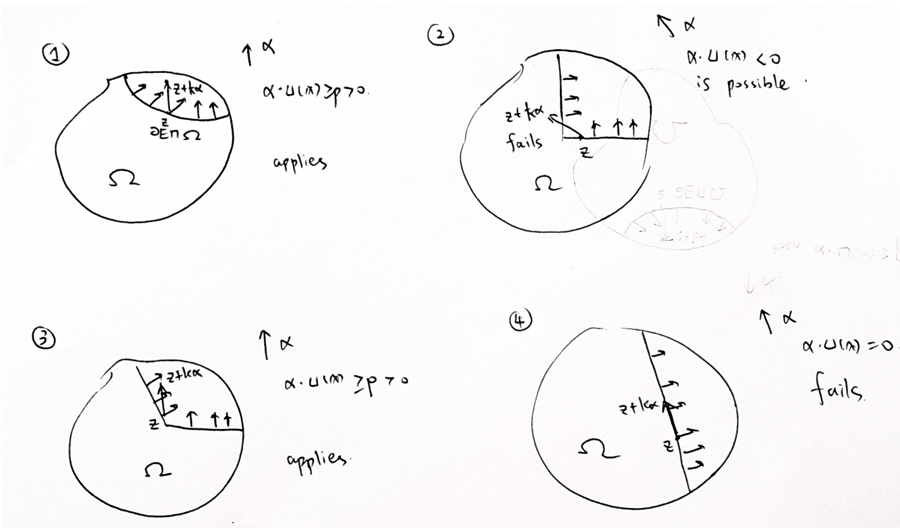

Figure 2.2: 4 illustrations for Lemma 2.2.7

Proof. One starts with showing for $k > 0$.

- First assume for contradiction that there exists $z \in \partial E \cap \Omega$ and $k > 0$ s.t. the line segment $z + \tau\alpha \in \Omega$ for any $0 < \tau \leq k$, yet $z + k\alpha$ does not lie in $\mathring{E}$. We wish to argue hence $[z, z + k\alpha] \subset \partial E$.
  1. Suppose there exists $0 < \tau \leq k$ s.t. $z + \tau\alpha \in \Omega \setminus \overline{E}$. Then one may choose $\rho > 0$ s.t. $B_\rho(z+\tau\alpha) \subset \Omega \setminus \overline{E}$. Using (2.51) and (2.50) one obtain

     $$0 \leq \int_0^\tau \int_{B_\rho(z+t\alpha)} D_\alpha \varphi_E \, dt = |E \cap B_\rho(z + \tau\alpha)| - |E \cap B_\rho(z)| = -|E \cap B_\rho(z)|$$

     Notice we're using E under Lemma 2.1.1 with the same topological boundary and measure theoretic boundary, in particular, (2.1). So for $z \in \partial E$

     $$0 \leq -|E \cap B_\rho(z)| < 0$$

     and we have a contradiction. Thus $[z, z + k\alpha] \subset \overline{E}$.

2. Notice we're assuming $z+k\alpha \notin \mathring{E}$. Hence $z+k\alpha \in \partial E$. Now suppose there exists $0<\tau\leq k$ s.t. $z+\tau\alpha \in \mathring{E}$. One may choose $\rho>0$ s.t. $B_\rho(z+\tau\alpha)\subset \mathring{E}$. This implies $|B_\rho(z+\tau\alpha)\cap E| = \omega_n\rho^n$. Now using again (2.51) with (2.50)

$$0 \leq \int_\tau^k \int_{B_\rho(z+t\alpha)} D_\alpha\varphi_E\, dt = |E\cap B_\rho(z+k\alpha)| - |E\cap B_\rho(z+\tau\alpha)| = |E\cap B_\rho(z+k\alpha)| - \omega_n\rho^n$$

Again notice under Lemma 2.1.1 with (2.1). So for $z+k\alpha\in\partial E$

$$0\leq |E\cap B_\rho(z+k\alpha)| - \omega_n\rho^n < \omega_n\rho^n - \omega_n\rho^n = 0$$

and we have a contradiction. Thus $[z, z+k\alpha]\subset \partial E$.

- We wish to further argue $[z,z+k\alpha]\subset\partial E$ leads to contradiction with the strict positivity in (2.51). Note we assume at first $[z,z+k\alpha]\subset\Omega$, hence $[z,z+k\alpha]\subset\partial E\cap\Omega$. We may choose ρ_0 so that for any $0<\rho\leq\rho_0$ and $0\leq t\leq k$, one has $B_\rho(z+t\alpha)\subset\Omega$. Then using (2.7) $D\varphi_E = \nu|D\varphi_E|$ agree Borel-a.e.

$$\begin{aligned}\int_{B_\rho(z+t\alpha)} D_\alpha\varphi_E &= \int_{B_\rho(z+t\alpha)} \alpha\cdot D\varphi_E = \int_{B_\rho(z+t\alpha)}\alpha\cdot\nu|D\varphi_E| \\ &= \int_{B_\rho(z+t\alpha)} \lim_{\rho\to 0}\frac{\int_{B_\rho(z+t\alpha)} D_\alpha\varphi_E}{\int_{B_\rho(z+t\alpha)}|D\varphi_E|}|D\varphi_E| \geq p\int_{B_\rho(z+t\alpha)}|D\varphi_E|\end{aligned}$$

Now notice (2.51) satisfies assumption for Uniform Density Estimate (2.9), one may apply (2.12)

$$\int_{B_\rho(z+t\alpha)}|D\varphi_E|\geq C\rho^{n-1}$$

for some constant $C>0$. Hence for any $0<\rho\leq\rho_0$ and for any $0\leq t\leq k$

$$\int_{B_\rho(z+t\alpha)} D_\alpha\varphi_E \geq Cp\,\rho^{n-1}$$

Thus apply (2.50)

$$|E\cap B_\rho(z+k\alpha)| - |E\cap B_\rho(z)| = \int_0^k\int_{B_\rho(z+t\alpha)} D_\alpha\varphi_E\,dt \geq Ckp\,\rho^{n-1}$$

Yet again by Lemma 2.1.1, since both z, $z+k\alpha\in\partial E$

$$|E\cap B_\rho(z+k\alpha)| + |E\cap B_\rho(z)| \leq 2\omega_n\rho^n$$

from which we may conclude for any $0<\rho\leq\rho_0$

$$2\omega_n\rho^n\geq Ckp\,\rho^{n-1} \implies \rho\geq\frac{Ckp}{2\omega_n}>0$$

But RHS is independent of ρ so we take $\rho\to 0$ on LHS and reach a contradiction.

For $k<0$, redefine $\tilde{\alpha}:=-\alpha$ and use (2.51) with strictly negative inequality. The same argument applies. □

Now we wish to show if $\nu(x)$ does not vary too much, then set E has Lipschitz continuous boundary. Upon rotating we consider ν varying not much and pointing upwards in x_n-axis direction.

Theorem 2.2.2 (Lipschitz Regularity for ∂E)**.** *$\Omega\subset\mathbb{R}^n$ open, convex and E Caccioppoli in Ω. Suppose there exists constant $1\geq q>\frac{\sqrt{2}}{2}$ s.t.*

$$\nu_n(x) = \lim_{\rho\to 0}\frac{\int_{B_\rho(x)} D_n\varphi_E}{\int_{B_\rho(x)}|D\varphi_E|}\geq q>\frac{\sqrt{2}}{2} \tag{2.52}$$

for $|D\varphi_E|$-a.e. $x\in\Omega$. Then there exists an open set $A\subset\mathbb{R}^{n-1}$ and a function $f:A\to\mathbb{R}$ s.t.

$$\partial E\cap\Omega = \{(y,t)\in A\times\mathbb{R}\mid f(y)=t\} \tag{2.53}$$

and for any y, $y'\in A$

$$|f(y)-f(y')|\leq\frac{\sqrt{1-q^2}}{q}|y-y'| \tag{2.54}$$

Proof. Consider any unit vector α with $\alpha_n > 0$. Then using (2.52)

$$D_\alpha \varphi_E = \alpha_n D_n \varphi_E + \sum_{i=1}^{n-1} \alpha_i D_i \varphi_E \geq \alpha_n D_n \varphi_E - \left(\sum_i^{n-1} \alpha_i^2\right)^{\frac{1}{2}} \left(\sum_i^{n-1} (D_i \varphi_E)^2\right)^{\frac{1}{2}}$$
$$\geq q\alpha_n |D\varphi_E| - \sqrt{1-\alpha_n^2} \left(\sum_i^{n-1} (D_i \varphi_E)^2\right)^{\frac{1}{2}}$$

But one may calculate

$$|D\varphi_E|^2 = \sum_{i=1}^{n} (D_i \varphi_E)^2 = \sum_{i=1}^{n-1} (D_i \varphi_E)^2 + (D_n \varphi_E)^2 \geq \sum_{i=1}^{n-1} (D_i \varphi_E)^2 + q^2 |D\varphi_E|^2 \implies \sqrt{1-q^2}|D\varphi_E| \geq \left(\sum_{i=1}^{n-1} (D_i \varphi_E)^2\right)^{\frac{1}{2}}$$

Hence

$$D_\alpha \varphi_E \geq \left(q\alpha_n - \sqrt{(1-\alpha_n^2)(1-q^2)}\right) |D\varphi_E|$$

One may choose $\alpha_n > \sqrt{1-q^2}$ so that $1-\alpha_n^2 < q^2$ and thus

$$D_\alpha \varphi_E > \left(q\sqrt{1-q^2} - q\sqrt{1-q^2}\right) |D\varphi_E| = 0$$

We may apply Lemma 2.2.7 so that for any $z \in \partial E \cap \Omega$ and for any α unit vector with $\alpha_n > \sqrt{1-q^2} \geq 0$ we have points in Ω of the form $z + t\alpha \in \mathring{E}$ and points in Ω of the form $z - t\alpha \in (\Omega \setminus E)^\circ$. Notice Lemma 2.2.7 is applicable due to convexity of Ω, ensuring all line segments connecting z and $z \pm t\alpha$ lie within Ω. Now we wish to choose in particular the α unit vector. If let $\alpha_n = q$, we indeed require

$$q > \sqrt{1-q^2} \iff 2q^2 > 1 \text{ and } q > 0 \iff q > \frac{\sqrt{2}}{2}$$

which satisfies our assumption $q > \frac{\sqrt{2}}{2}$. Hence choosing α with $\alpha_n = q$ so that $\left(\sum_{i=1}^{n-1} \alpha_i^2\right)^{\frac{1}{2}} = \sqrt{1-q^2}$ is plausible. For any $x \in \Omega$ s.t. $x = z + t\alpha \iff (x-z) \cdot \alpha = t$, if $t > 0$, by Lemma 2.2.7 we have $x \in \mathring{E}$. But $t > 0$ is equivalent to the condition that

$$(x_n - z_n)\alpha_n + \sum_{i=1}^{n-1} (x_i - z_i)\alpha_i > 0$$

Notice

$$(x_n - z_n)\alpha_n + \sum_{i=1}^{n-1} (x_i - z_i)\alpha_i \geq (x_n - z_n)\alpha_n - \left(\sum_{i=1}^{n-1} (x_i - z_i)^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} \alpha_i^2\right)^{\frac{1}{2}}$$

