[Luk] Introduction to Nonlinear Wave Equations

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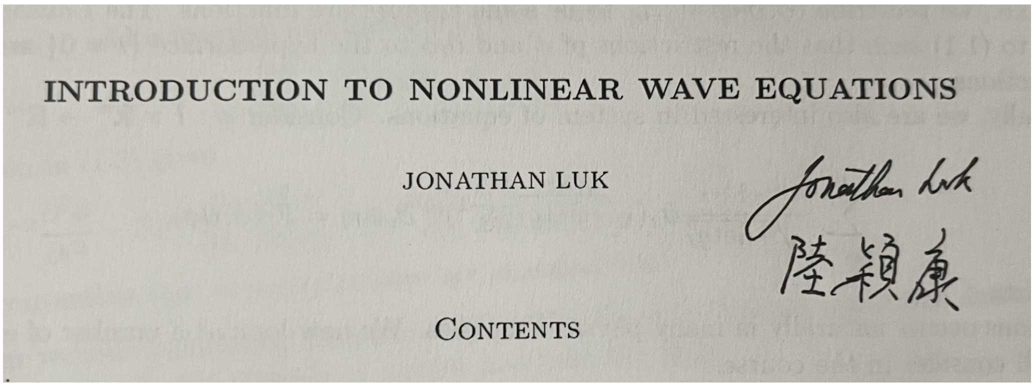
Contents

We earnestly follow the set of notes 'Introduction to Nonlinear Wave Equations' by Jonathan Luk. In fact, the author has his signature from Analysis Seminar at Courant.



Chapter 1

Constant Coefficient Linear Wave Equation

Consider Constant Coefficient Linear Wave Equation with solution $\phi : I \times \mathbb{R}^n \subset \mathbb{R}^{n+1} \to \mathbb{R}$ where $0 \in I$

$$\Box\,\phi = -\partial_t^2\phi + \sum_{i=1}^{n} \partial_{x_i}^2\phi = 0 \tag{1.1}$$

This is model by D' Alembert in 1749. We wish to study the initial value problem with given initial data $(\phi_0,\, \phi_1)$ on $I \times \mathbb{R}^n$

$$\begin{cases} \Box\,\phi = 0 \\ (\phi,\, \partial_t\phi)|_{t=0} = (\phi_0,\, \phi_1) \end{cases} \tag{1.2}$$

1.1 Fundamental Solution and Representation Formula

1.1.1 Distribution Theory

One start with distribution theory.

Definition 1.1.1 (Distribution, Restriction, Support, Distributional Derivative)**.** *Given open set $U \subset \mathbb{R}^n$.*

- *$u \in \mathcal{D}'(U)$ is a distribution in U if $u : C_c^\infty(U) \to \mathbb{R}$ is a linear map on the space of smooth and compactly supported functions, and for every $K \subset U$ compact, there exits $C = C(K) > 0$ and $k = k(K) \in \mathbb{N}$ s.t.*

$$|u(\varphi)| \equiv |\langle u, \varphi\rangle| \leq C \sum_{|\alpha|\leq k} \sup_{x\in K} |\partial^\alpha \varphi(x)| \qquad \forall\ \varphi \in C_c^\infty(K)$$

- *Let $V \subset U \subset \mathbb{R}^n$, and $u \in \mathcal{D}'(U)$. Define the restriction of u to V as u_V*

$$u_V(\varphi) := u(\varphi) \qquad \forall\ \varphi \in C_c^\infty(V)$$

- *$u \in \mathcal{D}'(U)$. Define the support of u as* $\mathrm{supp}(u)$

$$\mathrm{supp}(u) := U \setminus \bigcup \{A \subset U \text{ open} \mid u_A = 0\}$$

- *$u \in \mathcal{D}'(U)$. Define the distributional derivative for any $\alpha \in \mathbb{N}^n$ as $\partial^\alpha u$*

$$\langle \partial^\alpha u, \varphi\rangle := (-1)^{|\alpha|}\langle u, \partial^\alpha \varphi\rangle$$

Lemma 1.1.1 (Approximation with C_c^∞, Composition with C^∞, Chain Rule)**.** *Given open set $U \subset \mathbb{R}^n$*

- *$u \in \mathcal{D}'(U)$. There exists a sequence $\{u_j\} \subset C_c^\infty(U)$ s.t. $u_j \to u$ in $\mathcal{D}'(U)$, i.e.*

$$\int_U u_j \varphi\, dx = \langle u_j,\, \varphi\rangle \to \langle u,\, \varphi\rangle \qquad \forall\ \varphi \in C_c^\infty(U) \tag{1.3}$$

- *$f \in C^\infty(U)$ that is a submersion, i.e. $df \neq 0$. Define composition of $u \in \mathcal{D}'(\mathbb{R})$ with f as $u \circ f \in \mathcal{D}'(U)$*

(a) *There exists unique pushforward* $f^*: \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(U)$ *s.t.*

$$f^* u = u \circ f$$

for any $u \in C(\mathbb{R})$.

(b) *If* $u \in \mathcal{D}'(\mathbb{R})$, *there exists* $\{u_j\} \subset C_c^\infty(\mathbb{R})$ *s.t.* $\langle u_j, \phi\rangle \to \langle u, \phi\rangle$ *for any* $\phi \in C_c^\infty(\mathbb{R})$ *as in* (1.3). *Define*

$$f^* u = u \circ f := \lim_{j\to\infty} u_j \circ f \tag{1.4}$$

- $u \in \mathcal{D}'(\mathbb{R})$. $f \in C^\infty(U)$ *submersion. Then distributional derivative satisfies chain rule*

$$\partial(u \circ f) = (\partial f)(u' \circ f) \tag{1.5}$$

Proof. For any $\phi \in C_c^\infty(U)$, fix $\partial = \partial_i$ w.r.t. $i \in \{1, \cdots, n\}$

$$\begin{aligned}\langle \partial_i(u \circ f), \phi\rangle &= -\langle u \circ f, \partial_i \phi\rangle = -\lim_{j\to\infty}\langle u_j \circ f, \partial_i\phi\rangle = \lim_{j\to\infty}\langle \partial_i(u_j \circ f), \phi\rangle \\ &= \lim_{j\to\infty}\langle \partial_i f(u_j' \circ f), \phi\rangle = \langle \partial_i f(u' \circ f), \phi\rangle\end{aligned}$$

induct on i yields (1.5). □

Definition 1.1.2 (Convergence in C_c^∞ and in C^∞)**.** *we first specify convergence in* C_c^∞ *and in* C^∞

- $\varphi_j \subset C_c^\infty(\mathbb{R}^n)$ *converges to* φ *in* C_c^∞ *if there exists* $K \subset \mathbb{R}^n$ *compact s.t. all* $\operatorname{supp}(\varphi_j) \subset K$ *and*

$$\sup_{x\in K} |\partial^\alpha(\varphi(x) - \varphi(x))| \to 0 \qquad \forall\ \alpha \in \mathbb{N}^n$$

- $\varphi_j \subset C^\infty(\mathbb{R}^n)$ *converges to* φ *in* C^∞ *if for any* $K \subset \mathbb{R}^n$ *compact*

$$\sup_{x\in K} |\partial^\alpha(\varphi(x) - \varphi(x))| \to 0 \qquad \forall\ \alpha \in \mathbb{N}^n$$

Lemma 1.1.2 (Convolution)**.** $u \in \mathcal{D}'(\mathbb{R}^n)$ *and* $\varphi \in C_c^\infty(\mathbb{R}^n)$. *One define convolution of* u *and* φ *as*

$$(u * \varphi)(x) := \langle u,\ \varphi(x - \cdot)\rangle$$

- $u * \varphi \in C^\infty(\mathbb{R}^n)$ *with*

$$\partial(u * \varphi) = (\partial u) * \varphi = u * (\partial\varphi) \qquad \operatorname{supp}(u * \varphi) \subset \operatorname{supp}(u) + \operatorname{supp}(\varphi)$$

- *Let* $U : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ *be linear map s.t. for any* $\varphi_j \to 0$ *in* C_c^∞, *one has* $U(\varphi_j) \to 0$ *in* C^∞. *If* U *commutes with all translations, i.e. for any* $h \in \mathbb{R}^n$ *and* $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$U(\tau_h(\varphi)) = \tau_h(U(\varphi)) \qquad \textit{where } \tau_h : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n) \textit{ s.t. } \tau_h(\psi)(x) := \psi(x - h)$$

Then there exists unique distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ *s.t.* $u * \varphi = U(\varphi)$ *for all* $\varphi \in C_c^\infty(\mathbb{R}^n)$.

Proof. Define $u(\varphi) := (U(\tilde{\varphi}))(0)\ \ \forall\ \varphi \in C_c^\infty(\mathbb{R}^n)$ where $\tilde{\varphi}(x) := \varphi(-x)$. Then for any $h \in \mathbb{R}^n$

$$\begin{aligned}(U(\varphi))(-h) &= \tau_h(U(\varphi))(0) = U(\tau_h(\varphi))(0) = u(\widetilde{\tau_h(\varphi)}) \\ &= \langle u(y), \varphi\widetilde{(y - h)}\rangle = \langle u(y), \varphi(-h - y)\rangle = (u * \varphi)(-h)\end{aligned}$$

that u is unique is due to its explicit expression. And $u \in \mathcal{D}'(\mathbb{R}^n)$ due to continuity requirement on U. □

- $u_1,\, u_2 \in \mathcal{D}'(\mathbb{R}^n)$. *If* $(-\operatorname{supp}(u_1)) \cap (\operatorname{supp}(u_2) + K)$ *is compact for any compact set* K. *Then there exists unique distribution* $u \in \mathcal{D}'(\mathbb{R}^n)$ *s.t.*

$$u * \varphi = u_1 * (u_2 * \varphi) \qquad \textit{for every } \varphi \in C_c^\infty(\mathbb{R}^n)$$

One define $u_1 * u_2 := u \in \mathcal{D}'(\mathbb{R}^n)$ *as convolution of* $u_1,\, u_2$.

Proof. Define $U(\varphi) := u_1 * (u_2 * \varphi)$ for any $\varphi \in C_c^\infty(\mathbb{R}^n)$. We show this is well-defined. For any $x \in \mathbb{R}^n$

$(x - \text{supp}(u_1)) \cap \text{supp}(u_2 * \varphi) \subset (x - \text{supp}(u_1)) \cap (\text{supp}(u_2) + \text{supp}(\varphi))$ *is compact for any* $\text{supp}(\varphi)$ *compact*

Hence the quantity

$$U(\varphi)(x) = u_1 * (u_2 * \varphi)(x) = \langle u_1(y), u_2 * \varphi(x-y)\rangle < \infty \qquad \forall\ x \in \mathbb{R}^n$$

as convolution, $U(\varphi)$ is smooth. Linearity of U follows from convolution with φ and duality pairing. For any $\varphi_j \to 0$ in C_c^∞, there exists $K \subset \mathbb{R}^n$ compact s.t. $\text{supp}(\varphi_j) \subset K$ for any j and

$$\sup_{x\in K} |\partial^\alpha \varphi_j(x)| \to 0 \qquad \forall\ \alpha \in \mathbb{N}^n$$

Now for any $\tilde{K} \subset \mathbb{R}^n$ compact and for any $\alpha \in \mathbb{N}^n$

$$\sup_{x\in\tilde{K}} |\partial^\alpha(U(\varphi))(x)| = \sup_{x\in\tilde{K}\cap K} |u_1 * (u_2 * (\partial^\alpha \varphi(x)))| \le C \sup_{x\in K} |\partial^\alpha \varphi_j(x)| \to 0$$

Hence $U : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ defines continuous linear map. Moreover,

$$\begin{aligned} U(\tau_h(\varphi))(x) &= u_1 * (u_2 * (\tau_h\varphi))(x) = \langle u_1(y), (u_2 * (\tau_h\varphi))(x-y)\rangle = \langle u_1(y), \langle u_2(z), (\tau_h\varphi)(x-y-z)\rangle\rangle \\ &= \langle u_1(y), \langle u_2(z), \varphi(x-y-z-h)\rangle\rangle = \langle u_1(y), \tau_h(u_2 * \varphi)(x-y)\rangle \\ &= \tau_h(u_1 * (u_2 * \varphi))(x) = \tau_h(U(\varphi))(x) \end{aligned}$$

Hence U defines unique distribution $u_1 * u_2$ s.t. for any $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$(u_1 * u_2) * \varphi = u_1 * (u_2 * \varphi)$$

□

Now we discuss homogeneous distribution of degree a and the particular example χ_+^a.

Definition 1.1.3 (Homogeneous Distribution)**.** *One generalize homogeneous function to distributions.*

- $h : \mathbb{R}^n \to \mathbb{R}$ *function is (positive) homogeneous of degree* a *if*

$$\lambda^{-a} h(\lambda x) = h(x) \qquad \forall\ \lambda > 0 \tag{1.6}$$

- $h \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ *or* $\mathcal{D}'(\mathbb{R}^n)$ *is homogeneous of degree* a *if*

$$\langle h(x), \varphi(x)\rangle = \lambda^a \langle h(x), \lambda^n \varphi(\lambda x)\rangle \qquad \forall\ \varphi \in C_c^\infty(\mathbb{R}^n \setminus \{0\}) \text{ or } C_c^\infty(\mathbb{R}^n) \quad \forall\ \lambda > 0$$

Notice this is indeed, abuse of notation, a generalization of (1.6)

$$\langle h(x), \varphi(x)\rangle = \lambda^a \langle h(x), \lambda^n \varphi(\lambda x)\rangle = \lambda^a \langle h(\frac{y}{\lambda}), \varphi(y)\rangle = \langle t^{-a} h(ty), \varphi(y)\rangle$$

Lemma 1.1.3 (Homogeneous Extension to the origin)**.** $h \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ *is homogeneous of degree* a*, where* $a > -n$*. Then* h *has a unique and continuous extension to a homogeneous distribution* $h \in \mathcal{D}'(\mathbb{R}^n)$ *of degree* a*.*

Example 1.1.1 (Space-time distance in special relativity)**.** *For* $(t, x) \in \mathbb{R}^{1+n}$*, define* $s^2(t, x) := t^2 - |x|^2$*. Then* s^2 *is homogeneous function of degree* 2*.*

Proof. $s^2(\lambda t, \lambda x) = \lambda^2 s^2(t, x)$ for any $\lambda > 0$. □

Example 1.1.2 (□ drops degree by 2)**.** *For* $\phi \in \mathcal{D}'(\mathbb{R}^{1+n})$ *and suppose* ϕ *is homogeneous of degree* a*. Then* $\Box\phi \in \mathcal{D}'(\mathbb{R}^{1+n})$ *is homogeneous of degree* $a - 2$*.*

Proof. For any $\varphi \in C_c^\infty(\mathbb{R}^{1+n})$ and $\lambda > 0$

$$\begin{aligned} \langle \Box\, \phi(x), \lambda^n \varphi(\lambda x)\rangle &= \langle \phi(x), \lambda^n \Box\, (\varphi(\lambda x))\rangle = \langle \phi(x), \lambda^{n+2} (\Box\, \varphi)(\lambda x)\rangle \\ &= \lambda^{-a+2} \langle \phi(x), (\Box\, \varphi)(x)\rangle = \lambda^{-(a-2)} \langle \Box\, \phi(x), \varphi(x)\rangle \end{aligned}$$

Hence that ϕ is degree a implies $\Box\, \phi$ is degree $a - 2$. □

Lemma 1.1.4. *One has immediate homogeneity calculations for derivative and composition*

- *In general, distributional derivatives* $\frac{d^k}{dx^k}$ *of order* k *drop degree of homogeneity by* k *by same argument from Example 1.1.2.*

- *Composition of a distribution* $u \in \mathcal{D}'(\mathbb{R})$ *with* $f \in C^\infty(U)$ *where* $df \neq 0$, $U \subset \mathbb{R}^n$ *multiplies degree of homogeneity. If* $u \in \mathcal{D}'(\mathbb{R})$ *is homogeneous of degree* a_1 *and* $f \in C^\infty(U)$ *homogeneous of degree* a_2*, then* $u \circ f \in \mathcal{D}'(U)$ *is homogeneous of degree* $a_1 \cdot a_2$.

  Proof. For any $\varphi \in C_c^\infty(U)$ and $\lambda > 0$, choose $\{u_j\} \subset C_c^\infty(\mathbb{R})$ s.t. $u_j \to u$ in $\mathcal{D}'(\mathbb{R})$ (note we may choose sequence with same homogeneity as u due to its construction as convolution and multiplication with distribution)

$$\begin{aligned}\langle u \circ f(x), \lambda^n \varphi(\lambda x)\rangle &= \lim_{j\to\infty} \langle u_j \circ f(x), \lambda^n \varphi(\lambda x)\rangle = \lim_{j\to\infty} \langle u_j \circ f(\frac{y}{\lambda}), \varphi(y)\rangle = \lim_{j\to\infty} \langle u_j \circ (\lambda^{a_1} f(y)), \varphi(y)\rangle \\ &= \lambda^{a_1\cdot a_2} \lim_{j\to\infty} \langle u_j \circ f(y), \varphi(y)\rangle = \lambda^{a_1\cdot a_2}\langle u \circ f, \varphi\rangle\end{aligned}$$

  □

Example 1.1.3 ($\delta_0 \in \mathcal{D}'(\mathbb{R}^n)$)**.** *Dirac Delta* $\delta_0 \in \mathcal{D}'(\mathbb{R}^n)$ *is homogeneous of degree* $-n$.

Proof. For any $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$\langle \delta_0(x), \lambda^n \varphi(\lambda x)\rangle = \lambda^n \langle \delta_0(x), \varphi(\lambda x)\rangle = \lambda^n \varphi(0) = \lambda^n \langle \delta_0(x), \varphi(x)\rangle$$

□

Example 1.1.4 (χ_+^a)**.** $n = 1$. *For any* $a \in \mathbb{C}$.

- *If* $Re(a) > -1$, $x_+^a := \mathbb{1}_{\{x\geq 0\}}\, x^a$ *is homogeneous of degree* a.

  Proof. For any $\varphi \in C_c^\infty(\mathbb{R} \setminus \{0\})$

$$\langle x_+^a, \varphi\rangle = \int_0^\infty x^a \varphi(x)\, dx = \int_0^\infty (\lambda x)^a \varphi(\lambda x)\, d\lambda x = \lambda^a \int_0^\infty x^a \lambda\varphi(\lambda x)\, dx = \lambda^a \langle x_+^a, \lambda\varphi(\lambda x)\rangle$$

  □

- *If* $Re(a) > -1$, $\frac{d}{dx} x_+^a = a x_+^{a-1}$.

  Proof. For any $\varphi \in C_c^\infty(\mathbb{R} \setminus \{0\})$

$$\langle \frac{d}{dx} x_+^a, \varphi\rangle = -\langle x_+^a, \frac{d}{dx}\varphi(x)\rangle = -\int_0^\infty x^a \varphi'(x)\, dx = \int_0^\infty a x^{a-1} \varphi(x)\, dx = \langle a x_+^{a-1}, \varphi\rangle$$

  □

  One may think of defining $x_+^a := \frac{1}{a+1}\frac{d}{dx} x_+^{a+1}$ *for* $Re(a) > -2$*. But at* $a = -1$*, there is pole of order* 1.

- *Instead, define* $\chi_+^a(x) := \frac{1}{\Gamma(a+1)} x_+^a$ *where* $\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt$ *so* $\Gamma(a+1) = a\Gamma(a)$ *for any* $a \in \mathbb{C}$*, and*

$$\frac{d}{dx}\chi_+^a(x) = \chi_+^{a-1}(x) \qquad \forall\ Re(a) > -1 \tag{1.7}$$

  Proof. For any $\varphi \in C_c^\infty(\mathbb{R} \setminus \{0\})$

$$\langle \frac{d}{dx}\chi_+^a, \varphi\rangle = -\langle \chi_+^a, \frac{d}{dx}\varphi(x)\rangle = -\int_0^\infty \frac{1}{\Gamma(a+1)} x^a \varphi'(x)\, dx = \int_0^\infty \frac{1}{\Gamma(a)} x^{a-1}\varphi(x)\, dx = \langle \chi_+^{a-1}, \varphi\rangle$$

  □

  Hence one may define

$$\chi_+^a := \frac{d}{dx}\chi_+^{a+1} = \frac{d^k}{dx^k}\chi_+^{a+k} \qquad -k-1 < Re(a) \leq -k \quad k \in \mathbb{N}^+ \tag{1.8}$$

  and χ_+^a *is analytically continued from* $\mathbb{C} \setminus \{-1, -2, \cdots\}$ *to* $\mathbb{C}$.

