

An Introduction to Obstacle and Alt-Caffarelli Free Boundary Problem

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Q What is the Study of Obstacle / One-Phase Free Boundary Problem?

- Obstacle
1977 Caffarelli
minimize $J^1(u, B_1) := \int_{B_1} \frac{1}{2} |\nabla u|^2 + u_+$ among $u \in H^1(B_1)$, $u = \varphi|_{\partial B_1} \geq 0$
- One phase Bernoulli
1981 Alt-Caffarelli
minimize $J^0(u, B_1) := \int_{B_1} |\nabla u|^2 + \chi_{\{u>0\}}$
- Alt-Phillips
1986 Alt-Phillips
minimize $J^\gamma(u, B_2) := \int_{B_1} \frac{1}{2} |\nabla u|^2 + u^\gamma \chi_{\{u>0\}}$ $\gamma \in (0, 2)$
2022 DeSilva-Savin
for $\gamma \in (-2, 0)$

One easily obtain existence of minimizer $u \geq 0$ to the above problems.

What shall we expect of minimizers? $\left\{ \begin{array}{l} \text{harmonicity? sub/superharmonicity?} \\ \text{best regularity?} \\ \text{what equations they satisfy?} \end{array} \right.$

What's happening if we try to minimize the above?

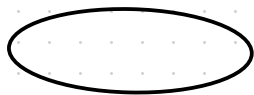
$10u^2$ Dirichlet energy $\rightarrow$ u try to stay "flat" or "linear".

$u^r \chi_{\{u>0\}}$ Potential $\rightarrow$ u try to be "zero"
 u^r denotes the weighted penalty for "positivity".

Classical example for one-phase Bernoulli

Boundary data $\bar{\Psi} = M > 0$. $\xrightarrow{\text{minimize } u}$ $M > M_0$

① for $M \gg 1$



② for certain threshold M_0



③ for $0 < M < M_0$



If we look at the Euler-Lagrange Equations

We're actually solving for both solution u & free Boundary $\partial\{u>0\}$.

Obstacle Problem $\Delta u = \chi_{\{u>0\}}$ B_1

Why? take any $\varphi \in C_0^\infty(B_2)$. consider perturbation $u + \varepsilon \varphi$

$$\int_{B_1} \frac{1}{2} |\nabla u|^2 + u_+ = J^2(u, B_2) \leq J^2(u + \varepsilon \varphi, B_2) = \int_{B_1} \frac{1}{2} |\nabla u|^2 + \varepsilon \nabla u \cdot \nabla \varphi + \frac{\varepsilon^2}{2} |\nabla \varphi|^2 + (u + \varepsilon \varphi)_+$$

match at ε order $0 \leq \int \nabla u \cdot \nabla \varphi + \int_{\{u>0\}} \varphi + \int_{\{u=0\}} \varphi_+$

if $\varphi \geq 0$ upward perturbation $\Rightarrow \Delta u \leq 1$ in B_1

if $\varphi \leq 0$ downward perturbation $\Rightarrow \int_{\{u>0\}} -\varphi \leq \int \nabla u \cdot \nabla \varphi \quad \varphi \leq 0 \Rightarrow \chi_{\{u>0\}} \leq \Delta u$

thus $\Delta u \in L^\infty \Rightarrow u \in W_{loc}^{2,p}(B_2) \quad \forall 1 < p < \infty \Rightarrow D^2 u$ makes sense a.e.

$$\Rightarrow \Delta u = 0 \text{ on } \{u=0\} \Rightarrow \Delta u = \chi_{\{u>0\}} \quad \square$$

Regularity given for free: $C_{loc}^{1,1-\varepsilon}(B_2) \quad \forall 0 < \varepsilon < 1.$

One-Phase Bernoulli

$$\begin{cases} \Delta u = 0 & \{u > 0\} \\ |\nabla u| = 1 & \partial\{u > 0\} \end{cases}$$

Let's discuss interior equation.

take $\varphi \in C_0^\infty(B_2)$ consider perturbation $(u - \varepsilon\varphi)_+$ $\varphi \geq 0$.

