An Introduction to Obstacle and Alt-Caffarelli Free Boundary Problem

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Q What is the Study of Obstacle / One-Phase Free Boundary Problem?

- Obstacle
  1977 Caffarelli

  minimize $J^1(u, B_1) := \int_{B_1} \frac{1}{2}|\nabla u|^2 + u_+$ among $u \in H^1(B_1)$. $u = \varphi|_{\partial B_1} \geq 0$

- One phase Bernoulli
  1981 Alt-Caffarelli

  minimize $J^0(u, B_1) := \int_{B_1} |\nabla u|^2 + \chi_{\{u>0\}}$

- Alt-Phillips
  1986 Alt-Phillips

  minimize $J^\gamma(u, B_1) := \int_{B_1} \frac{1}{2}|\nabla u|^2 + u^\gamma \chi_{\{u>0\}}$ $\gamma \in (0, 2)$

  2022 DeSilva-Savin for $\gamma \in (-2, 0)$

One easily obtain existence of minimizer $u \geq 0$ to the above problems.

What shall we expect of minimizers? $\begin{cases} \text{harmonicity? sub/superharmonicity?} \\ \text{best regularity?} \\ \text{What equations they satisfy?} \end{cases}$

What's happening if we try to minimize the above?

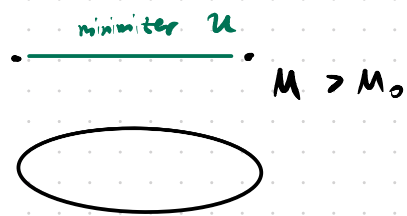
$|\nabla u|^2$ Dirichlet energy $\rightsquigarrow$ u try to stay "flat" or "linear".

$u^\gamma \chi_{\{u>0\}}$ Potential $\longrightarrow$ u try to be "zero"

u^γ denotes the weighted penalty for "positivity".

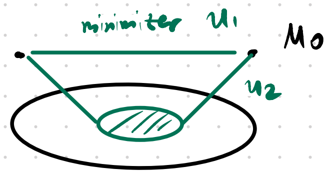
Classical example for one-phase Bernoulli

Boundary data $\Phi = M > 0$.



① for $M \gg 1$

② for certain threshold M_0



③ for $0 < M < M_0$

If we look at the Euler-Lagrange Equations

We're actually solving for both solution u & free Boundary $\partial\{u>0\}$.

Obstacle Problem $\Delta u = \chi_{\{u>0\}}$ B_1

Why? take any $\varphi \in C_0^\infty(B_1)$. consider perturbation $u+\varepsilon\varphi$

$$J^1(u, B_1) \le J^1(u+\varepsilon\varphi, B_1) = \int_{B_1} \tfrac{1}{2}|\nabla u|^2 + \varepsilon \nabla u \cdot \nabla\varphi + \tfrac{\varepsilon^2}{2}|\nabla\varphi|^2 + (u+\varepsilon\varphi)^+.$$

match at ε order $\quad 0 \le \int \nabla u \cdot \nabla\varphi + \int_{\{u>0\}} \varphi + \int_{\{u=0\}} \varphi^+$

if $\varphi \ge 0$ upward perturbation $\Rightarrow$ $\Delta u \le 1$

if $\varphi \le 0$ downward perturbation $\Rightarrow$ $\int_{\{u>0\}} -\varphi \le \int \nabla u \cdot \nabla\varphi$ $\varphi \le 0 \Rightarrow \chi_{\{u>0\}} \le \Delta u$

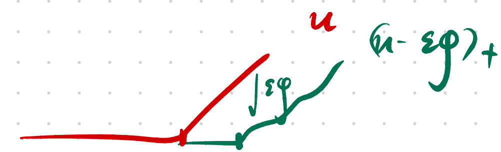
thus $\Delta u \in L^\infty \Rightarrow u \in W^{2,p}_{loc}(B_1)$ $\forall$ $1<p<\infty \Rightarrow D^2 u$ a.e. makes sense.