- *One has identity for any $a \in \mathbb{C}$*

$$x\chi_+^a(x) = (a+1)\chi_+^{a+1} \tag{1.9}$$

Proof. If $Re(a) > -1$,

$$x\chi_+^a(x) = x\frac{1}{\Gamma(a+1)}\mathbb{1}_{\{x\geq 0\}}x^a = \frac{1}{\Gamma(a+1)}\mathbb{1}_{\{x\geq 0\}}x^{a+1} = \frac{a+1}{\Gamma(a+2)}\mathbb{1}_{\{x\geq 0\}}x^{a+1} = (a+1)\chi_+^{a+1}(x)$$

Hence it holds for any $a \in \mathbb{C}$ by analytic continuation. □

Lemma 1.1.5 (Negative Integers and half Integers for χ_+^a)**.** *For any $k \in \mathbb{N}^+$.*

- *Denote δ_0 dirac delta so $\langle \delta_0, \varphi(x)\rangle = \varphi(0)$*

$$\chi_+^{-k}(x) = \delta_0^{(k-1)}(x) = (\frac{d}{dx})^{k-1}\delta_0(x) \tag{1.10}$$

Proof. It suffices to show $\frac{d}{dx}\chi_+^0 = \delta_0$. For any $\varphi \in C_c^\infty(\mathbb{R}\setminus\{0\})$

$$\langle \frac{d}{dx}\chi_+^0, \varphi\rangle = -\langle x_+^0, \varphi'\rangle = -\int_0^\infty \varphi'(x)\,dx = \varphi(0) = \langle \delta_0, \varphi\rangle$$

then conclude using (1.8). □

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$$\chi_+^{-k-\frac{1}{2}}(x) = \frac{1}{\sqrt{\pi}}(\frac{d}{dx})^k(x_+^{-\frac{1}{2}}) \tag{1.11}$$

Proof. It suffices to compute $\chi_+^{-\frac{1}{2}} = \frac{1}{\Gamma(\frac{1}{2})}x_+^{-\frac{1}{2}}$. Note Euler's Reflection formula for Gamma

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}$$

hence $\Gamma(\frac{1}{2})^2 = \pi$. so $\chi_+^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi}}x_+^{-\frac{1}{2}}$. Conclude using (1.8). □

1.1.2 Fundamental Solution

Consider constant coefficient linear wave equation (1.1) on $\mathbb{R}\times\mathbb{R}^n = \mathbb{R}^{1+n}$.

Definition 1.1.4 (Forward Fundamental Solution)**.** *E_+ is forward fundamental solution to (1.1) on $\mathbb{R}^{1+n}$ if*

- *$\Box E_+ = \delta_0$ in the sense of distribution where $\delta_0(\varphi) = \varphi(t=0, x=0)$.*
- $\mathrm{supp}(E_+) \subset \{(t,x) \in \mathbb{R}\times\mathbb{R}^n \mid 0 \leq |x| \leq t\}$.

Proposition 1.1.1 (Uniqueness of Forward Fundamental Solution)**.** *If forward fundamental solution E_+ to (1.1) exists, it is unique.*

Proof. Suppose both E and E_+ are forward fundamental solutions. Since $\mathrm{supp}\{\delta_0\} = \{0\} \in \mathbb{R}^{n+1}$ is compact, one may convolve this with E and E_+ in any order as suggested by Lemma 1.1.2. Notice for any $\varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^n)$

$$(E*\delta_0)*\varphi(x) = E*((\delta_0*\varphi)(x)) = E*(\langle\delta(y), \varphi(x-y)\rangle) = E*\varphi(x)$$

hence $E*\delta_0 = E$. Similarly $\delta_0 * E_+ = E_+$. Notice by supports of E and E_+, the convolutions $E*E_+$, $E*(\Box E_+)$ and $(\Box E)*E_+$ are well-defined. Hence for any $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$

$$\langle E, \varphi\rangle = \langle E*\delta_0, \varphi\rangle = \langle E*(\Box E_+), \varphi\rangle = \langle E*E_+, \Box\varphi\rangle = \langle(\Box E)*E_+, \varphi\rangle = \langle\delta_0*E_+, \varphi\rangle = \langle E_+, \varphi\rangle$$

concluding $E = E_+$. □

Proposition 1.1.2 (Representation Formula to Cauchy Problem)**.** *E_+ be forward fundamental solution to (1.1) on $\mathbb{R}^{n+1}$. Given initial values $(\phi_0, \phi_1) \in C^\infty(\mathbb{R}^n)\times C^\infty(\mathbb{R}^n)$*

- *the unique forward solution ϕ to (1.2) writes*

$$\phi(t,x) = -(\phi_1\delta_{\{t=0\}})*E_+ - (\phi_0\delta_{\{t=0\}})*(\partial_t E_+) \tag{1.12}$$

- *In general for nonhomogeneous equation $\Box\phi = F$ with $F \in C^\infty(\mathbb{R}^n)$, unique forward solution ϕ writes*

$$\phi(t,x) = -(\phi_1\delta_{\{t=0\}}) * E_+ - (\phi_0\delta_{\{t=0\}}) * (\partial_t E_+) + (F\mathbb{1}_{t\geq 0}) * E_+$$

Proof. Suppose ϕ solves $\Box\phi = F$ with initial values (ϕ_0, ϕ_1). Then $\phi\mathbb{1}_{t\geq 0}$ denotes the forward solution. To represent the solution, one convolve with $\delta_0 = \Box E_+$ then throw all derivatives to ϕ

$$\begin{aligned}
\phi\mathbb{1}_{t\geq 0} &= (\phi\mathbb{1}_{t\geq 0}) * \delta_0 = (\phi\mathbb{1}_{t\geq 0}) * \Box E_+ = (\phi\mathbb{1}_{t\geq 0}) * (-\partial_t^2 E_+) + (\Delta\phi\mathbb{1}_{t\geq 0}) * E_+ \\
&= (\phi\mathbb{1}_{t\geq 0}) * (-\partial_t^2 E_+) + (\partial_t^2\phi\mathbb{1}_{t\geq 0}) * E_+ + (F\mathbb{1}_{t\geq 0}) * E_+ \\
&= -(\partial_t(\phi\mathbb{1}_{t\geq 0})) * (\partial_t E_+) + (\partial_t^2\phi\mathbb{1}_{t\geq 0}) * E_+ + (F\mathbb{1}_{t\geq 0}) * E_+ \\
&= -(\partial_t\phi\mathbb{1}_{t\geq 0}) * (\partial_t E_+) - (\phi\delta_{\{t=0\}}) * (\partial_t E_+) + (\partial_t^2\phi\mathbb{1}_{t\geq 0}) * E_+ + (F\mathbb{1}_{t\geq 0}) * E_+ \\
&= -(\partial_t\phi\delta_{\{t=0\}}) * E_+ - (\phi\delta_{\{t=0\}}) * (\partial_t E_+) + (F\mathbb{1}_{t\geq 0}) * E_+ \\
&= -(\phi_1\delta_{\{t=0\}}) * E_+ - (\phi_0\delta_{\{t=0\}}) * (\partial_t E_+) + (F\mathbb{1}_{t\geq 0}) * E_+
\end{aligned}$$

□

Now one wish to find the Forward Fundamental Solution E_+ to (1.1) on $\mathbb{R}^{1+n}$. Notice symmetries of $\Box$ that fix the origin $\{0,0\} \in \mathbb{R}^{1+n}$ are Lorentze transformations, leaving invariant the quantity $s^2(t,x) = t^2 - |x|^2$. On the RHS, δ_0 is invariant under these symmetries as well. It is hence natural to look for solutions invariant under Lorentze transformation, and possibly with s^2 built in. Now notice

$$\Box E_+ = \delta_0$$

From Example 1.1.3 we know δ_0 is degree $-1-n$, and since $\Box$ drops degree by 2 from Example 1.1.2 it is natural to look for E_+ with degree $1-n$. From lemma 1.1.4, since we wish to build in s^2 which is degree of homogeneity 2 as in Example 1.1.1, we're forced to look for $E_+ = u \circ s^2$ where $u \in \mathcal{D}'(\mathbb{R})$ with degree of homogeneity $\frac{1-n}{2}$. Due to supp(E_+) requirement, we need $t \geq |x| \geq 0$, so choosing $u = \chi_+^{\frac{1-n}{2}}$ indeed guarantees a homogeneous $\frac{1-n}{2}$ distribution defines on $\mathcal{D}'(\mathbb{R})$ (it is originally defined on $\mathcal{D}'(\mathbb{R}\setminus\{0\})$ but by Lemma 1.1.3 one may apply homogeneous extension to $\mathcal{D}'(\mathbb{R})$) where $t \geq |x|$ makes sense in its support. To ensure we're dealing with forward solution in time $t \geq 0$, one simply multiply by $\mathbb{1}_{\{t\geq 0\}}$ which is itself homogeneous degree 0 and invariant under Lorentze transformations.

Proposition 1.1.3 (Forward Fundamental Solution)**.** *The unique forward fundamental solution to* (1.1) *over* $\mathbb{R}^{1+n}$ *is given by*

$$E_+(t,x) = -\frac{\pi^{\frac{1-n}{2}}}{2}\mathbb{1}_{\{t\geq 0\}}\chi_+^{\frac{1-n}{2}}(t^2 - |x|^2) \tag{1.13}$$

Proof. One first compute for $(t,x) \neq (0,0) \in \mathbb{R}^{1+n}$. Then, as a distribution in $\mathcal{D}'(\mathbb{R}^{1+n}\setminus\{(0,0)\})$, using chain rule (1.5) and (1.7) and (1.9) at last

$$\begin{aligned}
\Box(\mathbb{1}_{\{t\geq 0\}}\chi_+^{\frac{1-n}{2}}(t^2-|x|^2)) &= \mathbb{1}_{\{t\geq 0\}}\Box(\chi_+^{\frac{1-n}{2}}(t^2-|x|^2)) \\
&= \mathbb{1}_{\{t\geq 0\}}\left(-\partial_t((2t\chi_+^{\frac{-1-n}{2}})(t^2-|x|^2)) + \sum_{i=1}^n \partial_{x_i}((-2x_i\chi_+^{\frac{-1-n}{2}})(t^2-|x|^2))\right) \\
&= \mathbb{1}_{\{t\geq 0\}}\left(-2\chi_+^{\frac{-1-n}{2}}(t^2-|x|^2)) - 4t^2\chi_+^{\frac{-3-n}{2}}(t^2-|x|^2))\right) \\
&+ \mathbb{1}_{\{t\geq 0\}}\sum_{i=1}^n\left(-2\chi_+^{\frac{-1-n}{2}}(t^2-|x|^2) + 4x_i^2\chi_+^{\frac{-3-n}{2}}(t^2-|x|^2)\right) \\
&= \mathbb{1}_{\{t\geq 0\}}\left(-2(n+1)\chi_+^{\frac{-1-n}{2}}(t^2-|x|^2) - 4(t^2-|x|^2)\chi_+^{\frac{-3-n}{2}}(t^2-|x|^2)\right) \\
&= \mathbb{1}_{\{t\geq 0\}}\left(-2(n+1)\chi_+^{\frac{-1-n}{2}}(t^2-|x|^2) - 4\frac{-1-n}{2}\chi_+^{\frac{-1-n}{2}}(t^2-|x|^2)\right) = 0
\end{aligned}$$

Thus $\Box E_+$ is distribution supported at $\{0\}$, hence a linear combination of δ_0 and its derivatives. But as we've seen, $\Box E_+$ and δ_0 are manually constructed with the same degree of homogeneity, then we must have $\Box E_+ = c_n\delta_0$ for some constant $c_n > 0$. For computation of $c_n = -\frac{\pi^{\frac{1-n}{2}}}{2}$, see Appendix B of 'Lecture notes on linear wave equation' by Sung-jin Oh . □

We now apply formulas for E_+ to derive representation formulas in $n = 1$ and $n = 3$.

Proposition 1.1.4 (D'Alembert's formula)**.** *Let $(\phi_0, \phi_1) \in C_c^\infty(\mathbb{R}) \times C_c^\infty(\mathbb{R})$ be initial data to* (1.2) *in* $\mathbb{R}^{1+1}$. *Then unique solution ϕ writes*

$$\phi(t,x) = \frac{1}{2}(\phi_0(x+t) - \phi_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t}\phi_1(y)\,dy$$

Proof. (1.12) with $n = 1$ and using (1.13) writes

$$\phi(t,x) = (\phi_1 \delta_{\{t=0\}}) * (\frac{1}{2}\mathbb{1}_{\{t\geq 0\}}\chi_+^0(t^2 - |x|^2)) + (\phi_0 \delta_{\{t=0\}}) * (\frac{1}{2}\partial_t(\mathbb{1}_{\{t\geq 0\}}\chi_+^0(t^2 - |x|^2)))$$

Notice in fact $\mathbb{1}_{\{t\geq 0\}}\chi_+^0(t^2 - |x|^2) = \mathbb{1}_{\{t^2-|x|^2\geq 0\}}$, and

$$(\phi_1\delta_{\{t=0\}}) * (\mathbb{1}_{\{t^2-|x|^2\geq 0\}}) = \int_{-\infty}^{\infty} \phi_1(y)\mathbb{1}_{\{t^2-|x-y|^2\geq 0\}}\, dy = \int_{x-t}^{x+t} \phi_1(y)\, dy$$

$$(\phi_0\delta_{\{t=0\}}) * (\partial_t(\mathbb{1}_{\{t^2-|x|^2\geq 0\}})) = \partial_t\left(\phi_0\delta_{\{t=0\}} * \mathbb{1}_{\{t^2-|x|^2\geq 0\}}\right) = \partial_t(\int_{x-t}^{x+t} \phi_0(y)\, dy) = \phi_0(x+t) - \phi_0(x-t)$$

□

Proposition 1.1.5 (Kirchhoff's formula)**.** *Let* $(\phi_0, \phi_1) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3)$ *be initial data to* $\mathbb{R}^{1+3}$*. Then unique solution* ϕ *writes*

$$\phi(t,x) = \frac{1}{4\pi t}\int_{\{|y-x|=t\}} \phi_1(y)\, d\sigma_{\{|y-x|=t\}}(y) + \frac{d}{dt}\left(\frac{1}{4\pi t}\int_{\{|y-x|=t\}} \phi_0(y)\, d\sigma_{\{|y-x|=t\}}(y)\right) \tag{1.14}$$

Proof. (1.12) with $n = 3$ and using (1.13) writes

$$\phi(t,x) = (\phi_1\delta_{t=0}) * (\frac{1}{2\pi}\mathbb{1}_{\{t\geq 0\}}\chi_+^{-1}(t^2 - |x|^2)) + (\phi_0\delta_{\{t=0\}}) * (\partial_t(\frac{1}{2\pi}\mathbb{1}_{\{t\geq 0\}}\chi_+^{-1}(t^2 - |x|^2)))$$

one needs to calculate $\mathbb{1}_{\{t\geq 0\}}\chi_+^{-1}(t^2 - |x|^2)) = \mathbb{1}_{\{t\geq 0\}}\delta_0(t^2 - |x|^2)$. Indeed one use a sequence of $h_j \in C_c^\infty(\mathbb{R})$ to approximate $\delta_0(t^2 - |x|^2)$ as composition. To comprehend, one first make the following observations

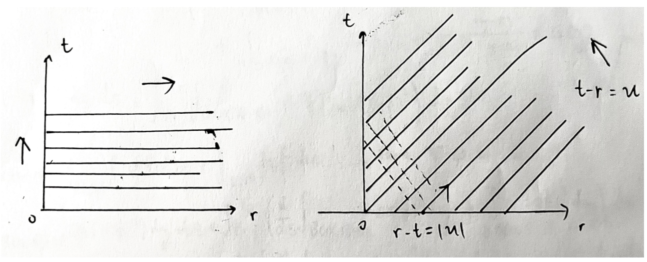
1. $\delta_0(t^2 - |x|^2)$ is supported in $\{t^2 = |x|^2\}$ the light cone. One first apply change into spherical coordinates to spatial variables to obtain $r = |x|$, $\omega = \frac{x}{|x|}$. Hence $\delta_0(t^2 - |x|^2)$ supported in $\{t = r\}$.
2. By change into null variables $\begin{cases} u = t - r \\ v = t + r \end{cases}$, $\delta_0(t^2 - |x|^2)$ is supported in $\{u = 0\}$.
3. To obtain limiting behavior $h_j(t^2 - |x|^2) = h_j(uv)$, one change into $\begin{cases} \overline{u} = uv \\ v = v \end{cases}$, so $h_j(uv) = h_j(\overline{u}) \to \delta_{\{\overline{u}=0\}}$.
4. One need the following Jacobian computations.

$$\begin{aligned} dx &= r^2 dr d\sigma_{\mathbb{S}^2}(\omega) \ where \ d\sigma_{\mathbb{S}^2}(\omega) \ denotes \ surface \ measure \ of \ \mathbb{S}^2 \\ drdt &= \frac{1}{2} dvdu \qquad if \ fix \ u = 0 \iff r = t \quad then \ dv = 2dt \\ dvdu &= \frac{1}{v} dvd\overline{u} \end{aligned}$$

For any $\varphi \in C_c^\infty(\mathbb{R}^{1+3})$, one calculate explicitly $\mathbb{1}_{\{t\geq 0\}}\delta_0(t^2 - |x|^2)$ by testing its limiting sequence $h_j(t^2 - |x|^2)$ against φ and then pass to ∞

$$\begin{aligned} \langle \mathbb{1}_{\{t\geq 0\}} h_j(t^2 - |x|^2), \varphi(t,x)\rangle &= \int_0^\infty \int_{\mathbb{R}^3} h_j(t^2 - |x|^2)\varphi(t,x)\, dxdt = \int_0^\infty \int_0^\infty \int_{\mathbb{S}^2} h_j(t^2 - r^2)\varphi(t,r,\omega) r^2\, d\sigma_{\mathbb{S}^2}(\omega) drdt \\ &= \int_{-\infty}^\infty \int_{|u|}^\infty \int_{\mathbb{S}^2} h_j(uv)\varphi(u,v,\omega)\frac{(v-u)^2}{8}\, d\sigma_{\mathbb{S}^2}(\omega) dvdu \end{aligned}$$

here $\int_0^\infty \int_0^\infty drdt = \int_{-\infty}^\infty \int_{|u|}^\infty \frac{1}{2} dvdu$ is illustrated as below via Fubini.



$$\begin{aligned}\langle \mathbb{1}_{\{t\geq 0\}} h_j(t^2-|x|^2), \varphi(t,x)\rangle &= \int_{-\infty}^{\infty}\int_{|u|}^{\infty}\int_{\mathbb{S}^2} h_j(\overline{u})\varphi(\frac{\overline{u}}{v}, v, \omega)\frac{(v-\frac{\overline{u}}{v})^2}{8}\frac{1}{v}\, d\sigma_{\mathbb{S}^2}(\omega) dv d\overline{u} \\ &\longrightarrow \int_0^{\infty}\int_{\mathbb{S}^2}\varphi(0,v,\omega)\frac{v}{8}\, d\sigma_{\mathbb{S}^2}(\omega)dv \qquad as\ j\to\infty \\ &= \int_0^{\infty}\int_{\mathbb{S}^2}\varphi(t, r=t,\omega)\frac{t}{2}\, d\sigma_{\mathbb{S}^2}(\omega)dt = \langle \mathbb{1}_{\{t\geq 0\}}\delta_0(t^2-|x|^2), \varphi(t,x)\rangle\end{aligned}$$

Hence convolution of $\mathbb{1}_{\{t\geq 0\}}\delta_0(t^2-|x|^2)$ with $\phi_1\delta_{\{t=0\}}$ yields, abuse of notation

$$\begin{aligned}\mathbb{1}_{\{t\geq 0\}}\delta_0(t^2-|x|^2) * \phi_1\delta_{\{t=0\}}(t,x) &= \frac{t}{2}\int_{\mathbb{S}^2}\phi_1(x+t\omega)\, d\sigma_{\mathbb{S}^2}(\omega) \\ &= \frac{1}{2t}\int_{\{|y-x|=t\}}\phi_1(y)\, d\sigma_{\{|y-x|=t\}}(y)\end{aligned}$$

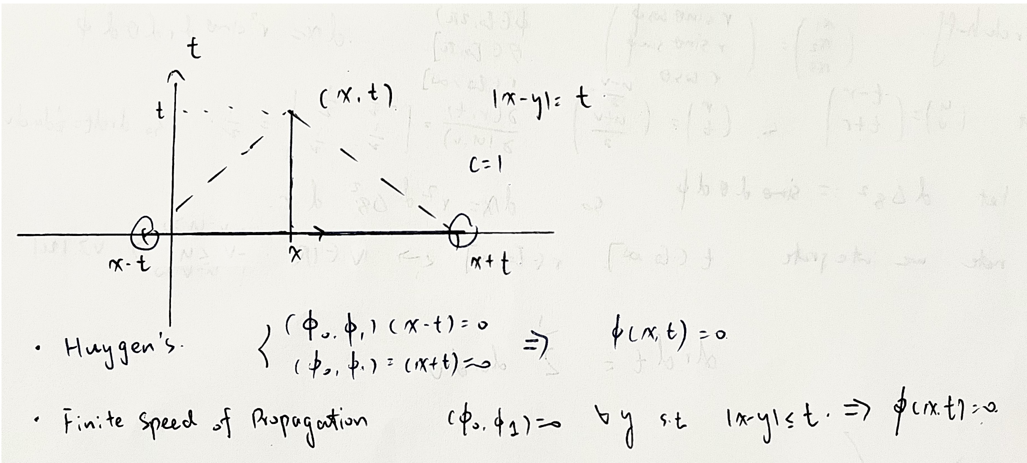
and the second term

$$\begin{aligned}\partial_t(\mathbb{1}_{\{t\geq 0\}}\delta_0(t^2-|x|^2)) * \phi_0\delta_{\{t=0\}}(t,x) &= \partial_t(\mathbb{1}_{\{t\geq 0\}}\delta_0(t^2-|x|^2) * \phi_0\delta_{\{t=0\}})(t,x) \\ &= \partial_t(\frac{1}{2t}\int_{\{|y-x|=t\}}\phi_0(y)\, d\sigma_{\{|y-x|=t\}}(y))\end{aligned}$$

summing up with constant $\frac{1}{2\pi}$ at the front yields (1.14). □

1.1.3 Properties of Solution

One has the characterizing properties of wave equations: Finite Speed of Propagation.