Hence requiring that

$$(x_n - z_n)\alpha_n > \left(\sum_{i=1}^{n-1} (x_i - z_i)^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} \alpha_i^2\right)^{\frac{1}{2}} \iff q(x_n - z_n) > \sqrt{1-q^2} \left(\sum_{i=1}^{n-1} (x_i - z_i)^2\right)^{\frac{1}{2}}$$

is an overkill condition that is sufficient for $x \in \mathring{E}$. Thus the cone

$$C_z := \left\{ x \in \Omega \mid x_n - z_n > \frac{\sqrt{1-q^2}}{q} \left(\sum_{i=1}^{n-1} (x_i - z_i)^2\right)^{\frac{1}{2}} \right\} \subset \mathring{E}$$

and for exact same reasoning

$$C'_z := \left\{ x \in \Omega \mid x_n - z_n < -\frac{\sqrt{1-q^2}}{q} \left(\sum_{i=1}^{n-1} (x_i - z_i)^2\right)^{\frac{1}{2}} \right\} \subset (\Omega \setminus E)^\circ$$

Hence we have

$$\partial E \cap \Omega \subset \Omega \setminus (C_z \cup C'_z) = \Omega \cap C_z^c \cap C_z'^{,c}$$

for any $z \in \partial E \cap \Omega$. The LHS is independent of z so one has

$$\partial E \cap \Omega \subset \bigcap_{z \in \partial E \cap \Omega} (\Omega \cap C_z^c \cap C_z'^{,c})$$

Suppose there exists $(x_1, \cdots, x_{n-1}, x_n)$, $(x_1, \cdots, x_{n-1}, x_n') \in \partial E \cap \Omega$ with $x_n \neq x_n'$, one has

$$|x_n - x_n'| \leq \frac{\sqrt{1-q^2}}{q} \left(\sum_{i=1}^{n-1} (x_i - x_i)^2 \right)^{\frac{1}{2}} = 0$$

reaching contradiction. Hence each point $x = (x_1, \cdots, x_{n-1}, x_n) \in \partial E \cap \Omega$ has unique correspondence between $(x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$, i.e., this defines a function $x_n = f(x_1, \cdots, x_{n-1})$ for any $x \in \partial E \cap \Omega$. Moreover, for any x, $x' \in \partial E \cap \Omega$

$$|x_n - x_n'| = |f(x_1, \cdots, x_{n-1}) - f(x_1', \cdots, x_{n-1}')| \leq \frac{\sqrt{1-q^2}}{q} \left(\sum_{i=1}^{n-1} (x_i - x_i')^2 \right)^{\frac{1}{2}}$$

This defines a Lipschitz continuous function f over its domain. Since Ω is open, $f^{-1}(\Omega \cap (\{0\} \times \mathbb{R})) \subset \mathbb{R}^{n-1}$ is open due to continuity of f. In fact, $A := f^{-1} \circ pr_n(\Omega \cap (\{0\} \times \mathbb{R}))$ is the domain of definition for f (where pr_n denotes projection onto n-th coordinate). Hence (2.53) follows. Writing $\sup_{y, y' \in A} \left| \frac{f(y)-f(y')}{y-y'} \right| \leq \frac{\sqrt{1-q^2}}{q} < \infty$ gives Lipschitz continuous function (2.54). □

To upgrade to C^1 regularity, one needs tool that transits from Lipschitz continuity to C^1.

Lemma 2.2.8 (Rademacher's Theorem). *Let $\Omega \subset \mathbb{R}^n$ open and $f : \Omega \to \mathbb{R}$ locally Lipschitz continuous. Then f is differentiable $\mathcal{L}^n$-a.e. in Ω ($\mathcal{L}^n$ denotes n-dim Lebesgue measure).*

Remark 2.2.1. *Under same assumptions as Theorem 2.2.2, there exists $f : A \subset \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz continuous with $\partial E \cap \Omega = \{(y, t) \in A \times \mathbb{R} \mid f(y) = t\}$. And for $\mathcal{L}^{n-1}$-a.e. $y \in A$, for $i = 1, \cdots, n-1$*

$$\nu_i(x) = \frac{D_i f(y)}{\sqrt{1 + |Df(y)|^2}}, \qquad \nu_n(x) = \frac{1}{\sqrt{1 + |Df(y)|^2}} \qquad for\ x = (y, f(y)) \in \partial E \cap \Omega \tag{2.55}$$

In fact, this holds are $|D\varphi_E|$-a.e. $x = (y, f(y)) \in \partial E \cap \Omega$.

Proof. Since $f : A \subset \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz continuous, by Rademacher's 2.2.8, for $\mathcal{L}^{n-1}$-a.e. $y \in A$, $f(y)$ is differentiable. Hence for $\mathcal{L}^{n-1}$-a.e. $y \in A$, the quantities $D_i f(y)$ and $\sqrt{1 + |Df(y)|^2}$ are well-defined. But since $\partial E \cap \Omega$ has Lipschitz boundary, Trace Extension Property for $\varphi_E \in BV(\Omega)$ says in (2.8) that

$$D\varphi_E = n(x) dH_{n-1} \llcorner \partial E \qquad on\ Borel\ sets\ in\ \Omega$$

where n denotes the classical inner unit normal w.r.t. ∂E. This immediately implies

- classical inner normal n is well-defined on $H_{n-1} \llcorner \partial E$-a.e. $x \in \Omega$. This is because the set in $A \subset \mathbb{R}^{n-1}$ where Df does not exists has $\mathcal{L}^{n-1}$ measure zero, and by one to one correspondence via $x = (y, f(y))$, the set that $n(x)$ is not defined has $H_{n-1} \llcorner \partial E$ measure zero.
- for points $y \in A$ on which f is differentiable (from classical theory)

$$n(x) = n((y, f(y))) = \left(\frac{D_1 f(y)}{\sqrt{1 + |Df(y)|^2}}, \cdots, \frac{D_{n-1} f(y)}{\sqrt{1 + |Df(y)|^2}}, \frac{1}{\sqrt{1 + |Df(y)|^2}} \right) \tag{2.56}$$

One use the equivalence (2.7)

$$D\varphi_E = \nu(x) |D\varphi_E| \qquad on\ Borel\ sets\ in\ \mathbb{R}^n$$

to equate (2.8) with the Structure Theorem (2.34)

$$|D\varphi_E| = H_{n-1} \llcorner \partial^* E \qquad on\ Borel\ sets\ in\ \mathbb{R}^n$$

So for $H_{n-1} \llcorner \partial^* E = |D\varphi_E|$-a.e. $x \in \Omega$, $\nu(x) = n(x)$. Recall the set that $n(x)$ is not defined has $H_{n-1} \llcorner \partial E$ measure zero, hence $H_{n-1} \llcorner \partial^* E$-measure zero. We take the union of the two $H_{n-1} \llcorner \partial^* E$ measure zero sets where either $\nu = n$ or (2.56) fails. Their union is still of $H_{n-1} \llcorner \partial^* E$ measure zero. One project such set onto $\mathbb{R}^{n-1}$ to obtain a $\mathcal{L}^{n-1}$ measure zero set. Hence one has (2.55) for both $\mathcal{L}^{n-1}$-a.e. y and $|D\varphi_E|$-a.e. $x = (y, f(y)) \in \partial E \cap \Omega$. □

Theorem 2.2.3 (C^1 Regularity for ∂E). *$\Omega \subset \mathbb{R}^n$ open and E Caccioppoli set in Ω. If $\nu(x)$ exists for every $x \in \partial E \cap \Omega$ and is continuous. Then $\partial E \cap \Omega$ is C^1 hypersurface.*

Proof. By definition of reduced boundary, $|\nu(x)| = 1$ on $\partial^* E$. But by Structure Theorem (2.36), $\overline{\partial^* E} = \partial E$. Using $\nu \in C(\partial E \cap \Omega; \mathbb{R}^n)$ continuous, $|\nu(x)| = 1$ on ∂E. Now for any $z \in \partial E \cap \Omega$, one may cover z using B ball small enough s.t. (2.52) holds upon rotating so that ν_n is close to x_n axis pointing upwards. This is of course applicable due to continuity of ν. And since we're covering with balls that are convex objects, one may apply Theorem 2.2.2 to obtain locally f with Lipschitz regularity whose graph is the boundary $\partial E \cap B$. Applying Remark 2.2.1, one has for $\mathcal{L}^{n-1}$-a.e. y

$$D_i f(y) = \frac{\nu_i(x)}{\nu_n(x)} \quad for\ x = (y, f(y)) \in \partial E \cap B$$

Hence the derivative of f coincides a.e. y with a continuous function. This is equivalent to $f \in C^1$ locally. Now repeat for arbitrary point $z \in \partial E \cap \Omega$, one obtain $\partial E \cap \Omega$ with C^1 regularity. □

Chapter 3

De Giorgi's Lemma

In this chapter we develop the key lemma to tackle regularity theory for minimal sets, the De Giorgi's lemma.

Definition 3.0.1 (Minimal Set)**.** *For $\Omega \subset \mathbb{R}^n$ open, and E Caccioppoli Set. E is minimal in Ω if for any $F \subset \mathbb{R}^n$ Borel s.t. $F = E$ outside Ω, we have*

$$P(E, \Omega) = \int_\Omega |D\varphi_E| \leq \int_\Omega |D\varphi_F| = P(F, \Omega)$$

From Theorem 1.1.5 we know the existence of Caccioppoli Set with minimal perimeter within bounded open Ω. Given such minimizing set, we examine its regularity. In doing this, we need to approximate the minimal set, and we shall introduce a measure of how close a set is to being minimal.

Definition 3.0.2 (ν and ψ)**.** *For $\Omega \subset \mathbb{R}^n$ open, let $f \in BV(\Omega)$*

- $\nu(f, \Omega) := \inf\{\int_\Omega |Dg| \mid g \in BV(\Omega), \operatorname{supp}(g - f) \subset \Omega\}$ *and* $\psi(f, \Omega) := \int_\Omega |Df| - \nu(f, \Omega)$.
- *If $\Omega = B_\rho$ we write $\nu(f, \rho) := \nu(f, B_\rho)$ and $\psi(f, \rho) := \psi(f, B_\rho)$.*
- *If $f = \varphi_E$ for some Caccioppoli set E, we write $\nu(E, \Omega) := \nu(\varphi_E, \Omega)$ and $\psi(E, \Omega) := \psi(\varphi_E, \Omega)$.*

Remark 3.0.1. *If E is a minimal set in bounded open Ω, then $\psi(E, \Omega) = 0$.*

Proof. If suffices to show $\nu(E, \Omega) = \inf\{\int_\Omega |Dg| \mid g \in BV(\Omega), \operatorname{supp}(g - \varphi_E) \subset \Omega\} = \int_\Omega |D\varphi_E| = P(E, \Omega)$. But indeed $\varphi_E \in BV(\Omega)$ due to Caccioppoli, and $\operatorname{supp}(\varphi_E - \varphi_E) = \varnothing \subset \Omega$, so $\nu(E, \Omega) \leq \int_\Omega |D\varphi_E|$. On the other hand, for any $\varepsilon > 0$, there exists $g \in BV(\Omega)$ and $\operatorname{supp}(g - \varphi_E) \subset \Omega$ s.t.

$$\nu(E, \Omega) + \varepsilon \geq \int_\Omega |Dg|$$

Now using Coarea formula (1.28), we may write

$$\int_\Omega |Dg| = \int_{-\infty}^{\infty} \int_\Omega |D\varphi_{\{x \in \Omega | g(x) < t\}}| \, dt$$