$\Rightarrow$ $\Delta u = 0$ on $\{u=0\}$ $\Rightarrow$ $\Delta u = \chi_{\{u>0\}}$ $\blacksquare$

Regularity given for free : $C^{1,1-\varepsilon}_{loc}(B_1)$ $\forall$ $0<\varepsilon<1$.

One-Phase Bernoulli

$$\begin{cases} \Delta u = 0 & \{u>0\} \\ |\nabla u| = 1 & \partial\{u>0\} \end{cases}$$



Let's discuss interior equation.

take $\varphi \in C_0^\infty(B_1)$ consider perturbation $(u-\varepsilon\varphi)_+$. $\varphi \geq 0$.

$J^0(u, B_1) \leq J^0((u-\varepsilon\varphi)_+, B_1) \Rightarrow \Delta u \geq 0$ subharmonic because "positive set shrinks".

for subharmonic functions. one has pointwise representation $\begin{cases} \tilde{u}(x_0) = \lim_{r\to 0} \fint_{B_r(x_0)} u \\ \tilde{u} \text{ is lower semi-continuous.} \end{cases}$

Q How to obtain $\Delta u \leq 0$ in $\{u>0\}$?

if we can do upward perturbations in $B_r(x_0) \subseteq \{u>0\}$ we're happy.

↳ so that positive set does not enlarge!

to do this: one need "continuity" of u.

so that $\{u>0\}$ is open set. and then one may squeeze in open balls.

Is u continuous?

Proposition let u minimize one phase. then $\forall\ 0<\alpha<1$, $u \in C^{0,\alpha}_{loc}(B_1)$.

key lemma (Campanato) $\exists\ \delta = \delta(n,\alpha) > 0$ s.t. for u minimize to $J^0_\delta(u) = \int_{B_1} |\nabla u|^2 + \delta\, \chi_{\{u>0\}}$ and $\fint_{B_1} |\nabla u|^2 \le 1$. one has $\rho^{2-2\alpha} \fint_{B_\rho} |\nabla u|^2 \le 1$ for $\rho = \rho(n,\alpha) \in (0,1)$.

proof consider harmonic replacement ϕ of u in B_1. compare $J^0_\delta(u) \le J^0_\delta(\phi)$

$$\int_{B_1} |\nabla(u-\phi)|^2 \le \delta \cdot |B_1| \quad \text{Also} \quad |\nabla \phi(x)|^2 \le \fint_{B_{1/2}} |\nabla \phi|^2 \le \fint_{B_1} |\nabla u|^2 \lesssim 1.$$

$$\text{so } \rho^{2-2\alpha} \fint_{B_\rho} |\nabla u|^2 \lesssim \rho^{2-2\alpha-n} \cdot \delta + \rho^{2-2\alpha} \le 1.$$

first choose ρ small
then chose δ small. ∎

Now rescale $u_\rho(x) = \frac{1}{\rho^\alpha} u(\rho x) \Rightarrow u_\rho$ minimize $J^0_{\rho^{2-2\alpha}\delta}$ → even smaller perturbation

$\Rightarrow$ keep iterating $r^{2-2\alpha} \fint_{B_r} |\nabla u|^2 \le 1 \quad \forall\ 0<r<1 \Rightarrow u \in C^{0,\alpha}_{loc}$.

Now for u minimizer to J^0. do $\tilde{u} = \frac{1}{\|u\|_{L^2}} \delta \cdot u \Rightarrow \tilde{u}$ minimize J^0_δ ∎

this gives us interior equation ! $\Delta u = 0 \quad \{u>0\}$.