Proposition 1.1.6 (Finite Speed of Propagation). *$\phi\in C^\infty(\mathbb{R}^{1+n})$ solution to (1.2) with initial data $(\phi_0,\,\phi_1)$. Fix $(t,x)\in\mathbb{R}^{1+n}$, if $(\phi_0,\,\phi_1)=(0,0)$ in $\{y\in\mathbb{R}^n \mid |x-y|\leq t\}$ then $\phi(t,x)=0$.*

Proof. Note $\phi(t,x) = -(\phi_1\delta_{\{t=0\}}) * E_+ - (\phi_0\delta_{\{t=0\}}) * (\partial_t E_+)$ as in (1.12). And due to support requirement $\text{supp}(E_+)\subset\{(t,x)\in\mathbb{R}^{1+n}\mid 0\leq|x|\leq t\}$

$$\begin{aligned}\text{supp}(\phi(t,x)) &\subset \text{supp}((\phi_1\delta_{\{t=0\}}) * E_+)\cup\text{supp}((\phi_0\delta_{\{t=0\}}) * (\partial_t E_+)) \\ &\subset (\{0\}\times\text{supp}(\phi_1)+\text{supp}(E_+))\cup(\{0\}\times\text{supp}(\phi_0)+\text{supp}(E_+))\end{aligned}$$

so if $\phi(t,x)\neq 0$, then either $(t,x)\in\{0\}\times\text{supp}(\phi_1)+\text{supp}(E_+)$ or $(t,x)\in\{0\}\times\text{supp}(\phi_0)+\text{supp}(E_+)$. WLOG assume the latter. Then $(t,x)-\{0\}\times\text{supp}(\phi_0)\subset\text{supp}(E_+)$, i.e., there exists $y\in\text{supp}(\phi_0)\subset\mathbb{R}^n$ s.t.

$$(t,x-y)\in\text{supp}(E_+) \implies |x-y|\leq t$$

and that $\phi_0(y)\neq 0$ contradicts our assumption. □

Proposition 1.1.7 (Strong Huygens Principle). *$n\geq 3$ odd integer. $\phi\in C^\infty(\mathbb{R}^{1+n})$ solution to (1.2) with initial data $(\phi_0,\,\phi_1)$. Fix $(t,x)\in\mathbb{R}^{1+n}$, if $(\phi_0,\,\phi_1)=(0,0)$ in $\{y\in\mathbb{R}^n\mid|x-y|=t\}$ then $\phi(t,x)=0$.*

Proof. For odd integers $n \geq 3$, $\frac{1-n}{2}$ gives negative integer. Hence (1.10) says $\chi_+^{\frac{1-n}{2}}$, as derivatives of Dirac Delta, has support at $\{0\}$. Hence E_+ has support at $\{(t,x) \in \mathbb{R}^{1+n} \mid |x|^2 = t^2\}$. Repeat as in Prop 1.1.6, for any (t,x) s.t. $\phi(t,x) \neq 0$, WLOG let $(t,x) - \{0\} \times \mathrm{supp}(\phi_0) \subset \mathrm{supp}(E_+)$, there exists $y \in \mathrm{supp}(\phi_0)$ s.t.

$$(t, x-y) \in \mathrm{supp}(E_+) \implies |x-y| = t$$

and that $\phi_0(y) \neq 0$ contradicts our assumption. □

Now we discuss the decay property using representation formula.

Proposition 1.1.8 (Decay Property using Kirchhoff)**.** *$n = 3$. $(\phi_0, \phi_1) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3)$ and ϕ unique solution to (1.2) in $\mathbb{R}^{1+3}$, with explicit formula as in (1.14). Then there exists $C = C(\phi_0, \phi_1) > 0$ s.t.*

$$\sup_{x\in\mathbb{R}^3} |\phi(t,x)| \leq C\frac{1}{1+t}$$

Proof.

$$\phi(t,x) = \frac{1}{4\pi t}\int_{\{|y-x|=t\}} \phi_1(y)\, d\sigma_{\{|y-x|=t\}}(y) + \frac{d}{dt}\left(\frac{1}{4\pi t}\int_{\{|y-x|=t\}} \phi_0(y)\, d\sigma_{\{|y-x|=t\}}(y)\right) =: \mathbf{I} + \mathbf{II}$$

For $t \in [0,1]$, note $|\{|y-x| = t\}| = 4\pi t^2$, so using compact support of ϕ_1

$$\sup_{x\in\mathbb{R}^3} |\mathbf{I}| \leq \frac{1}{4\pi t} 4\pi t^2 \sup_{y\in\mathbb{R}^3} |\phi_1(y)| \leq Ct \leq C$$

while for $t > 1$, since $\{|y-x| = t\} \cap \mathrm{supp}(\phi_1)$ is compact, one may bound $|\{|y-x| = t\} \cap \mathrm{supp}(\phi_1)| \leq C$.

$$\sup_{x\in\mathbb{R}^3} |\mathbf{I}| \leq \frac{1}{4\pi t} C \sup_{y\in\mathbb{R}^3} |\phi_1(y)| \leq C\frac{1}{t}$$

Now since ϕ_0 smooth, one may write $\mathbf{II}$

$$\begin{aligned}
\mathbf{II} &= \frac{d}{dt}\left(\frac{1}{4\pi t}\int_{\{|y-x|=t\}} \phi_0(y)\, d\sigma_{\{|y-x|=t\}}(y)\right) = \frac{d}{dt}\left(\frac{1}{4\pi t}\int_{\mathbb{S}^2} \phi_0(x+t\omega)\, t^2\, d\sigma_{\mathbb{S}^2}(\omega)\right) \\
&= \frac{d}{dt}\left(\frac{1}{4\pi} t \int_{\mathbb{S}^2} \phi_0(x+t\omega)\, d\sigma_{\mathbb{S}^2}(\omega)\right) = \frac{1}{4\pi}\int_{\mathbb{S}^2} \phi_0(x+t\omega)\, d\sigma_{\mathbb{S}^2}(\omega) + \frac{1}{4\pi} t \int_{\mathbb{S}^2} \nabla\phi_0(x+t\omega)\cdot\omega\, d\sigma_{\mathbb{S}^2}(\omega) \\
&= \frac{1}{4\pi t^2}\int_{\{|y-x|=t\}} \phi_0(y)\, d\sigma_{\{|y-x|=t\}}(y) + \frac{1}{4\pi t}\int_{\{|y-x|=t\}} \nabla\phi_0(y)\cdot\frac{y-x}{t}\, d\sigma_{\{|y-x|=t\}}(y) =: \mathbf{III} + \mathbf{IV}
\end{aligned}$$

For $t \in [0,1]$

$$\begin{aligned}
\sup_{x\in\mathbb{R}^3} |\mathbf{III}| &\leq \sup_{y\in\mathbb{R}^3} |\phi_0(y)| \leq C \\
\sup_{x\in\mathbb{R}^3} |\mathbf{IV}| &\leq t \sup_{y\in\mathbb{R}^3} |D\phi_0(y)| \leq Ct \leq C
\end{aligned}$$

whereas for $t > 1$ one may bound $|\{|y-x| = t\} \cap \mathrm{supp}(\phi_0)| \leq C$.

$$\begin{aligned}
\sup_{x\in\mathbb{R}^3} |\mathbf{III}| &\leq \frac{1}{4\pi t^2} C \sup_{y\in\mathbb{R}^3} |\phi_0(y)| \leq C\frac{1}{t^2} \leq C\frac{1}{t} \\
\sup_{x\in\mathbb{R}^3} |\mathbf{IV}| &\leq \frac{1}{4\pi t} C \sup_{y\in\mathbb{R}^3} |D\phi_0(y)| \leq C\frac{1}{t}
\end{aligned}$$

□

Chapter 2

Non-constant Coefficient Linear Wave Equation

Consider Non-constant Coefficient Linear Wave Equation with solution $\phi : I \times \mathbb{R}^n \to \mathbb{R}$ where $0 \in I$

$$\partial_\alpha(a^{\alpha\beta}\partial_\beta\phi) = \sum_{\alpha=0}^{n} \partial_\alpha \left(\sum_{\beta=0}^{n} a^{\alpha\beta}\partial_\beta\phi \right) = F \tag{2.1}$$

with conditions

- a is symmetric $(1+n) \times (1+n)$ matrix on $I \times \mathbb{R}^n$ with values in $\mathbb{R}$ s.t.

$$\sum_{\alpha,\beta=0}^{n} |a^{\alpha\beta} - m^{\alpha\beta}| < \frac{1}{10} \tag{2.2}$$

  for constant $(1+n) \times (1+n)$ Minkowski matrix $(m^{\alpha\beta}) = diag(1, -1, \cdots, -1)$.

- $F : I \times \mathbb{R}^n \to \mathbb{R}$

We're interested in Initial Value Problem with initial data $(\phi_0, \phi_1) : \mathbb{R}^n \to \mathbb{R}^2$

$$\begin{cases} \partial_\alpha(a^{\alpha\beta}\partial_\beta\phi) = F \\ (\phi, \partial_t\phi)|_{t=0} = (\phi_0, \phi_1) \end{cases} \tag{2.3}$$

2.1 Energy Estimates

One needs the Grönwall's Lemma as tool

Lemma 2.1.1 (Grönwall's Lemma)**.** *Fix $T > 0$. Let $f \in C(\mathbb{R}; (0, \infty))$ and $g \in L^1(\mathbb{R}; (0, \infty))$ s.t. for some $A \geq 0$*

$$f(t) \leq A + \int_0^t f(s)\, g(s)\, ds \qquad \forall\, t \in [0, T] \tag{2.4}$$

Then

$$f(t) \leq A \exp(\int_0^t g(s)\, ds) \qquad \forall\, t \in [0, T] \tag{2.5}$$

Proof of (2.5) *Method 1.* Differentiate and using $f > 0$

$$\frac{d}{dt}\left(A + \int_0^t f(s)g(s)\, ds \right) = f(t)g(t) \leq g(t)\left(A + \int_0^t f(s)g(s)\, ds \right)$$
$$\implies \frac{1}{A + \int_0^t f(s)g(s)\, ds} \frac{d}{dt}\left(A + \int_0^t f(s)g(s)\, ds \right) \leq g(t)$$

Then integrate from 0 to t to obtain

$$\log\left(A + \int_0^t f(s)g(s)\, ds \right) - \log A \leq \int_0^t g(s)\, ds$$
$$\implies \frac{A + \int_0^t f(s)g(s)\, ds}{A} \leq \exp\left(\int_0^t g(s)\, ds \right)$$
$$\implies f(t) \leq A + \int_0^t f(s)g(s)\, ds \leq A \exp\left(\int_0^t g(s)\, ds \right)$$

□

Proof of (2.5) *Method 2 (Bootstrap Method).* For every $\varepsilon > 0$, consider the condition

$$f(t) \leq (1+\varepsilon) A \exp\left((1+\varepsilon)\int_0^t g(s)\, ds\right) \tag{2.6}$$

define $B_\varepsilon := \{t \in [0,T] \mid (2.6)\ holds\, \forall\ s \in [0,t]\}$. Using connectedness of $[0,T]$, one wish to prove that

- B_ε is nonempty. This is trivial as $f(0) \leq A$ so $0 \in B_\varepsilon$.
- B_ε is closed. This is trivial as f is continuous.
- B_ε is open is left to see. If so, then $B_\varepsilon = [0,T]$.

To argue B_ε open, it suffices to argue in fact

$$f(t) \leq A \exp\left((1+\varepsilon)\int_0^t g(s)\, ds\right)$$

so that if one extend t beyond for small portion that contributes less than ε, the extension still satisfies (2.6). To see the bound improvement, for any $t \in B_\varepsilon$

$$\begin{aligned}
f(t) &\leq A + \int_0^t f(s) g(s)\, ds \leq A + (1+\varepsilon) A \int_0^t g(s) \exp\left((1+\varepsilon)\int_0^s g(r)\, dr\right) ds \\
&\leq A\left(1 + \int_0^t ((1+\varepsilon) g(s)\,) \exp\left((1+\varepsilon)\int_0^s g(r)\, dr\right) ds\right) \\
&= A\left(1 + \int_0^t \frac{d}{ds}\left(\exp\left((1+\varepsilon)\int_0^s g(r)\, dr\right)\right) ds\right) = A\left(1 + \exp\left((1+\varepsilon)\int_0^t g(s)\, ds\right) - 1\right) \\
&= A \exp\left((1+\varepsilon)\int_0^t g(s)\, ds\right)
\end{aligned}$$

Hence $B_\varepsilon = [0,T]$ for any $\varepsilon > 0$. Since LHS of (2.6) is independent of ε, one take $\varepsilon \to 0$ to obtain (2.5). □

We write $|\partial\phi|^2 := (\partial_t \phi)^2 + \sum_{i=1}^n \partial_{x_i} \phi^2$.

Theorem 2.1.1 (A priori Energy Estimate for non-constant linear wave). *Fix $T \in I$. Assume the following regularities:*

- *$a \in L^\infty([0,T] \times \mathbb{R}^n)$ and $\partial a \in L^1([0,T]; L^\infty(\mathbb{R}^n))$.*
- *$F \in L^1([0,T]; L^2(\mathbb{R}^n))$.*
- *Initial data $(\phi_0, \phi_1) \in \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.*

Then, if $(\phi, \partial_t \phi) \in L^\infty([0,T]; H^1(\mathbb{R}^n)) \times L^\infty([0,T]; L^2(\mathbb{R}^n))$ is solution to (2.3), there exists constant $C(n) > 0$ s.t.

$$\sup_{t \in [0,T]} \|\partial\phi\|_{L^2(\mathbb{R}^n)}(t) \leq C\left(\|(\phi_0, \phi_1)\|_{\dot{H}^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} + \int_0^T \|F\|_{L^2(\mathbb{R}^n)}(t)\, dt\right) \exp\left(C\int_0^T \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\, dt\right) \tag{2.7}$$

Proof. One makes use of the observation

$$\partial_t \phi \left(\partial_\alpha (a^{\alpha\beta} \partial_\beta \phi) - F\right) = 0$$

and wish to compute at first the quantities $\partial_t \phi\, \partial_\alpha (a^{\alpha\beta} \partial_\beta \phi)$. For $\alpha = \beta = 0$, i.e. w.r.t. t, by product rule

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^n} \partial_t \phi\, \partial_t (a^{00} \partial_t \phi)\, dx dt &= \int_0^T \int_{\mathbb{R}^n} (\partial_t \phi)^2\, \partial_t a^{00} + \frac{1}{2} \partial_t ((\partial_t \phi)^2) a^{00}\, dx dt \\
&= \int_0^T \int_{\mathbb{R}^n} (\partial_t \phi)^2\, \partial_t a^{00}\, dx dt + \frac{1}{2}\int_{\mathbb{R}^n} \left((\partial_t \phi)^2 a^{00}(T) - (\partial_t \phi)^2 a^{00}(0) - \int_0^T (\partial_t \phi)^2 \partial_t a^{00}\, dt\right) dx \\
&= \frac{1}{2}\int_0^T \int_{\mathbb{R}^n} (\partial_t \phi)^2\, \partial_t a^{00}\, dx dt + \frac{1}{2}\int_{\mathbb{R}^n} (\partial_t \phi)^2 a^{00}(T)\, dx - \frac{1}{2}\int_{\mathbb{R}^n} (\partial_t \phi)^2 a^{00}(0)\, dx
\end{aligned}$$

For $i,\, j = 1, \cdots, n$, using symmetry of a^{ij}, and then Product Rule

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^n} \partial_t\phi\, \partial_i(a^{ij}\partial_j\phi)\, dxdt &= -\int_0^T \int_{\mathbb{R}^n} \partial_i(\partial_t\phi)\, a^{ij}\partial_j\phi\, dxdt \\ &= -\frac{1}{2}\int_0^T \int_{\mathbb{R}^n} \partial_i(\partial_t\phi)\, a^{ij}\partial_j\phi\, dxdt - \frac{1}{2}\int_0^T \int_{\mathbb{R}^n} \partial_j(\partial_t\phi)\, a^{ji}\partial_i\phi\, dxdt \\ &= -\frac{1}{2}\int_0^T \int_{\mathbb{R}^n} \left(\partial_i(\partial_t\phi)\partial_j\phi + \partial_j(\partial_t\phi)\partial_i\phi\right)\ a^{ij} dxdt \\ &= -\frac{1}{2}\int_0^T \int_{\mathbb{R}^n} \partial_t\left(\partial_i\phi\, \partial_j\phi\right)\ a^{ij} dxdt \\ &= -\frac{1}{2}\int_{\mathbb{R}^n} \partial_i\phi\, \partial_j\phi\, a^{ij}(T) - \partial_i\phi\, \partial_j\phi\, a^{ij}(0)\, dx + \frac{1}{2}\int_0^T \int_{\mathbb{R}^n} \partial_i\phi\, \partial_j\phi\, \partial_t a^{ij}\, dxdt\end{aligned}$$

For $i = 1, \cdots, n$ and $\beta = 0$ w.r.t t, and vice versa, we again use symmetry of $a^{\alpha\beta}$

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^n} \partial_t\phi\, \partial_i(a^{i0}\partial_t\phi) + \partial_t\phi\, \partial_t(a^{0i}\partial_i\phi)\, dxdt &= \int_0^T \int_{\mathbb{R}^n} (\partial_t\phi)^2\partial_i a^{i0} + \partial_t\phi\partial_i\partial_t\phi(a^{i0} + a^{0i}) + \partial_t\phi\partial_i\phi\partial_t a^{0i}\, dxdt \\ &= \int_0^T \int_{\mathbb{R}^n} (\partial_t\phi)^2\partial_i a^{i0} + \partial_i((\partial_t\phi)^2)a^{i0} + \partial_t\phi\partial_i\phi\partial_t a^{i0}\, dxdt \\ &= \int_0^T \int_{\mathbb{R}^n} (\partial_t\phi)^2\partial_i a^{i0} - (\partial_t\phi)^2\partial_i a^{i0} + \partial_t\phi\partial_i\phi\partial_t a^{i0}\, dxdt \\ &= \int_0^T \int_{\mathbb{R}^n} \partial_t\phi\partial_i\phi\partial_t a^{i0}\, dxdt\end{aligned}$$

Since we require integrability, terms as $|x| \to \infty$ in space necessarily vanishes, so weak derivatives make sense in the above duality pairings. Now using

$$\int_0^T \int_{\mathbb{R}^n} \partial_t\phi(\partial_\alpha(a^{\alpha\beta}\partial_\beta\phi))\, dxdt = \int_0^T \int_{\mathbb{R}^n} \partial_t\phi\, F\, dxdt$$

One may put time T terms on the LHS and write, using Hölder in force F and direct bound in $\|\partial a\|_{L^\infty}$

$$\begin{aligned}\frac{1}{2}\left|\int_{\mathbb{R}^n} (\partial_t\phi)^2 a^{00}(T) - \partial_i\phi\partial_j\phi\, a^{ij}(T)\, dx\right| &\le \frac{1}{2}\left|\int_{\mathbb{R}^n} (\partial_t\phi)^2 a^{00}(0) - \partial_i\phi\partial_j\phi\, a^{ij}(0)\, dx\right| \\ &\quad + C_0\int_0^T \left(\|\partial\phi\|_{L^2(\mathbb{R}^n)}(t)\, \|F\|_{L^2(\mathbb{R}^n)}(t) + \|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(t)\, \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\right) dt\end{aligned}$$

for some $C_0 > 0$. But due to L^∞ bounds on $a^{\alpha\beta}$ (2.2), both LHS and first term on RHS are comparable to $\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}$. So there exists constant $C_1,\, C_2 > 0$ s.t.

$$\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(T) \le C_1\frac{1}{2}\left|\int_{\mathbb{R}^n} (\partial_t\phi)^2 a^{00}(T) - \partial_i\phi\partial_j\phi\, a^{ij}(T)\, dx\right|$$

and

$$\frac{1}{2}\left|\int_{\mathbb{R}^n} (\partial_t\phi)^2 a^{00}(0) - \partial_i\phi\partial_j\phi\, a^{ij}(0)\, dx\right| \le C_2\, \|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(0)$$

So rewriting $C = \max\{C_1C_2,\, C_0\} > 0$

$$\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(T) \le C\, \|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(0) + C\int_0^T \left(\|\partial\phi\|_{L^2(\mathbb{R}^n)}(t)\, \|F\|_{L^2(\mathbb{R}^n)}(t) + \|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(t)\, \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\right) dt$$

But up to performing the same argument in smaller interval of time, one in fact obtain uniform estimate in t

$$\sup_{0\le t\le T} \|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(t) \le C\, \|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(0) + C\int_0^T \left(\|\partial\phi\|_{L^2(\mathbb{R}^n)}(t)\, \|F\|_{L^2(\mathbb{R}^n)}(t) + \|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(t)\, \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\right) dt$$

Now we strive to bounded RHS independent of $\partial\phi$. The philosophy is as follows

- Notice our target is to bound in $\|\partial\phi\|^2_{L^2}$ and RHS consists of both $\|\partial\phi\|_{L^2}$ and $\|\partial\phi\|^2_{L^2}$.
- For $\|\partial\phi\|_{L^2}$ we may sacrifice 1 power by using ε-Youngs and absorb the small term to the LHS.

- For the only remaining $\|\partial\phi\|_{L^2}^2$ one may use Grönwall to eliminate but sacrificing a raise to exponential.