$J^0(u, B_2) \leq J^0((u - \varepsilon\varphi)_+, B_2) \Rightarrow \Delta u \geq 0$ **subharmonic** because "positive set shrinks".

for subharmonic functions one has **pointwise representation** $\tilde{u}(x_0) = \lim_{r \rightarrow 0} \int_{B_r(x_0)} u$

$\tilde{u}$ is lower semi-continuous.

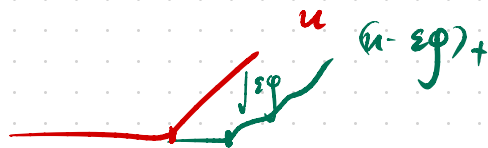
Q How to obtain $\Delta u \leq 0$ in $\{u > 0\}$?

if we can do upward perturbations in $\underline{B_r(x_0)} \subseteq \{u > 0\}$ we're happy.

to do this, one needs "continuity" of u . $\hookrightarrow$ so that positive set does not enlarge!

so that $\{u > 0\}$ is open set and then one may squeeze in open balls.

Is u continuous?



Proposition let u minimize **one-phase** then $\forall 0 < \alpha < 1$. $u \in C_{loc}^{0,\alpha}(B_2)$.

key lemma (Cianfrani) $\exists \delta = \delta(n, \alpha) > 0$ s.t. for u minimize to $J_\delta^0(u) = \int_{B_1} |\nabla u|^2 + \delta \chi_{\{u > 0\}}$
and $\int_{B_2} |\nabla u|^2 \leq 1$ one has $\int_{B_1} |\nabla u|^2 \leq 1$ for $\delta = \delta(n, \alpha) \in (0, 1)$

proof consider harmonic replacement ϕ of u in B_1 . compare $J_\delta^0(u) \leq J_\delta^0(\phi)$
 $\int_{B_1} |\nabla(u - \phi)|^2 \leq \delta |B_1|$ Also $|\nabla \phi(x)|^2 \leq \int_{B_{1/2}} |\nabla \phi|^2 \leq \int_{B_1} |\nabla u|^2 \leq 1$.
so $\int_{B_1} |\nabla u|^2 \leq \int_{B_1} |\nabla \phi|^2 + \delta |B_1| \leq 1$ first choose δ small
then choose δ small. $\square$

Now rescale $u_r(x) = \frac{1}{r^\alpha} u(rx) \Rightarrow u_r$ minimize $J_{\frac{\delta}{r^{2-2\alpha}}}^0$ $\rightarrow$ even smaller perturbation
 $\Rightarrow$ keep iterating $r^{2-2\alpha} \int_{B_r} |\nabla u|^2 \leq 1 \quad \forall 0 < r < 1 \Rightarrow u \in C_{loc}^{0,\alpha}$

Now for u minimize to J^0 . do $\tilde{u} = \frac{1}{\|u\|_{L^2}} u \Rightarrow \tilde{u}$ minimize $J_\delta^0 \quad \square$

The above gives us the interior equation $\Delta u = 0 \quad \{u > 0\}$

What about Boundary? We do **Domain Deformation**. $y = x + \varepsilon \zeta(x) \quad \zeta \in C_0^\infty(B_1; \mathbb{R}^n)$

$$\text{so } D_x y = \mathbb{I} + \varepsilon D \zeta \quad D_y x = \mathbb{I} - \varepsilon D \zeta + o(\varepsilon) \quad \det(D_x y) = 1 + \varepsilon \operatorname{div}(\zeta) + o(\varepsilon)$$

consider competitor $u_\varepsilon(y) = u(x)$

$$\begin{aligned} \int_{B_1} |\nabla u_\varepsilon(y)|^2 dy &= \int_{B_1} \nabla u^T (1 - \varepsilon D \zeta + o(\varepsilon))^T (1 - \varepsilon D \zeta + o(\varepsilon)) \nabla (1 + \varepsilon \operatorname{div}(\zeta) + o(\varepsilon)) dx \\ &= \int_{B_1} |\nabla u|^2 + \varepsilon \int_{B_1} \boxed{-2 \nabla u^T D \zeta \nabla u + \operatorname{div}(\zeta) |\nabla u|^2} + o(\varepsilon) \end{aligned} \quad (*)$$

$$\int_{B_1} \chi_{\{u < 0\}} dy = \int_{B_1} \chi_{\{u > 0\}} + \varepsilon \int \chi_{\{u > 0\}} \operatorname{div}(\zeta) dx \quad (**)$$