What about Boundary ?. We do Domain Deformation. $y = x + \varepsilon \xi(x)$. $\xi \in C_0^\infty(B_1; \mathbb{R}^n)$

so $D_x y = I + \varepsilon D\xi$. $D_y x = I - \varepsilon D\xi + o(\varepsilon)$. $\det(D_x y) = 1 + \varepsilon \operatorname{div}(\xi) + o(\varepsilon)$

consider competitor $u_\varepsilon(y) = u(x)$.

$$\int_{B_1} |\nabla u_\varepsilon(y)|^2 dy = \int_{B_1} \nabla u^T (1 - \varepsilon D\xi + o(\varepsilon))^T (1 - \varepsilon D\xi + o(\varepsilon)) \nabla u \left(1 + \varepsilon \operatorname{div}\xi + o(\varepsilon)\right) dx$$

$$= \int_{B_1} |\nabla u|^2 + \varepsilon \boxed{\int_{B_1} -2\nabla u^T D\xi \nabla u + \operatorname{div}(\xi) |\nabla u|^2} + o(\varepsilon)$$

(*)

$$\int_{B_1} \chi_{\{u_\varepsilon > 0\}} dy = \int_{B_1} \chi_{\{u>0\}} + \varepsilon \int \chi_{\{u>0\}} \operatorname{div}(\xi)\, dx \quad (**)$$

If we're able to integrate by parts.

$$(*) = \int_{B_1} \operatorname{div}\left(-2(\nabla u \cdot \xi)\nabla u + |\nabla u|^2 \xi\right) dx = \int_{\partial\{u>0\}} \left(-2(\nabla u \cdot \xi)\nabla u + |\nabla u|^2 \xi\right) \cdot \nu \, d\mathcal{H}^{n-1}$$

But $\nu = -\dfrac{\nabla u}{|\nabla u|}$ outwards normal $= \displaystyle\int_{\partial\{u>0\}} -|\nabla u|^2 \xi \cdot \nu \, d\mathcal{H}^{n-1}$

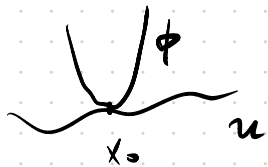
On the other hand. $(**) = \displaystyle\int_{\partial\{u>0\}} \xi \cdot \nu \, dx \Rightarrow |\nabla u| = 1$ on $\partial\{u>0\}$.

Another Way to interpret the Boundary : Viscosity sense.

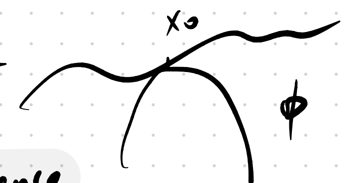
We say $u \in C(B_1)$ is viscosity solution to one-phase Bernoulli

if • $\Delta u = 0$ in $\{u>0\}$ in viscosity sense

i.e.,



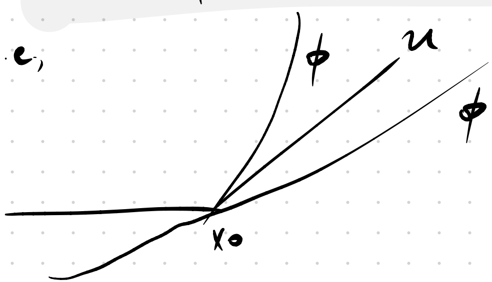
$\Rightarrow \Delta\phi(x_0) \ge 0$ and



$\Rightarrow \Delta\phi(x_0) \le 0$

• $|\nabla u| = 1$ on $\partial\{u>0\}$ in viscosity sense

i.e.,



$\forall\ \phi \in C^2$, ϕ^+ touch u from above at $x_0 \in \partial\{u>0\}$.

then $|\nabla\phi(x_0)| \ge 1$

ϕ touch u from below at $x_0 \in \partial\{u>0\}$

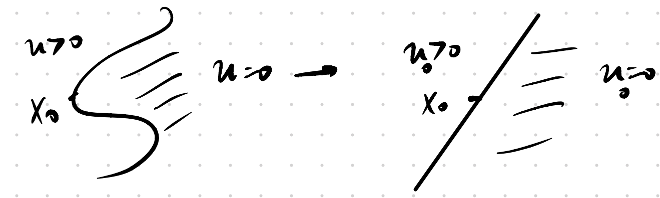
then $|\nabla\phi(x_0)| \le 1$

it's often easier to make sense as viscosity solution.

So far we've discussed minimizer u.

People are moreover interested in how smooth $\partial\{u>0\}$ is. and its Singular set.

