Optimal Regularity for Alt-Phillips Type Free Boundary Problem via Dichotomy Argument.

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Alt-Caffarelli-Phillips Free Boundary Problem

Given $\gamma \in (-2,2)$ study non-negative minimizers $u \in H^1(B_1)$ to

$$J^{\gamma}(u, B_1) := \int_{B_1} |\nabla u|^2 + u^{\gamma} \chi_{\{u>0\}} \qquad B_1 \subseteq \mathbb{R}^d.$$

- $\gamma = 1$ $\qquad \Delta u = \frac{1}{2} \chi_{\{u>0\}}$

Obstacle Problem
[Caffarelli 1977]

- $\gamma = 0$ $\qquad \begin{cases} \Delta u = 0 & \{u>0\} \\ |\nabla u| = 1 & \partial\{u>0\}. \end{cases}$

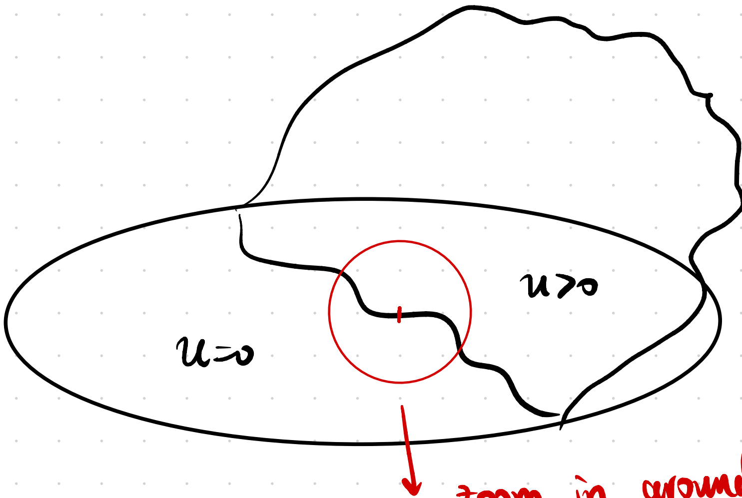
One Phase Bernoulli Problem
[Alt-Caffarelli 1981]

- $\gamma \in (0,2)$
- $\gamma \in (-2,0)$

$$\Delta u = \frac{1}{2} \gamma \cdot u^{\gamma-1} \cdot \chi_{\{u>0\}}$$

[Alt-Phillips 1986]
[De Silva - Savin 2022]

Notice $\partial\{u>0\} \cap B_1$ is unknown!