If we're able to integrate by parts.

$$(*) = \int_{B_1} \operatorname{div}(-2(\nabla u \cdot \zeta) \nabla u + |\nabla u|^2 \zeta) dx = \int_{\partial\{u > 0\}} (-2(\nabla u \cdot \zeta) \nabla u + |\nabla u|^2 \zeta) \cdot \nu d\mathcal{H}^{n-1}$$

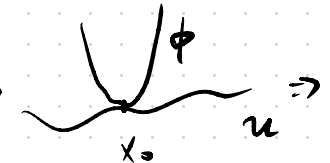
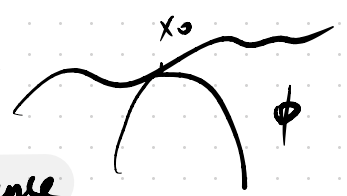
$$\text{But } \nu = -\frac{\nabla u}{|\nabla u|} \text{ outwards normal} \quad = \int_{\partial\{u > 0\}} -|\nabla u|^2 \zeta \cdot \nu d\mathcal{H}^{n-1}$$

$$\text{On the other hand. } (**) = \int_{\partial\{u > 0\}} \zeta \cdot \nu dx \Rightarrow |\nabla u| = 1 \text{ on } \partial\{u > 0\}$$

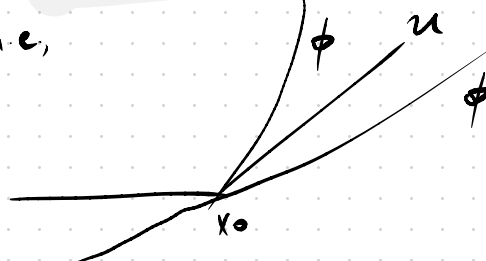
Another way to interpret the Boundary: **Viscosity sense.**

We say $u \in C(B_1)$ is **viscosity solution to one-phase Bernoulli**

if $\Delta u = 0$ in $\{u > 0\}$ in viscosity sense

i.e.,  $\Delta \phi(x_0) \geq 0$ and  $\Delta \phi(x_0) \leq 0$

$|\nabla u| = 1$ on $\partial\{u > 0\}$ in viscosity sense

i.e., 

$\forall \phi \in C^2$ ϕ^+ touch u from above at $x_0 \in \partial\{u > 0\}$

then $|\nabla \phi(x_0)| \geq 1$

ϕ touch u from below at $x_0 \in \partial\{u > 0\}$

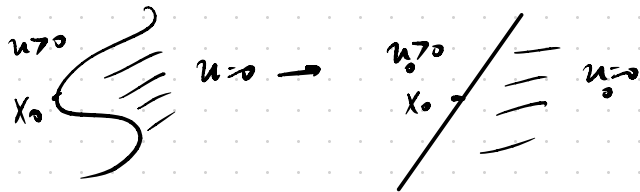
then $|\nabla \phi(x_0)| \leq 1$

it's often easier to make sense as viscosity solution.

So far we've discussed minimizer u .

People are moreover interested in how smooth $\partial\{u>0\}$ is. and its Singular set.

Key idea one may blow up at a given point $x_0 \in \partial\{u>0\}$. Flatness implies $C^{1,\alpha}$



if $\partial\{u_0>0\}$ is flat. then

for small neighborhood around x_0 . Boundary is $C^{1,\alpha}$.

Some questions to ask.

- Why can one blow up?
- What do blow ups look like?
- Can we classify Blow ups in some sense?
- When do we see "flat" blow ups?

OK... before everything. What do we mean by blow up?

$$u_r(x) = \frac{1}{r^\alpha} u(rx)$$

↪ dilation

choose α so that u_r remains minimizer in B_1 .