Key idea one may blow up at a given point $x_0 \in \partial\{u>0\}$. Flatness implies $C^{1,\alpha}$



if $\partial\{u_0>0\}$ is flat. then

for small neighborhood around x_0. Boundary is $C^{1,\alpha}$.

Some questions to ask.

- Why can one blow up?
- What do blow ups look like?
- Can we Classify Blow ups in some sense?
- When do we see "flat" blow ups?

OK... before everything. What do we mean by blow up?

$$u_r(x) = \frac{1}{r^\alpha} u(rx)$$

↳ dilation

choose α so that u_r remains minimizer in B_1.

We compute natural rescaling for obstacle & one phase Bernoulli.

rewrite $r^{\alpha} u_r\left(\frac{x}{r}\right) = u(x)$

$$J^1\left(r^{\alpha} u_r\left(\tfrac{x}{r}\right), B_r\right) = \int_{B_r} r^{2\alpha-2} |\nabla u_r|^2 + r^{\alpha} (u_r)_+ \Rightarrow 2\alpha - 2 = \alpha \Rightarrow \alpha = 2.$$

$$J^0\left(r^{\alpha} u_r\left(\tfrac{x}{r}\right), B_r\right) = \int_{B_r} r^{2\alpha-2} |\nabla u_r|^2 + \chi_{\{u_r > 0\}} \Rightarrow 2\alpha - 2 = 0 \Rightarrow \alpha = 1.$$

Q: What allows us to blow up?

Optimal Growth Rate & Non-degeneracy

- obstacle $\quad c r^2 \le \sup_{B_r} u \le C \cdot r^2 \quad 0 \in \partial\{u > 0\}$
- one-phase Bernoulli $\quad c r \le \sup_{r} u \le C \cdot r \quad 0 \in \partial\{u > 0\}$

The above yields the following

for x close to $\partial\{u>0\}$

① Optimal Regularity

- obstacle $r^2 \|D^2 u\|_{L^\infty(B_r(x))} \leq C \cdot \left(\|u\|_{L^\infty(B_r(x))} + r^2 \right) \leq C \cdot r^2 \Rightarrow |D^2 u(x)| \leq C$

  $u \in C^{1,1}$

- one-phase $\forall x \in \{u>0\}$. $r = \text{dist}(x, \{u=0\})$.

  $r \|\nabla u\|_{L^\infty(B_{r/2}(x))} \leq C \cdot \left(\|u\|_{L^\infty(B_{3r/4}(x))} \right) \leq C \cdot r \Rightarrow |\nabla u(x)| \leq C.$

  $u \in C^{0,1}$

② existence of nontrivial blowup limit

- obstacle
  - $\|u_r\|_{L^\infty(B_1)} \leq C$.
  - $\|D^2 u_r\|_{L^\infty(B_1)} \leq C$.
- one-phase
  - $\|u_r\|_{L^\infty(B_1)} \leq C$
  - $\|\nabla u_r\|_{L^\infty(B_1)} \leq C$.

$\Bigg\}$ Ascoli-Arzela $\Bigg\{$ $u_r \xrightarrow{C^{1,\alpha}} u_0 \in C^{1,1}$; $u_r \xrightarrow{C^{0,\alpha}} u_0 \in C^{0,1}$

moreover using non-degeneracy. $0 \in \partial\{u_0>0\}$.

u_0 are nontrivial, global minimizers.

- obstacle
$$\begin{cases} \Delta u_0 = \chi_{\{u_0>0\}} & \mathbb{R}^n \\ u_0 \geq 0 \\ 0 \in \partial\{u_0>0\} \end{cases} \qquad u_0 \in C^{1,1}(\mathbb{R}^n)$$

- one-phase
$$\begin{cases} \Delta u_0 = 0 & \{u_0>0\} \\ u_0 \geq 0 \\ 0 \in \partial\{u_0>0\} \end{cases} \qquad u_0 \in C^{0,1}(\mathbb{R}^n)$$

that u_r are local minimizers ensure u_0 is local minimizer in $\mathbb{R}^n$

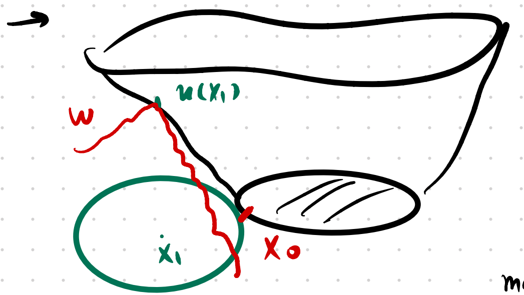
Q How to obtain such optimal growth Rate & Nondegeneracy ?