To do so, first apply ε-Young's (or Cauchy-Schwarz) so for $\delta > 0$ sufficiently small and some $C_3 > 0$

$$\int_0^T \|\partial\phi\|_{L^2(\mathbb{R}^n)}(t)\,\|F\|_{L^2(\mathbb{R}^n)}(t) \le \delta \sup_{0\le t\le T}\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(t) + \frac{C_3}{\delta}\left(\int_0^T \|F\|_{L^2(\mathbb{R}^n)}(t)\,dt\right)^2$$

Absorb the first term on RHS to the LHS so that, rewriting $C \ge \frac{C_3}{\delta}$ and the original constant C,

$$\sup_{0\le t\le T}\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(t) \le C\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(0) + C\left(\int_0^T \|F\|_{L^2(\mathbb{R}^n)}(t)\,dt\right)^2 + C\int_0^T \|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(t)\,\|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\,dt$$

Now one apply Grönwall's (2.5) so that

$$\sup_{0\le t\le T}\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(t) \le C\left(\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(0) + \left(\int_0^T \|F\|_{L^2(\mathbb{R}^n)}(t)\,dt\right)^2\right)\exp\left(C\int_0^T \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\,dt\right)$$

Recover by lifting the power. Notice lifting $\frac{1}{2}$ on exp arrives at $(\exp)^{\frac{1}{2}} = \exp^{\frac{1}{2}}$ hence we absorb it into C. And up to another constant $C > 0$, we have (2.7)

$$\sup_{0\le t\le T}\|\partial\phi\|_{L^2(\mathbb{R}^n)}(t) \le C\left(\|\partial\phi\|_{L^2(\mathbb{R}^n)}(0) + \int_0^T \|F\|_{L^2(\mathbb{R}^n)}(t)\,dt\right)\exp\left(C\int_0^T \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\,dt\right)$$

$\square$

One may rewrite (2.7) as

$$\sup_{t\in[0,T]}\|(\phi,\,\partial_t\phi)\|_{\dot{H}^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}(t) \le C\left(\|(\phi_0,\,\phi_1)\|_{\dot{H}^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} + \int_0^T \|F\|_{L^2(\mathbb{R}^n)}(t)\,dt\right)\exp\left(C\int_0^T \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\,dt\right) \tag{2.8}$$

Notice we're only controlling L^2 norms of spatial derivatives for ϕ, but not L^2 norm of ϕ itself. Yet a control on $\|\phi\|_{L^2(\mathbb{R}^n)}$ is indeed possible if we restrict to fixed time interval $[0,T]$.

Lemma 2.1.2 (Control on $\|\phi\|_{L^2(\mathbb{R}^n)}$). *Given assumptions as in Theorem 2.1.1, there exists $C(n) > 0$ s.t.*

$$\sup_{t\in[0,T]}\|\phi\|_{L^2(\mathbb{R}^n)}(t) \le C\left(\|\phi_0\|_{L^2(\mathbb{R}^n)} + \int_0^T \|\partial_t\phi\|_{L^2(\mathbb{R}^n)}(t)\,dt\right) \tag{2.9}$$

Proof. By assumption $(\phi,\,\partial_t\phi) \in L^\infty([0,T];H^1(\mathbb{R}^n)) \times L^\infty([0,T];L^2(\mathbb{R}^n))$. By Rademacher's ϕ is C^1 in time for a.e. $t \in [0,T]$. Using density of $C_0^\infty(\mathbb{R}^n)$ in $H^1(\mathbb{R}^n)$ and in $L^2(\mathbb{R}^n)$, one choose a sequence $\phi_n \in \mathrm{Lip}([0,T];C_0^\infty(\mathbb{R}^n))$ s.t. $\|\phi_n - \phi\|_{H^1(\mathbb{R}^n)}(t) \to 0$ and $\|\partial_t\phi_n - \partial_t\phi\|_{L^2(\mathbb{R}^n)}(t) \to 0$ for all $t \in [0,T]$. Hence the Fundamental Theorem of Calculus holds

$$\begin{aligned}
\phi_n(t) &= \phi_n(t) - \phi_n(0) + \phi_n(0)\\
&= \int_0^t \partial_t\phi_n(s)\,ds + \phi_n(0) \qquad \textit{for a.e. } x \in \mathbb{R}^n\\
\implies \|\phi_n\|_{L^2(\mathbb{R}^n)}(t) &= \left\|\int_0^t \partial_t\phi_n(s)\,ds + \phi_n(0)\right\|_{L^2(\mathbb{R}^n)}\\
&\le \int_0^t \|\partial_t\phi_n\|_{L^2(\mathbb{R}^n)}(s)\,ds + \|\phi_n(0)\|_{L^2(\mathbb{R}^n)}\\
\implies \|\phi\|_{L^2(\mathbb{R}^n)}(t) &\le C\left(\|\phi_0\|_{L^2(\mathbb{R}^n)} + \int_0^t \|\partial_t\phi\|_{L^2(\mathbb{R}^n)}(s)\,ds\right)\\
\sup_{t\in[0,T]}\|\phi\|_{L^2(\mathbb{R}^n)}(t) &\le C\left(\|\phi_0\|_{L^2(\mathbb{R}^n)} + \int_0^T \|\partial_t\phi\|_{L^2(\mathbb{R}^n)}(t)\,dt\right)
\end{aligned}$$

$\square$

Notice the RHS of (2.9) explodes to ∞ as one take $T \to \infty$.

Corollary 2.1.1 (A priori Energy Estimate for non-constant linear wave with $\|\phi\|_{L^2(\mathbb{R}^n)}$ control)**.** *Given assumptions as in Theorem 2.1.1, there exists $C(n) > 0$ s.t.*

$$\sup_{t\in[0,T]} \|(\phi,\, \partial_t\phi)\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}(t) \le C(1+T)\left(\|(\phi_0,\, \phi_1)\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} + \int_0^T \|F\|_{L^2(\mathbb{R}^n)}(t)\,dt\right) \times \exp\left(C\int_0^T \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\,dt\right) \tag{2.10}$$

Proof. Add up (2.8) and (2.9). Then notice we may control again using (2.8)

$$\int_0^T \|\partial_t\phi\|_{L^2(\mathbb{R}^n)}(t)\,dt \le CT \sup_{t\in[0,T]} \|\partial_t\phi\|_{L^2(\mathbb{R}^n)} \le CT\cdot(2.8)$$

□

Differentiating the equation (2.1), one may obtain control in higher order derivatives of ϕ in L^2.

Corollary 2.1.2 (A priori Energy Estimate for non-constant linear wave with higher derivatives)**.** *Fix $T \in I$. Let $k \ge 1$ be natural number. Assume the following regularities:*

- *$a \in L^\infty([0,T]\times\mathbb{R}^n)$ and $\partial a \in L^1([0,T]; H^{k-1}(\mathbb{R}^n))$.*
- *$F \in L^1([0,T]; H^{k-1}(\mathbb{R}^n))$.*
- *Initial data $(\phi_0,\, \phi_1) \in H^k(\mathbb{R}^n)\times H^{k-1}(\mathbb{R}^n)$.*

Then, if $(\phi, \partial_t\phi) \in L^\infty([0,T]; H^k(\mathbb{R}^n)) \times L^\infty([0,T]; H^{k-1}(\mathbb{R}^n))$ is solution to (2.3), there exists constant $C(n) > 0$ s.t.

$$\begin{aligned}&\sup_{t\in[0,T]} \|(\phi,\, \partial_t\phi)\|_{H^k(\mathbb{R}^n)\times H^{k-1}(\mathbb{R}^n)}(t)\\ &\le C(1+T)\exp\left(C\int_0^T \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\,dt\right)\times\\ &\left(\|(\phi_0,\, \phi_1)\|_{H^k(\mathbb{R}^n)\times H^{k-1}(\mathbb{R}^n)} + \int_0^T \|F\|_{H^{k-1}}(t)\,dt + C\sum_{0\le|\gamma|+|\delta|\le k-2}\int_0^T \left(\left\|\partial\partial_x^\gamma a\partial\partial_x^\delta\phi\right\|_{L^2}(t) + \left\|\partial_x^\gamma a\partial\partial\partial_x^\delta\phi\right\|_{L^2}(t)\right)\,dt\right)\end{aligned} \tag{2.11}$$

Proof. For any $\gamma \in \mathbb{N}^n$ with $|\gamma| = \sum_{i=1}^n \gamma_i \le k-1$, differentiate equation (2.1) using ∂_x^γ

$$\begin{aligned}\partial_\alpha\left(\partial_x^\gamma\left(a^{\alpha\beta}\partial_\beta\phi\right)\right) &= \partial_x^\gamma F\\ \partial_\alpha\left(\sum_{0\le|\delta|\le|\gamma|} \partial_x^{\gamma-\delta}a^{\alpha\beta}\partial_x^\delta\partial_\beta\phi\right) &= \partial_x^\gamma F\\ \partial_\alpha(a^{\alpha\beta}\partial_\beta(\partial_x^\gamma\phi)) + \sum_{0\le|\delta|\le|\gamma|-1} \partial_\alpha\left(\partial_x^{\gamma-\delta}a^{\alpha\beta}\partial_x^\delta\partial_\beta\phi\right) &= \partial_x^\gamma F\\ \partial_\alpha(a^{\alpha\beta}\partial_\beta(\partial_x^\gamma\phi)) &= \partial_x^\gamma F - \sum_{0\le|\delta|\le|\gamma|-1} \partial_\alpha\left(\partial_x^{\gamma-\delta}a^{\alpha\beta}\partial_x^\delta\partial_\beta\phi\right)\end{aligned}$$

viewing $\partial_x^\gamma\phi$ as solution, RHS as forcing, along with initial data

$$(\partial_x^\gamma\phi, \partial_t\partial_x^\gamma\phi)|_{t=0} = (\partial_x^\gamma\phi_0, \partial_x^\gamma\phi_1) \in H^{k-|\gamma|}(\mathbb{R}^n)\times H^{k-1-|\gamma|}(\mathbb{R}^n)$$

one apply Theorem 2.1.1 to obtain

$$\begin{aligned}&\sup_{t\in[0,T]} \|(\partial_x^\gamma\phi,\, \partial_t\partial_x^\gamma\phi)\|_{\dot{H}^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}(t)\\ &\le C\left(\|(\partial_x^\gamma\phi_0,\, \partial_x^\gamma\phi_1)\|_{\dot{H}^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} + \int_0^T \|\partial_x^\gamma F\|_{L^2(\mathbb{R}^n)}(t)\,dt + \sum_{0\le|\delta|\le|\gamma|-1}\int_0^T \left\|\partial_\alpha\left(\partial_x^{\gamma-\delta}a^{\alpha\beta}\partial_x^\delta\partial_\beta\phi\right)\right\|_{L^2(\mathbb{R}^n)}(t)\,dt\right)\\ &\times\exp\left(C\int_0^T \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\,dt\right)\end{aligned}$$

Now summing over all possible $|\gamma| \leq k-1$

$$\sum_{|\gamma|\leq k-1} \sup_{t\in[0,T]} \|(\partial_x^\gamma \phi,\, \partial_t\partial_x^\gamma \phi)\|_{\dot{H}^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}(t)$$

$$\leq C\left(\|(\phi_0,\,\phi_1)\|_{\dot{H}^k(\mathbb{R}^n)\times H^{k-1}(\mathbb{R}^n)} + \int_0^T \|F\|_{H^{k-1}(\mathbb{R}^n)}(t)\,dt + \sum_{0\leq|\gamma|+|\delta|\leq k-2}\sum_{\alpha,\beta=0}^n \int_0^T \left\|\partial_\alpha\left(\partial_x^\gamma a^{\alpha\beta}\,\partial_\beta\partial_x^\delta\phi\right)\right\|_{L^2(\mathbb{R}^n)}(t)\,dt\right)$$

$$\times \exp\left(C\int_0^T \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\,dt\right)$$

In particular for the mixed term, one needs to do a product rule so that

$$\sum_{0\leq|\gamma|+|\delta|\leq k-2}\int_0^T \left\|\partial_\alpha\left(\partial_x^\gamma a^{\alpha\beta}\,\partial_\beta\partial_x^\delta\phi\right)\right\|_{L^2(\mathbb{R}^n)}(t)\,dt$$

$$\leq \sum_{0\leq|\gamma|+|\delta|\leq k-2}\int_0^T \left\|\left(\partial_\alpha\partial_x^\gamma a^{\alpha\beta}\right)\partial_\beta\partial_x^\delta\phi\right\|_{L^2(\mathbb{R}^n)}(t)\,dt + \sum_{0\leq|\gamma|+|\delta|\leq k-2}\int_0^T \left\|\partial_x^\gamma a^{\alpha\beta}\left(\partial_\alpha\partial_\beta\partial_x^\delta\phi\right)\right\|_{L^2(\mathbb{R}^n)}(t)\,dt$$

$$= \int_0^T \sum_{0\leq|\gamma|+|\delta|\leq k-2}\left(\left\|\left(\partial_\alpha\partial_x^\gamma a^{\alpha\beta}\right)\partial_\beta\partial_x^\delta\phi\right\|_{L^2(\mathbb{R}^n)}(t) + \left\|\partial_x^\gamma a^{\alpha\beta}\left(\partial_\alpha\partial_\beta\partial_x^\delta\phi\right)\right\|_{L^2(\mathbb{R}^n)}(t)\right)dt$$

$$\leq \int_0^T \sum_{0\leq|\gamma|+|\delta|\leq k-2}\left(\left\|\partial\partial_x^\gamma a\partial\partial_x^\delta\phi\right\|_{L^2(\mathbb{R}^n)}(t) + \left\|\partial_x^\gamma a\partial\partial\partial_x^\delta\phi\right\|_{L^2(\mathbb{R}^n)}(t)\right)dt$$

where we use the short hand notation

$$\partial a = \sum_{|\eta|=1}\partial^\eta a = \partial_t a + \sum_{i=1}^n \partial_{x_i} a$$

Notice we're once again left with L^2 norm of ϕ. Hence add (2.9) to above and conclude as in (2.10). □

2.2 Existence of Solution

Consider Non-constant Coefficient Linear Wave Equation initial value problem (2.3) with assumptions on a (2.2) and that all derivatives of a are uniformly bounded on $[0,T]\times\mathbb{R}^n$. One wish to argue, given suitable function class assumptions on initial data (ϕ_0,ϕ_1), one has unique solution to (2.3) in the distributional sense, which can further be improved to the classical sense. For simplicity, let

$$\mathcal{L}\phi := \partial_\alpha\left(a^{\alpha\beta}\partial_\beta\phi\right)$$

Notice the formal adjoint of $\mathcal{L}$, denoted as $\mathcal{L}^*$, coincides with $\mathcal{L}$. To see this, for any $\phi,\,\psi\in C_0^\infty((0,T)\times\mathbb{R}^n)$.

$$\begin{aligned}\langle\mathcal{L}\phi,\psi\rangle &= \langle\partial_\alpha\left(a^{\alpha\beta}\partial_\beta\phi\right),\psi\rangle = (-1)^{1+n}\langle a^{\alpha\beta}\partial_\beta\phi,\partial_\alpha\psi\rangle = (-1)^{1+n}\langle\partial_\beta\phi,(a^T)^{\beta\alpha}\partial_\alpha\psi\rangle\\ &= (-1)^{2+2n}\langle\phi,\partial_\beta\left(a^{\beta\alpha}\partial_\alpha\psi\right)\rangle = \langle\phi,\partial_\beta\left(a^{\beta\alpha}\partial_\alpha\psi\right)\rangle = \langle\phi,\mathcal{L}^*\psi\rangle\end{aligned}$$

where we've used the symmetry of $a^{\alpha\beta}$.

Definition 2.2.1 (Weak Solution to (2.3))**.** *$\phi\in\mathcal{D}'((-\infty,T)\times\mathbb{R}^n)$ is a weak solution to (2.3) with initial data $(\phi_0,\,\phi_1)$ if for any $\psi\in C_0^\infty((-\infty,T)\times\mathbb{R}^n)$*

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n}\psi F\,dxdt &= \int_0^T\int_{\mathbb{R}^n}\mathcal{L}^*\psi\,\phi\,dxdt\\ &\quad -\int_{\mathbb{R}^n}\left(a^{00}\phi_1(x)+\sum_{i=1}^n a^{0i}\partial_{x_i}\phi_0(x)\right)\psi(0,x)\,dx + \int_{\mathbb{R}^n}\left(a^{00}\partial_t\psi+\sum_{i=1}^n a^{i0}\partial_{x_i}\psi\right)(0,x)\phi_0(x)\,dx\end{aligned}$$

Proof that $\phi\in C^2([0,T)\times\mathbb{R}^n)$ solution to (2.3) satisfies weak solution definition. For $\phi\in C^2([0,T)\times\mathbb{R}^n)$ as solution, for any $\psi\in C_0^\infty((-\infty,T)\times\mathbb{R}^n)$

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n}\psi F\,dxdt &= \int_0^T\int_{\mathbb{R}^n}\psi\mathcal{L}(\phi)\,dxdt = \int_0^T\int_{\mathbb{R}^n}\psi\,\partial_\alpha\left(a^{\alpha\beta}\partial_\beta\phi\right)\,dxdt\\ &= \int_0^T\int_{\mathbb{R}^n}\psi\,\partial_t\left(a^{00}\partial_t\phi\right)\,dxdt + \sum_{i=1}^n\int_0^T\int_{\mathbb{R}^n}\psi\,\left(\partial_t\left(a^{0i}\partial_{x_i}\phi\right)+\partial_{x_i}\left(a^{i0}\partial_t\phi\right)\right)\,dxdt\\ &\quad + \sum_{i,j=1}^n\int_0^T\int_{\mathbb{R}^n}\psi\partial_{x_i}\left(a^{ij}\partial_{x_j}\phi\right)\,dxdt =: \mathbf{I}+\mathbf{II}+\mathbf{III}\end{aligned}$$

Here

$$\begin{aligned}
\mathbf{I} &= \int_{\mathbb{R}^n} (\psi a^{00}\partial_t\phi)(T,x) - (\psi a^{00}\partial_t\phi)(0,x)\,dx - \int_0^T\int_{\mathbb{R}^n} \partial_t\psi a^{00}\partial_t\phi\,dxdt \\
&= -\int_{\mathbb{R}^n} (\psi a^{00}\partial_t\phi)(0,x)\,dx - \int_0^T\int_{\mathbb{R}^n} \partial_t\psi a^{00}\partial_t\phi\,dxdt \\
&= -\int_{\mathbb{R}^n} (\psi a^{00}\partial_t\phi)(0,x)\,dx + \int_0^T\int_{\mathbb{R}^n} \partial_t\left(a^{00}\partial_t\psi\right)\phi\,dxdt + \int_{\mathbb{R}^n} a^{00}\partial_t\psi\phi(0,x)\,dx \\
\mathbf{II} &= \sum_{i=1}^n \left(\int_{\mathbb{R}^n} (\psi a^{0i}\partial_{x_i}\phi)(T) - (\psi a^{0i}\partial_{x_i}\phi)(0)\,dx - \int_0^T\int_{\mathbb{R}^n} \partial_{x_i}\psi a^{0i}\partial_t\phi\,dxdt\right) - \sum_{i=1}^n\int_0^T\int_{\mathbb{R}^n} \partial_{x_i}\psi a^{i0}\partial_t\phi\,dxdt \\
&= -\sum_{i=1}^n \left(\int_{\mathbb{R}^n} (\psi a^{0i}\partial_{x_i}\phi)(0)\,dx + \int_0^T\int_{\mathbb{R}^n} \partial_t\psi a^{0i}\partial_{x_i}\phi\,dxdt\right) - \sum_{i=1}^n\int_0^T\int_{\mathbb{R}^n} \partial_{x_i}\psi a^{i0}\partial_t\phi\,dxdt \\
&= -\sum_{i=1}^n \left(\int_{\mathbb{R}^n} (\psi a^{0i}\partial_{x_i}\phi)(0)\,dx - \int_0^T\int_{\mathbb{R}^n} \partial_{x_i}\left(a^{0i}\partial_t\psi\right)\phi\,dxdt\right) \\
&\quad -\sum_{i=1}^n \left(\int_{\mathbb{R}^n} \partial_{x_i}\psi a^{i0}\phi(T,x) - \partial_{x_i}\psi a^{i0}\phi(0,x)\,dx - \int_0^T\int_{\mathbb{R}^n} \partial_t\left(a^{i0}\partial_{x_i}\psi\right)\phi\,dxdt\right) \\
&= -\sum_{i=1}^n \left(\int_{\mathbb{R}^n} (\psi a^{0i}\partial_{x_i}\phi)(0) - \partial_{x_i}\psi a^{i0}\phi(0,x)\,dx\right) + \sum_{i=1}^n\int_0^T\int_{\mathbb{R}^n} \left(\partial_t\left(a^{i0}\partial_{x_i}\psi\right)\phi + \partial_t\left(a^{i0}\partial_{x_i}\psi\right)\phi\right)\,dxdt \\
\mathbf{III} &= \sum_{i,j=1}^n \int_0^T\int_{\mathbb{R}^n} \partial_{x_i}\left(a^{ij}\partial_{x_j}\psi\right)\phi\,dxdt
\end{aligned}$$

Thus summing up gives

$$\begin{aligned}
\mathbf{I}+\mathbf{II}+\mathbf{III} &= \int_0^T\int_{\mathbb{R}^n} \partial_\alpha\left(a^{\alpha\beta}\partial_\beta\psi\right)\phi\,dxdt - \int_{\mathbb{R}^n}\left(a^{00}\partial_t\phi + \sum_{i=1}^n a^{0i}\partial_{x_i}\phi\right)(0,x)\psi(0,x)\,dx \\
&\quad + \int_{\mathbb{R}^n}\left(a^{00}\partial_t\psi + \sum_{i=1}^n a^{i0}\partial_{x_i}\psi\right)(0,x)\phi(0,x)\,dx \\
&= \int_0^T\int_{\mathbb{R}^n} \mathcal{L}^*\psi\,\phi\,dxdt \\
&\quad - \int_{\mathbb{R}^n}\left(a^{00}\phi_1(x) + \sum_{i=1}^n a^{0i}\partial_{x_i}\phi_0(x)\right)\psi(0,x)\,dx + \int_{\mathbb{R}^n}\left(a^{00}\partial_t\psi + \sum_{i=1}^n a^{i0}\partial_{x_i}\psi\right)(0,x)\phi_0(x)\,dx
\end{aligned}$$

using initial data $(\phi, \partial_t\phi)|_{t=0} = (\phi_0,\, \phi_1)$. □

One needs an essential lemma, which is essentially achieved by commutator estimate.