We compute natural rescaling for obstacle & one phase Bernoulli.

rewrite $r^\alpha u_r(\frac{x}{r}) = u(x)$

$$J^1(r^\alpha u_r(\frac{x}{r}), B_r) = \int_{B_r} r^{2\alpha-2} |\nabla u_r|^2 + r^\alpha (u_r)_+ \Rightarrow 2\alpha-2=\alpha \Rightarrow \alpha=2$$

$$J^0(r^\alpha u_r(\frac{x}{r}), B_r) = \int_{B_r} r^{2\alpha-2} |\nabla u_r|^2 + \chi\{u_r > 0\} \Rightarrow 2\alpha-2=0 \Rightarrow \alpha=1$$

Q: What allows us to blow up?

Optimal Growth Rate & Non-degeneracy

• obstacle

$$cr^2 \leq \sup_{B_r} u \leq C \cdot r^2$$

$$0 \in \partial\{u > 0\}$$

• one-phase Bernoulli

$$cr \leq \sup_{B_r} u \leq C \cdot r$$

$$0 \in \partial\{u > 0\}$$



Assume for the above lower/upper bounds. Then one has

① Optimal Regularity

• obstacle $r^2 \|D^2 u\|_{L^\infty(B_r(x))} \leq C \cdot (\|u\|_{L^\infty(B_r(x))} + r^2) \leq C \cdot r^2 \Rightarrow \begin{cases} \text{for } x \text{ close to } \partial\{u>0\} \\ |D^2 u(x)| \leq C \\ u \in C^{1,1} \end{cases}$

• one-phase $\forall x \in \{u>0\} \quad r = \text{dist}(x, \{u=0\})$

$r \| \nabla u \|_{L^\infty(B_{r/2}(x))} \leq C \cdot (\|u\|_{L^\infty(B_{3r/4}(x))}) \leq C \cdot r \Rightarrow \begin{cases} |\nabla u(x)| \leq C \\ u \in C^{0,1} \end{cases}$

② existence of nontrivial blowing up limit

• obstacle $\begin{cases} \|u_r\|_{L^\infty(B_1)} \leq C \\ \|D^2 u_r\|_{L^\infty(B_1)} \leq C \end{cases}$

• one-phase $\begin{cases} \|u_r\|_{L^\infty(B_1)} \leq C \\ \|\nabla u_r\|_{L^\infty(B_1)} \leq C \end{cases}$

} Ascoli-Arzelà

$\begin{cases} u_r \xrightarrow{C^{1,\alpha}} u_0 \in C^{1,1} \\ u_r \xrightarrow{C^{0,\alpha}} u_0 \in C^{0,1} \end{cases}$

more over using non-degeneracy $0 \in \partial\{u_0>0\}$

$u_0: \mathbb{R}^n \rightarrow [0, \infty)$ are nontrivial, global minimizers.

• obstacle

$$\begin{cases} \Delta u_0 = \chi_{\{u_0 > 0\}} & \mathbb{R}^n \\ u_0 \geq 0 \\ 0 \in \partial \{u_0 > 0\} \end{cases} \quad u_0 \in C^1(\mathbb{R}^n)$$

• one-phase

$$\begin{cases} \Delta u_0 = 0 & \{u_0 > 0\} \\ u_0 \geq 0 \\ 0 \in \partial \{u_0 > 0\} \end{cases} \quad u_0 \in C^1(\mathbb{R}^n)$$

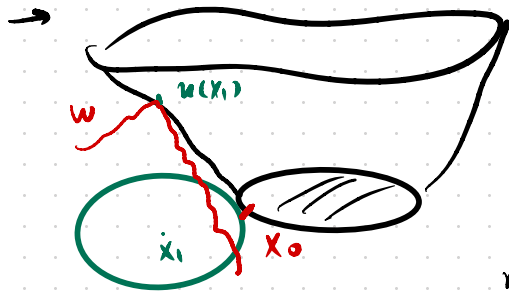
that u_r are local minimizers ensure
 u_0 is local minimizer in $\mathbb{R}^n$

Q How to obtain such optimal growth Rate & Nondegeneracy?