Hence

$$\begin{aligned}
\nu(E, \Omega) + \varepsilon \geq \int_\Omega |Dg| &= \int_{-\infty}^{\infty} \int_\Omega |D\varphi_{\{x \in \Omega | g(x) < t\}}| \, dt \\
&\geq \int_0^1 \int_\Omega |D\varphi_{\{x \in \Omega | g(x) < t\}}| \, dt \\
&= \int_0^1 \int_\Omega |D\varphi_{\{x \in \Omega | g(x) > t\}}| \, dt \\
&\geq \int_0^1 \int_\Omega |D\varphi_{\{x \in \mathbb{R}^n | g(x) > t\}}| \, dt \\
&\geq \int_0^1 \int_\Omega |D\varphi_E| \, dt = \int_\Omega |D\varphi_E|
\end{aligned}$$

where we've used $\operatorname{supp}(g - \varphi_E) \subset \Omega \implies g = \varphi_E$ outside Ω, so $F_t := \{x \in \mathbb{R}^n \mid g(x) > t\}$ satisfies $F_t = E$ outside Ω for any $t \in (0, 1)$. This is necessary so we may apply minimality of E. We also used that $\{x \in \Omega \mid g(x) < t\}$ and $\{x \in \Omega \mid g(x) > t\}$ mutually disjoint so they give same perimeter. □

We may now state the De Giorgi's Lemma

Theorem 3.0.1 (De Giorgi's Lemma). *For any $n \geq 2$ and $\alpha \in (0,1)$, there exists positive constant $\sigma = \sigma(n,\alpha) > 0$ s.t. for any $E \subset \mathbb{R}^n$ Caccioppoli and $\rho > 0$ satisfying the following*

$$\psi(E,\rho) := \int_{B_\rho} |D\varphi_E| - \inf\{\int_{B_\rho} |Dg| \mid g \in BV(B_\rho), \operatorname{supp}(g - \varphi_E) \subset B_\rho\} = 0 \tag{3.1}$$

$$\int_{B_\rho} |D\varphi_E| - |\int_{B_\rho} D\varphi_E| < \sigma(n,\alpha)\rho^{n-1} \tag{3.2}$$

Then

$$\int_{B_{\alpha\rho}} |D\varphi_E| - |\int_{B_{\alpha\rho}} D\varphi_E| < \alpha^n \left(\int_{B_\rho} |D\varphi_E| - |\int_{B_\rho} D\varphi_E| \right) \tag{3.3}$$

Notice the important term

Definition 3.0.3 (Excess). $\Lambda(E,\rho) := \frac{1}{\rho^{n-1}} \left\{ \int_{B_\rho} |D\varphi_E| - |\int_{B_\rho} D\varphi_E| \right\}$.

Remark 3.0.2. *Using Structure Theorem for Caccioppoli Set (2.34) and (2.35), one may write*

$$\int_{B_\rho} |D\varphi_E| = H_{n-1}(\partial^* E \cap B_\rho)$$

$$\int_{B_\rho} D\varphi_E = \int_{\partial^* E \cap B_\rho} \nu(x)\, dH_{n-1}$$

Hence the Excess writes

$$\Lambda(E,\rho) = \frac{1}{\rho^{n-1}} \left\{ H_{n-1}(\partial^* E \cap B_\rho) - |\int_{\partial^* E \cap B_\rho} \nu(x)\, dH_{n-1}| \right\}$$

as a measure of how much the direction of $\nu(x)$ changes in $B_\rho \cap \partial^* E$. If $\Lambda(E,\rho)$ is small, then $\nu(x)$ must remain approximately in a constant direction and thus we expect results from Theorem 2.2.2.

3.1 Approximation of Minimal Sets - C^1 Caccioppoli Sets

If $E \cap B_r$ are locally graphs of C^1 function, i.e. $E \cap B_r = \{(y,t) \in A \times \mathbb{R} \mid f(y) < t\} \cap B_r$ where $A \subset \mathbb{R}^{n-1}$ and $f \in C^1(A)$, then the boundary of our set E writes

$$\partial E \cap B_r = \{(y,t) \in \mathbb{R}^n \mid y \in A,\, t = f(y)\}$$

and our measures $|D\varphi_E|$, $D\varphi_E$ writes, recalling (2.55)

$$\int_{B_r} |D\varphi_E| = \int_{\partial E \cap B_r} 1\, dH_{n-1} = \int_A \sqrt{1 + |Df(y)|^2}\, dy \tag{3.4}$$

$$\begin{aligned} \int_{B_r} D\varphi_E &= \int_{\partial E \cap B_r} \nu(x)\, dH_{n-1} \\ &= \left(\int_A D_1 f(y)\, dy, \cdots, \int_A D_{n-1} f(y)\, dy, \int_A 1\, dy \right) \end{aligned} \tag{3.5}$$

Then the quantity

$$\begin{aligned} |\int_{B_r} D\varphi_E| &= \left(\sum_{j=1}^{n-1} \left(\int_A D_j f(y)\, dy \right)^2 + |A|^2 \right)^{\frac{1}{2}} \\ &= |A| \left(\sum_{j=1}^{n-1} \left(\frac{1}{|A|} \int_A D_j f(y)\, dy \right)^2 + 1 \right)^{\frac{1}{2}} \end{aligned}$$

If we're considering sets A approximated using $\mathscr{B}_\rho \subset \mathbb{R}^{n-1}$, we may introduce notation for $f \in C(\mathscr{B}_\rho)$

$$(f)_\rho := \frac{1}{|\mathscr{B}_\rho|} \int_{\mathscr{B}_\rho} f\, dx$$

and rewrite the above as

$$|\int_{B_r} D\varphi_E| \sim |\mathscr{B}_\rho| \left(\sum_{j=1}^{n-1} \left(\frac{1}{|\mathscr{B}_\rho|} \int_{\mathscr{B}_\rho} D_j f(y)\, dy \right)^2 + 1 \right)^{\frac{1}{2}}$$
$$= |\mathscr{B}_\rho| \sqrt{1 + |(Df)_\rho|^2} = \int_{\mathscr{B}_\rho} \sqrt{1 + |(Df)_\rho|^2}\, dy$$

Thus the quantity to the LHS of (3.2) writes

$$\int_{B_r} |D\varphi_E| - |\int_{B_r} D\varphi_E| \sim \int_{\mathscr{B}_\rho} \sqrt{1 + |Df(y)|^2}\, dy - \int_{\mathscr{B}_\rho} \sqrt{1 + |(Df)_\rho|^2}\, dy \tag{3.6}$$

With simplification using C^1 boundary, it suffices to study quantities in the RHS of (3.6). Moreover, notice the RHS is always non-negative due to choice of $(Df)_\rho$ as the average of Df over $\mathscr{B}_\rho$. Since both terms on RHS are finite measures locally, their difference, which is non-negative, again defines a measure on $\mathbb{R}^{n-1}$ that is locally finite.
Now to minimize as in (3.1) the perimeter of E in B_r

$$\int_{B_r} |D\varphi_E|$$

in C^1 boundary case, we're essentially minimizing

$$\int_A \sqrt{1 + |Df(y)|^2}\, dy$$

among all functions $f \in C^1(A)$. Yet for $|Df|$ small, i.e., ∂E nearly flat in B_ρ, $\sqrt{1 + |Df|^2}$ is roughly $1 + \frac{1}{2}|Df|^2$ via Taylor Expansion, so f minimize

$$I(f) = \int_A |Df|^2\, dx$$

That is, f must be nearly harmonic. Hence it is important to obtain estimates for harmonic functions which approximate sequences of surfaces tending to a minimum. We have the analogue of the De Giorgi Lemma for harmonic functions.

Lemma 3.1.1 (De Giorgi's Lemma for harmonic functions)**.** *Suppose $\mathscr{B}_\rho \subset \mathbb{R}^m$ and $u \in C^1(\mathscr{B}_\rho)$ harmonic in $\mathscr{B}_\rho$. Then for every $\alpha \in (0,1)$*

$$\int_{\mathscr{B}_{\alpha\rho}} \left(|Du|^2 - |(Du)_\rho|^2\right)\, dx \le \alpha^{m+2} \int_{\mathscr{B}_\rho} \left(|Du|^2 - |(Du)_\rho|^2\right)\, dx \tag{3.7}$$

Proof. A harmonic function $u \in C^1(\mathscr{B}_\rho)$ is automatically analytic in $\mathscr{B}_\rho$. Hence it may be written as Homogeneous Expansion, i.e., as series of homogeneous harmonic polynomials

$$u(x) = \sum_{i=0}^{\infty} V_i(x) \qquad \forall x \in \mathscr{B}_\rho$$

Here

- Each V_i is a harmonic polynomial homogeneous of degree i
- Since we're restricting to ball $\mathscr{B}_\rho$, V_i are orthogonal.
- In particular, for $j \neq k$

$$\int_{\mathscr{B}_{\alpha\rho}} \langle DV_j, DV_k \rangle dx = \int_{\mathscr{B}_\rho} \langle DV_j,\, DV_k \rangle dx = 0$$

- Since DV_i preserves harmonicity, for any $j \geq 2$

$$\int_{\mathscr{B}_{\alpha\rho}} DV_j dx = \int_{\mathscr{B}_\rho} DV_j dx = 0$$

  Hence by Mean Value Property

$$(Du)_\rho = \frac{1}{|\mathscr{B}_\rho|} \int_{\mathscr{B}_\rho} Du\, dx = \frac{1}{|\mathscr{B}_\rho|} \int_{\mathscr{B}_\rho} \sum_{i=0}^{\infty} DV_i\, dx = \frac{1}{\mathscr{B}_\rho} \int_{\mathscr{B}_\rho} DV_1\, dx = DV_1(0) = DV_1$$

Combining above and using orthogonality, one has

$$\int_{\mathscr{B}_\rho} \left(|Du|^2 - |(Du)_\rho|^2\right) dx = \sum_{j=2}^{\infty} \int_{\mathscr{B}_\rho} |DV_j|^2 \, dx$$

$$\int_{\mathscr{B}_{\alpha\rho}} \left(|Du|^2 - |(Du)_\rho|^2\right) dx = \sum_{j=2}^{\infty} \int_{\mathscr{B}_{\alpha\rho}} |DV_j|^2 \, dx$$

Finally noting

$$\begin{aligned}\sum_{j=2}^{\infty} \int_{\mathscr{B}_{\alpha\rho}} |DV_j|^2(x) \, dx &= \sum_{j=2}^{\infty} \int_{\mathscr{B}_\rho} |DV_j|^2(\alpha y) \, \alpha^m dy = \alpha^m \sum_{j=2}^{\infty} \int_{\mathscr{B}_\rho} \alpha^{2j-2} |DV_j|^2(y) \, dy \\ &\leq \alpha^{m+2} \sum_{j=2}^{\infty} \int_{\mathscr{B}_\rho} |DV_j|^2(y) \, dy\end{aligned}$$

□

One has an analogue of the De Giorgi Lemma for sequence of C^1 functions whose

- gradients tend to zero
- Excess has control via β_j positive constants, and do not differ much from harmonic functions in the sense that their defined surfaces are close.