Lemma 2.2.1 (Commutator Estimate)**.** *For any $\gamma \in \mathbb{N}^n$, for some $C(\gamma,\, a) > 0$*

$$|[\mathcal{L},\, \partial_x^\gamma]\phi| := |\mathcal{L}(\partial_x^\gamma\phi) - \partial_x^\gamma\mathcal{L}(\phi)| \le C \sum_{0\le|\delta|\le|\gamma|+1} |\partial_x^\delta\phi| \tag{2.12}$$

Proof.

$$\begin{aligned}
[\mathcal{L},\, \partial_x^\gamma]\phi &= \partial_\alpha\left(a^{\alpha\beta}\partial_\beta\partial_x^\gamma\phi\right) - \partial_x^\gamma\left(\partial_\alpha\left(a^{\alpha\beta}\partial_\beta\phi\right)\right) \\
&= \partial_\alpha\left(a^{\alpha\beta}\partial_\beta\partial_x^\gamma\phi\right) - \partial_\alpha\left(\partial_x^\gamma\left(a^{\alpha\beta}\partial_\beta\phi\right)\right) \\
&= \partial_\alpha\left(a^{\alpha\beta}\partial_\beta\partial_x^\gamma\phi\right) - \partial_\alpha\left(\sum_{0\le|\delta|\le|\gamma|} \partial_x^{\gamma-\delta}a^{\alpha\beta}\partial_\beta\partial_x^\delta\phi\right) \\
&= -\partial_\alpha\left(\sum_{0\le|\delta|\le|\gamma|-1} \partial_x^{\gamma-\delta}a^{\alpha\beta}\partial_\beta\partial_x^\delta\phi\right)
\end{aligned}$$

Hence

$$|[\mathcal{L},\, \partial_x^\gamma]\phi| \leq C \left| \sum_{1\leq|\eta|\leq|\gamma|} \partial\partial_x^\eta a \right| \left| \sum_{0\leq|\delta|\leq|\gamma|-1} \partial\partial_x^\delta \phi \right|$$
$$\leq C \sum_{0\leq|\delta|\leq|\gamma|+1} |\partial_x^\delta \phi|$$

□

Our essential lemma bounds ψ in H^m norm using the integral in time of $\mathcal{L}^*\psi$ in H^{m-1} norm. This is highly nontrivial as $\mathcal{L}^*$ should in principle reduce regularity by order 2. Yet the commutator estimate saves us.

Lemma 2.2.2. *For any $m \in \mathbb{Z}$, there exists $C = C(m, T, a) > 0$ s.t. for any $\psi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n)$ and $0 \leq t \leq T$*

$$\|\psi\|_{H^m(\mathbb{R}^n)}(t) \leq C \int_t^T \|\mathcal{L}^*\psi\|_{H^{m-1}(\mathbb{R}^n)}(s)\, ds \tag{2.13}$$

Proof. For $m \geq 1$, since $\|\psi\|_{H^m(\mathbb{R}^n)}(T) = 0$, one apply higher derivative estimate (2.11) and Grönwall's (2.5) starting at initial time $t = T$ and solving backwards so that

$$\|\psi\|_{H^m(\mathbb{R}^n)}(t) \leq C \left(\int_t^T \|F\|_{H^{m-1}(\mathbb{R}^n)}(s)\, ds + C \int_t^T \sum_{|\gamma|+|\delta|\leq m-2} \left(\left\|\partial\partial_x^\gamma a \partial\partial_x^\delta \psi\right\|_{L^2(\mathbb{R}^n)}(t) + \left\|\partial_x^\gamma a \partial\partial\partial_x^\delta \psi\right\|_{L^2(\mathbb{R}^n)}(t) \right) \right)$$
$$\leq C \left(\int_t^T \|F\|_{H^{m-1}(\mathbb{R}^n)}(s)\, ds + C \sup_{s\in[0,t]} \|\partial a\|_{H^{m-1}(\mathbb{R}^n)} \int_t^T \sum_{|\delta|\leq m-2} \left(\left\|\partial\partial_x^\delta \psi\right\|_{L^2}(t) + \left\|\partial\partial\partial_x^\delta \psi\right\|_{L^2}(t) \right) \right)$$
$$\leq C \left(\int_t^T \|F\|_{H^{m-1}(\mathbb{R}^n)}(s)\, ds + C \sup_{s\in[0,t]} \|\partial a\|_{H^{m-1}(\mathbb{R}^n)}(s) \int_t^T \|\psi\|_{H^m(\mathbb{R}^n)}(s)\, ds \right)$$
$$\leq C \int_t^T \|F\|_{H^{m-1}(\mathbb{R}^n)}(s)\, ds \exp\left((T-t) \sup_{s\in[0,t]} C \|\partial a\|_{H^{m-1}(\mathbb{R}^n)}(s) \right)$$
$$\leq C(m, T, a) \int_t^T \|F\|_{H^{m-1}(\mathbb{R}^n)}(s)\, ds = C \int_t^T \|\mathcal{L}^*\psi\|_{H^{m-1}(\mathbb{R}^n)}(s)\, ds$$

To deal with $m \leq 0$, one use induction. Assume the result holds for $m + 2$, then we wish to prove (2.13) for m. To do so, define

$$\Psi(t, x) := (1 - \Delta)^{-1}\psi(t, x) := \mathcal{F}^{-1}\left((1 + 4\pi|\xi|^2)^{-1}\hat{\psi}(\xi) \right)(t, x)$$

where the Fourier Transform is only w.r.t. spatial variable. Then for some $C = C(m, a) > 0$ varying from line to line

$$\|\mathcal{L}^*\Psi\|_{H^{m+1}(\mathbb{R}^n)}(t) \leq C \|(1 - \Delta)\mathcal{L}^*\Psi\|_{H^{m-1}(\mathbb{R}^n)}(t)$$
$$= C \|\mathcal{L}^*\psi - \mathcal{L}^*\psi + (1 - \Delta)\mathcal{L}^*\Psi\|_{H^{m-1}(\mathbb{R}^n)}(t)$$
$$\leq C \|\mathcal{L}^*\psi\|_{H^{m-1}(\mathbb{R}^n)}(t) + C \|(1 - \Delta)\mathcal{L}^*\Psi - \mathcal{L}^*\psi\|_{H^{m-1}(\mathbb{R}^n)}(t)$$
$$= C \|\mathcal{L}^*\psi\|_{H^{m-1}(\mathbb{R}^n)}(t) + C \|(1 - \Delta)\mathcal{L}^*\Psi - \mathcal{L}^*(1 - \Delta)\Psi\|_{H^{m-1}(\mathbb{R}^n)}(t)$$
$$= C \|\mathcal{L}^*\psi\|_{H^{m-1}(\mathbb{R}^n)}(t) + C \|[1 - \Delta,\, \mathcal{L}^*]\Psi\|_{H^{m-1}(\mathbb{R}^n)}(t)$$

Now one may apply Commutator Estimate (2.12)

$$\|[1 - \Delta,\, \mathcal{L}^*]\Psi\|_{H^{m-1}(\mathbb{R}^n)}(t) \leq C \|\Psi\|_{H^{m+2}(\mathbb{R}^n)}(t)$$

Hence

$$\|\mathcal{L}^*\Psi\|_{H^{m+1}(\mathbb{R}^n)}(t) \leq C \left(\|\mathcal{L}^*\psi\|_{H^{m-1}(\mathbb{R}^n)}(t) + \|\Psi\|_{H^{m+2}(\mathbb{R}^n)}(t) \right)$$

But by induction hypothesis, one may induct using $\Phi \in C_0^\infty$ and then Grönwall's (2.5)

$$\|\Psi\|_{H^{m+2}(\mathbb{R}^n)}(t) \leq C \int_t^T \|\mathcal{L}^*\Psi\|_{H^{m+1}(\mathbb{R}^n)}(s)\, ds$$
$$\leq C \int_t^T \|\mathcal{L}^*\psi\|_{H^{m-1}(\mathbb{R}^n)}(s) + \|\Psi\|_{H^{m+2}(\mathbb{R}^n)}(s)\, ds$$
$$\implies \|\Psi\|_{H^{m+2}(\mathbb{R}^n)}(t) \leq C \int_t^T \|\mathcal{L}^*\psi\|_{H^{m-1}(\mathbb{R}^n)}(s)\, ds \exp\left(\int_t^T 1\, ds\right)$$
$$\leq C(T) \int_t^T \|\mathcal{L}^*\psi\|_{H^{m-1}(\mathbb{R}^n)}(s)\, ds$$

Hence the constant $C = C(m, T, a) > 0$, and

$$\|\psi\|_{H^m(\mathbb{R}^n)}(t) = \|(1-\Delta)\Psi\|_{H^m(\mathbb{R}^n)}(t) \le C\,\|\Psi\|_{H^{m+2}(\mathbb{R}^n)}(t) \le C\int_t^T \|\mathcal{L}^*\psi\|_{H^{m-1}(\mathbb{R}^n)}(s)\,ds$$

By induction, we may truncate for all $m \in \mathbb{Z}$. □

To deal with Existence, one needs the Hahn-Banach theorem.

Lemma 2.2.3 (Hahn-Banach)**.** *X normed vector space and $Y \subset X$ subspace that inherits the same norm $\|y\|_Y = \|y\|_X$ for any $y \in Y$. If $f \in Y^*$ is bounded linear functional on Y, then there exists an extension $\tilde{f} \in X^*$ s.t.*

$$\tilde{f}\Big|_Y = f \qquad \textit{and} \qquad \left\|\tilde{f}\right\|_{X^*} = \|f\|_Y$$

Theorem 2.2.1 (Existence and Uniqueness of Solutions to (2.3))**.** *Fix $T \in I$. Let $k \ge 1$ be natural number. Assume the following regularities:*

- *$a \in L^\infty([0,T]\times\mathbb{R}^n)$ and $\partial a \in L^\infty([0,T]; H^{k-1}(\mathbb{R}^n))$.*
- *$F \in L^1([0,T]; H^{k-1}(\mathbb{R}^n))$.*
- *Initial data $(\phi_0,\, \phi_1) \in H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)$.*

Then there exists unique solution $(\phi,\, \partial_t\phi) \in C\left([0,T]; H^k(\mathbb{R}^n)\right) \times C\left([0,T]; H^{k-1}(\mathbb{R}^n)\right)$ to (2.3).

Proof. (i) One first see uniqueness. If $(\psi_1,\, \partial_t\psi_1)$ and $(\psi_2,\, \partial_t\psi_2)$ are two sets of solution prescribing the same initial data, then $(\psi_1 - \psi_2, \partial_t(\psi_1-\psi_2))$ solves (2.3) with $F = 0$ and initial data 0. Hence applying higher derivatives estimates (2.11) and Grönwall's (2.5), to achieve

$$\|\psi_1 - \psi_2\|_{H^k(\mathbb{R}^n)}(t) \le C\int_0^t \|\mathcal{L}(\psi_1-\psi_2)\|_{H^{k-1}(\mathbb{R}^n)}(s)\,ds = 0$$

Hence $\psi_1 = \psi_2$ a.e., and we have uniqueness.

(ii) First assume $(\phi_0,\, \phi_1) = (0,0)$. Consider the map Φ that maps from $\mathcal{L}^*(C_0^\infty((-\infty,T)\times\mathbb{R}^n))$ to $\mathbb{R}$ s.t. for any $\psi \in C_0^\infty((-\infty,T)\times\mathbb{R}^n)$

$$\Phi : \mathcal{L}^*\psi \mapsto \int_0^T\int_{\mathbb{R}^n} \psi F\,dxdt =: \langle F,\, \psi\rangle$$

Due to uniqueness we've just shown, the map Φ is well-defined. Now using $F \in L^1([0,T]; H^{k-1}(\mathbb{R}^n))$ and Lemma (2.13)

$$\begin{aligned}
|\langle F,\psi\rangle| &= |\int_0^T\int_{\mathbb{R}^n} \psi F\,dxdt| \\
&\le C\int_0^T \|F\|_{H^{k-1}(\mathbb{R}^n)}(t)\,dt \sup_{t\in[0,T]} \|\psi\|_{H^{-k+1}(\mathbb{R}^n)}(t) \\
&\le C\int_0^T \|F\|_{H^{k-1}(\mathbb{R}^n)}(t)\,dt \int_0^T \|\mathcal{L}^*\psi\|_{H^{-k}(\mathbb{R}^n)}(t)\,dt
\end{aligned}$$

Apply Hahn-Banach Lemma 2.2.3 to $Y = \mathcal{L}^*(C_0^\infty((-\infty,T)\times\mathbb{R}^n)) \subset C_0^\infty((-\infty,T)\times\mathbb{R}^n)$ and $X = L^1((-\infty,T); H^{-k}(\mathbb{R}^n))$ with bounded linear functional $f := \Phi \in Y^*$. Then there exists a bounded linear functional

$$\phi \in X^* = \left(L^1((-\infty,T); H^{-k}(\mathbb{R}^n))\right)^* = L^\infty((-\infty,T); H^k(\mathbb{R}^n))$$

that extends Φ, i.e. for any $\psi \in C_0^\infty((-\infty,T)\times\mathbb{R}^n)$

$$\langle F,\psi\rangle = \Phi(\mathcal{L}^*\psi) = \langle \phi, \mathcal{L}^*\psi\rangle$$

Note $\phi(t) = 0$ for $t < 0$ necessarily, otherwise RHS might be nonzero for ψ supported at negative time intervals. Finally, since we're assuming $(\phi_0,\, \phi_1) = (0,0)$, according to definition 2.2.1 $\mathcal{L}\phi = F$ in the sense of distributions.

(iii) One wish to upgrade regularity of ϕ in time to agree with initial condition. To do so, we need Schauder theory from Elliptic Regularity. Since $\mathcal{L}\phi = F$

$$\begin{aligned}
\partial_\alpha\left(a^{\alpha\beta}\partial_\beta\phi\right) &= F \\
\sum_{i,j=1}^n \partial_{x_i}\left(a^{ij}\partial_{x_j}\phi\right) &= F - \sum_{i=1}^n \left(\partial_t(a^{0i}\partial_{x_i}\phi) + \partial_{x_i}(a^{i0}\partial_t\phi)\right) - \partial_t(a^{00}\partial_t\phi)
\end{aligned}$$

due to constraint on a (2.2), LHS is uniformly elliptic. Hence Schauder Theory tells us the force in RHS and the equation in LHS lie in the same function space. On LHS, $\partial_{x_i x_j}\phi \in H^{k-2}(\mathbb{R}^n)$, hence RHS $\partial_{x_i}(\partial_t\phi) \in H^{k-2}(\mathbb{R}^n)$, thus $\partial_t\phi \in H^{k-1}(\mathbb{R}^n)$. Moreover, $\partial_t^2\phi \in H^{k-2}$, implying $\partial_t\phi$ is Lipschitz in time. By Rademacher's theorem, a Lipschitz function is C^1 a.e., so $\partial_t\phi$ is C^1 a.e. This further implies $\phi \in C^1$. Thus

$$\phi \in C\left([0,T]; H^k(\mathbb{R}^n)\right) \cap C^1\left([0,T]; H^{k-1}(\mathbb{R}^n)\right)$$

Then necessarily $(\phi,\, \partial_t\phi)|_{t=0} = (0,0)$.

(iv) For general $(\phi_0,\, \phi_1) \in H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)$. First assume $(\phi_0,\, \phi_1) \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$ smooth and compactly supported. Then define

$$u(t,x) := \phi_0(x) + t\phi_1(x)$$

Hence $(u,\, \partial_t u)|_{t=0} = (\phi_0,\, \phi_1)$ satisfied Cauchy Data. Now let η solve

$$\begin{cases} \mathcal{L}\eta = F - \mathcal{L}u \\ (\eta,\, \partial\eta)|_{t=0} = (0,0) \end{cases}$$

According to our previous argument, $\eta \in C\left([0,T]; H^k(\mathbb{R}^n)\right) \cap C^1\left([0,T]; H^{k-1}(\mathbb{R}^n)\right)$. Hence via superposition, $\phi := u + \eta$ solves (2.3) with $(\phi_0,\, \phi_1) \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$. For $(\phi_0,\, \phi_1) \in H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)$, choose $(\phi_0^\varepsilon,\, \phi_1^\varepsilon) \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$ s.t.

$$\|\phi_0 - \phi_0^\varepsilon\|_{H^k(\mathbb{R}^n)} \to 0 \qquad \textit{and} \qquad \|\phi_1 - \phi_1^\varepsilon\|_{H^{k-1}(\mathbb{R}^n)} \to 0$$

and similarly $F^\varepsilon \in C_0^\infty((-\infty,T) \times \mathbb{R}^n)$ s.t.

$$\|F - F^\varepsilon\|_{L^1([0,T];H^{k-1}(\mathbb{R}^n))} \to 0$$

Define $u^\varepsilon := \phi_0^\varepsilon(x) + t\phi_1^\varepsilon(x)$ and the corresponding η^ε solutions to

$$\begin{cases} \mathcal{L}\eta^\varepsilon = F^\varepsilon - \mathcal{L}u^\varepsilon \\ (\eta^\varepsilon,\, \partial\eta^\varepsilon)|_{t=0} = (0,0) \end{cases}$$

Thus $\{u^\varepsilon + \eta^\varepsilon\}$ is a Cauchy sequence in $C\left([0,T]; H^k(\mathbb{R}^n)\right) \cap C^1\left([0,T]; H^{k-1}(\mathbb{R}^n)\right)$ by Lemma (2.13). Hence there exists unique limit $\phi = \lim_{\varepsilon\to 0} u^\varepsilon + \eta^\varepsilon$ in the same function class.

□

Indeed, if the initial data $(\phi_0,\, \phi_1)$ are smooth enough, via Sobolev Embedding, one recover classical solution.

Lemma 2.2.4 (Sobolev Embedding). *For any $s > \frac{n}{2}$, there exists $C = C(n,s) > 0$ s.t.*

$$\|\phi\|_{L^\infty(\mathbb{R}^n)} \le C\, \|\phi\|_{H^s(\mathbb{R}^n)} \qquad \forall\ \phi \in H^s(\mathbb{R}^n) \tag{2.14}$$

Corollary 2.2.1. *For $(\phi_0,\, \phi_1) \in \bigcap_{k=1}^\infty H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)$ and a, F as in assumptions from Theorem 2.2.1, there exists*

$$(\phi,\, \partial_t\phi) \in C\left([0,T]; C^\infty(\mathbb{R}^n)\right) \times C\left([0,T]; C^\infty(\mathbb{R}^n)\right)$$

to (2.3).

Chapter 3

Quasi-linear Wave Equations

We consider the quasi-linear wave equations for $\phi : I \times \mathbb{R}^n \subset \mathbb{R}^{1+n} \to \mathbb{R}$ where $0 \in I$

$$\partial_\alpha (a^{\alpha\beta}(\phi)\partial_\beta \phi) = \sum_{\alpha=0}^{n} \partial_\alpha \left(\sum_{\beta=0}^{n} a^{\alpha\beta}(\phi)\partial_\beta \phi \right) = F(\phi,\, \partial\phi) \tag{3.1}$$

with conditions

- a is smooth, symmetric $(1+n) \times (1+n)$ matrix on $\mathbb{R}$ with values in $\mathbb{R}$ s.t.

$$\sum_{\alpha,\beta=0}^{n} |a^{\alpha\beta} - m^{\alpha\beta}| < \frac{1}{10} \tag{3.2}$$

  for constant $(1+n) \times (1+n)$ Minkowski matrix $(m^{\alpha\beta}) = diag(1, -1, \cdots, -1)$. Also assume

$$a(0) = 0$$

- $F : \mathbb{R} \times \mathbb{R}^{1+n} \to \mathbb{R}$ is smooth function s.t.

$$F(0, 0) = 0$$

- Due to smoothness of a and F, we assume bounds for any $N \in \mathbb{N}$ and $A > 0$ s.t.

$$\sum_{\alpha,\beta=0}^{n} \sum_{|\gamma| \le N} \sup_{|x| \le A} |\partial_x^\gamma (a^{\alpha,\beta})|(x) \le C_{A,N} \tag{3.3}$$

$$\sum_{|\gamma| \le N} \sup_{|x|, \|p\| \le A} |\partial_{x,p}^\gamma F|(x, p) \le C_{A,N} \tag{3.4}$$

We're interested in Initial Value Problem with initial data $(\phi_0, \phi_1) : \mathbb{R}^n \to \mathbb{R}^2$

$$\begin{cases} \partial_\alpha (a^{\alpha\beta}(\phi)\partial_\beta \phi) = F(\phi,\, \partial\phi) \\ (\phi,\, \partial_t \phi)|_{t=0} = (\phi_0,\, \phi_1) \end{cases} \tag{3.5}$$

3.1 Hadamard Well-Posedness

3.1.1 Existence of Local-in-time Solution

In this subsection we prove the Local Existence Theorem.

Theorem 3.1.1 (Local Existence)**.** *Given smooth a and F as in (3.2) to (3.4). If*

$$(\phi_0,\, \phi_1) \in H^{n+2}(\mathbb{R}^n) \times H^{n+1}(\mathbb{R}^n)$$

Then there exists

$$T = T(\|\phi_0\|_{H^{n+2}},\, \|\phi_1\|_{H^{n+1}},\, n,\, a,\, F) > 0$$

and solution

$$(\phi, \partial_t \phi) \in L^\infty([0, T]; H^{n+2}(\mathbb{R}^n)) \times L^\infty([0, T]; H^{n+1}(\mathbb{R}^n))$$

to (3.5).