- Obstacle

→ $\Delta u = \chi_{\{u>0\}}$ for $0 \in \partial\{u>0\}$. Harnack yields

$$\sup_{B_r} u \le C\cdot\left(\inf_{B_r} u + r^2\right) \le C r^2$$

→



$w(x) := u(x) - \frac{1}{4n}|X - X_1|^2$ in $B_r(X_1)$ $\quad r = \mathrm{dist}(X_1, \{u=0\})$

Let $X_0 \in \partial\{u>0\}$ achieve $|X_1 - X_0| = r$.

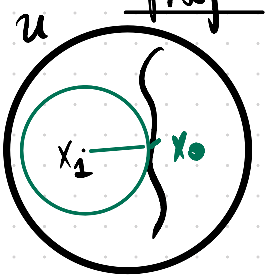
$\Delta w = \Delta u - \frac{1}{2} \ge \frac{1}{2}$ in $B_r(X_1) \subseteq \{u>0\}$.

maximum principle $0 < u(X_1) = w(X_1) \le \sup_{\partial B_r(X_1)} w \le \sup_{\partial B_r(X_1)} u - \frac{1}{4n} r^2$

$$\Rightarrow \quad c_n r^2 \le \sup_{B_r(X_0)} u \quad \text{for } X_0 \in \partial\{u>0\}.$$

- One-phase

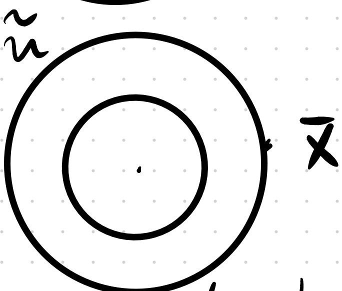
Let's do optimal growth rate via "viscosity approach".



Proof: take $x_1 \in \{u>0\}$. $r = \mathrm{dist}(x_1, \{u=0\})$, rescale $\tilde u(x) = \frac{1}{r} u(x_1 + rx)$

Problem reduces to showing $\tilde u(0) \le C$. Let $\tilde u(\bar x) = \frac{1}{r} u(x_0) = 0$

Idea: Build v s.t. $\begin{cases} \tilde u \ge v & \partial B_1 \cup \partial B_{1/2} \\ \Delta \tilde u = \Delta v & B_1 \setminus B_{1/2} \end{cases} \Rightarrow \tilde u \ge v \quad B_1 \setminus B_{1/2}$



Also require that $v = 0$ ∂B_1. Since $\tilde u(\bar x) = 0$

$\Rightarrow$ v touch $\tilde u$ from below at $\bar x \in \partial\{\tilde u > 0\}$.

use $\tilde u$ is viscosity solution. $|\nabla v(\bar x)| \le 1$.

↳ want to encode useful information

to make this useful, note $\begin{cases} \tilde u \ge c_0 \sup_{B_{1/2}} \tilde u \ge c_0 \tilde u(0) & \partial B_{1/2} \\ \tilde u \ge 0 & \partial B_1 \end{cases}$

Refine $v(x) = C_1 \cdot c_0 \tilde u(0) \left(|x|^{2-n} - 1 \right)$ for C_1 s.t. $v|_{\partial B_{1/2}} = c_0 \tilde u(0)$

Q : Can we classify Blow ups in some sense ?

Blowups are **Homogeneous** !

