Optimal Regularity for Multiple Membranes Alt-Phillips Free Boundary Problem

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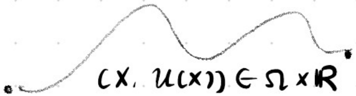
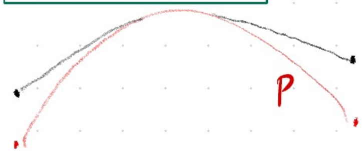
motivations



Consider elastic membrane, where $u : \Omega \to \mathbb{R}$ models its displacement.

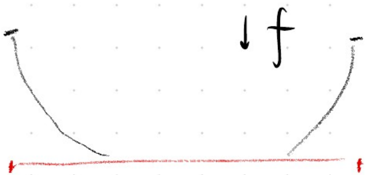
minimize surface tension $\displaystyle \min_{u=\varphi\ \partial\Omega} \int_\Omega \sqrt{1+|\nabla u|^2}\, dx \xrightarrow{|\nabla u| \ll 1} \min_{u=\varphi\ \partial\Omega} \int_\Omega \frac{1}{2}|\nabla u|^2\, dx$.

Obstacle Problem



Consider membrane u lying on top of obstacle P

one minimize w.r.t. constraint $\displaystyle \min_{\substack{u=\varphi\ \partial\Omega \\ u \geq P}} \int_\Omega \frac{1}{2}|\nabla u|^2\, dx$



Equivalently consider $v = u - P$, $f := -\Delta P > 0$

one minimize w.r.t. zero obstacle $\displaystyle \min_{\substack{u=\varphi\ \partial\Omega \\ u \geq 0}} \int_\Omega \frac{1}{2}|\nabla u|^2 + f u \ dx$

One-Phase Bernoulli Problem

consider "adhesive property" for our membrane.

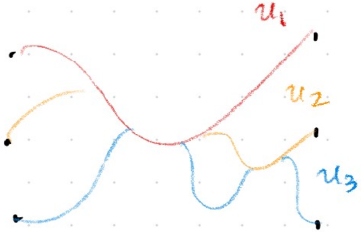
one minimize w.r.t. constraint as well as uniform penalty for detaching. $\displaystyle \min_{\substack{u=\varphi\ \partial\Omega \\ u \geq 0}} \int_\Omega |\nabla u|^2 + \chi_{\{u>0\}}\ dx$

Alt-Phillips Problem

$\displaystyle \min_{\substack{u=\varphi\ \partial\Omega \\ u \geq 0}} \int_\Omega |\nabla u|^2 + u^\gamma \chi_{\{u>0\}}\ dx. \quad \gamma \in (0,1)$

the Membranes Problem

consider "N" elastic membranes $u_i : \Omega \to \mathbb{R}$.



One would like to minimize energy among admissible set

$$\mathcal{A} = \left\{ U = (u_1, \cdots, u_N) \in H^1(B_1)^N \;\middle|\; u_1 \geq u_2 \geq \cdots \geq u_N,\ U - \Phi \in \left(H_0^1(B_1)\right)^N \right\}$$

non-penetration condition.

N-Membranes Problem (1985 Chipot-Vergara Caffarelli, 2019 Savin-Yu)

Given force field $f_1 > \cdots > f_N$.

$$\min_{U \in \mathcal{A}} \int_{B_1} \sum_{i=1}^{N} \left(|\nabla u_i|^2 + 2 f_i u_i \right) dx$$

Harmonic Graphs with Junction (2024 De Silva-Savin)

$$\min_{U \in \mathcal{A}} \int_{B_1} \left(\sum_{i=1}^{N} |\nabla u_i|^2 + \sum_{i=1}^{N-1} \chi_{\{u_i > u_{i+1}\}} \right) dx$$

In our work, we study on "Alt-Phillips" analog of the membranes problem.

Derivation of the model We consider energy for $\Omega \subseteq \mathbb{R}^d$. $\gamma \in (0,1)$

$$J^{\gamma}_{N,d}(U, \Omega) = \int_\Omega |\nabla U|^2 + W^{\gamma}_N(U) := \int_\Omega \sum_{i=1}^{N} |\nabla u_i|^2 + \sum_{i=1}^{N-1} \omega_i (u_i - u_{i+1})^{\gamma} \chi_{\{u_i > u_{i+1}\}}.$$

where $w_i > 0$ $\forall i = 1, \dots, N-1$, and $w_i > \frac{1}{2}(w_{i-1} + w_{i+1})$ $\forall i$, with convention $w_0 = w_N = 0$.