Remark 3.1.1. *By Sobolev Embedding (2.14), since $n+1 > \frac{n}{2}$, $(\phi, \partial_t\phi)$ are in fact L^∞, hence they're solutions to (3.5) of the Classical sense.*

Proof. To show existence, we use Picard's Iteration. By density of $\mathcal{S}$ in Sobolev Spaces over $\mathbb{R}^n$, it suffices to assume $(\phi_0,\, \phi_1) \in \mathcal{S}\times\mathcal{S}$. We define a sequence of smooth functions $\phi^{(i)}$ for $i \geq 1$ s.t.

- $\phi^{(1)} = 0$.
- $\phi^{(i)}$ for $i \geq 2$ is defined iteratively as the unique solution to

$$\begin{cases} \partial_\alpha\left(a^{\alpha\beta}(\phi^{(i-1)})\partial_\beta\phi^{(i)}\right) = F(\phi^{(i-1)}, \partial\phi^{(i-1)}) \\ (\phi^{(i)},\, \partial_t\phi^{(i)})\big|_{t=0} = (\phi_0,\, \phi_1) \end{cases} \tag{3.6}$$

  Indeed the solution $(\phi^{(i)},\, \partial_t\phi^{(i)})$ exists uniquely due to Theorem 2.2.1 and since $(\phi_0,\, \phi_1) \in \mathcal{S}\times\mathcal{S}$, using Corollary 2.2.1, the solution $(\phi^{(i)}, \partial_t\phi^{(i)})$ are smooth in space.

The argument strives to show 2 Lemmas describing certain properties of $(\phi^{(i)}, \partial_t\phi^{(i)})$.

Lemma 3.1.1 (Uniform Boundedness in $H^{n+2}\times H^{n+1}$)**.** *For*

$$T = T(\|\phi_0\|_{H^{n+2}},\, \|\phi_1\|_{H^{n+1}},\, n,\, a,\, F) > 0$$

sufficiently small, the sequence $(\phi^{(i)}, \partial_t\phi^{(i)})$ is uniformly bounded in $L^\infty([0,T]; H^{n+2}(\mathbb{R}^n))\times L^\infty([0,T]; H^{n+1}(\mathbb{R}^n))$, i.e. for some $A = A(\|\phi_0\|_{H^{n+2}},\, \|\phi_1\|_{H^{n+1}},\, n) > 0$, we have bound uniformly in i

$$\left\|(\phi^{(i)}, \partial_t\phi^{(i)})\right\|_{L^\infty([0,T];H^{n+2}(\mathbb{R}^n))\times L^\infty([0,T];H^{n+1}(\mathbb{R}^n))} \leq A \tag{3.7}$$

Lemma 3.1.2 (Cauchy Sequence in $H^1\times L^2$)**.** *For (rechosen to be smaller if necessary)*

$$T = T(\|\phi_0\|_{H^{n+2}},\, \|\phi_1\|_{H^{n+1}},\, n,\, a,\, F) > 0$$

sufficiently small, the sequence $(\phi^{(i)}, \partial_t\phi^{(i)})$ Cauchy in $L^\infty([0,T]; H^1(\mathbb{R}^n))\times L^\infty([0,T]; L^2(\mathbb{R}^n))$.

Assume for now we've shown Lemma 3.1.1 and 3.1.2 for some common $T > 0$. Then

- Since $(\phi^{(i)}, \partial_t\phi^{(i)})$ is Cauchy in $L^\infty([0,T]; H^1(\mathbb{R}^n))\times L^\infty([0,T]; L^2(\mathbb{R}^n))$ Banach Space, there exists limiting function
$$(\phi,\, \partial_t\phi) \in L^\infty([0,T]; H^1(\mathbb{R}^n))\times L^\infty([0,T]; L^2(\mathbb{R}^n))$$
- Since $(\phi^{(i)}, \partial_t\phi^{(i)})$ is uniformly bounded in $L^\infty([0,T]; H^{n+2}(\mathbb{R}^n))\times L^\infty([0,T]; H^{n+1}(\mathbb{R}^n))$, then for a.e. $t\in[0,T]$, the sequence
$$\left\|(\phi^{(i)}, \partial_t\phi^{(i)})\right\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)}(t) \leq A < \infty$$
  is uniformly bounded in i. Since $H^{n+2}\times H^{n+1}$ are reflexive Banach Space, by Banach-Alaoglu, upon passing to subsequence there exists a weak limit in $H^{n+2}\times H^{n+1}$ for a.e. $t\in[0,T]$.
- By uniqueness of limits, the 2 limits at t agree for a.e. $t\in[0,T]$. Hence the limiting solution
$$(\phi, \partial_t\phi) \in L^\infty([0,T]; H^{n+2}(\mathbb{R}^n))\times L^\infty([0,T]; H^{n+1}(\mathbb{R}^n))$$

In fact, the solution $(\phi, \partial_t\phi)$ is unique. We leave the uniqueness as a consequence of the next subsection for continuity dependence on initial data. □

In what follows, we respectively prove Lemma 3.1.1 and Lemma 3.1.2.

Proof of Lemma 3.1.1

To prove Lemma 3.1.1, we use Induction and Bootstrap Method. First of all, there exists $A > 0$ and $T > 0$ independent of i s.t.

$$0 = \left\|(\phi^{(1)}, \partial_t\phi^{(1)}\right\|_{L^\infty([0,T];H^{n+2}(\mathbb{R}^n))\times L^\infty([0,T];H^{n+1}(\mathbb{R}^n))} < A$$

as base case. Then assume for induction that we've shown for $i\geq 2$ that

$$\left\|(\phi^{(i-1)}, \partial_t\phi^{(i-1)})\right\|_{L^\infty([0,T];H^{n+2}(\mathbb{R}^n))\times L^\infty([0,T];H^{n+1}(\mathbb{R}^n))} \leq A \tag{3.8}$$

Then of course, if we're able to prove for the next iteration along with the same bound

$$\left\|(\phi^{(i)}, \partial_t\phi^{(i)})\right\|_{L^\infty([0,T];H^{n+2}(\mathbb{R}^n))\times L^\infty([0,T];H^{n+1}(\mathbb{R}^n))} \leq A \tag{3.9}$$

we're done. Also, since the solutions $(\phi^{(i)}, \partial_t\phi^{(i)})$ are lies in $\bigcap_{k=1}^{\infty} C\left([0,T]; H^k(\mathbb{R}^n)\right) \times C\left([0,T]; H^{k-1}(\mathbb{R}^n)\right)$ due to Theorem 2.2.1, we at least know they're bounded at ith iteration. Then we're eligible to make the following Bootstrap assumption

$$\left\|(\phi^{(i)}, \partial_t\phi^{(i)})\right\|_{L^\infty([0,T];H^{n+2}(\mathbb{R}^n))\times L^\infty([0,T];H^{n+1}(\mathbb{R}^n))} \leq 4A \tag{3.10}$$

and wish to improve the bound to A. Hence we make the assumptions

- Inductive Assumption (3.8).
- Bootstrap Assumption (3.10).

and wish to show for (3.9). In particular, if we're able to apply (2.11) to ith iteration with $k = n+2$ and denote

$$\mathbf{I} = \int_0^T \left\|F(\phi^{(i-1)}, \partial\phi^{(i-1)})\right\|_{H^{n+1}(\mathbb{R}^n)} (t)\, dt \tag{3.11}$$

$$\mathbf{II} = \sum_{0\leq|\gamma|+|\delta|\leq n} \int_0^T \left(\left\|\partial\partial_x^\gamma a(\phi^{(i-1)})\partial\partial_x^\delta\phi^{(i)}\right\|_{L^2}(t) + \left\|\partial_x^\gamma a(\phi^{(i-1)})\partial\partial\partial_x^\delta\phi^{(i)}\right\|_{L^2}(t)\right) dt \tag{3.12}$$

$$\mathbf{III} = \int_0^T \left\|\partial a(\phi^{(i-1)})\right\|_{L^\infty(\mathbb{R}^n)} (t)\, dt \tag{3.13}$$

then we have

$$\sup_{t\in[0,T]} \left\|(\phi^{(i)},\, \partial_t\phi^{(i)}\right\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)} (t) \leq C(1+T)\left(\|(\phi_0,\, \phi_1)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)} + \mathbf{I} + C\mathbf{II}\right) \times \exp\left(C\mathbf{III}\right)$$

To ensure we're eligible to bound as above, we make use of the following Claim

Lemma 3.1.3 (Bounds on **I**, **II** and **III**)**.** *There exists $B = B(A, n, a, F) > 0$ s.t. for any $t \in [0,T]$*

$$\sum_{0\leq|\alpha|\leq n+1} \left\|\partial_x^\alpha F(\phi^{(i-1)}, \partial\phi^{(i-1)})\right\|_{L^2(\mathbb{R}^n)} (t) \leq B \tag{3.14}$$

$$\sum_{0\leq|\gamma|+|\delta|\leq n}\sum_{\alpha,\beta=0}^{n} \left(\left\|\partial_\alpha\partial_x^\gamma a^{\alpha\beta}(\phi^{(i-1)})\partial_\beta\partial_x^\delta\phi^{(i)}\right\|_{L^2}(t) + \left\|\partial_x^\gamma a^{\alpha\beta}(\phi^{(i-1)})\partial_\alpha\partial_\beta\partial_x^\delta\phi^{(i)}\right\|_{L^2}(t)\right) \leq B \tag{3.15}$$

$$\sum_{|\gamma|=1} \left\|\partial^\gamma a(\phi^{(i-1)})\right\|_{L^\infty(\mathbb{R}^n)} (t) \leq B \tag{3.16}$$

Assume Lemma 3.1.3 holds, then

$$\sup_{t\in[0,T]} \left\|(\phi^{(i)},\, \partial_t\phi^{(i)}\right\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)} (t) \leq C(1+T)\left(\|(\phi_0,\, \phi_1)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)} + BT + CBT\right)\exp(CBT)$$

$$\leq C\left((1+T)\,\|(\phi_0,\, \phi_1)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)} + B(1+C)T + B(1+C)T^2\right)\exp(CBT) \tag{3.17}$$

WLOG assume the universal constant $C \geq 2$. The point is: While B may be large depending on A, we may choose T to be sufficiently small depending on A, B, C, ϕ_0, ϕ_1. We choose $T > 0$ small s.t.

- $T\,\|(\phi_0,\, \phi_1)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)} + B(1+C)T + B(1+C)T^2 \leq \|(\phi_0,\, \phi_1)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)}$
- $\exp(CBT) \leq 2$

Hence we have

$$\left\|(\phi^{(i)},\, \partial_t\phi^{(i)}\right\|_{L^\infty([0,T];H^{n+2}(\mathbb{R}^n))\times L^\infty([0,T];H^{n+1}(\mathbb{R}^n))} \leq 4C\,\|(\phi_0,\, \phi_1)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)}$$

Now we define

$$A := 4C\,\|(\phi_0,\, \phi_1)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)}$$

and indeed we've shown that for ith iteration (3.9) holds. In particular

$$\begin{aligned} T &= T(A, B, C, \|\phi_0\|_{H^{n+2}}, \|\phi_1\|_{H^{n+1}}) = T(B, C, \|\phi_0\|_{H^{n+2}}, \|\phi_1\|_{H^{n+1}}) \\ &= T(C, n, a, F, \|\phi_0\|_{H^{n+2}}, \|\phi_1\|_{H^{n+1}}) = T(\|\phi_0\|_{H^{n+2}}, \|\phi_1\|_{H^{n+1}}, n, a, F) > 0 \end{aligned}$$

Now we need to prove the Claim.

Proof of Lemma 3.1.3. We illustrate (3.14), then sketch (3.15) and (3.16).

(i) We unpack using product rule and use Bound on F (3.4)

$$\sum_{0\leq|\alpha|\leq n+1}\left\|\partial_x^\alpha F(\phi^{(i-1)},\partial\phi^{(i-1)})\right\|_{L^2(\mathbb{R}^n)}(t)$$
$$\leq C\left(\sum_{0\leq|\alpha|\leq n+1}\left\|\partial\partial_x^\alpha\phi^{(i-1)}\right\|_{L^2}(t)+\sum_{0\leq|\alpha_1|+|\alpha_2|\leq n+1}\left\|\partial\partial_x^{\alpha_1}\phi^{(i-1)}\partial\partial_x^{\alpha_2}\phi^{(i-1)}\right\|_{L^2}(t)+\text{cubic}+\cdots+(n+1)\,\text{order}\right)$$

Consider the quadratic term. Since $0\leq|\alpha_1|+|\alpha_2|\leq n+1$, we know either $|\alpha_1|$ or $|\alpha_2|$ is smaller than or equal to $\frac{n+1}{2}$. Assume WLOG $|\alpha_1|\leq\frac{n+1}{2}$. We know that using Sobolev Inequality (2.14) we may embed $H^{\frac{n}{2}}$ into L^∞. Since $\frac{n+1}{2}+\frac{n}{2}=\frac{2n+1}{2}<n+1$, we may control the L^∞ norm of terms with derivatives order less than or equal to $\frac{n+1}{2}$ using $H^{\frac{n}{2}}$ norm, and do not exceed H^{n+1}. In particular

$$\begin{aligned}\sum_{0\leq|\alpha_1|\leq\frac{n+1}{2}}\left\|\partial\partial_x^{\alpha_1}\phi^{(i-1)}\right\|_{L^\infty}(t)&\leq C(n)\sum_{0\leq|\alpha_1|\leq\frac{n+1}{2}}\left\|\partial\partial_x^{\alpha_1}\phi^{(i-1)}\right\|_{H^{\frac{n}{2}}}(t)\\&\leq C(n)\sum_{0\leq|\alpha_1|\leq\frac{n+1}{2}}\sum_{0\leq|\alpha_2|\leq\frac{n}{2}}\left\|\partial\partial_x^{\alpha_2}\partial_x^{\alpha_1}\phi^{(i-1)}\right\|_{L^2}(t)\\&=C(n)\sum_{0\leq|\alpha|\leq\frac{2n+1}{2}}\left\|\partial\partial_x^{\alpha}\phi^{(i-1)}\right\|_{L^2}(t)\\&\leq C(n)\sum_{0\leq|\alpha|\leq n+1}\left\|\partial\partial_x^{\alpha}\phi^{(i-1)}\right\|_{L^2}(t)\end{aligned}$$

Hence using the trivial bound

$$\begin{aligned}&\sum_{0\leq|\alpha_1|+|\alpha_2|\leq n+1}\left\|\partial\partial_x^{\alpha_1}\phi^{(i-1)}\partial\partial_x^{\alpha_2}\phi^{(i-1)}\right\|_{L^2}(t)\leq\sum_{0\leq|\alpha_1|+|\alpha_2|\leq n+1}\left\|\partial\partial_x^{\alpha_1}\phi^{(i-1)}\right\|_{L^\infty}(t)\left\|\partial\partial_x^{\alpha_2}\phi^{(i-1)}\right\|_{L^2}(t)\\&\leq\left(\sum_{0\leq|\alpha_1|\leq\frac{n+1}{2}}\left\|\partial\partial_x^{\alpha_1}\phi^{(i-1)}\right\|_{L^\infty}(t)\right)\left(\sum_{0\leq|\alpha_2|\leq n+1}\left\|\partial\partial_x^{\alpha_2}\phi^{(i-1)}\right\|_{L^2}(t)\right)\\&\leq C(n)\left(\sum_{0\leq|\alpha|\leq n+1}\left\|\partial\partial_x^{\alpha}\phi^{(i-1)}\right\|_{L^2}(t)\right)\left(\sum_{0\leq|\alpha_2|\leq n+1}\left\|\partial\partial_x^{\alpha_2}\phi^{(i-1)}\right\|_{L^2}(t)\right)\\&=C(n)\left(\sum_{0\leq|\alpha|\leq n+1}\left\|\partial\partial_x^{\alpha}\phi^{(i-1)}\right\|_{L^2}(t)\right)^2\end{aligned}$$

Hence dealing with cubic and higher order terms in a similar fashion, and using Inductive assumption (3.8)

$$\sum_{0\leq|\alpha|\leq n+1}\left\|\partial_x^\alpha F(\phi^{(i-1)},\partial\phi^{(i-1)})\right\|_{L^2(\mathbb{R}^n)}(t)$$
$$\leq C(F)\left(1+\left\|\phi^{(i-1)}\right\|_{L^2(\mathbb{R}^n)}(t)+\sum_{0\leq|\alpha|\leq n+1}\left\|\partial\partial_x^\alpha\phi^{(i-1)}\right\|_{L^2}(t)\right)^{n+1}\leq\tilde{C}(n,F)A^{n+1}=:B(A,n,F)$$

Hence (3.14) follows.

(ii) (3.15) follows from Bounds on a (3.3) and Sobolev Embedding (2.14). The second term might involve 2 partial derivatives on t, where we need to use the equation to exchange into $\partial_t\partial_x$ and ∂_x^2 and then use (3.14). We need Bootstrap (3.10) at the final step, redefining B.

(iii) (3.16) follows from Bounds on a (3.3) and Sobolev Embedding (2.14). We again need Bootstrap (3.10) at the final step, redefining B.

□

Proof of Lemma 3.1.2

Proof. For any $i \geq 3$, we consider the sequence $\{\phi^{(i)} - \phi^{(i-1)}\}$. In particular, substracting $i-1$th iteration from ith iteration, we establish

$$\partial_\alpha \left(a^{\alpha\beta}(\phi^{(i-1)})\partial_\beta\phi^{(i)} - a^{\alpha\beta}(\phi^{(i-2)})\partial_\beta\phi^{(i-1)}\right) = F(\phi^{(i-1)}, \partial\phi^{(i-1)}) - F(\phi^{(i-2)}, \partial\phi^{(i-2)})$$
$$\left(\phi^{(i)} - \phi^{(i-1)}, \partial_t(\phi^{(i)} - \phi^{(i-1)})\right)\Big|_{t=0} = (0,0)$$

Substract and Add the term $\partial_\alpha(a^{\alpha\beta}(\phi^{(i-1)})\partial_\beta\phi^{(i-1)})$ from the first equation yields

$$\begin{aligned}&\partial_\alpha \left(a^{\alpha\beta}(\phi^{(i-1)})\partial_\beta(\phi^{(i)} - \phi^{(i-1)})\right)\\ &= -\partial_\alpha\left((a^{\alpha\beta}(\phi^{(i-1)}) - a^{\alpha\beta}(\phi^{(i-2)}))\partial_\beta\phi^{(i-1)}\right) + F(\phi^{(i-1)}, \partial\phi^{(i-1)}) - F(\phi^{(i-2)}, \partial\phi^{(i-2)})\end{aligned} \tag{3.18}$$

We need to control the 2 terms at the RHS of (3.18). For the first, do product rule and use Mean Value Theorem for the latter.

$$\begin{aligned}&\left|\partial_\alpha\left((a^{\alpha\beta}(\phi^{(i-1)}) - a^{\alpha\beta}(\phi^{(i-2)}))\partial_\beta\phi^{(i-1)}\right)\right|\\ &\leq \left|\partial_\alpha((a^{\alpha\beta}(\phi^{(i-1)}) - a^{\alpha\beta}(\phi^{(i-2)}))\partial_\beta\phi^{(i-1)}\right| + \left|(a^{\alpha\beta}(\phi^{(i-1)}) - a^{\alpha\beta}(\phi^{(i-2)}))\partial_\alpha\partial_\beta\phi^{(i-1)}\right|\\ &\leq C(a)|\partial\phi^{(i-1)} - \partial\phi^{(i-2)}||\partial\phi^{(i-1)}| + C(a)|\phi^{(i-1)} - \phi^{(i-2)}||\partial\partial\phi^{(i-1)}|\end{aligned}$$

Recall we have Bounds on a (3.3) and result from Lemma 3.1.1 (3.7). Hence going to norms, for any $t \in [0,T]$

$$\left\|\partial_\alpha\left((a^{\alpha\beta}(\phi^{(i-1)}) - a^{\alpha\beta}(\phi^{(i-2)}))\partial_\beta\phi^{(i-1)}\right)\right\|_{L^2}(t) \leq C(a,A)\left(\left\|\partial\phi^{(i-1)} - \partial\phi^{(i-2)}\right\|_{L^2}(t) + \left\|\phi^{(i-1)} - \phi^{(i-2)}\right\|_{L^2}(t)\right) \tag{3.19}$$

Similarly for F, the second term at RHS of (3.18), one directly use Mean Value Theorem

$$\left|F(\phi^{(i-1)}, \partial\phi^{(i-1)}) - F(\phi^{(i-2)}, \partial\phi^{(i-2)})\right| \leq C(F)\left(\left|\phi^{(i-1)} - \phi^{(i-2)}\right| + \left|\partial\phi^{(i-1)} - \partial\phi^{(i-2)}\right|\right)$$

Hence going to L^2 norm, for any $t \in [0,T]$ we have

$$\left\|F(\phi^{(i-1)}, \partial\phi^{(i-1)}) - F(\phi^{(i-2)}, \partial\phi^{(i-2)})\right\|_{L^2(\mathbb{R}^n)}(t) \leq C(F)\left(\left\|\partial\phi^{(i-1)} - \partial\phi^{(i-2)}\right\|_{L^2}(t) + \left\|\phi^{(i-1)} - \phi^{(i-2)}\right\|_{L^2}(t)\right) \tag{3.20}$$

Now treating the RHS at (3.18) as forcing term for equation with solution $\phi^{(i)} - \phi^{(i-1)}$. We add up energy estimates (2.7) and (2.9) directly so that using bounds on the forcing (3.19) and (3.20)

$$\begin{aligned}&\sup_{t\in[0,T]}\left\|(\phi^{(i)} - \phi^{(i-1)}, \partial_t\phi^{(i)} - \partial_t\phi^{(i-1)})\right\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}(t)\\ &\leq C(n,a,A,F)\cdot T\cdot \sup_{t\in[0,T]}\left\|(\phi^{(i-1)} - \phi^{(i-2)}, \partial_t\phi^{(i-1)} - \partial_t\phi^{(i-2)})\right\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}(t)\end{aligned}$$

Notice we've additionally used

- (3.16) to bound $\exp\left(\int_0^T \left\|\partial(a^{\alpha\beta}(\phi^{(i-1)}))\right\|_{L^\infty(\mathbb{R}^n)}\right) \leq \exp\left(T\cdot B(A,n,a,F)\right)$ independent of i.
- The key point: $\left\|(\phi^{(i-1)} - \phi^{(i-2)}, \partial_t\phi^{(i-1)} - \partial_t\phi^{(i-2)})|_{t=0}\right\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} = 0$ due to zero initial data.