Key tool : Weiss Monotonicity formula (Weiss 1999)

- obstacle u_0 2-homogeneous $u_0(\lambda x) = \lambda^2 u_0(x)$

$$W_u(r) := \int_{B_1} \frac{1}{2}|\nabla u_r|^2 + u_{r+} - \int_{\partial B_1} u_r^2 \, d\mathcal{H}^{n-1} \quad \text{for } u_r(x) = \frac{1}{r^2} u(rx)$$

$$\frac{d}{dr} W_u(r) = \frac{1}{r}\int_{\partial B_1} |\nabla u_r \cdot x - 2u_r|^2 \geq 0 \Rightarrow W_u \nearrow \text{ monotone quantity}$$

- one-phase u_0 1-homogeneous $u_0(\lambda x) = \lambda u_0(x)$

$$W_u(r) := \int_{B_1} |\nabla u_r|^2 + \chi_{\{u_r > 0\}} - \int_{\partial B_1} u_r^2 \, d\mathcal{H}^{n-1} \quad \text{for } u_r(x) = \frac{1}{r} u(rx)$$

$$\frac{d}{dr} W_u(r) = \frac{2}{r}\int_{\partial B_1} |\nabla u_r \cdot x - u_r|^2 \geq 0$$

Why does monotonicity of Weiss energy imply homogeneity?

say for obstacle problem. Use that $\lim_{r\to 0} W_u(r) = W_u(0^+)$ exists.

$\forall \rho \in (0,1)$ $\quad W_u(0^+) = \lim_{r\to 0} W_u(\rho r) = \lim_{r\to 0} W_{u_r}(\rho) = W_{u_0}(\rho)$

where u_0 is the blow up.

But now RHS is constant in ρ $\Rightarrow \frac{d}{d\rho} W_{u_0} = 0$.

$\Rightarrow \quad \nabla (u_0)_\rho \cdot x - 2 (u_0)_\rho = 0 \quad \forall \rho$.

$= \nabla u_0(\rho x) \cdot \frac{x}{\rho} - 2 \frac{1}{\rho^2} u_0(\rho x)$

$\Rightarrow \nabla u_0(y) \cdot y - 2\, u_0(y) = 0 \quad \forall y \in \partial B_r \quad \forall r \in (0,1)$

$\Rightarrow$ homogeneous degree 2.

let's compute $\frac{d}{dr} W_u(r)$ for obstacle problem

$$\frac{d}{dr} W_u(r) = \int_{B_1} \nabla u_r \cdot \nabla \frac{d}{dr} u_r + \frac{d}{dr} u_r \chi_{\{u_r>0\}} - \int_{\partial B_1} 2 u_r \frac{d}{dr} u_r \, d\mathcal{H}^{n-1}$$

$$= \int_{B_1} \cancelto{0}{-\Delta u_r \cdot \frac{d}{dr} u_r + \frac{d}{dr} u_r \chi_{\{u_r>0\}}} + \int_{\partial B_1} (\nabla u_r \cdot x) \frac{d}{dr} u_r - \int_{\partial B_1} 2 u_r \frac{d}{dr} u_r \, d\mathcal{H}^{n-1}$$

$$= \int_{\partial B_1} \frac{d}{dr} u_r \left(\nabla u_r \cdot x - 2 u_r \right) d\mathcal{H}^{n-1} \qquad (*)$$

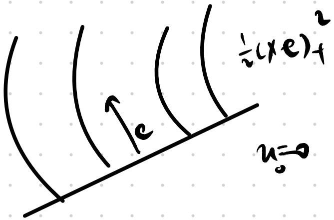
now $$\frac{d}{dr} u_r = \frac{d}{dr}\left(\frac{1}{r^2} u(rx) \right) = -\frac{2}{r^3} u(rx) + \frac{1}{r^2} \nabla u(rx) \cdot x$$

$$= -\frac{2}{r} u_r + (\nabla u_r) \cdot \frac{x}{r}$$

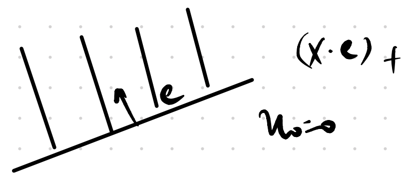
$$\Rightarrow \quad (*) = \frac{1}{r} \int_{\partial B_1} |\nabla u_r \cdot x - 2 u_r|^2 \, d\mathcal{H}^{n-1} \geq 0 \qquad \blacksquare$$

Q: What are our "flat solutions"?