Why? assume $f_1 > \cdots > f_N$, constants, and that $\sum_{i=1}^{N} f_i = 0$.

then the choice of $w_i = 2\sum_{j=1}^{i} f_j \iff f_i = \frac{1}{2}(w_i - w_{i-1})$ satisfies $\sum_{i=1}^{N} 2 f_i u_i = \sum_{i=1}^{N-1} \omega_i (u_i - u_{i+1})$

For later purposes, we introduce for $K > 0$,

$$\mathcal{G}_K = \left\{ G : \Omega \times \mathbb{R}^N \to \mathbb{R} \,\middle|\, G(x,0) = 0,\ |G(x,U) - G(x,V)| \leq K|U - V| \quad \forall\, U, V \in \mathbb{R}^N \right\}$$

$\boxed{\text{Multiple Membranes Alt-Phillips Problem (2026 M)}}$ for $G \in \mathcal{G}_K$

$$\min_{U \in \mathcal{A}} J^{\gamma, G}_{N,d}(U, B_1) = \int_{B_1} |\nabla U|^2 + W^{\gamma}_N(U) + G(x,U)\, dx$$

Our main focus for today : Optimal Regularity

Theorem $\forall\ 0<\gamma<1$ $u \in C^{1,\beta-1}_{loc}(B_1)$, $\beta = \frac{2}{2-\gamma} \in (1,2)$

and for $C = C(d, N, \gamma, K, \omega) > 0$

$$\|u\|_{C^{1,\beta-1}(\overline{B}_{1/2})} \le C\left(1 + \|u\|_{L^2(B_1)}\right)$$

Why $\beta = \frac{2}{2-\gamma}$? this is Alt-Phillips invariant scale.

$$\text{for } u_r(x) = \frac{1}{r^\beta} u(rx) \iff u(x) = r^\beta u_r\left(\frac{x}{r}\right)$$

then

$$|\nabla u|^2 = r^{2\beta-2} |\nabla u_r|^2 \Rightarrow \text{choose } 2\beta - 2 = \gamma\beta$$

$$W_N^r(u) = r^{\gamma\beta}\, W_N^\gamma(u_r)$$

u_r minimizes $J_{N,d}^{\gamma, G_r}$ in B_1 with $G_r(x,V) = r^{2-2\beta} G(rx, r^\beta V) \in \mathcal{G}_K$

Hölder Regularity $\quad \forall \alpha \in (0,1)$

Let U minimize $\mathcal{F}^{\delta}(U, B_1) = \int_{B_1} |\nabla U|^2 + \delta F(x, U)$ for $F(x,U) \leq K(1 + |U|^2)$ and assume $⨍_{B_1} |U|^2 \leq 1$.

There exists $\delta_1 = \delta_1(d, N, \alpha, K) \in (0,1)$, $\rho = \rho(d, N, \alpha) \in (0, 1/2)$ s.t. $\forall\, 0 < \delta \leq \delta_1$.

$$\left(⨍_{B_\rho} |U - M|^2 \right)^{1/2} \leq \rho^\alpha \quad \text{for some } |M| \leq C(d, N)$$

Proof Consider Φ as harmonic replacement of U.

$$\int_{B_1} |\nabla (U - \Phi)|^2 \leq \delta \int_{B_1} F(x, \Phi) - F(x, U) \lesssim K \cdot \delta \cdot \left(1 + \int_{B_1} |\Phi|^2 \right)$$

But $\int_{B_1} |\Phi|^2 \lesssim \int |U - \Phi|^2 + \int |U|^2 \lesssim 1 + \int |\nabla (U - \Phi)|^2$

One may choose δ small so $\int |U - \Phi|^2 + \int |\nabla (U - \Phi)|^2 \lesssim K \cdot \delta$

in particular $M := \Phi(0)$ Then
$$\begin{cases} |M|^2 \leq ⨍_{B_1} |\Phi|^2 \lesssim \int |U - \Phi|^2 + \int |U|^2 \leq C(d, N) \\ ⨍_{B_\rho} |\Phi - M|^2 \lesssim \rho^2 ⨍_{B_1} |\Phi - M|^2 \lesssim \rho^2 \end{cases}$$