From Lemma 3.1.1, we know for $A = A(\|\phi_0\|_{H^{n+2}}, \|\phi_1\|_{H^{n+1}}, n) > 0$, the initial iteration

$$\sup_{t\in[0,T]}\left\|(\phi^{(2)} - \phi^{(1)}, \partial_t\phi^{(2)} - \partial_t\phi^{(1)})\right\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}(t) \leq 2A$$

Now we shrink $T = T(n,a,A,F) = T(\|\phi_0\|_{H^{n+2}}, \|\phi_1\|_{H^{n+1}}, n, a, F) > 0$ sufficiently small so that

$$\sup_{t\in[0,T]}\left\|(\phi^{(i)} - \phi^{(i-1)}, \partial_t\phi^{(i)} - \partial_t\phi^{(i-1)})\right\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}(t) \leq \frac{1}{2}\sup_{t\in[0,T]}\left\|(\phi^{(i-1)} - \phi^{(i-2)}, \partial_t\phi^{(i-1)} - \partial_t\phi^{(i-2)})\right\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}(t)$$

Hence we form a geometric decreasing sequence

$$\sup_{t\in[0,T]}\left\|(\phi^{(i)} - \phi^{(i-1)}, \partial_t\phi^{(i)} - \partial_t\phi^{(i-1)})\right\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}(t) \leq \left(\frac{1}{2}\right)^{i-2} 2A$$

Hence $(\phi^{(i)} - \phi^{(i-1)}, \partial_t\phi^{(i)} - \partial_t\phi^{(i-1)})$ is a Cauchy Sequence in $L^\infty\left([0,T]; H^1(\mathbb{R}^n)\right) \times L^\infty\left([0,T]; L^2(\mathbb{R}^n)\right)$. □

3.1.2 Continuous Dependence on Initial Data

In this subsection we prove Continuous Dependence on Initial Data. As a corollary, the uniqueness of local solution follows. To begin, note an interpolation Lemma.

Lemma 3.1.4 (Sobolev Interpolation). *Let $0 \leq s_1 \leq s_2$. If $\eta \in H^{s_1}(\mathbb{R}^n) \cap H^{s_2}(\mathbb{R}^n)$, then for any*

$$s \in [s_1, s_2] \qquad s.t. \quad \theta_1 s_1 + \theta_2 s_2 = s \qquad for \quad \theta_1 + \theta_2 = 1$$

we have interpolation for some $C = C(s_1, s_2, s, n) > 0$

$$\|\eta\|_{H^s(\mathbb{R}^n)} \leq C \|\eta\|_{H^{s_1}(\mathbb{R}^n)}^{\theta_1} \|\eta\|_{H^{s_2}(\mathbb{R}^n)}^{\theta_2} \tag{3.21}$$

Theorem 3.1.2 (Continuous Dependence of Initial Data). *Let $(\phi, \partial_t\phi)$ be solution to (3.5) arising from initial data $(\phi_0, \phi_1) \in H^{n+2}(\mathbb{R}^n) \times H^{n+1}(\mathbb{R}^n)$. For any $(\phi_0^{(i)}, \phi_1^{(i)})$ sequence of pairs of functions s.t.*

$$\left\|\phi_0^{(i)} - \phi_0\right\|_{H^{n+2}(\mathbb{R}^n)} \to 0 \qquad \left\|\phi_1^{(i)} - \phi_1\right\|_{H^{n+1}(\mathbb{R}^n)} \to 0 \qquad as\ i \to \infty$$

Denote $(\phi^{(i)}, \partial_t\phi^{(i)})$ as solution to (3.5) arising from initial data $(\phi_0^{(i)},\, \phi_1^{(i)})$. Then there exists

$$T = T(\|\phi_0\|_{H^{n+2}},\, \|\phi_1\|_{H^{n+1}},\, n,\, a,\, F) > 0$$

sufficiently small independent of i s.t. for any $1 \leq s < n+2$

$$\left\|(\phi^{(i)} - \phi, \partial_t(\phi^{(i)} - \phi))\right\|_{L^\infty([0,T];H^s(\mathbb{R}^n)) \times L^\infty([0,T];H^{s-1}(\mathbb{R}^n))} \to 0 \qquad as\ i \to \infty \tag{3.22}$$

Proof. We formulate the IVP (3.5) that $(\phi, \partial_t\phi)$ and $(\phi^{(i)}, \partial_t\phi^{(i)})$ solves respectively.

$$\begin{cases} \partial_\alpha\left(a^{\alpha\beta}(\phi)\partial_\beta\phi\right) = F(\phi, \partial\phi) \\ (\phi,\, \partial_t\phi)|_{t=0} = (\phi_0,\, \phi_1) \end{cases} \qquad \text{and} \qquad \begin{cases} \partial_\alpha\left(a^{\alpha\beta}(\phi^{(i)})\partial_\beta\phi^{(i)}\right) = F(\phi^{(i)}, \partial\phi^{(i)}) \\ (\phi^{(i)},\, \partial_t\phi^{(i)})|_{t=0} = (\phi_0^{(i)},\, \phi_1^{(i)}) \end{cases}$$

Substracting one from the other, then substract and add the term $\partial_\alpha(a^{\alpha\beta}(\phi^{(i)})\partial_\beta\phi)$, we obtain the IVP

$$\begin{cases} \partial_\alpha\left(a^{\alpha\beta}(\phi^{(i)})\partial_\beta(\phi^{(i)} - \phi)\right) = -\partial_\alpha\left(\left(a^{\alpha\beta}(\phi^{(i)}) - a^{\alpha\beta}(\phi)\right)\partial_\beta\phi\right) + F(\phi^{(i)}, \partial\phi^{(i)}) - F(\phi, \partial\phi) \\ (\phi^{(i)} - \phi,\, \partial_t(\phi^{(i)} - \phi))|_{t=0} = (\phi_0^{(i)} - \phi_0,\, \phi_1^{(i)} - \phi_1) \end{cases} \tag{3.23}$$

We first make several remarks and observations on dependence of T on initial data $(\phi_0^{(i)} - \phi_0,\, \phi_1^{(i)} - \phi_1)$.

- Let

  $$T = T(\|\phi_0\|_{H^{n+2}},\, \|\phi_1\|_{H^{n+1}}, n, a, F) > 0 \qquad T_i = T(\left\|\phi_0^{(i)}\right\|_{H^{n+2}}, \left\|\phi_1^{(i)}\right\|_{H^{n+1}}, n, a, F) > 0$$

  be the local time of existence for IVP with data (ϕ_0, ϕ_1) and with data $(\phi_0^{(i)}, \phi_1^{(i)})$ respectively. In principle, the IVP (3.23) only has well-defined classical solution up to time $\min\{T, T_i\} \leq T$. But there might be problem that $T_i \to 0$ as $i \to \infty$, and then we're discussing nonsense. To rule out the situation, we make use of $\left\|(\phi_0^{(i)} - \phi_0,\, \phi_1^{(i)} - \phi_1)\right\|_{H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \to 0$.

- For any $\varepsilon > 0$, take N large enough so that for all $i \geq N$, one has

  $$\left\|(\phi_0^{(i)} - \phi_0,\, \phi_1^{(i)} - \phi_1)\right\|_{H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} < \varepsilon$$

  We need to do Picard's Iteration as in Theorem 3.1.1. In particular, at step (3.17), on the right hand side we have

  $$\begin{aligned} & C\left((1+T)\left\|(\phi_0^{(i)} - \phi_0,\, \phi_1^{(i)} - \phi_1)\right\|_{H^{n+2}(\mathbb{R}^n) \times H^{n+1}(\mathbb{R}^n)} + B(1+C)T + B(1+C)T^2\right)\exp(CBT) \\ & \leq C\left(\varepsilon + T\left(\varepsilon + B(1+C) + B(1+C)T\right)\right)\exp(CBT) \end{aligned}$$

  uniformly for $i \geq N$. Then shrinking $T = T(\varepsilon, n, a, F) > 0$ we improve to the same bound for the next Picard Iteration. In particular Bounds of F via $B = B(\varepsilon, n, a, F)$ follows from Lemma 3.1.3 (3.14), with forcing term writing as in (3.19) and (3.20). There we obtain estimate on L^2 bounds for both terms.

- Now shrinking T depending on ε, we obtain same T for solutions to IVP (3.23) uniformly for $i \geq N$.

From there we respectively use for each of the following lines

- energy estimate for $H^1 \times L^2$ (2.10)
- $H^1 \times L^2$ bounds on L^2 forcing (3.19) and (3.20)
- Grönwalls

so that for $C = C(n) > 0$ from energy estimate

$$\begin{aligned}
&\sup_{t\in[0,T]} \left\|(\phi^{(i)} - \phi,\, \partial_t(\phi^{(i)} - \phi))\right\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}(t) \\
&\leq \left(\left\|(\phi_0^{(i)} - \phi_0,\, \phi_1^{(i)} - \phi_1)\right\|_{H^1\times L^2} + \int_0^T \left\|-\partial_\alpha\left(\left(a^{\alpha\beta}(\phi^{(i)}) - a^{\alpha\beta}(\phi)\right)\partial_\beta\phi\right) + F(\phi^{(i)}, \partial\phi^{(i)}) - F(\phi, \partial\phi)\right\|_{L^2}(t)\,dt\right) \\
&\times C(n)(1+T)\exp\left(C\int_0^T \left\|\partial a(\phi^{(i)})\right\|_{L^\infty(\mathbb{R}^n)}(t)\,dt\right) \\
&\leq \left(\left\|(\phi_0^{(i)} - \phi_0,\, \phi_1^{(i)} - \phi_1)\right\|_{H^1\times L^2} + C(n,a,F)\int_0^T \left\|(\phi^{(i)} - \phi, \partial_t\phi^{(i)} - \partial_t\phi)\right\|_{H^1\times L^2}(s)\,ds\right)\cdot C(n,T)\exp\left(T\cdot C(a)\right) \\
&\leq C(\varepsilon, n, a, F, T)\left\|(\phi_0^{(i)} - \phi_0,\, \phi_1^{(i)} - \phi_1)\right\|_{H^{`}(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \qquad (3.24)
\end{aligned}$$

Luckily, the RHS of (3.24) goes to 0 as $i \to \infty$. Hence

$$\left\|(\phi^{(i)} - \phi, \partial_t(\phi^{(i)} - \phi))\right\|_{L^\infty([0,T];H^1(\mathbb{R}^n))\times L^\infty([0,T];L^2(\mathbb{R}^n))} \to 0 \qquad as\ i \to \infty$$

Finally by interpolation (3.21) between $s_1 = 1$ and $s_2 = n+2$, for any $1 \leq s \leq n+2$,

$$\frac{n+2-s}{n+1}\cdot 1 + \frac{s-1}{n+1}\cdot(n+2) = s$$

Note, similarly

$$\frac{n+2-s}{n+1}\cdot 0 + \frac{s-1}{n+1}\cdot(n+1) = s-1$$

We do interpolation simultaneously between $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ and $H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)$ so that for $C = C(s,n) > 0$

$$\begin{aligned}
&\left\|(\phi^{(i)} - \phi, \partial_t(\phi^{(i)} - \phi))\right\|_{L^\infty([0,T];H^s(\mathbb{R}^n))\times L^\infty([0,T];H^{s-1}(\mathbb{R}^n))} \\
&\leq C(s,n)\left\|(\phi^{(i)} - \phi, \partial_t(\phi^{(i)} - \phi))\right\|_{L^\infty([0,T];H^1(\mathbb{R}^n))\times L^\infty([0,T];L^2(\mathbb{R}^n))}^{\frac{n+2-s}{n+1}}\left\|(\phi^{(i)} - \phi, \partial_t(\phi^{(i)} - \phi))\right\|_{L^\infty([0,T];H^{n+2}(\mathbb{R}^n))\times L^\infty([0,T];H^{n+1}(\mathbb{R}^n))}^{\frac{s-1}{n+1}}
\end{aligned}$$

But only for $s < n+2$ can we put positive power on the first term, which goes to 0 as $i \to \infty$ as we've shown in (3.24). □

Corollary 3.1.1 (Local Uniqueness). *Given smooth a and F as in (3.2) to (3.4). If*

$$(\phi_0,\, \phi_1) \in H^{n+2}(\mathbb{R}^n) \times H^{n+1}(\mathbb{R}^n)$$

Then the solution to (3.5) as in Theorem 3.1.1 is unique up to $T = T(\|\phi_0\|_{H^{n+2}}, \|\phi_1\|_{H^{n+1}}, n, a, F) > 0$.

Proof. Let $(\phi^{(1)}, \partial_t\phi^{(1)})$ and $(\phi^{(2)}, \partial_t\phi^{(2)})$ be solutions w.r.t. same initial data. Then from (3.24), the RHS is 0, hence LHS is 0. □

3.2 Maximal Time of Existence

3.2.1 Persistence of Regularity

Definition 3.2.1 (Maximal Time of Existence). *With fixed initial data (ϕ_0, ϕ_1), we define the maximal time of existence*

$$T_* := \sup\left\{T > 0 \mid \text{there exists unique}\,(\phi, \partial_t\phi) \in L^\infty([0,T]; H^{n+2}(\mathbb{R}^n)) \times L^\infty([0,T]; H^{n+1}(\mathbb{R}^n))\ \text{solution to}\,(3.5)\right\} \qquad (3.25)$$

In particular, for any $(\phi_0, \phi_1) \in H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)$, from Local Existence 3.1.1 and Uniqueness Theory 3.1.1, we have $T_* > 0$ because the set is non-empty.

However, we'd like to argue that 'all higher regularities of initial data are propagated', i.e., if the initial data in fact lies in function class of higher regularity, the unique solution exists up to the same maximal time of existence.

Theorem 3.2.1 (Persistence of Regularity)**.** *Given initial data* $(\phi_0, \phi_1) \in H^{n+2}(\mathbb{R}^n) \times H^{n+1}(\mathbb{R}^n)$.

(i) If $(\phi_0, \phi_1) \in H^m(\mathbb{R}^n) \times H^{m-1}(\mathbb{R}^n)$ *for some* $m \in \mathbb{N}$ *s.t.* $m > n+2$*, then the solution*

$$(\phi, \partial_t\phi) \in L^\infty([0,T]; H^m(\mathbb{R}^n)) \times L^\infty([0,T]; H^{m-1}(\mathbb{R}^n)) \tag{3.26}$$

for any $T < T_*$.

(ii) If $(\phi_0, \phi_1) \in \bigcap_{m \geq n+2} \left(H^m(\mathbb{R}^n) \times H^{m-1}(\mathbb{R}^n)\right)$*, then the solution*

$$(\phi, \partial_t\phi) \in L^\infty([0,T]; C^\infty(\mathbb{R}^n)) \times L^\infty([0,T]; C^\infty(\mathbb{R}^n))$$

for any $T < T_*$.

Proof. We prove (i) using Induction.

- For $m = n+2$, indeed (3.26) holds due to 3.1.1.
- Assume for $m-1 \geq n+2$, we wish to prove for m. We make the inductive assumption that for any $T < T_*$, there exists some number $A(T) > 0$ s.t.

$$\sup_{t\in[0,T]} \|(\phi, \partial_t\phi)\|_{H^{m-1}(\mathbb{R}^n)\times H^{m-2}(\mathbb{R}^n)}(t) \leq A(T) \tag{3.27}$$

Fix any $T < T_*$. Using higher order Energy Estimate (2.11), for $C = C(n) > 0$

$$\begin{aligned}
&\sup_{t\in[0,T]} \|(\phi,\, \partial_t\phi)\|_{H^m(\mathbb{R}^n)\times H^{m-1}(\mathbb{R}^n)}(t) \\
&\leq C(1+T)\exp\left(C\int_0^T \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\,dt\right) \times \\
&\left(\|(\phi_0,\, \phi_1)\|_{H^m(\mathbb{R}^n)\times H^{m-1}(\mathbb{R}^n)} + \int_0^T \|F\|_{H^{m-1}}(t)\,dt + C\sum_{0\leq|\gamma|+|\delta|\leq m-2}\int_0^T \left(\left\|\partial\partial_x^\gamma a\partial\partial_x^\delta\phi\right\|_{L^2}(t) + \left\|\partial_x^\gamma a\partial\partial\partial_x^\delta\phi\right\|_{L^2}(t)\right)dt\right)
\end{aligned} \tag{3.28}$$

We wish to bound for $C = C(m, n, A(T), F, a) > 0$ s.t. for any $0 < t \leq T$

$$\sum_{0\leq|\alpha|\leq m-1} \|\partial_x^\alpha F(\phi, \partial\phi)\|_{L^2(\mathbb{R}^n)}(t) \leq C\,\|(\phi, \partial_t\phi)\|_{H^m(\mathbb{R}^n)\times H^{m-1}(\mathbb{R}^n)}(t) \tag{3.29}$$

$$\sum_{0\leq|\gamma|+|\delta|\leq m-2}\ \sum_{\alpha,\beta=0}^{n} \left(\left\|\partial_\alpha\partial_x^\gamma a^{\alpha\beta}(\phi)\partial_\beta\partial_x^\delta\phi\right\|_{L^2}(t) + \left\|\partial_x^\gamma a^{\alpha\beta}(\phi)\partial_\alpha\partial_\beta\partial_x^\delta\phi\right\|_{L^2}(t)\right) \leq C\,\|(\phi, \partial_t\phi)\|_{H^m(\mathbb{R}^n)\times H^{m-1}(\mathbb{R}^n)}(t) \tag{3.30}$$

Assume for now we've obtained (3.29) and (3.30). Note we have trivial bounds for

$$\exp\left(\int_0^T \left\|\partial(a^{\alpha\beta}(\phi))\right\|_{L^\infty(\mathbb{R}^n)}(s)\,ds\right) \leq \exp\left(T\cdot C(a, A(T))\right)$$

using induction (3.27) and bounds on a (3.3). Then plugging the above into (3.28), we obtain for $C = C(T, A(T), n, m, a, F) > 0$

$$\begin{aligned}
&\sup_{t\in[0,T]} \|(\phi,\, \partial_t\phi)\|_{H^m(\mathbb{R}^n)\times H^{m-1}(\mathbb{R}^n)}(t) \\
&\leq C\left(\|(\phi_0,\, \phi_1)\|_{H^m(\mathbb{R}^n)\times H^{m-1}(\mathbb{R}^n)} + C\int_0^T \|(\phi, \partial_t\phi)\|_{H^m(\mathbb{R}^n)\times H^{m-1}(\mathbb{R}^n)}(t)\,dt\right)
\end{aligned}$$

Using Grönwalls we arrive at

$$\sup_{t\in[0,T]} \|(\phi,\, \partial_t\phi)\|_{H^m(\mathbb{R}^n)\times H^{m-1}(\mathbb{R}^n)}(t) \leq C(T, n, m, a, F)\,\|(\phi_0,\, \phi_1)\|_{H^m(\mathbb{R}^n)\times H^{m-1}(\mathbb{R}^n)} < \infty$$

Thus we proved for m. It suffices to check (3.29) and (3.30). We illustrate F

$$\begin{aligned}&\sum_{0\le|\alpha|\le m-1}\|\partial_x^\alpha F(\phi,\partial\phi)\|_{L^2(\mathbb{R}^n)}(t)\\&\le C\left(\sum_{0\le|\alpha|\le m-1}\|\partial\partial_x^\alpha\phi\|_{L^2}(t)+\sum_{0\le|\alpha_1|+|\alpha_2|\le m-1}\|\partial\partial_x^{\alpha_1}\phi\partial\partial_x^{\alpha_2}\phi\|_{L^2}(t)+\text{cubic}+\cdots+(m-1)\,\text{order}\right)\end{aligned}$$

Consider the quadratic term. Since $0\le|\alpha_1|+|\alpha_2|\le m-1$, we know either $|\alpha_1|$ or $|\alpha_2|$ is smaller than or equal to $\frac{m-1}{2}$. Assume WLOG $|\alpha_1|\le\frac{m-1}{2}$. We know that using Sobolev Inequality (2.14) we may embed $H^{\frac{n}{2}}$ into L^∞. Since in our inductive assumption we let $m-1\ge n+2 \implies m-3\ge n$, we have

$$\frac{m-1}{2}+\frac{n}{2}=\frac{m+n-1}{2}\le m-2$$

Thus we may control the L^∞ norm of terms with derivatives order less than or equal to $\frac{m-1}{2}$ using $H^{\frac{n}{2}}$ norm, and do not exceed H^{m-2}. In particular

$$\begin{aligned}\sum_{0\le|\alpha_1|\le\frac{m-1}{2}}\|\partial\partial_x^{\alpha_1}\phi\|_{L^\infty}(t)&\le C(n)\sum_{0\le|\alpha_1|\le\frac{m-1}{2}}\|\partial\partial_x^{\alpha_1}\phi\|_{H^{\frac{n}{2}}}(t)\\&\le C(n)\sum_{0\le|\alpha_1|\le\frac{m-1}{2}}\sum_{0\le|\alpha_2|\le\frac{n}{2}}\|\partial\partial_x^{\alpha_2}\partial_x^{\alpha_1}\phi\|_{L^2}(t)\\&=C(n)\sum_{0\le|\alpha|\le\frac{m+n-1}{2}}\|\partial\partial_x^{\alpha}\phi\|_{L^2}(t)\\&\le C(n)\sum_{0\le|\alpha|\le m-2}\|\partial\partial_x^{\alpha}\phi\|_{L^2}(t)\end{aligned}$$

Hence using the trivial bound

$$\begin{aligned}&\sum_{0\le|\alpha_1|+|\alpha_2|\le m-1}\|\partial\partial_x^{\alpha_1}\phi\partial\partial_x^{\alpha_2}\phi\|_{L^2}(t)\le\sum_{0\le|\alpha_1|+|\alpha_2|\le m-1}\|\partial\partial_x^{\alpha_1}\phi\|_{L^\infty}(t)\,\|\partial\partial_x^{\alpha_2}\phi\|_{L^2}(t)\\&\le\left(\sum_{0\le|\alpha_1|\le\frac{m-1}{2}}\|\partial\partial_x^{\alpha_1}\phi\|_{L^\infty}(t)\right)\left(\sum_{0\le|\alpha_2|\le m-1}\|\partial\partial_x^{\alpha_2}\phi\|_{L^2}(t)\right)\\&\le C(n)\left(\sum_{0\le|\alpha|\le m-2}\|\partial\partial_x^{\alpha}\phi\|_{L^2}(t)\right)\left(\sum_{0\le|\alpha_2|\le m-1}\|\partial\partial_x^{\alpha_2}\phi\|_{L^2}(t)\right)\\&\le C(n)\,\|(\phi,\,\partial_t\phi)\|_{H^{m-1}(\mathbb{R}^n)\times H^{m-2}(\mathbb{R}^n)}(t)\cdot\|(\phi,\,\partial_t\phi)\|_{H^m(\mathbb{R}^n)\times H^{m-1}(\mathbb{R}^n)}(t)\end{aligned}$$

Using Induction hypothesis (3.27), for any $T<T_*$

$$\|(\phi,\,\partial_t\phi)\|_{H^{m-1}(\mathbb{R}^n)\times H^{m-2}(\mathbb{R}^n)}(t)\le A(T)$$

Thus

$$\sum_{0\le|\alpha_1|+|\alpha_2|\le m-1}\|\partial\partial_x^{\alpha_1}\phi\partial\partial_x^{\alpha_2}\phi\|_{L^2}(t)\le C(A(T),n)\,\|(\phi,\,\partial_t\phi)\|_{H^m(\mathbb{R}^n)\times H^{m-1}(\mathbb{R}^n)}(t)$$

One repeat for each term and conclude (3.29). For (3.30), conclude using a similar method with Bounds on a (3.3), Sobolev Inequality (2.14) and inductive assumption (3.27) for all lower order terms compared to m. (ii) follows from Sobolev Embedding (2.14). □

3.2.2 Breakdown Criteria

With Local Theory, we know that if $T_*<\infty$, then some norms must blow up as the time T_* is approached.