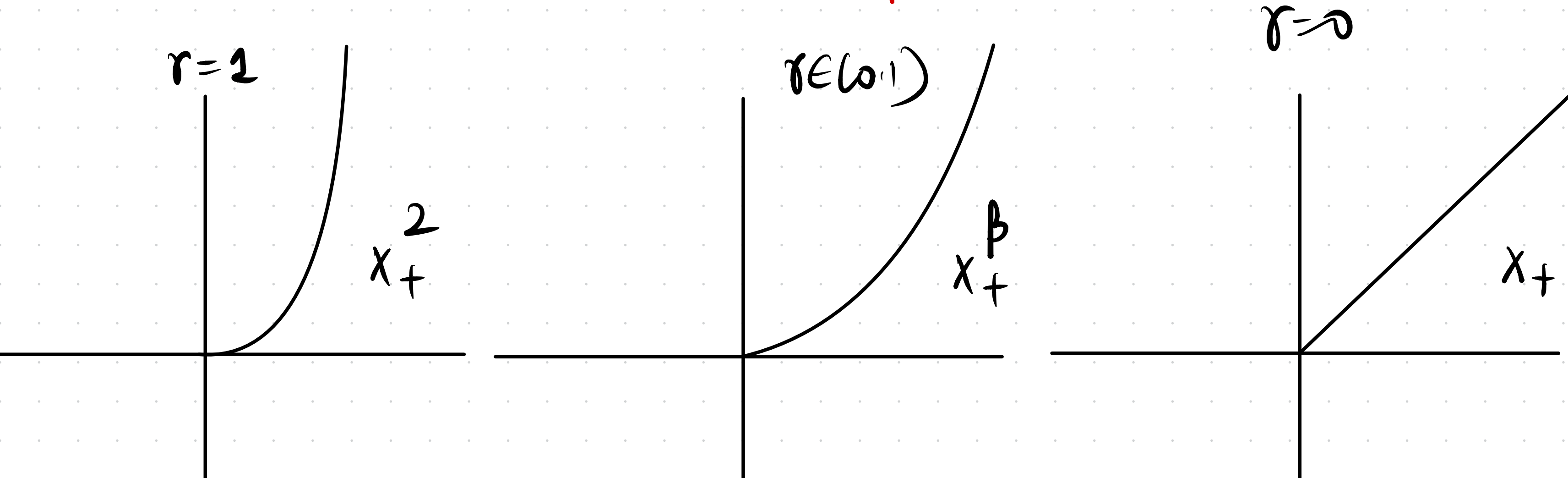
zoom in around this point

rescale $u_r(x) := \dfrac{u(rx)}{r^\beta}$

then $r^\beta u_r\left(\frac{x}{r}\right) = u(x)$ minimize

$$J^\gamma(u, B_1) = \int_{B_1} r^{2\beta-2}\left|\nabla u_r\left(\tfrac{x}{r}\right)\right|^2 + r^{\beta\gamma} u_r^\gamma\left(\tfrac{x}{r}\right)_+$$

match $2\beta - 2 = \beta\gamma \Rightarrow \beta = \dfrac{2}{2-\gamma}$



What's the Story of these problems?

- About minimizers
  1) Existence, Basic Estimates.
  2) upper / lower bounds on rate of growth from free boundary $\partial\{u>0\}$.
     - optimal regularity
     - non-degeneracy
  3) rescaling and blow up $\Rightarrow$ nontrivial global solution
  4) Weiss monotonicity formula $\Rightarrow$ homogeneous global solution.
  5) How to classify nontrivial global homogeneous solutions? { flat / singular
- About free boundaries
  1) density estimates
  2) improvement of flatness $\Rightarrow$ Regularity of free boundary
  3) Structure of Singular set? Hausdorff dimension, Rectifiability.

For "Optimal" Regular. What's the key?

look at $\gamma = 1$. so $\Delta u = \chi\{u>0\}$.

- By elliptic regularity $\Delta u \in L^\infty \Rightarrow u \in C^{1,\alpha}_{loc}$ $\forall$ $0<\alpha<1$.

- By Harnack take $x_0 \in \partial\{u>0\}$.

$$\sup_{B_r(x_0)} u \le C(d)\Big(\inf_{B_r(x_0)} u + r^2 \|\chi\{u>0\}\|_{L^\infty(B_r(x_0))}\Big) \le C(d)\, r^2$$

optimal growth rate

$\Rightarrow$ How to conclude that in fact $u \in C^{1,1}_{loc}$?

$\forall$ $x \in \{u>0\} \cap B_{1/2}$. take $x_0 \in \partial\{u>0\}$ s.t. $|x-x_0| = \mathrm{dist}(x, \partial\{u>0\}) = 2\rho$.

apply Schauder in $B_\rho(x)$. $\|D^2 u\|_{L^\infty(B_\rho(x))} \le C(d)\Big(\frac{1}{\rho^2}\|u\|_{L^\infty(B_{\rho}(x))} + 1\Big) \le C(d)$.

∎

What about Alt-Phillips?

equation $\Delta u = \gamma \cdot u^{\gamma-1} \chi_{\{u>0\}}$. right hand side explodes as $u \downarrow 0$.

- Assume we're able to obtain

① β-growth rate from free boundary, i.e., $\forall x_0 \in \partial\{u>0\}$. $|u(x)| \le C \cdot |x-x_0|^{\beta}$.

② Harnack Inequality, i.e., $\forall x \in \{u>0\}$. $c_0 u(x) \le u(y) \le C_0 u(x)$ $\forall |y| \le \eta_0 (u(x))^{\frac{1}{\beta}}$.

then to conclude $u \in C^{1,\beta-1}_{loc}$. $\forall x \in \{u>0\} \cap B_{1/2}$. we work with $B_{\eta_0 u(x)^{\frac{1}{\beta}}} \subseteq \{u>0\}$.