Lemma 3.1.2 (De Giorgi's Lemma for C^1 functions approximating harmonic functions)**.** *Suppose $\mathscr{B}_\rho \subset \mathbb{R}^m$. Let $\omega_j \in C^1(\overline{\mathscr{B}}_\rho)$ be sequence of C^1 functions and $u_j \in C^1(\overline{\mathscr{B}}_\rho)$ harmonic functions s.t.*

$$u_j = \omega_j \qquad \text{on } \partial\mathscr{B}_\rho$$

Suppose for $\{\beta_j\} \subset \mathbb{R}_+$ sequence of positive numbers s.t.

$$\lim_{j\to\infty} \sup_{x\in\mathscr{B}_\rho} |D\omega_j(x)| = 0 \tag{3.8}$$

$$\int_{\mathscr{B}_\rho} \left(\sqrt{1+|D\omega_j|^2} - \sqrt{1+|(D\omega_j)_\rho|^2}\right) dx \leq \beta_j \tag{3.9}$$

$$\limsup_{j\to\infty} \frac{1}{\beta_j} \int_{\mathscr{B}_\rho} \left(\sqrt{1+|D\omega_j|^2} - \sqrt{1+|Du_j|^2}\right) dx = 0 \tag{3.10}$$

Then for any $\alpha \in (0,1)$

$$\limsup_{j\to\infty} \frac{1}{\beta_j} \int_{\mathscr{B}_{\alpha\rho}} \left(\sqrt{1+|D\omega_j|^2} - \sqrt{1+|(D\omega_j)_{\alpha\rho}|^2}\right) dx \leq \alpha^{m+2} \tag{3.11}$$

Proof. Before we start, we first derive some inequalities that we need. For any $x, y \in \mathbb{R}_+$, we taylor expand $\sqrt{1+y}$ at the point x. This gives us for some ξ between x and y

$$\sqrt{1+y} = \sqrt{1+x} + \frac{y-x}{2\sqrt{1+x}} - \frac{(y-x)^2}{8(1+\xi)^{\frac{3}{2}}}$$

Write $x = B^2$ and $y = A^2$, hence ξ is between A^2 and B^2, and thus nonnegative. We obtain

$$\sqrt{1+A^2} - \sqrt{1+B^2} \leq \frac{A^2 - B^2}{2\sqrt{1+B^2}} \tag{3.12}$$

now for $B^2 < 3$, we have $4(1+\xi)^{\frac{3}{2}} \geq 4 > \sqrt{1+B^2}$, hence $-\frac{1}{8(1+\xi)^{\frac{3}{2}}} > -\frac{1}{2\sqrt{1+B^2}}$ and

$$\sqrt{1+A^2} - \sqrt{1+B^2} - \frac{A^2 - B^2}{2\sqrt{1+B^2}} = -\frac{(A^2-B^2)^2}{8(1+\xi)^{\frac{3}{2}}} > -\frac{(A^2-B^2)^2}{2\sqrt{1+B^2}} \tag{3.13}$$

On the other hand, let A, B, $C \in \mathbb{R}^m$, for any $\varepsilon > 0$

$$\begin{aligned}|A-B|^2 &= \sum_{i=1}^{m} (A_i - B_i)^2 = \sum_{i=1}^{m} (A_i - C_i + C_i - B_i)^2 \\ &\leq \sum_{i=1}^{m} (A_i - C_i)^2 + 2\sum_{i=1}^{m} (A_i - C_i)(C_i - B_i) + \sum_{i=1}^{m} (C_i - B_i)^2\end{aligned}$$

$$\begin{aligned}2(A_i - C_i)(C_i - B_i) &= \frac{\sqrt{2}}{\sqrt{\varepsilon}} (A_i - C_i)\sqrt{2}\sqrt{\varepsilon}(C_i - B_i) \\ &\leq \frac{1}{\varepsilon}(A_i - C_i)^2 + \varepsilon(C_i - B_i)^2\end{aligned}$$

Hence

$$|A-B|^2 \leq (1+\frac{1}{\varepsilon})\sum_{i=1}^m (A_i - C_i)^2 + (1+\varepsilon)\sum_{i=1}^m (C_i - B_i)^2 = (1+\frac{1}{\varepsilon})|A-C|^2 + (1+\varepsilon)|B-C|^2 \tag{3.14}$$

(i) Our main goal is to bound

$$\limsup_{j\to\infty} \frac{1}{\beta_j}\int_{\mathscr{B}_{\alpha\rho}} \left(\sqrt{1+|D\omega_j|^2} - \sqrt{1+|(D\omega_j)_{\alpha\rho}|^2}\right) dx$$

in terms of α^{m+2}. In the first step we reduce the LHS to

$$\frac{1}{2}\limsup_{j\to\infty} \frac{1}{\beta_j}\int_{\mathscr{B}_{\alpha\rho}} |D\omega_j - (D\omega_j)_\rho|^2\, dx$$

To do so, apply (3.12) with $A = |D\omega_j|$ and $B = |(D\omega_j)_{\alpha\rho}|$ so that

$$\sqrt{1+|D\omega_j|^2} - \sqrt{1+|(D\omega_j)_{\alpha\rho}|^2} \leq \frac{|D\omega_j|^2 - |(D\omega_j)_{\alpha\rho}|^2}{2\sqrt{1+|(D\omega_j)_{\alpha\rho}|^2}}$$

Notice that

$$\begin{aligned}
|D\omega_j - (D\omega_j)_{\alpha\rho}|^2 &= \sum_{i=1}^m \left(D_i\omega_j - \frac{1}{|\mathscr{B}_{\alpha\rho}|}\int_{\mathscr{B}_{\alpha\rho}} D_i\omega_j\, dx\right)^2 \\
&= \sum_{i=1}^m \left(D_i\omega_j^2 - \frac{2}{|\mathscr{B}_{\alpha\rho}|}\left(\int_{\mathscr{B}_{\alpha\rho}} D_i\omega_j\right) D_i\omega_j + \frac{1}{|\mathscr{B}_{\alpha\rho}|^2}\left(\int_{\mathscr{B}_{\alpha\rho}} D_i\omega_j\right)^2\right) \\
\implies \int_{\mathscr{B}_{\alpha\rho}} |D\omega_j - (D\omega_j)_{\alpha\rho}|^2 &= \sum_{i=1}^m \int_{\mathscr{B}_{\alpha\rho}} D_i\omega_j^2 - \frac{2}{|\mathscr{B}_{\alpha\rho}|}\sum_{i=1}^m \left(\int_{\mathscr{B}_{\alpha\rho}} D_i\omega_j\right)^2 + \frac{1}{|\mathscr{B}_{\alpha\rho}|}\sum_{i=1}^m \left(\int_{\mathscr{B}_{\alpha\rho}} D_i\omega_j\right)^2 \\
&= \int_{\mathscr{B}_{\alpha\rho}} |D\omega_j|^2 - \sum_{i=1}^m \frac{1}{|\mathscr{B}_{\alpha\rho}|}\left(\int_{\mathscr{B}_{\alpha\rho}} D_i\omega_j\right)^2 \\
&= \int_{\mathscr{B}_{\alpha\rho}} |D\omega_j|^2 - |\mathscr{B}_{\alpha\rho}|\sum_{i=1}^m ((D_i\omega_j)_{\alpha\rho})^2 \\
&= \int_{\mathscr{B}_{\alpha\rho}} |D\omega_j|^2 - \int_{\mathscr{B}_{\alpha\rho}} |(D\omega_j)_{\alpha\rho}|^2
\end{aligned}$$

Also note that due to minimization of $(D\omega_j)_{\alpha\rho}$ over $\mathscr{B}_{\alpha\rho}$, one has

$$\int_{\mathscr{B}_{\alpha\rho}} |D\omega_j - (D\omega_j)_{\alpha\rho}|^2\, dx \leq \int_{\mathscr{B}_{\alpha\rho}} |D\omega_j - (D\omega_j)_\rho|^2\, dx$$

Hence one obtain bound

$$\begin{aligned}
\int_{\mathscr{B}_{\alpha\rho}} \sqrt{1+|D\omega_j|^2} - \sqrt{1+|(D\omega_j)_{\alpha\rho}|^2}\, dx &\leq \int_{\mathscr{B}_{\alpha\rho}} \frac{|D\omega_j|^2 - |(D\omega_j)_{\alpha\rho}|^2}{2\sqrt{1+|(D\omega_j)_{\alpha\rho}|^2}}\, dx \\
&\leq \int_{\mathscr{B}_{\alpha\rho}} \frac{|D\omega_j|^2 - |(D\omega_j)_{\alpha\rho}|^2}{2}\, dx \\
&= \int_{\mathscr{B}_{\alpha\rho}} \frac{|D\omega_j - (D\omega_j)_{\alpha\rho}|^2}{2}\, dx \\
&\leq \frac{1}{2}\int_{\mathscr{B}_{\alpha\rho}} |D\omega_j - (D\omega_j)_\rho|^2\, dx
\end{aligned}$$

(ii) In the second step we bound our previous term $\int_{\mathscr{B}_{\alpha\rho}} |D\omega_j - (D\omega_j)_\rho|^2\, dx$ part using $|D\omega_j - (D\omega_j)_\rho|^2$ on the unscaled ball $\mathscr{B}_\rho$ and the other part using $|D\omega_j - Du_j|^2$ on $\mathscr{B}_\rho$. The key is to go from scaled domain to the unscaled domain using (3.7) for harmonic functions. This throws the scaling parameter α from the domain to coefficients of the RHS. To do so, first use (3.14) for $A = D\omega_j$, $B = (D\omega_j)_\rho$ and $C = Du_j$. Fix any $\varepsilon > 0$

$$\begin{aligned}
\mathbf{III}_{\alpha\rho} := \int_{\mathscr{B}_{\alpha\rho}} |D\omega_j - (D\omega_j)_\rho|^2\, dx &\leq \left(1+\frac{1}{\varepsilon}\right)\int_{\mathscr{B}_{\alpha\rho}} |D\omega_j - Du_j|^2\, dx + (1+\varepsilon)\int_{\mathscr{B}_{\alpha\rho}} |Du_j - (D\omega_j)_\rho|^2\, dx \\
&=: \left(1+\frac{1}{\varepsilon}\right)\mathbf{I}_{\alpha\rho} + (1+\varepsilon)\,\mathbf{II}_{\alpha\rho}
\end{aligned} \tag{3.15}$$

Since $u_j = \omega_j$ on $\partial\mathscr{B}_\rho$, we may apply Gauss-Green Theorem, denoting ν as outer unit normal on $\partial\mathscr{B}_\rho$

$$(D\omega_j)_\rho = \frac{1}{|\mathscr{B}_\rho|}\int_{\partial\mathscr{B}_\rho} \omega_j \nu(x)\, dH_{n-1} = \frac{1}{|\mathscr{B}_\rho|}\int_{\partial\mathscr{B}_\rho} u_j \nu(x)\, dH_{n-1} = (Du_j)_\rho$$

hence using (3.7)

$$\begin{aligned}\mathbf{II}_{\alpha\rho} = \int_{\mathscr{B}_{\alpha\rho}} |Du_j - (D\omega_j)_\rho|^2\, dx &= \int_{\mathscr{B}_{\alpha\rho}} |Du_j - (Du_j)_\rho|^2\, dx \\ &\leq \alpha^{m+2}\int_{\mathscr{B}_\rho} |Du_j - (Du_j)_\rho|^2\, dx \\ &= \alpha^{m+2}\int_{\mathscr{B}_\rho} |Du_j - (D\omega_j)_\rho|^2\, dx =: \alpha^{m+2}\mathbf{II}_\rho\end{aligned}$$

Again applying (3.14) with $A = Du_j$, $B = (D\omega_j)_\rho$ and $C = D\omega_j$ to the last term $\int_{\mathscr{B}_\rho} |Du_j - (D\omega_j)_\rho|^2\, dx$ gives us

$$\begin{aligned}\mathbf{II}_\rho = \int_{\mathscr{B}_\rho} |Du_j - (D\omega_j)_\rho|^2\, dx &\leq \left(1+\frac{1}{\varepsilon}\right)\int_{\mathscr{B}_\rho} |Du_j - D\omega_j|^2\, dx + (1+\varepsilon)\int_{\mathscr{B}_\rho} |D\omega_j - (D\omega_j)_\rho|^2\, dx \\ &= \left(1+\frac{1}{\varepsilon}\right)\mathbf{I}_\rho + (1+\varepsilon)\,\mathbf{III}_\rho\end{aligned}$$