• Obstacle

→ $\Delta u = \chi_{\{u > 0\}}$ for $0 \in \partial\{u > 0\}$. Harnack yields

$$\sup_{B_r} u \leq C \left(\inf_{B_r} u + r^2 \right) \leq C r^2$$



$w(x) := u(x) - \frac{1}{4n} |x - x_1|^2$ in $B_r(x_1)$. $r = \text{dist}(x_1, \{u=0\})$

let $x_0 \in \partial\{u > 0\}$ achieve $|x_1 - x_0| = r$.

$\Delta w = \Delta u - \frac{1}{2} = \frac{1}{2}$ in $B_r(x_1) \subseteq \{u > 0\}$.

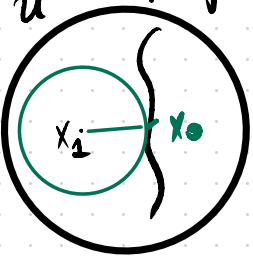
maximum principle $0 < u(x_1) = w(x_1) \leq \sup_{\partial B_r(x_1)} w \leq \sup_{\partial B_r(x_1)} u - \frac{1}{4n} r^2$

$$\Rightarrow C_n r^2 \leq \sup_{B_r(x_0)} u \quad \text{for } x_0 \in \partial\{u > 0\}. \quad \square$$

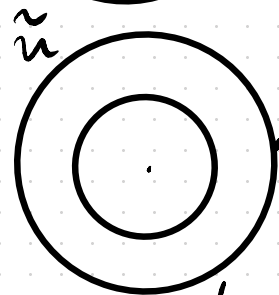
One-phase

let's do optimal growth rate via "viscosity approach".

Proof take $x_1 \in \{u > 0\}$, $r = \text{dist}(x_1, \{u=0\}) = |x_1 - x_0|$. Rescale $\tilde{u}(x) = \frac{1}{r} u(x_1 + rx)$
 Problem reduces to showing $\tilde{u}(0) \leq C$. let $\tilde{u}(\bar{x}) = \frac{1}{r} u(x_0) = 0$



Idea Build v s.t. $\begin{cases} \tilde{u} \geq v & \partial B_1 \cup \partial B_{1/2} \\ \Delta \tilde{u} = \Delta v & B_1 \setminus B_{1/2} \end{cases} \Rightarrow \tilde{u} \geq v \quad B_1 \setminus B_{1/2}$



Also require that $v \geq 0$ on ∂B_1 . Since $\tilde{u}(\bar{x}) = 0$

$\Rightarrow v$ touch $\tilde{u}$ from below at $\bar{x} \in \partial\{\tilde{u} > 0\}$.

use $\tilde{u}$ is viscosity solution. $|\nabla v(\bar{x})| \leq 1$.

$\hookrightarrow$ want to encode useful information.

to make this useful, note $\begin{cases} \tilde{u} \geq \inf_{B_{1/2}} \tilde{u} \geq \inf_{B_1} \tilde{u} & \partial B_{1/2} \\ \tilde{u} \geq 0 & \partial B_1 \end{cases}$

Define $v(x) = (1 - \inf_{B_1} \tilde{u})(|x|^{2n} - 1)$ for C_1 s.t. $v|_{\partial B_{1/2}} = \inf_{B_1} \tilde{u}$ $\square$

Q: Can we classify Blow ups in some sense?

Blowups are **Homogeneous**!

Key tool: Weiss Monotonicity formula (Weiss 1999)

• **obstacle** u_0 2-homogeneous $u_0(\lambda x) = \lambda^2 u_0(x)$

$$W_u(r) := \int_{B_1} \frac{1}{2} |\nabla u_r|^2 + u_r^2 - \int_{\partial B_1} u_r^2 d\mathcal{H}^{n-1} \quad \text{for } u_r(x) = \frac{1}{r^2} u(rx)$$

$$\frac{d}{dr} W_u(r) = \frac{1}{r} \int_{\partial B_1} |\nabla u_r \cdot X - 2 u_r|^2 \geq 0 \quad \Rightarrow \quad W_u \nearrow \text{monotone quantity}$$

• **one-phase** u_0 1-homogeneous $u_0(\lambda x) = \lambda u_0(x)$

$$W_u(r) := \int_{B_1} |\nabla u_r|^2 + \chi_{\{u_r > 0\}} - \int_{\partial B_1} u_r^2 d\mathcal{H}^{n-1} \quad \text{for } u_r(x) = \frac{1}{r} u(rx)$$

$$\frac{d}{dr} W_u(r) = \frac{2}{r} \int_{\partial B_1} |\nabla u_r \cdot X - u_r|^2 \geq 0$$