Denote $\rho = \frac{1}{2} \eta_0 u(x)^{\frac{1}{\beta}}$ then

$$\|\nabla u\|_{C^{0,\beta-1}(B_\rho(x))} \le C(d,\gamma) \frac{1}{\rho^{\beta}} \left(\|u\|_{L^\infty(B_\rho(x))} + \rho^2 \|u^{\gamma-1}\|_{L^\infty(B_\rho(x))} \right)$$

$$\le C(d,\gamma,c_0,C_0,\eta_0) \left(1 + \rho^{2-\beta} \cdot \rho^{\beta(\gamma-1)} \right) \le C(d,\gamma)$$

☒

Question: How to obtain β-growth rate & a Harnack type Inequality?

the original method was established in [Phillips 1983].

But the proof was quite constructive ...... hard to apply to other problems.

We use a Dichotomy Argument originated from [De Silva - Savin 2020]

Idea let u be minimizer.

If $\left(\fint_{B_1} u^2\right)^{\frac{1}{2}} = a \geq a_*$ sufficiently large.

then the rescaling $\tilde{u} = \frac{1}{a} u$ minimizes

$$J_\delta^r(\tilde{u}) = \int_{B_1} |\nabla \tilde{u}|^2 + a^{r-2} \tilde{u}_+^r = \int_{B_1} |\nabla \tilde{u}|^2 + \delta F(x, \tilde{u})$$

where $\delta = a^{-1}$ is small perturbation and $0 \leq F(x, \tilde{u}) \leq 1 + \tilde{u}^2$.

So J_δ^r is "morally" a perturbed Dirichlet energy.

the most natural competitor to choose is thus h harmonic replacement of $\tilde{u} = \frac{1}{a} u$.

We compare energies! $J_\delta^r(\tilde{u}, B_1) \leq J_\delta^r(h, B_1)$

$$\int_{B_1} |\nabla(\tilde{u}-h)|^2 = \int_{B_1} |\nabla \tilde{u}|^2 - |\nabla h|^2 \leq \delta \int_{B_1} 1 + \underline{h^2}.$$

Can I make this small, universal ?

yes! $\int_{B_1} h^2 \leq 2\int_{B_1} (\tilde{u}-h)^2 + \tilde{u}^2 \leq C\left(\int_{B} |\nabla(\tilde{u}-h)|^2 + 1\right)$

choose δ small to hide this "$\int_{B_1} |\nabla(\tilde{u}-h)|^2$"

thus $\int_{B_1} |\nabla(\tilde{u}-h)|^2 + |\tilde{u}-h|^2 \leq C \cdot \delta$

Our $\tilde{u}$ really is close to h in H^1 sense. !

Can we upgrade such "closedness to harmonic replacement" as one step inside?

We distinguish into two cases

① $0 < h(0) \le c_0$ for some c_0 small universal to choose

② $c_0 < h(0) \le C_0$. (note the upper bound is always true!)

Let's say we're in the case ① $h(0) \le c_0$ is small!

using h is harmonic for any $\zeta \in (0, 1/2)$.

$$\fint_{B_\zeta} |h - h(0)|^2 \le \sup_{B_\zeta} |\nabla h|^2 \cdot \zeta^2 \le \Big(\sup_{B_{3\zeta/2}} h\Big)^2 \le C\, h(0)^2 \le C \cdot c_0^2.$$

thus $$\fint_{B_\zeta} \tilde{u}^2 \le 2\Big(\fint_{B_\zeta} |\tilde{u} - h|^2 + |h - h(0)|^2\Big) \lesssim \zeta^{-d} \cdot \delta + c_0^2$$

scale back. $$\zeta^{-2\beta} \fint_{B_\zeta} u^2 \lesssim \big(\zeta^{-2\beta} \zeta^{-d} a^{-1} + \zeta^{-2\beta} c_0^2\big)\, a^2 \le \frac{1}{4} a^2$$

What can we say in the other case ② $c_0 < h(0) \le c_0$?

using h is harmonic. $\fint_{B_\eta} |h - h(0)|^2 \le C \cdot \eta^2 \fint_{B_1} |h - h(0)|^2 \le C \cdot \eta^2$

$$\fint_{B_\eta} |\tilde{u} - h(0)|^2 \le 2 \fint_{B_\eta} |\tilde{u} - h|^2 + |h - h(0)|^2 \lesssim \eta^{-d} \cdot \delta + \eta^2 .$$