- obstacle $\quad u_0(x) = \frac{1}{2}(x\cdot e)_+^2$



- one phase $\quad u_0(x) = (x\cdot e)_+$



Flatness implies $C^{1,\alpha}$

Theorem (1981 Alt-Caffarelli; 2009 De Silva)

$\exists\ \varepsilon>0,\ \delta>0$ s.t. for $u\geq 0$ $\begin{cases}\text{minimizer}\\ \text{viscosity solution}\end{cases}$ to one-phase $\begin{cases}\Delta u = 0 & \{u>0\}\\ |\nabla u| = 1 & \partial\{u>0\}\end{cases}$

that is ε-flat in B_1, i.e.

$$(x_n-\varepsilon)_+ \leq u(x) \leq (x_n+\varepsilon)_+ \qquad \forall\, x\in B_1$$

Then $\partial\{u>0\}$ is $C^{1,\alpha}$ graph in $B_\delta'\times(-\delta,\delta)$

Theorem (1977 Caffarelli) Let u be minimizer (solution) to obstacle.

If $\limsup_{r \to 0} \frac{|\{u>0\} \cap B_r|}{|B_r|} > 0$ (So $\exists r_k \to 0$ $u_{r_k} \to \frac{1}{2}(x \cdot e)_+^2$)

then $\partial\{u>0\}$ is $C^{1,\alpha}$ locally.

Q: Will the free Boundary $\partial\{u>0\}$ be smooth all the time?

this is equivalent to asking whether all minimizing global cones are "flat".

One Phase	Obstacle
smooth:	
• $n=2$ 1981 Alt-Caffarelli	• $n \geq 2$ 1974 Schaeffer
• $n=3$ 2004 Caffarelli-Jerison-Kenig	there are already singular minimizing cones
• $n=4$ 2015 Jerison-Savin	But! We can classify all Blowups!
$n=5, 6$ is open!	
singular minimizing cone: $n \geq 7$ 2009 De Silva-Jerison	

Q: Classification of Blow up for Obstacle Problem?

There is one piece of additional information one get for Global Solutions to obstacle.

Convexity !

<u>Theorem</u> (1977 Caffarelli) let $u_0 \in C^{1,1}$ $\begin{cases} \Delta u_0 = \chi_{\{u_0>0\}} & \mathbb{R}^n \\ u_0 \geq 0 & \mathbb{R}^n \end{cases}$, $\|D^2 u\|_{L^\infty(\mathbb{R}^n)} < \infty$

then u_0 is convex. and $\{u_0 = 0\}$ is convex.

<u>Rmk</u> the proof works for Global solutions! Not necessarily Blowups.

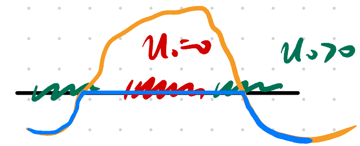
But for sake of simplicity, we argue assuming 2-homogeneity.

<u>Proof (assume u_0 2-homogeneous)</u> let $e \in S^{n-1}$. $w := \min\{\partial_{ee} u_0, 0\}$

<u>Claim</u> $\boxed{w \text{ is superharmonic.}}$

Consider 2nd difference quotient $\delta_t^2 u_0 = \frac{1}{t^2}\left(u_0(x+te) + u_0(x-te) - 2u_0(x)\right)$

then $\begin{cases} \delta_t^2 u_0 \geq 0 & \{u_0 = 0\} \\ \Delta \delta_t^2 u_0 \leq 0 & \{u_0 > 0\} \end{cases}$



$\Rightarrow$ $\min\{\delta_t^2 u_0, 0\}$ superharmonic. send $t \to 0$.

$\Rightarrow$ superharmonicity implies $w(x) = \sup_{r>0} \fint_{B_r(x)} w$ decreasing average.

Now. notice w is in addition 0-homogeneous

$\Rightarrow$ w is constant along all lines passing through origin.

thus $\exists$ global infimum!