Hence the above summarizes to

$$\mathbf{II}_{\alpha\rho} \leq \alpha^{m+2}\mathbf{II}_\rho \leq \alpha^{m+2}\left(1+\frac{1}{\varepsilon}\right)\mathbf{I}_\rho + \alpha^{m+2}(1+\varepsilon)\,\mathbf{III}_\rho$$

Also notice the trivial bound

$$\mathbf{I}_{\alpha\rho} = \int_{\mathscr{B}_{\alpha\rho}} |D\omega_j - Du_j|^2\, dx \leq \int_{\mathscr{B}_\rho} |D\omega_j - Du_j|^2\, dx = \mathbf{I}_\rho$$

So plugging the two into (3.15) yields

$$\begin{aligned}&\mathbf{III}_{\alpha\rho} = \int_{\mathscr{B}_{\alpha\rho}} |D\omega_j - (D\omega_j)_\rho|^2\, dx \\ &\leq \left(1+\frac{1}{\varepsilon}\right)\int_{\mathscr{B}_\rho} |D\omega_j - Du_j|^2\, dx + (1+\varepsilon)\,\alpha^{m+2}\int_{\mathscr{B}_\rho} |Du_j - (D\omega_j)_\rho|^2\, dx = \left(1+\frac{1}{\varepsilon}\right)\mathbf{I}_\rho + (1+\varepsilon)\,\alpha^{m+2}\mathbf{II}_\rho \\ &\leq \left(1+\frac{1}{\varepsilon}\right)\int_{\mathscr{B}_\rho} |D\omega_j - Du_j|^2\, dx + (1+\varepsilon)\,\alpha^{m+2}\left((1+\frac{1}{\varepsilon})\int_{\mathscr{B}_\rho} |Du_j - D\omega_j|^2\, dx + (1+\varepsilon)\int_{\mathscr{B}_\rho} |D\omega_j - (D\omega_j)_\rho|^2\, dx\right) \\ &= \left(1+\frac{1}{\varepsilon}\right)\mathbf{I}_\rho + (1+\varepsilon)\,\alpha^{m+2}\left(1+\frac{1}{\varepsilon}\right)\mathbf{I}_\rho + (1+\varepsilon)^2\,\alpha^{m+2}\mathbf{III}_\rho \\ &= (1+\varepsilon)^2\,\alpha^{m+2}\int_{\mathscr{B}_\rho} |D\omega_j - (D\omega_j)_\rho|^2\, dx + \left(1+\frac{1}{\varepsilon}\right)\left(1+(1+\varepsilon)\,\alpha^{m+2}\right)\int_{\mathscr{B}_\rho} |D\omega_j - Du_j|^2\, dx \\ &= (1+\varepsilon)^2\,\alpha^{m+2}\mathbf{III}_\rho + \left(1+\frac{1}{\varepsilon}\right)\left(1+(1+\varepsilon)\,\alpha^{m+2}\right)\mathbf{I}_\rho \\ &=: (1+\varepsilon)^2\,\alpha^{m+2}\int_{\mathscr{B}_\rho} |D\omega_j - (D\omega_j)_\rho|^2\, dx + Q(\varepsilon,\alpha,m)\int_{\mathscr{B}_\rho} |D\omega_j - Du_j|^2\, dx\end{aligned}$$

Hence we've arrived at

$$\mathbf{III}_{\alpha\rho} \leq (1+\varepsilon)^2\,\alpha^{m+2}\mathbf{III}_\rho + Q(\varepsilon,\alpha,m)\mathbf{I}_\rho$$

i.e.

$$\int_{\mathscr{B}_{\alpha\rho}} |D\omega_j - (D\omega_j)_\rho|^2\, dx \leq (1+\varepsilon)^2\,\alpha^{m+2}\int_{\mathscr{B}_\rho} |D\omega_j - (D\omega_j)_\rho|^2\, dx + Q(\varepsilon,\alpha,m)\int_{\mathscr{B}_\rho} |D\omega_j - Du_j|^2\, dx \tag{3.16}$$

(iii) In the third step we bound the first term on RHS of (3.16) upon dividing by β_j, using our assumptions. Recall, as in Step 1

$$\int_{\mathscr{B}_\rho} |D\omega_j - (D\omega_j)_\rho|^2\, dx = \int_{\mathscr{B}_\rho} |D\omega_j|^2 - |(D\omega_j)_\rho|^2\, dx$$

and it suffices to bound the RHS. We apply (3.13) with $A = |D\omega_j|$ and $B = |(D\omega_j)_\rho|$ so that

$$\sqrt{1+|D\omega_j|^2} - \sqrt{1+|(D\omega_j)_\rho|^2} > \frac{1}{2\sqrt{1+|(D\omega_j)_\rho|^2}}\left(|D\omega_j|^2 - |(D\omega_j)_\rho|^2 - \left(|D\omega_j|^2 - |(D\omega_j)_\rho|^2\right)^2\right)$$

Note (3.13) is valid for j large enough as (3.8) ensures $\lim_{j\to\infty} |(D\omega_j)_\rho| = 0$ hence $B = |(D\omega_j)_\rho| < 3$. We need to deal with the second term on RHS $\left(|D\omega_j|^2 - |(D\omega_j)_\rho|^2\right)^2$. Note the model for arbitrary $x,\, y \in \mathbb{R}^m$

$$\begin{aligned}
|x|^2 - |y|^2 &= \sum_{i=1}^m (x_i^2 - y_i^2) = \sum_{i=1}^m (x_i + y_i)(x_i - y_i) \\
&= \sum_{i=1}^m (x_i)(x_i - y_i) + \sum_{i=1}^m (y_i)(x_i - y_i) \\
&\le \left(\sum_{i=1}^m (x_i)^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^m (x_i - y_i)^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^m (y_i)^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^m (x_i - y_i)^2\right)^{\frac{1}{2}} \\
&= (|x| + |y|)\,|x - y| \\
\implies (|x|^2 - |y|^2)^2 &\le (|x| + |y|)^2\, |x - y|^2
\end{aligned}$$

Let $x = D\omega_j$ and $y = (D\omega_j)_\rho$, we have

$$\left(|D\omega_j|^2 - |(D\omega_j)_\rho|^2\right)^2 \le \left(\sup_{\mathscr{B}_\rho} |D\omega_j| + |(D\omega_j)_\rho|\right)^2 |D\omega_j - (D\omega_j)_\rho|^2 =: m_j |D\omega_j - (D\omega_j)_\rho|^2$$

Thus under the integral over $\mathscr{B}_\rho$

$$\begin{aligned}
\int_{\mathscr{B}_\rho} \sqrt{1+|D\omega_j|^2} - \sqrt{1+|(D\omega_j)_\rho|^2} &> \int_{\mathscr{B}_\rho} \frac{1}{2\sqrt{1+|(D\omega_j)_\rho|^2}}\left(|D\omega_j|^2 - |(D\omega_j)_\rho|^2 - \left(|D\omega_j|^2 - |(D\omega_j)_\rho|^2\right)^2\right) \\
&\ge \int_{\mathscr{B}_\rho} \frac{1}{2\sqrt{1+|(D\omega_j)_\rho|^2}}\left(|D\omega_j|^2 - |(D\omega_j)_\rho|^2 - m_j |D\omega_j - (D\omega_j)_\rho|^2\right) \\
&= \int_{\mathscr{B}_\rho} \frac{1}{2\sqrt{1+|(D\omega_j)_\rho|^2}}\left(|D\omega_j|^2 - |(D\omega_j)_\rho|^2 - m_j \left(|D\omega_j|^2 - |(D\omega_j)_\rho|^2\right)\right) \\
&= \frac{1 - m_j}{2\sqrt{1+|(D\omega_j)_\rho|^2}} \int_{\mathscr{B}_\rho} \left(|D\omega_j|^2 - |(D\omega_j)_\rho|^2\right)
\end{aligned}$$

Reversing the inequality and using (3.9)

$$\begin{aligned}
\mathbf{III}_\rho = \int_{\mathscr{B}_\rho} \left(|D\omega_j|^2 - |(D\omega_j)_\rho|^2\right) &\le \frac{2\sqrt{1+|(D\omega_j)_\rho|^2}}{1 - m_j} \int_{\mathscr{B}_\rho} \sqrt{1+|D\omega_j|^2} - \sqrt{1+|(D\omega_j)_\rho|^2} \\
&\le \frac{2\sqrt{1+|(D\omega_j)_\rho|^2}}{1 - m_j} \beta_j
\end{aligned} \tag{3.17}$$

Thus taking $\limsup$ on both sides gives

$$\limsup_{j\to\infty} \frac{1}{\beta_j} \mathbf{III}_\rho = \limsup_{j\to\infty} \frac{1}{\beta_j} \int_{\mathscr{B}_\rho} \left(|D\omega_j|^2 - |(D\omega_j)_\rho|^2\right) \le 2 \tag{3.18}$$

(iv) In the fourth step we bound the second term on RHS of (3.16) upon dividing by β_j. Again, as in Step 1

$$\begin{aligned}
|D\omega_j - Du_j|^2 &= \sum_{i=1}^m \left(D_i\omega_j - D_i u_j\right)^2 \\
&= \sum_{i=1}^m \left(D_i\omega_j^2 - 2D_i\omega_j D_i u_j + D_i u_j^2\right)
\end{aligned}$$

Notice that we may apply Green's first identity. Using that u_j is harmonic in $\mathscr{B}_\rho$ hence $\Delta u_j = 0$ in $\mathscr{B}_\rho$, and that $u_j = \omega_j$ on $\partial\mathscr{B}_\rho$

$$\begin{aligned}
\int_{\mathscr{B}_\rho} \sum_{i=1}^{m} D_i\omega_j D_i u_j &= \int_{\partial\mathscr{B}_\rho} \omega_j \frac{\partial u_j}{\partial \nu}\, dH_{n-1} - \int_{\mathscr{B}_\rho} \Delta u_j\, \omega_j\, dx \\
&= \int_{\partial\mathscr{B}_\rho} \omega_j \frac{\partial u_j}{\partial \nu}\, dH_{n-1} = \int_{\partial\mathscr{B}_\rho} u_j \frac{\partial u_j}{\partial \nu}\, dH_{n-1} \\
\int_{\mathscr{B}_\rho} \sum_{i=1}^{m} D_i u_j^2\, dx &= \int_{\partial\mathscr{B}_\rho} u_j \frac{\partial u_j}{\partial \nu}\, dH_{n-1} - \int_{\mathscr{B}_\rho} \Delta u_j\, u_j\, dx \\
&= \int_{\partial\mathscr{B}_\rho} u_j \frac{\partial u_j}{\partial \nu}\, dH_{n-1} \\
\implies \mathbf{I}_\rho = \int_{\mathscr{B}_\rho} |D\omega_j - Du_j|^2 &= \int_{\mathscr{B}_\rho} |D\omega_j|^2 - 2\int_{\partial\mathscr{B}_\rho} u_j \frac{\partial u_j}{\partial \nu}\, dH_{n-1} + \int_{\partial\mathscr{B}_\rho} u_j \frac{\partial u_j}{\partial \nu}\, dH_{n-1} \\
&= \int_{\mathscr{B}_\rho} |D\omega_j|^2 - \int_{\partial\mathscr{B}_\rho} u_j \frac{\partial u_j}{\partial \nu}\, dH_{n-1} \\
&= \int_{\mathscr{B}_\rho} |D\omega_j|^2 - \int_{\mathscr{B}_\rho} |Du_j|^2
\end{aligned}$$