Why does monotonicity of Weiss energy imply homogeneity?

say for obstacle problem use that $\lim_{r \rightarrow 0} W_n(r) = W_n(0^+)$ exists.

$$\forall f \in (0,1) \quad W_n(0^+) = \lim_{r \rightarrow 0} W_n(fr) = \lim_{r \rightarrow 0} W_n(f) = W_{u_0}(f)$$

where u_0 is the blow up

But now RHS is constant in $f \Rightarrow \frac{d}{df} W_{u_0} = 0$

$$\Rightarrow \nabla(u_0) \cdot X - 2(u_0)_f = 0 \quad \forall f$$

$$= \nabla u_0(fX) \cdot \frac{X}{f} - 2 \int \nabla u_0(fX)$$

$$\Rightarrow \nabla u_0(y) \cdot y - 2 u_0(y) = 0 \quad \forall y \in \partial B_r \quad \forall r \in (0,1)$$

$\Rightarrow$ homogeneous degree 2.

let's compute $\frac{d}{dr} W_u(r)$ for obstacle problem

$$\begin{aligned}
 \frac{d}{dr} W_u(r) &= \int_{B_1} \nabla u_r \cdot \nabla \frac{d}{dr} u_r + \frac{d}{dr} u_r \chi_{\{u_r > 0\}} - \int_{\partial B_1} 2u_r \frac{d}{dr} u_r d\mathcal{H}^{n-1} \\
 &= \int_{B_1} -\Delta u_r \frac{d}{dr} u_r + \frac{d}{dr} u_r \chi_{\{u_r > 0\}} + \int_{\partial B_1} (\nabla u_r \cdot X) \frac{d}{dr} u_r - \int_{\partial B_1} 2u_r \frac{d}{dr} u_r d\mathcal{H}^{n-1} \\
 &= \int_{\partial B_1} \frac{d}{dr} u_r (\nabla u_r \cdot X - 2u_r) d\mathcal{H}^{n-1} \quad (*)
 \end{aligned}$$

$$\begin{aligned}
 \text{now } \frac{d}{dr} u_r &= \frac{d}{dr} \left(\frac{1}{r^2} u(r, x) \right) = -\frac{2}{r^3} u(r, x) + \frac{1}{r^2} \nabla u(r, x) \cdot X \\
 &= -\frac{2}{r} u_r + (\nabla u_r) \cdot \frac{X}{r}
 \end{aligned}$$

$$\Rightarrow (*) = \frac{1}{r} \int_{\partial B_1} |\nabla u_r \cdot X - 2u_r|^2 d\mathcal{H}^{n-1} \geq 0$$

□

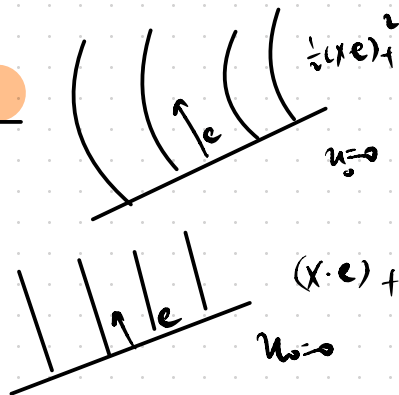
Q: What are our "flat solutions"?

• obstacle

$$u_0(x) = \frac{1}{2} (x \cdot e)_+^2$$

• one phase

$$u_0(x) = (x \cdot e)_+$$



Flatness implies $C^{1,\alpha}$

Theorem (1981 Alt-Caffarelli; 2008 deSilva)

$\exists \varepsilon > 0, \delta > 0$ s.t. for $u \geq 0$ $\left\{ \begin{array}{l} \text{minimizer} \\ \text{viscosity solution} \end{array} \right.$ to one-phase $\left\{ \begin{array}{l} \Delta u = 0 \quad \{u > 0\} \\ |Du| = 1 \quad \partial\{u > 0\} \end{array} \right.$

that is ε -flat in B_1 , i.e.