↳ which is 0 (otherwise there's nothing to prove)

now superharmonicity forces w to stay 0.

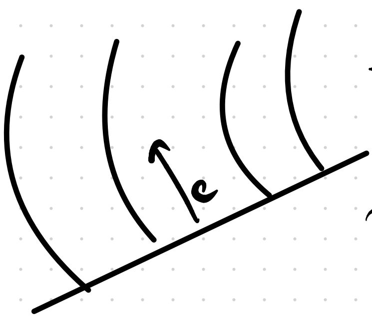
$\Rightarrow$ $w \equiv 0$ $\Rightarrow$ $\partial_{ee} u_0 \geq 0$ $\Rightarrow$ convexity ∎

Using Homogeneity + Convexity to Classify Blowups

Theorem: let u_0 be blow up limit for obstacle problem. $0 \in \partial\{u_0 > 0\}$.

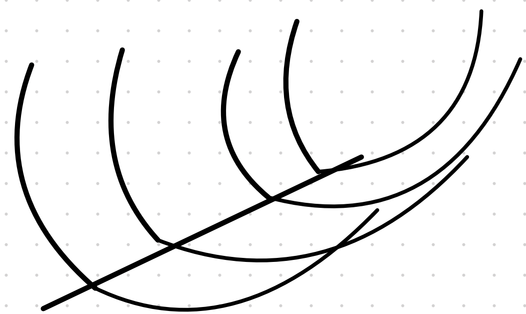
if $\{u_0 = 0\}$ has nonempty interior: $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$ for $e \in S^{n-1}$.

if $\{u_0 = 0\}$ has empty interior: $u_0(x) = \frac{1}{2} x^T A x$ for $A \geq 0$, $\operatorname{tr}(A) = 1$.



$\frac{1}{2}(x \cdot e)_+^2$

$u_0 = 0$



$\frac{1}{2} x^T A x$

$\hookrightarrow \ker(A)$

$W_{u_0} = \frac{1}{2}\alpha_n$

$W_{u_0} = \alpha_n$

Proof if $\{u_0=0\}$ nonempty interior, it is closed, convex cone.

$\forall \tau \in S^{n-1}$ s.t. $-\tau \in \{u_0=0\}^\circ$. We claim $\boxed{\partial_\tau u_0 \geq 0}$.

indeed. $g(t) = u_0(x+t\tau)$. $g''(t) \geq 0 \Rightarrow g'(t) \nearrow$. But $g'(t)=0$ for $t \ll -1$.

Now let $v(x) = \partial_\tau u_0$.

v solves $\begin{cases} \Delta v = 0 & \{u_0>0\} \\ v > 0 & \{u_0>0\} \\ v = 0 & \{u_0=0\} \end{cases}$ and v homogeneous degree 1

$\Rightarrow$ $\{u_0=0\}$ has to be half space.

moreover. since u_0 convex and contains straight line $\Rightarrow$ u_0 one-dimensional

assume $u_0(x) = U(x\cdot e)$ $\begin{cases} \Delta u_0 = U''(x\cdot e) = 1 \\ U'(0) = 0 \\ U(0) = 0 \end{cases}$ $\Rightarrow u_0(x) = \frac{1}{2}(x\cdot e)_+^2$

if $\{u_0=0\}$ empty interior

$\{u_0=0\}$ convex $\subseteq \partial H$ hyperplane

But $u_0 \in C^{1,1}$ and $\Delta u_0 = 1$ on $\{u_0>0\} \supseteq \mathbb{R}^n \setminus \partial H$.

$\Rightarrow$ $\Delta u_0 = 1$ $\mathbb{R}^n$ globally

Now by Liouville all second order derivatives bdd & harmonic

$\Rightarrow$ $u_0(x) = \frac{1}{2}x^T A x + b\cdot x + c$

conclude using $u_0(0) = \nabla u_0(0) = 0$

$u_0 \geq 0 \Rightarrow A \geq 0$

$\Delta u_0 = 1 \Rightarrow tr(A) = 1$

$\square$