It suffices to bound $\int_{\mathscr{B}_\rho} |D\omega_j|^2 - \int_{\mathscr{B}_\rho} |Du_j|^2$. To do so, write

$$\mathbf{I}_\rho = \int_{\mathscr{B}_\rho} |D\omega_j|^2 - \int_{\mathscr{B}_\rho} |Du_j|^2 = \int_{\mathscr{B}_\rho} \left(|D\omega_j|^2 - |(D\omega_j)_\rho|^2\right) + \int_{\mathscr{B}_\rho} \left(|(D\omega_j)_\rho|^2 - |Du_j|^2\right) =: \mathbf{III}_\rho - \mathbf{II}_\rho$$

For $\mathbf{III}_\rho$, apply (3.17) from Step 3 so

$$\mathbf{III}_\rho \le \frac{2\sqrt{1+|(D\omega_j)_\rho|^2}}{1-m_j} \int_{\mathscr{B}_\rho} \left(\sqrt{1+|D\omega_j|^2} - \sqrt{1+|(D\omega_j)_\rho|^2}\right) dx$$

For $\mathbf{II}_\rho$, we apply (3.12) with $A = |Du_j|$ and $B = |(D\omega_j)_\rho|$ to obtain

$$\begin{aligned}
\sqrt{1+|Du_j|^2} - \sqrt{1+|(D\omega_j)_\rho|^2} &\le \frac{|Du_j|^2 - |(D\omega_j)_\rho|^2}{2\sqrt{1+|(D\omega_j)_\rho|^2}} \\
|(D\omega_j)_\rho|^2 - |Du_j|^2 &\le 2\sqrt{1+|(D\omega_j)_\rho|^2}\left(\sqrt{1+|(D\omega_j)_\rho|^2} - \sqrt{1+|Du_j|^2}\right) \\
\implies -\mathbf{II}_\rho = \int_{\mathscr{B}_\rho} |(D\omega_j)_\rho|^2 - |Du_j|^2 &\le 2\sqrt{1+|(D\omega_j)_\rho|^2}\int_{\mathscr{B}_\rho}\left(\sqrt{1+|(D\omega_j)_\rho|^2} - \sqrt{1+|Du_j|^2}\right) dx
\end{aligned}$$

Thus combining above gives

$$\begin{aligned}
&\mathbf{I}_\rho = \int_{\mathscr{B}_\rho} |D\omega_j|^2 - \int_{\mathscr{B}_\rho} |Du_j|^2 = \int_{\mathscr{B}_\rho} \left(|D\omega_j|^2 - |(D\omega_j)_\rho|^2\right) + \int_{\mathscr{B}_\rho} \left(|(D\omega_j)_\rho|^2 - |Du_j|^2\right) = \mathbf{III}_\rho - \mathbf{II}_\rho \\
&\le \frac{2\sqrt{1+|(D\omega_j)_\rho|^2}}{1-m_j} \int_{\mathscr{B}_\rho} \left(\sqrt{1+|D\omega_j|^2} - \sqrt{1+|(D\omega_j)_\rho|^2}\right) \\
&+ 2\sqrt{1+|(D\omega_j)_\rho|^2}\int_{\mathscr{B}_\rho}\left(\sqrt{1+|(D\omega_j)_\rho|^2} - \sqrt{1+|Du_j|^2}\right) \\
&= 2\sqrt{1+|(D\omega_j)_\rho|^2}\left(\frac{1}{1-m_j}\int_{\mathscr{B}_\rho}\left(\sqrt{1+|D\omega_j|^2} - \sqrt{1+|(D\omega_j)_\rho|^2}\right) + \int_{\mathscr{B}_\rho}\left(\sqrt{1+|(D\omega_j)_\rho|^2} - \sqrt{1+|Du_j|^2}\right)\right)
\end{aligned}$$

For the last term, write

$$\begin{aligned}
&\int_{\mathscr{B}_\rho}\left(\sqrt{1+|(D\omega_j)_\rho|^2} - \sqrt{1+|Du_j|^2}\right) \\
&= \int_{\mathscr{B}_\rho}\left(\sqrt{1+|(D\omega_j)_\rho|^2} - \sqrt{1+|Dw_j|^2}\right) + \int_{\mathscr{B}_\rho}\left(\sqrt{1+|D\omega_j|^2} - \sqrt{1+|Du_j|^2}\right) \\
&= -\int_{\mathscr{B}_\rho}\left(\sqrt{1+|Dw_j|^2} - \sqrt{1+|(D\omega_j)_\rho|^2}\right) + \int_{\mathscr{B}_\rho}\left(\sqrt{1+|D\omega_j|^2} - \sqrt{1+|Du_j|^2}\right)
\end{aligned}$$

observe

$$\frac{1}{1-m_j} - 1 = \frac{1-(1-m_j)}{1-m_j} = \frac{m_j}{1-m_j}$$

so we add back

$$\mathbf{I}_\rho = \int_{\mathscr{B}_\rho} |D\omega_j|^2 - \int_{\mathscr{B}_\rho} |Du_j|^2$$
$$= 2\sqrt{1+|(D\omega_j)_\rho|^2}\left(\frac{m_j}{1-m_j}\int_{\mathscr{B}_\rho}\left(\sqrt{1+|D\omega_j|^2}-\sqrt{1+|(D\omega_j)_\rho|^2}\right)+\int_{\mathscr{B}_\rho}\left(\sqrt{1+|D\omega_j|^2}-\sqrt{1+|Du_j|^2}\right)\right)$$
$$\leq 2\sqrt{1+|(D\omega_j)_\rho|^2}\left(\frac{m_j}{1-m_j}\beta_j+\int_{\mathscr{B}_\rho}\left(\sqrt{1+|D\omega_j|^2}-\sqrt{1+|Du_j|^2}\right)\right)$$

Now divide by β_j

$$\frac{1}{\beta_j}\left(\int_{\mathscr{B}_\rho} |D\omega_j|^2 - \int_{\mathscr{B}_\rho} |Du_j|^2\right) \leq 2\sqrt{1+|(D\omega_j)_\rho|^2}\left(\frac{m_j}{1-m_j}+\frac{1}{\beta_j}\int_{\mathscr{B}_\rho}\left(\sqrt{1+|D\omega_j|^2}-\sqrt{1+|Du_j|^2}\right)\right)$$

Notice due to $\sup_{\mathscr{B}_\rho}|D\omega_j| + |(D\omega_j)_\rho| = m_j \to 0$ from (3.8)

$$\limsup_{j\to\infty}\frac{m_j}{1-m_j} = 0$$

So taking lim sup on both sides and using (3.10) gives

$$\limsup_{j\to\infty}\frac{1}{\beta_j}\mathbf{I}_\rho = \limsup_{j\to\infty}\frac{1}{\beta_j}\left(\int_{\mathscr{B}_\rho}|D\omega_j|^2 - \int_{\mathscr{B}_\rho}|Du_j|^2\right) \leq 0 \tag{3.19}$$

(v) In our final step, we put things together. In particular, we plug (3.18) and (3.19) into (3.16)

$$\int_{\mathscr{B}_{\alpha\rho}} |D\omega_j - (D\omega_j)_\rho|^2 \leq (1+\varepsilon)^2\alpha^{m+2}\int_{\mathscr{B}_\rho}|D\omega_j-(D\omega_j)_\rho|^2 + Q(\varepsilon,\alpha,m)\int_{\mathscr{B}_\rho}|D\omega_j - Du_j|^2$$
$$\frac{1}{\beta_j}\int_{\mathscr{B}_{\alpha\rho}} |D\omega_j - (D\omega_j)_\rho|^2 \leq \frac{1}{\beta_j}(1+\varepsilon)^2\alpha^{m+2}\int_{\mathscr{B}_\rho}|D\omega_j-(D\omega_j)_\rho|^2 + \frac{1}{\beta_j}Q(\varepsilon,\alpha,m)\int_{\mathscr{B}_\rho}|D\omega_j - Du_j|^2$$

Taking lim sup on both sides yields

$$\limsup_{j\to\infty}\frac{1}{\beta_j}\int_{\mathscr{B}_{\alpha\rho}}|D\omega_j-(D\omega_j)_\rho|^2 \leq 2(1+\varepsilon)^2\alpha^{m+2}$$

Going back to Step 1 yields

$$\limsup_{j\to\infty}\frac{1}{\beta_j}\int_{\mathscr{B}_{\alpha\rho}}\left(\sqrt{1+|D\omega_j|^2}-\sqrt{1+|(D\omega_j)_{\alpha\rho}|^2}\right)dx$$
$$\leq \frac{1}{2}\limsup_{j\to\infty}\frac{1}{\beta_j}\int_{\mathscr{B}_{\alpha\rho}}|D\omega_j-(D\omega_j)_\rho|^2\,dx$$
$$\leq (1+\varepsilon)^2\alpha^{m+2}$$

For any $\varepsilon > 0$. Taking $\varepsilon \to 0$ on RHS and conclude (3.11).

□

Now instead of looking at C^1 functions, we look at the sets determined by the functions via (3.4). In particular, we replace the condition (3.10) with a condition saying that sets tend to a minimum.

- Let $\mathscr{B}_\rho \subset \mathbb{R}^m$ and $\omega_j \in C^1(\mathscr{B}_\rho)$
- $W_j := \{(x,t) \in \mathscr{B}_\rho \times \mathbb{R} \mid t < \omega_j(x)\}$
- $Q_j := \{(x,t) \in \mathscr{B}_\rho \times \mathbb{R} \mid \min_{x\in\mathscr{B}_\rho}\omega_j(x) - 1 < t < \max_{x\in\mathscr{B}_\rho}\omega_j(x)+1\}$

Lemma 3.1.3 (De Giorgi's Lemma for C^1 functions representing sets approximating flat boundary)**.** *Suppose $\mathscr{B}_\rho \subset \mathbb{R}^m$. Let $\omega_j \in C^1(\overline{\mathscr{B}}_\rho)$ be sequence of C^1 functions. Let $\{\beta_j\} \subset \mathbb{R}_+$ be sequence of positive numbers s.t.*

$$\lim_{j\to\infty}\sup_{x\in\mathscr{B}_\rho}|D\omega_j(x)| = 0 \tag{3.20}$$

$$\int_{\mathscr{B}_\rho}\left(\sqrt{1+|D\omega_j|^2}-\sqrt{1+|(D\omega_j)_\rho|^2}\right)dx \leq \beta_j \tag{3.21}$$

$$\lim_{j\to\infty}\frac{1}{\beta_j}\psi(W_j,Q_j) = 0 \tag{3.22}$$

Then for any $\alpha \in (0,1)$

$$\limsup_{j\to\infty} \frac{1}{\beta_j} \int_{\mathscr{B}_{\alpha\rho}} \left(\sqrt{1+|D\omega_j|^2} - \sqrt{1+|(D\omega)_{\alpha\rho}|^2}\right) dx \le \alpha^{m+2} \tag{3.23}$$

Proof. By Perron's Method, one may construct $u_j \in C^1(\overline{\mathscr{B}}_\rho)$ harmonic in $\mathscr{B}_\rho$ s.t. $u_j = \omega_j$ on $\partial\mathscr{B}_\rho$. Recall from (3.4)

$$P(W_j, Q_j) = \int_{Q_j} |D\varphi_{W_j}| = \int_{\mathscr{B}_\rho} \sqrt{1+|D\omega_j|^2}$$