We call $m := a\,h(0)$

so Rescale back. $\fint_{B_\eta} |u - m|^2 \lesssim (\eta^{-d}\delta + \eta^2)\, a^2 .$

Now for any $\varepsilon \in (0,1)$ of our choice

one may first choose η small depending on ε.

then choose $\delta = a^{-1}$ small depending on ε

So that $\left(\fint_{B_\eta} |u - m|^2\right)^{1/2} \le \varepsilon \cdot a$

Let's summarize our main Dichotomy Proposition

Prop for d, γ. $C_0 = C_0(d, \gamma)$ universal, $\forall \varepsilon \in (0,1)$ and $\forall \xi \in (0, \frac{1}{2})$

there exists $\eta = \eta(\varepsilon)$ small, $c_0 = c_0(\xi)$ small, and $a_* = a_*(\varepsilon, \xi)$ large

such that for any $u \in H^1(B_1)$, $u \geq 0$, $\left(\fint_{B_1} u^2\right)^{\frac{1}{2}} = a \geq a_*$

that minimizes J^γ.

either $$\xi^{-\beta}\left(\fint_{B_\xi} u^2\right)^{\frac{1}{2}} \leq \frac{1}{2} a \qquad (1)$$

or for certain $c_0 a \leq m \leq C_0 a$

$$\left(\fint_{B_\eta} |u - m|^2\right)^{\frac{1}{2}} \leq \varepsilon \cdot a \qquad (2)$$

How do we make use of this?

using a Hölder iteration under (2).

one may choose $\varepsilon > 0$ small so that for any

u as minimizer with $\left(\fint_{B_1} u^2\right)^{\frac{1}{2}} = a$.

one has $\frac{1}{4} C_0 a \le u(x) \le 4 C_0 a \quad \forall\, x \in B_{\eta^*}$

So u stays a distance away from 0

of height comparable to $\left(\fint_{B_1} u^2\right)^{\frac{1}{2}}$.

within a ball of universal radius.

Now we conclude our Proof.

Assume $\fint_{B_1} u^2 \le 1$.

Fix for example $\zeta = \frac{1}{2}$, we work with sequence 2^{-k}. Let $a_k := (2^{-k})^{-\beta} \left(\fint_{2^{-k}} u^2 \right)^{\frac{1}{2}}$

note $a_{k+1} \le \bar{C} a_k$ ok. For "A" large universal to choose

Consider the set $\mathcal{K} := \{ k \in \mathbb{N} \mid a_k \le \bar{C} A + 2^{-k} \}$.

1 $0 \in \mathcal{K}$; 2 if $\mathbb{N} = \mathcal{K}$. then a_k is uniformly bounded

thus $u(0) = 0$. and $\sup_{B_r(0)} u \le C \cdot r^\beta$. optimal growth.

3 otherwise consider first k st. $k \in \mathcal{K}$. but $k+1 \notin \mathcal{K}$.

then $a_k > A$. if not. $a_{k+1} \le \bar{C} a_k \le \bar{C} A$ contradiction

thus if choose $A \ge a_*$ from our Dichotomy Proposition. one may run the Dichotomy argument again !

Since $a_k \geq a_*$, either $a_{k+1} \leq \frac{1}{2} a_k$

or $\frac{1}{4} C_0 a_k \leq u_{2^{-k}}(x) = \frac{1}{2^{-k\beta}} u(2^{-k} x) \leq 4 C_0 a_k$ in B_{η^*}.

But the first alternative cannot happen, since then

$$a_{k+1} \leq \tfrac{1}{2} a_k \leq \tfrac{1}{2}(\bar{C}A + 2^{-k}) \leq \bar{C}A + 2^{-(k+1)}$$

contradicting $k+1 \notin \mathcal{K}$

Now in particular $u(0) \leq 4 C_0 a_k \cdot 2^{-k\beta} \leq \tilde{C} \cdot 2^{-k\beta}$.

So $(u(0))^{1/\beta} \lesssim C \cdot 2^{-k}$.

choose η_0 small universal so that $\eta_0 (u(x))^{1/\beta} \leq 2^{-k} \eta^*$.

thus, renaming c_0, C_0.

$$c_0 u(0) \leq u(x) \leq C_0 u(0) \quad \forall\ |x| \leq \eta_0 (u(x))^{\frac{1}{\beta}}$$

$\square$

THANK YOU.