Theorem 3.2.2 (Blowup Criteria)**.** *Given Initial data $(\phi_0,\,\phi_1)\in H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)$. Let T_* be maximal time of existence. If $T_*<\infty$, then*

$$\liminf_{t\to T_*}\|(\phi,\partial_t\phi)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)}(t)\to\infty \tag{3.31}$$

$$\limsup_{t\to T_*}\left(\sum_{|\alpha|\le\lfloor\frac{n+2}{2}\rfloor}\|\partial_x^\alpha\phi\|_{L^\infty(\mathbb{R}^n)}(t)+\sum_{|\alpha|\le\lfloor\frac{n}{2}\rfloor}\|\partial_t\partial_x^\alpha\phi\|_{L^\infty(\mathbb{R}^n)}(t)\right)\to\infty \tag{3.32}$$

Proof of (3.31). We argue by contradiction. Assume the statement is false, so there exists an increasing sequence $\{t_m\} \subset [0, T_*)$ s.t. $t_m \to T_*$, and $M > 0$ independent of m s.t.

$$\|(\phi, \partial_t \phi)\|_{H^{n+2}(\mathbb{R}^n), H^{n+1}(\mathbb{R}^n)}(t_m) \leq M \qquad \forall\, m$$

But recall our step in Picard Iteration (3.17) for Local Existence Theory. There, if we let the data at t_m be our initial condition, on the RHS we have

$$\begin{aligned}
&C\left((1+T)\|(\phi,\, \partial_t\phi)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)}(t_m) + B(1+C)T + B(1+C)T^2\right)\exp(CBT)\\
&\leq C\left((1+T)M + B(1+C)T + B(1+C)T^2\right)\exp(CBT)
\end{aligned}$$

uniformly in m. Then we shrink $T = T(M, n, a, F) > 0$ sufficiently small to get same bound for next iteration. In particular, we have no dependence of T on m, but only on M. Therefore, once we take m sufficiently large, the solution exists for $[t_m, t_m + T] \times \mathbb{R}^n$ where $t_m + T > T_*$ exceeds the maximal time of existence, which we've assumed to be finite. Now we arrive at contradiction. □

Proof of (3.32). We argue by contradiction. Assume the statement is false, so there exists constant $D > 0$ s.t.

$$\sup_{t\in[0,T_*)}\left(\sum_{|\alpha|\leq\lfloor\frac{n+2}{2}\rfloor}\|\partial_x^\alpha\phi\|_{L^\infty(\mathbb{R}^n)}(t) + \sum_{|\alpha|\leq\lfloor\frac{n}{2}\rfloor}\|\partial_t\partial_x^\alpha\phi\|_{L^\infty(\mathbb{R}^n)}(t)\right) \leq D \tag{3.33}$$

We wish to obtain uniform bound on

$$\sup_{t\in[0,T_*)}\|(\phi, \partial_t\phi)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)}(t)$$

which contradicts (3.31). To do so, we proceed in 2 steps. We first obtain control on

$$\sup_{t\in[0,T_*)}\|(\phi, \partial_t\phi)\|_{H^{n+1}(\mathbb{R}^n)\times H^{n}(\mathbb{R}^n)}(t)$$

and then use such bound to control

$$\sup_{t\in[0,T_*)}\|(\phi, \partial_t\phi)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)}(t)$$

(i) Control on $\sup_{t\in[0,T_*)}\|(\phi, \partial_t\phi)\|_{H^{n+1}(\mathbb{R}^n)\times H^{n}(\mathbb{R}^n)}(t)$. Recall energy estimate (2.11).

$$\begin{aligned}
&\sup_{t\in[0,T]}\|(\phi,\, \partial_t\phi)\|_{H^k(\mathbb{R}^n)\times H^{k-1}(\mathbb{R}^n)}(t)\\
&\leq C(1+T)\exp\left(C\int_0^T\|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\,dt\right)\times\\
&\left(\|(\phi_0,\, \phi_1)\|_{H^k\times H^{k-1}} + \int_0^T\|F\|_{H^{k-1}}(t)\,dt + C\sum_{0\leq|\gamma|+|\delta|\leq k-2}\int_0^T\left(\left\|\partial\partial_x^\gamma a\partial\partial_x^\delta\phi\right\|_{L^2}(t) + \left\|\partial_x^\gamma a\partial\partial\partial_x^\delta\phi\right\|_{L^2}(t)\right)dt\right)
\end{aligned} \tag{3.34}$$

Choose $k = n+1$ and $0 \leq T < T_*$. We wish to bound for $C = C(n, T, D, F, a) > 0$ s.t. for any $0 < t \leq T$

$$\sum_{0\leq|\alpha|\leq n}\|\partial_x^\alpha F(\phi, \partial\phi)\|_{L^2(\mathbb{R}^n)}(t) \leq C\|(\phi, \partial_t\phi)\|_{H^{n+1}(\mathbb{R}^n)\times H^n(\mathbb{R}^n)}(t) \tag{3.35}$$

$$\sum_{0\leq|\gamma|+|\delta|\leq n-1}\sum_{\alpha,\beta=0}^{n}\left(\left\|\partial_\alpha\partial_x^\gamma a^{\alpha\beta}(\phi)\partial_\beta\partial_x^\delta\phi\right\|_{L^2}(t) + \left\|\partial_x^\gamma a^{\alpha\beta}(\phi)\partial_\alpha\partial_\beta\partial_x^\delta\phi\right\|_{L^2}(t)\right) \leq C\|(\phi, \partial_t\phi)\|_{H^{n+1}(\mathbb{R}^n)\times H^n(\mathbb{R}^n)}(t) \tag{3.36}$$

Assume for now we've obtained (3.35) and (3.36). Note we have trivial bounds for

$$\exp\left(\int_0^T\left\|\partial(a^{\alpha\beta}(\phi))\right\|_{L^\infty(\mathbb{R}^n)}(s)\,ds\right) \leq \exp\left(T\cdot C(a, D)\right)$$

using contradiction assumption (3.33) and bounds on a (3.3). Then plugging the above into (3.34), we obtain for $C = C(T, D, n, a, F) > 0$

$$\begin{aligned}&\sup_{t\in[0,T]} \|(\phi,\, \partial_t\phi)\|_{H^{n+1}(\mathbb{R}^n)\times H^n(\mathbb{R}^n)}(t)\\ &\le C\left(\|(\phi_0,\, \phi_1)\|_{H^{n+1}\times H^n} + C\int_0^T \|(\phi, \partial_t\phi)\|_{H^{n+1}(\mathbb{R}^n)\times H^n(\mathbb{R}^n)}(t)\, dt\right)\end{aligned}$$

Using Grönwalls and take T to T_*, we have

$$\sup_{t\in[0,T_*)} \|(\phi,\, \partial_t\phi)\|_{H^{n+1}(\mathbb{R}^n)\times H^n(\mathbb{R}^n)}(t) \le C(T_*, D, n, a, F)\, \|(\phi_0,\, \phi_1)\|_{H^{n+1}(\mathbb{R}^n)\times H^n(\mathbb{R}^n)} =: \tilde{C} \tag{3.37}$$

Now to check (3.35) and (3.36) we use similar (but more straightforward) approach as before. We illustrate F.

$$\begin{aligned}&\sum_{0\le|\alpha|\le n} \|\partial_x^\alpha F(\phi, \partial\phi)\|_{L^2(\mathbb{R}^n)}(t)\\ &\le C\left(\sum_{0\le|\alpha|\le n} \|\partial\partial_x^\alpha\phi\|_{L^2}(t) + \sum_{0\le|\alpha_1|+|\alpha_2|\le n} \|\partial\partial_x^{\alpha_1}\phi\partial\partial_x^{\alpha_2}\phi\|_{L^2}(t) + \text{cubic} + \cdots + n\,\text{order}\right)\end{aligned}$$

Consider the quadratic term. Since $0 \le |\alpha_1| + |\alpha_2| \le n$, we know either $|\alpha_1|$ or $|\alpha_2|$ is smaller than or equal to $\lfloor \frac{n}{2}\rfloor$. Assume WLOG $|\alpha_1| \le \lfloor \frac{n}{2}\rfloor$. Note we have assumption (3.33)

$$\begin{aligned}&\sum_{0\le|\alpha_1|+|\alpha_2|\le n} \|\partial\partial_x^{\alpha_1}\phi\partial\partial_x^{\alpha_2}\phi\|_{L^2}(t) \le \sum_{0\le|\alpha_1|+|\alpha_2|\le n} \|\partial\partial_x^{\alpha_1}\phi\|_{L^\infty}(t)\, \|\partial\partial_x^{\alpha_2}\phi\|_{L^2}(t)\\ &\le \left(\sum_{0\le|\alpha_1|\le\lfloor\frac{n}{2}\rfloor} \|\partial\partial_x^{\alpha_1}\phi\|_{L^\infty}(t)\right)\left(\sum_{0\le|\alpha_2|\le n} \|\partial\partial_x^{\alpha_2}\phi\|_{L^2}(t)\right)\\ &\le D\cdot \|(\phi,\, \partial_t\phi)\|_{H^{n+1}(\mathbb{R}^n)\times H^n(\mathbb{R}^n)}(t)\end{aligned}$$

Thus we justified (3.37).

(ii) Control on $\sup_{t\in[0,T_*)} \|(\phi, \partial_t\phi)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)}(t)$. Recall energy estimate (3.34). Choose $k = n+2$ and $0 \le T < T_*$. We need to bound for

$$C = C(\|(\phi_0, \phi_1)\|_{H^{n+1}(\mathbb{R}^n)\times H^n(\mathbb{R}^n)}, n, T, D, F, a) > 0$$

s.t. for any $0 < t \le T$

$$\sum_{0\le|\alpha|\le n+1} \|\partial_x^\alpha F(\phi, \partial\phi)\|_{L^2(\mathbb{R}^n)}(t) \le C\, \|(\phi, \partial_t\phi)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)}(t) \tag{3.38}$$

$$\sum_{0\le|\gamma|+|\delta|\le n}\ \sum_{\alpha,\beta=0}^{n} \left(\left\|\partial_\alpha\partial_x^\gamma a^{\alpha\beta}(\phi)\partial_\beta\partial_x^\delta\phi\right\|_{L^2}(t) + \left\|\partial_x^\gamma a^{\alpha\beta}(\phi)\partial_\alpha\partial_\beta\partial_x^\delta\phi\right\|_{L^2}(t)\right) \le C\, \|(\phi, \partial_t\phi)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n=1}(\mathbb{R}^n)}(t) \tag{3.39}$$

Assume for now we've obtained (3.38) and (3.39). Note we again have trivial bounds for

$$\exp\left(\int_0^T \left\|\partial(a^{\alpha\beta}(\phi))\right\|_{L^\infty(\mathbb{R}^n)}(s)\, ds\right) \le \exp\left(T\cdot C(a, D)\right)$$

using contradiction assumption (3.33) and bounds on a (3.3). Then plugging the above into (3.34), we obtain for $C = C(\|(\phi_0, \phi_1)\|_{H^{n+1}(\mathbb{R}^n)\times H^n(\mathbb{R}^n)}, n, T, D, F, a) > 0$

$$\begin{aligned}&\sup_{t\in[0,T]} \|(\phi,\, \partial_t\phi)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)}(t)\\ &\le C\left(\|(\phi_0,\, \phi_1)\|_{H^{n+2}\times H^{n+1}} + C\int_0^T \|(\phi, \partial_t\phi)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)}(t)\, dt\right)\end{aligned}$$

Using Grönwalls and take T to T_*, we have

$$\begin{aligned}
&\sup_{t\in[0,T_*)} \|(\phi,\,\partial_t\phi)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)}(t) \\
&\le C(\|(\phi_0,\phi_1)\|_{H^{n+1}(\mathbb{R}^n)\times H^{n}(\mathbb{R}^n)}, T_*, D, n, a, F)\,\|(\phi_0,\,\phi_1)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)} < \infty
\end{aligned} \tag{3.40}$$

Now to check (3.38) and (3.39) we use similar (less straightforward) approach as before. We illustrate F.

$$\begin{aligned}
&\sum_{0\le|\alpha|\le n+1} \|\partial_x^\alpha F(\phi,\partial\phi)\|_{L^2(\mathbb{R}^n)}(t) \\
&\le C\left(\sum_{0\le|\alpha|\le n+1}\|\partial\partial_x^\alpha\phi\|_{L^2}(t) + \sum_{0\le|\alpha_1|+|\alpha_2|\le n+1}\|\partial\partial_x^{\alpha_1}\phi\partial\partial_x^{\alpha_2}\phi\|_{L^2}(t) + \text{cubic} + \cdots + (n+1)\,\text{order}\right)
\end{aligned}$$

Consider the quadratic term.

(a) Suppose n is odd. Then $n+1$ is even. For $0\le|\alpha_1|+|\alpha_2|\le n+1$, we have 2 cases. WLOG, let $|\alpha_1|\le|\alpha_2|$

- Either $|\alpha_1|\le\lfloor\frac{n}{2}\rfloor<\frac{n+1}{2}$
- Or both $|\alpha_1|\le\lfloor\frac{n}{2}\rfloor+1=\frac{n+1}{2}$ and $|\alpha_2|\le\lfloor\frac{n}{2}\rfloor+1=\frac{n+1}{2}$.

Hence

$$\begin{aligned}
&\sum_{0\le|\alpha_1|+|\alpha_2|\le n+1}\|\partial\partial_x^{\alpha_1}\phi\partial\partial_x^{\alpha_2}\phi\|_{L^2}(t) \le \sum_{0\le|\alpha_1|+|\alpha_2|\le n+1}\|\partial\partial_x^{\alpha_1}\phi\|_{L^\infty}(t)\,\|\partial\partial_x^{\alpha_2}\phi\|_{L^2}(t) \\
&\le \left(\sum_{0\le|\alpha_1|\le\lfloor\frac{n}{2}\rfloor}\|\partial\partial_x^{\alpha_1}\phi\|_{L^\infty}(t)\right)\left(\sum_{0\le|\alpha_2|\le n+1}\|\partial\partial_x^{\alpha_2}\phi\|_{L^2}(t)\right) \\
&+ \left(\sum_{0\le|\alpha_1|\le\frac{n+1}{2}}\|\partial\partial_x^{\alpha_1}\phi\|_{L^\infty}(t)\right)\left(\sum_{0\le|\alpha_2|\le\frac{n+1}{2}}\|\partial\partial_x^{\alpha_2}\phi\|_{L^2}(t)\right)
\end{aligned}$$

The reason why we do such separation is because for the first case we have contradiction assumption (3.33) as a direct bound, while for the second case, we use Sobolev Embedding for the first term and (3.37) for the second term. In particular

$$\left(\sum_{0\le|\alpha_1|\le\lfloor\frac{n}{2}\rfloor}\|\partial\partial_x^{\alpha_1}\phi\|_{L^\infty}(t)\right)\left(\sum_{0\le|\alpha_2|\le n+1}\|\partial\partial_x^{\alpha_2}\phi\|_{L^2}(t)\right)\le D\cdot\|(\phi,\partial_t\phi)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)}(t)$$

For the second term

- since $H^{\frac{n}{2}}$ embeds into L^∞

$$\frac{n+1}{2}+\frac{n}{2}=\frac{2n+1}{2}<n+1$$

  So

$$\begin{aligned}
\sum_{0\le|\alpha_1|\le\frac{n+1}{2}}\|\partial\partial_x^{\alpha_1}\phi\|_{L^\infty}(t) &\le \sum_{0\le|\alpha_1|\le\frac{n+1}{2}}\|\partial\partial_x^{\alpha_1}\phi\|_{H^{\frac{n}{2}}}(t) \\
&\le \sum_{0\le|\alpha_1|\le\frac{n+1}{2}}\sum_{0\le|\alpha_2|\le\frac{n}{2}}\|\partial\partial_x^{\alpha_2}\partial_x^{\alpha_1}\phi\|_{L^2(\mathbb{R}^n)}(t) \\
&\le \sum_{0\le|\alpha|\le n+1}\|\partial\phi\|_{L^2(\mathbb{R}^n)}(t) = \|(\phi,\partial_t\phi)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)}(t)
\end{aligned}$$

- Since

$$\frac{n+1}{2}+1\le n+1 \qquad \frac{n+1}{2}\le n$$

  we may use (3.37) s.t.

$$\left(\sum_{0\le|\alpha_2|\le\frac{n+1}{2}}\|\partial\partial_x^{\alpha_2}\phi\|_{L^2}(t)\right)\le\|(\phi,\partial_t\phi)\|_{H^{n+1}(\mathbb{R}^n)\times H^{n}(\mathbb{R}^n)}(t)\le\tilde{C}$$

Thus combining the 2 portions of the second term we have

$$\left(\sum_{0\le|\alpha_1|\le\frac{n+1}{2}} \|\partial\partial_x^{\alpha_1}\phi\|_{L^\infty}(t)\right)\left(\sum_{0\le|\alpha_2|\le\frac{n+1}{2}} \|\partial\partial_x^{\alpha_2}\phi\|_{L^2}(t)\right) \le \tilde{C}\,\|(\phi,\partial_t\phi)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)}(t)$$

Further combining with the first term gives

$$\sum_{0\le|\alpha_1|+|\alpha_2|\le n+1} \|\partial\partial_x^{\alpha_1}\phi\partial\partial_x^{\alpha_2}\phi\|_{L^2}(t) \le (\tilde{C}+D)\,\|(\phi,\partial_t\phi)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)}(t)$$

(b) Suppose n is even. Then $n+1$ is odd. For $0 \le |\alpha_1| + |\alpha_2| \le n+1$, WLOG let $|\alpha_1| \le |\alpha_2|$, we must have $|\alpha_1| \le \lfloor \frac{n}{2} \rfloor = \frac{n}{2}$. Then we simply bound by (3.33) so

$$\sum_{0\le|\alpha_1|+|\alpha_2|\le n+1} \|\partial\partial_x^{\alpha_1}\phi\partial\partial_x^{\alpha_2}\phi\|_{L^2}(t) \le D\cdot\|(\phi,\partial_t\phi)\|_{H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)}(t)$$

Hence (3.38) is proved. Proof for (3.39) is similar via Contradiction Assumption (3.33), our previous bound on $H^{n+1}\times H^n$ (3.37) and Sobolev Embedding.

Thus the bound (3.40) contradicts (3.31). Hence by contradiction, (3.33) fails, and so (3.32) holds. □

As a matter of fact, we have stronger result for blowup of L^∞ norms. Even the first order derivatives blowup.

Corollary 3.2.1 (Breakdown Criteria for L^∞)**.** *Given Initial data $(\phi_0, \phi_1) \in H^{n+2}(\mathbb{R}^n)\times H^{n+1}(\mathbb{R}^n)$. Let T_* be maximal time of existence. If $T_* < \infty$,*

$$\limsup_{t\to T_*} \sum_{0\le|\alpha|\le 1} \|\partial^\alpha\phi\|_{L^\infty(\mathbb{R}^n)}(t) \to \infty$$