Then

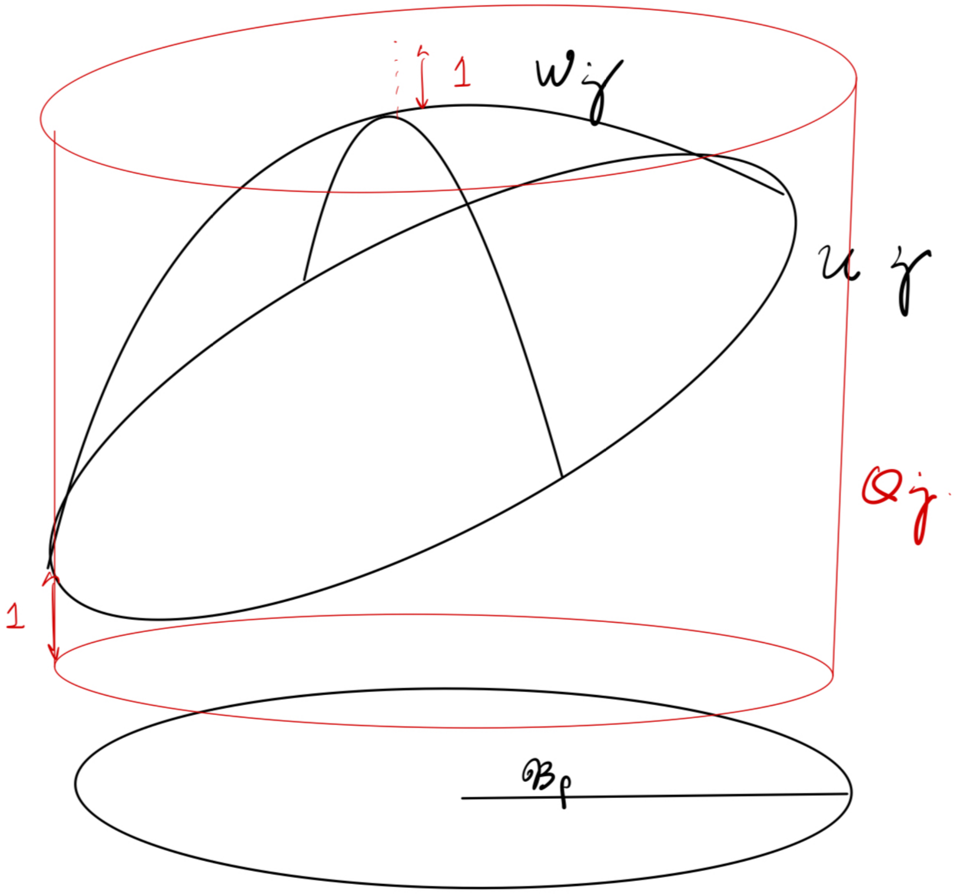

Figure 3.1: ω_j, u_j and Q_j

$$\int_{\mathscr{B}_\rho} \sqrt{1+|D\omega_j|^2} - \int_{\mathscr{B}_\rho} \sqrt{1+|Du_j|^2} \le \int_{Q_j} |D\varphi_{W_j}| - \inf\{\int_{Q_j} |Dg| \mid g \in BV(Q_j), \operatorname{supp}(g-\varphi_{W_j}) \subset Q_j\}$$
$$= \psi(W_j, Q_j)$$

Since u_j itself defines $g := \varphi_{\{(x,t)\in\mathscr{B}_\rho\times\mathbb{R} | t<u_j(x)\}}$, and indeed $\operatorname{supp}(g-\varphi_{W_j}) \subset Q_j$ due to construction of u_j. Now one may apply Lemma 3.1.2. □

Lemma 3.1.4 (De Giorgi's Lemma for C^1 Caccioppoli Sets)**.** *Let* $\{L_j\}$ *be sequence of Caccioppoli sets in* $\mathbb{R}^n$. *Let* $\{\beta_j\} \subset \mathbb{R}_+$ *be sequence of positive numbers, and* $\rho > 0$ *s.t.* $\partial L_j \cap B_\rho$ *is* C^1 *hypersurface, and*

$$\lim_{j\to\infty} \inf_{\partial L_j \cap B_\rho} \nu_n^j(x) = 1 \qquad \textit{for } \nu^j(x) \textit{ the normal to } L_j \textit{ at the point } x \tag{3.24}$$

$$\int_{B_\rho} |D\varphi_{L_j}| - |\int_{B_\rho} D\varphi_{L_j}| \le \beta_j \tag{3.25}$$

$$\lim_{j\to\infty} \frac{1}{\beta_j} \psi(L_j, \rho) = 0 \tag{3.26}$$

Then for any $\alpha \in (0,1)$

$$\limsup_{j\to\infty} \frac{1}{\beta_j} \left(\int_{B_{\alpha\rho}} |D\varphi_{L_j}| - |\int_{B_{\alpha\rho}} D\varphi_{L_j}|\right) \le \alpha^{n+1} \tag{3.27}$$

Proof. Assume for contradiction, i.e., there exists a sequence of Caccioppoli Sets $\{L_j\}$, a sequence $\{\beta_j\} \subset \mathbb{R}_+$ and $\rho > 0$ s.t. $\partial L_j \cap B_\rho$ is C^1 hypersurface and (3.24) to (3.26) holds, yet for some $\alpha \in (0,1)$, we obtain

$$\lim_{j\to\infty} \frac{1}{\beta_j}\left(\int_{B_{\alpha\rho}} |D\varphi_{L_j}| - |\int_{B_{\alpha\rho}} D\varphi_{L_j}|\right) > \alpha^{n+1} \tag{3.28}$$

(i) Since (3.24) ensures $\nu_n^j(x)$ converges to 1 for any $x \in \partial L_j \cap B_\rho$, we may assume $\nu_n^j(x) \geq q > \frac{\sqrt{2}}{2}$ for every $x \in \partial L_j \cap B_\rho$ for j sufficiently large. From Theorem 2.2.2 and Theorem 2.2.3, since ν_n^j exists everywhere and $\partial L_j \cap B_\rho$ are C^1 boundaries, we know there exists open set $A_j \subset \mathbb{R}^{n-1}$ and C^1 functions $\omega_j : A_j \subset \mathbb{R}^{n-1} \to \mathbb{R}$ s.t.

$$\partial L_j \cap B_\rho = \{(y,t) \in \mathbb{R}^{n-1}\times\mathbb{R} \mid y \in A_j,\, t = \omega_j(y)\}$$

Moreover, since $\nu_n^j > \frac{\sqrt{2}}{2}$ satisfies (2.52), we have bound on

$$|\omega_j(y) - \omega_j(y')| \leq \frac{\sqrt{1-q^2}}{q}|y - y'|$$

for any $y,\, y' \in A_j$. Thus as $j \to \infty$, (3.24) ensures $q \to 1$, and the bound yields

$$\sup_{y,\, y'\in A_j} |\frac{\omega_j(y) - \omega_j(y')}{y - y'}| \leq \frac{\sqrt{1-q^2}}{q} \implies \lim_{j\to\infty} \sup_{A_j}|D\omega_j| = 0 \tag{3.29}$$

On the other hand, since we're considering $\partial L_j \cap B_\rho$, it is guaranteed that $\sup_{A_j}|\omega_j| \leq \rho$ for any j large. Hence $\{\inf_{A_j}\omega_j\}_j \subset [-\rho, \rho]$ is bounded sequence in $\mathbb{R}$, by Bolzano Weierstrass, there exists a convergent subsequence and constant $c \in [-\rho, \rho]$ s.t.

$$\lim_{j\to\infty} \inf_{A_j}\omega_j = c \tag{3.30}$$

We claim that $c^2 < \rho^2$. If not, i.e., $c^2 = \rho^2$, then since $\lim_{j\to\infty} \sup_{A_j}|D\omega_j| = 0$, for j sufficiently large, we eventually reach $\lim_{j\to\infty} |\omega| = \rho$, i.e., $\partial L_j \cap B_{\alpha\rho} = \varnothing$ for $\alpha \in (0,1)$. But this contradicts assumption (3.28).

(ii) Due to (3.29) and (3.30), for any $\varepsilon > 0$, there exists $j_{\varepsilon,1} > 0$ s.t. for any $j \geq j_{\varepsilon,1}$, one has

$$|\omega_j(x) - c| < \varepsilon \qquad \forall\, x \in A_j \tag{3.31}$$

Since c is constant, let $\sigma^2 = \rho^2 - c^2 > 0$, there exists $j_\varepsilon \geq j_{\varepsilon,1}$ s.t. for balls $\mathscr{B}_{\sigma+\varepsilon}, \mathscr{B}_{\sigma-\varepsilon} \subset \mathbb{R}^{n-1}$

$$\mathscr{B}_{\sigma-\varepsilon} \subseteq A_j \subseteq \mathscr{B}_{\sigma+\varepsilon} \qquad \forall\, j \geq j_\varepsilon \geq j_{\varepsilon,1}$$

And on the set $(\mathscr{B}_{\sigma-\varepsilon}\times\mathbb{R}) \cap B_\rho$ with piecewise smooth boundary, we have from (1.4) that

$$\int_{(\mathscr{B}_{\sigma-\varepsilon}\times\mathbb{R})\cap B_\rho} D\varphi_{L_j} = \int_{\partial L_j\cap(\mathscr{B}_{\sigma-\varepsilon}\times\mathbb{R})\cap B_\rho} \nu^j(x) dH_{n-1} \qquad \forall\, j \geq j_\varepsilon$$

In particular, via change of variables and computations (3.4) and (3.5), for any $j \geq j_\varepsilon$

$$\int_{(\mathscr{B}_{\sigma-\varepsilon}\times\mathbb{R})\cap B_\rho} |D\varphi_{L_j}| = \int_{\mathscr{B}_{\sigma-\varepsilon}} \sqrt{1 + |D\omega_j(y)|^2}\, dy$$

$$\int_{(\mathscr{B}_{\sigma-\varepsilon}\times\mathbb{R})\cap B_\rho} D_i\varphi_{L_j} = \int_{\partial L_j\cap(\mathscr{B}_{\sigma-\varepsilon}\times\mathbb{R})\cap B_\rho} \nu_i^j(x) dH_{n-1} = \int_{\mathscr{B}_{\sigma-\varepsilon}} D_i\omega_j\, dy \qquad i = 1, \cdots, n-1$$

$$\int_{(\mathscr{B}_{\sigma-\varepsilon}\times\mathbb{R})\cap B_\rho} D_n\varphi_{L_j} = \int_{\mathscr{B}_{\sigma-\varepsilon}} 1\, dy = |\mathscr{B}_{\sigma-\varepsilon}|$$

$$\begin{aligned}
|\int_{(\mathscr{B}_{\sigma-\varepsilon}\times\mathbb{R})\cap B_\rho} D\varphi_{L_j}| &= \left(\sum_{i=1}^{n-1}\left(\int_{\mathscr{B}_{\sigma-\varepsilon}} D_i\omega_j\, dy\right)^2 + |\mathscr{B}_{\sigma-\varepsilon}|^2\right)^{\frac{1}{2}} \\
&= |\mathscr{B}_{\sigma-\varepsilon}|\left(\sum_{i=1}^{n-1}\left(\frac{1}{|\mathscr{B}_{\sigma-\varepsilon}|}\int_{\mathscr{B}_{\sigma-\varepsilon}} D_i\omega_j\, dy\right)^2 + 1\right)^{\frac{1}{2}} \\
&= |\mathscr{B}_{\sigma-\varepsilon}|\left(\sum_{i=1}^{n-1}((D_i\omega_j)_{\sigma-\varepsilon})^2 + 1\right)^{\frac{1}{2}} = |\mathscr{B}_{\sigma-\varepsilon}|\sqrt{1 + |(D\omega_j)_{\sigma-\varepsilon}|^2} \\
&= \int_{\mathscr{B}_{\sigma-\varepsilon}} \sqrt{1 + |(D\omega_j)_{\sigma-\varepsilon}|^2}\, dy
\end{aligned}$$