$$(x_n - \varepsilon)_+ \leq u(x) \leq (x_n + \varepsilon)_+ \quad \forall x \in B_1$$

Then $\partial\{u > 0\}$ is $C^{1,\alpha}$ graph in $B'_\delta \times (-\delta, \delta)$

Theorem (1977 Caffarelli) Let u be minimizer (solution) to obstacle

$$\text{If } \limsup_{r \rightarrow 0} \frac{|\{u > 0\} \cap B_r|}{|B_r|} > 0 \quad (\text{so } \exists r_k \rightarrow 0 \quad u_{r_k} \rightarrow \frac{1}{2} |x|_+^2)$$

then $\partial\{u > 0\}$ is $C^{1,\alpha}$ locally

Q: Will the free boundary $\partial\{u > 0\}$ be smooth all the time?

this is equivalent to asking whether all globally minimizing cones are "flat".

One Phase

- | | | | |
|--------------------------|---|------------|-------------------------------|
| Smooth | { | $n=2$ | 1981 Alt-Caffarelli |
| | | $n=3$ | 2004 Caffarelli-Jerison-Kenig |
| | | $n=4$ | 2015 Jerison-Savin |
| Singular minimizing cone | { | $n=5, 6$ | is open! |
| | | $n \geq 7$ | 2009 DeSilva-Jerison |

Obstacle

$n \geq 2$ 1974 Schaeffer
there are already singular minimizing cones.
But! We can classify all Blowups!

Q: Classification of Blow up for Obstacle Problem?

There is one piece of additional information one get for Global Solutions to obstacle.

Convexity!

Theorem (1977 Caffarelli) let $u_0 \in C^1$ $\left\{ \begin{array}{l} \Delta u_0 = \chi_{\{u_0 > 0\}} \mathbb{R}^n \\ u_0 \geq 0 \text{ on } \mathbb{R}^n \\ 0 \in \partial \{u_0 > 0\} \end{array} \right. \quad \|D^2 u_0\|_{L^\infty(\mathbb{R}^n)} < \infty$

then u_0 is convex, and $\{u_0 > 0\}$ is convex.

Rmk the proof works for Global solutions. Not necessarily Blow ups.

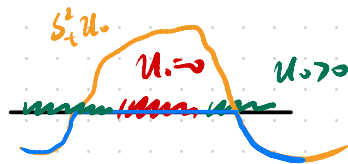
But for sake of simplicity, we argue assuming 2-homogeneity.

Proof (assume u_0 2-homogeneous) let $e \in S^{n-1}$ $w := \min\{\partial_{ee} u_0, 0\}$

Claim w is superharmonic.

Consider 2nd difference quotient $\delta_t^2 u_0 = \frac{1}{t^2} (u_0(x+te) + u_0(x-te) - 2u_0(x))$

then
$$\begin{cases} \delta_t^2 u_0 \geq 0 & \{u_0 = 0\} \\ \Delta \delta_t^2 u_0 \leq 0 & \{u_0 > 0\} \end{cases}$$



$\Rightarrow \min\{\delta_t^2 u_0, 0\}$ superharmonic $\Rightarrow$ send $t \rightarrow 0$, u is superharmonic

$\Rightarrow$ superharmonicity implies $u(x) = \sup_{r>0} \int_{B_r(x)} u$ decreasing average.

now, notice u is in addition 0 -homogeneous

$\Rightarrow u$ is constant along all lines passing through origin.

the value of u on ∂B_1 characterizes u . thus $\exists$ global infimum achieved by u .