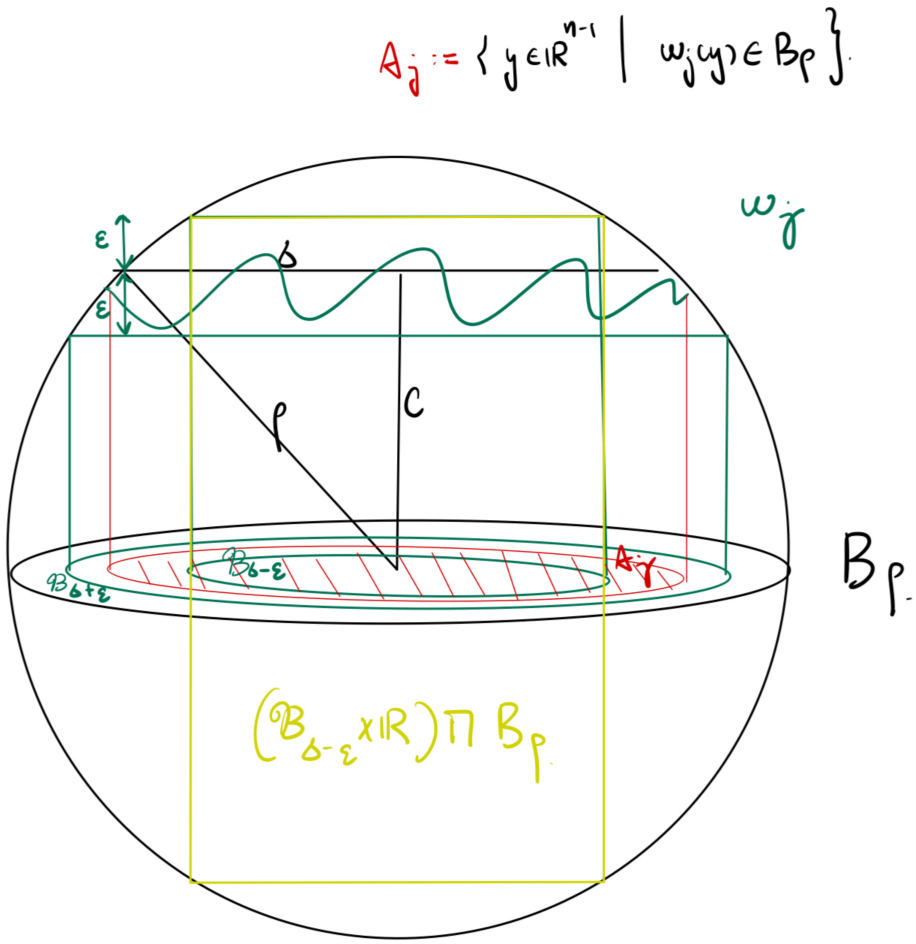

Figure 3.2: $\mathscr{B}_{\sigma-\varepsilon} \subseteq A_j \subseteq \mathscr{B}_{\sigma+\varepsilon}$

as in (3.6), we obtain

$$\int_{(\mathscr{B}_{\sigma-\varepsilon}\times\mathbb{R})\cap B_\rho} |D\varphi_{L_j}| - \left|\int_{(\mathscr{B}_{\sigma-\varepsilon}\times\mathbb{R})\cap B_\rho} D\varphi_{L_j}\right| = \int_{\mathscr{B}_{\sigma-\varepsilon}} \sqrt{1+|D\omega_j(y)|^2}\,dy - \int_{\mathscr{B}_{\sigma-\varepsilon}} \sqrt{1+|(D\omega_j)_{\sigma-\varepsilon}|^2}\,dy$$

Notice the above defines locally a finite measure (in particular, non-negative and monotonic) on $\mathbb{R}^n$. Hence

$$\begin{aligned}\int_{\mathscr{B}_{\sigma-\varepsilon}} \sqrt{1+|D\omega_j(y)|^2}\,dy - \int_{\mathscr{B}_{\sigma-\varepsilon}} \sqrt{1+|(D\omega_j)_{\sigma-\varepsilon}|^2}\,dy &= \int_{(\mathscr{B}_{\sigma-\varepsilon}\times\mathbb{R})\cap B_\rho} |D\varphi_{L_j}| - \left|\int_{(\mathscr{B}_{\sigma-\varepsilon}\times\mathbb{R})\cap B_\rho} D\varphi_{L_j}\right| \\ &\le \int_{B_\rho} |D\varphi_{L_j}| - \left|\int_{B_\rho} D\varphi_{L_j}\right| \le \beta_j\end{aligned}$$

Thus we're ready to apply Lemma 3.1.3 and obtain from (3.23) that for any $0<\gamma<1$

$$\limsup_{j\to\infty} \frac{1}{\beta_j}\int_{\mathscr{B}_{\gamma(\sigma-\varepsilon)}} \left(\sqrt{1+|D\omega_j(y)|^2} - \sqrt{1+|(D\omega)_{\gamma(\sigma-\varepsilon)}|^2}\right) dy \le \gamma^{n+1} \tag{3.32}$$

(iii) Recall definition for A_j. We may rewrite

$$A_j = \{y\in\mathbb{R}^{n-1} \mid (y,\omega_j(y))\in B_\rho\}$$

It's reasonable to consider its subset

$$C_j := \{y\in A_j \mid (y,\omega_j(y))\in B_{\alpha\rho}\}$$

The same argument from (3.31) indicates there exists $j_{\varepsilon,2}$ for any $j\ge j_{\varepsilon,2}$

$$|\omega_j(x)-c| < \alpha\varepsilon \qquad \forall\, x\in C_j\subset A_j$$

Also notice $c^2<\alpha^2\rho^2$ for the same reason as $c^2<\rho^2$. Hence

$$\alpha^2\sigma^2 = \alpha^2(\rho^2-c^2) > \alpha^2\rho^2 - c^2 > 0$$

In particular

$$\alpha\sigma > \sqrt{\alpha^2\rho^2 - c^2} \implies \alpha(\sigma+\varepsilon) > \sqrt{\alpha^2\rho^2 - c^2} + \alpha\varepsilon$$

so so there exists $j'_\varepsilon \geq j_{\varepsilon,2}$ s.t. for any $j \geq j'_\varepsilon$

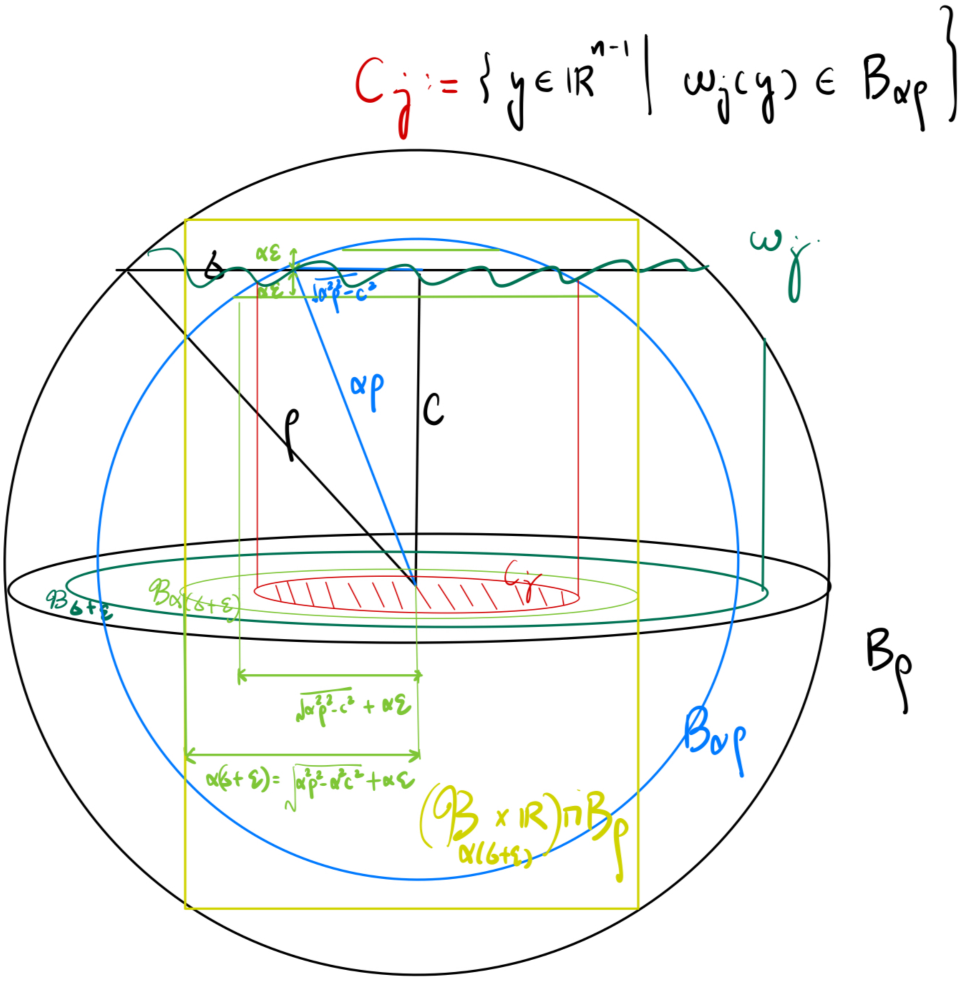

Figure 3.3: $C_j \subseteq \mathscr{B}_{\sqrt{\alpha^2\rho^2-c^2}+\alpha\varepsilon} \subseteq \mathscr{B}_{\alpha(\sigma+\varepsilon)}$

$$C_j \subseteq \mathscr{B}_{\sqrt{\alpha^2\rho^2-c^2}+\alpha\varepsilon} \subseteq \mathscr{B}_{\alpha(\sigma+\varepsilon)}$$

Now on the other hand

$$\begin{aligned}\int_{B_{\alpha\rho}} |D\varphi_{L_j}| - |\int_{B_{\alpha\rho}} D\varphi_{L_j}| &\leq \int_{(\mathscr{B}_{\alpha(\sigma+\varepsilon)}\times\mathbb{R})\cap B_\rho} |D\varphi_{L_j}| - |\int_{(\mathscr{B}_{\alpha(\sigma+\varepsilon)}\times\mathbb{R})\cap B_\rho} D\varphi_{L_j}| \\ &= \int_{\mathscr{B}_{\alpha(\sigma+\varepsilon)}} \sqrt{1+|D\omega_j(y)|^2}\,dy - \int_{\mathscr{B}_{\alpha(\sigma+\varepsilon)}} \sqrt{1+|(D\omega_j)_{\alpha(\sigma+\varepsilon)}|^2}\,dy\end{aligned}$$

Take $\gamma = \alpha\frac{\sigma+\varepsilon}{\sigma-\varepsilon}$ in (3.32) and combining with above, we obtain

$$\limsup_{j\to\infty} \frac{1}{\beta_j} \int_{B_{\alpha\rho}} |D\varphi_{L_j}| - |\int_{B_{\alpha\rho}} D\varphi_{L_j}| \leq \left(\alpha\frac{\sigma+\varepsilon}{\sigma-\varepsilon}\right)^{n+1} = \alpha^{n+1}\left(\frac{\sigma+\varepsilon}{\sigma-\varepsilon}\right)^{n+1}$$

Let $\varepsilon \to 0$ to reach a contradiction against (3.28).

□