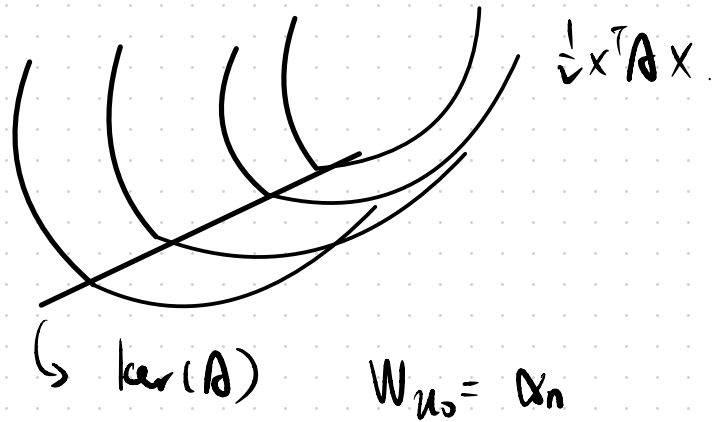
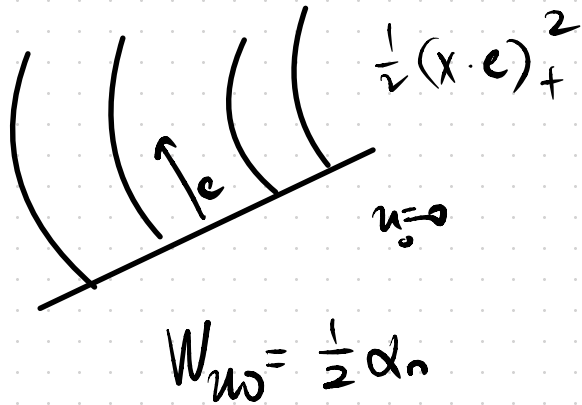
- if $\inf_{\partial B_1} u \geq 0$ then there is nothing to prove
- otherwise assume $\inf_{\partial B_1} u < 0$.

But using $\int_{B_r(x)} u$ is decreasing in r , in particular at point $u(x) = \inf_{\partial B_1} u$
 this forces u to stay constant, i.e., $u \equiv \inf_{\partial B_1} u < 0$. contradict $0 \in \partial\{u_0 > 0\}$



Using Homogeneity + Convexity to Classify Blowups

Theorem let u_0 be blow up limit for obstacle problem. $0 \in \partial\{u_0 > 0\}$.
 if $\{u_0 > 0\}$ has nonempty interior $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$ for $e \in S^{n-1}$.
 if $\{u_0 > 0\}$ has empty interior $u_0(x) = \frac{1}{2}x^T A x$ for $A \geq 0$ $\text{tr}(A) = 1$



Proof if $\{u_0=0\}$ nonempty interior, it is closed, convex, cone.

$\forall \tau \in S^{n-1}$ s.t. $-\tau \in \{u_0=0\}$. We claim $\partial_\tau u_0 \geq 0$.

indeed, $g(t) = u_0(x + t\tau)$. $g'(t) \geq 0 \Rightarrow g'(t) \nearrow$. But $g'(t) = 0$ for $t \ll -1$.

Now let $v(x) = \partial_\tau u_0$.

v solves $\begin{cases} \Delta v = 0 & \{u_0 > 0\} \\ v > 0 & \{u_0 > 0\} \\ v = 0 & \{u_0 = 0\} \end{cases}$ and v homogeneous degree 1

$\Rightarrow \{u_0 = 0\}$ has to be half space.

moreover, since u_0 convex and $\{u_0 = 0\}$ contains straight line $\Rightarrow u_0$ one-dimensional

assume $u_0(x) = U(x \cdot e)$ $\begin{cases} \Delta u_0 = U'(x \cdot e) = 1 \\ U'(0) = 0 \\ U(0) = 0 \end{cases} \Rightarrow u_0(x) = \frac{1}{2}(x \cdot e)^2$

if $\{u_0=0\}$ empty interior

$\{u_0=0\}$ convex $\subseteq \partial H$ hyperplane

But $u_0 \in C^1$ and $\Delta u_0 = 1$ on $\{u_0 > 0\} \subseteq \mathbb{R}^n \setminus \partial H$.

$\Rightarrow \Delta u_0 = 1$ $\mathbb{R}^n$ globally

Now by Liouville, all second order derivatives bdd & harmonic

$$\Rightarrow u_0(x) = \frac{1}{2} x^T A x + b \cdot x + c$$

conclude using $u_0(0) = \nabla u_0(0) = 0$

$$u_0 \geq 0 \Rightarrow A \geq 0$$

$$\Delta u_0 = 1 \Rightarrow \text{tr}(A) = 1$$