Hence $⨍_{B_\rho} |U - M|^2 \lesssim ⨍_{B_\rho} |U - \Phi|^2 + ⨍_{B_\rho} |\Phi - M|^2 \lesssim \rho^{-n} \cdot K \cdot \delta + \rho^2 \leq \rho^{2\alpha}$

first choose ρ small then δ small $\quad \square$

We want to iterate the above. set $U_0 := U$. $P_0 := 0$. $F_0 := F$

construct inductively $P_k := \sum_{j=1}^{k} \rho^{(j-1)\alpha} M_j$, $U_k(x) := \rho^{-k\alpha}(U(\rho^k x) - P_k)$, $F_k(x,V) := \rho^{2k(1-\alpha)} F(\rho^k x, P_k + \rho^{k\alpha} V)$

thus U_k minimizes $\int_{B_1} |\nabla U_k|^2 + \delta F_k(x, U_k)$ along with

$|P_k| \le \frac{C}{1-\rho^\alpha} = \tilde{C}$. $\fint_{B_1} |U_k|^2 \le 1$, $|F_k(x,V)| \le K(1 + |P_k + \rho^{k\alpha} V|^2) \le K(1 + 2\tilde{C}^2 + 2|V|^2) \le \tilde{K}(1 + |V|^2)$

apply previous lemma to $\delta \le \delta_1(\tilde{K})$ then $|M_{k+1}| \le C$. $\fint_{B_\rho} |U_k - M_{k+1}|^2 \le \rho^{2\alpha}$.

thus define $M_\infty := \lim_{k\to\infty} P_k$.

so that $\left(\fint_{B_{\rho^k}} |U - M_\infty|^2\right)^{\frac{1}{2}} \le C \cdot \rho^{k\alpha}$

Corollary $\forall \alpha \in (0,1)$ U minimize $J^{r,G}_{N,d}$.

then $U \in C^{0,\alpha}_{loc}(B_1)$ and $\|U\|_{C^{0,\alpha}(\overline{B_{1/2}})} \le C(d, N, \alpha, r, K, \omega)\left(1 + \|U\|_{L^2(B_1)}\right)$

Assume for $\sum_{i=1}^N u_i = 0$.

perturbing all membranes simultaneously gives $\Delta\left(\frac{1}{N}\sum_{i=1}^N u_i\right) = \frac{1}{2N}\sum_{i=1}^N \partial_{u_i} G(x, U) \in L^\infty \Rightarrow \frac{1}{N}\sum_i^N u_i \in C^{1,\alpha}$ $\forall\, 0<\alpha<1$

and $\bar{U} = U - \left(\frac{1}{N}\sum_{i=1}^N u_i\right)\mathbb{1}$ minimizes some other $J_{N,d}^{\gamma,\bar{G}}$ for $\bar{G} \in \mathcal{G}_K$

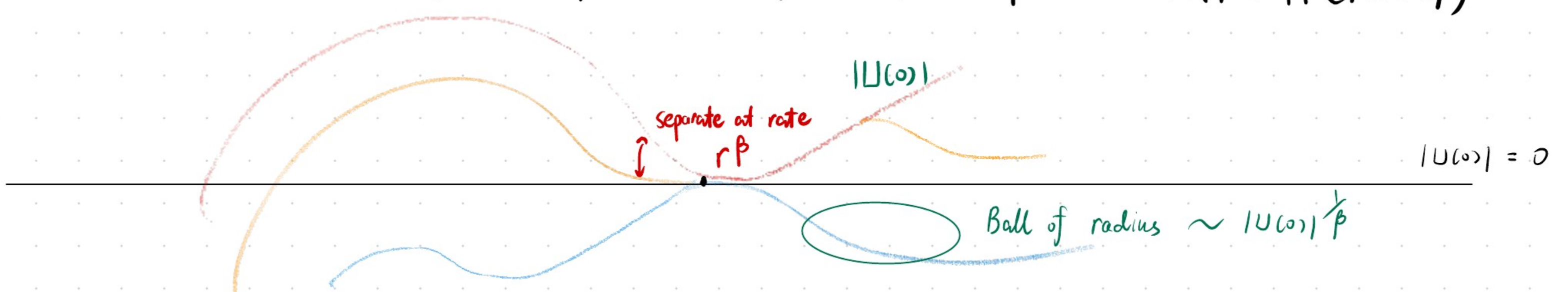
Key Proposition (β-growth rate & Harnack Inequality) Recall $\beta = \frac{2}{2-\gamma} \in (1,2)$.

There exists universal constants $c_H > 0$, $C_H \geq 1$, $r_H \in (0,1)$ depending only on d, N, γ, K, ω s.t.

for U as zero mean minimizer to $J_{N,d}^{\gamma, G}$ with $G \in \mathcal{G}_K$, and $\fint_{B_1} |U|^2 \leq 1$.

either $U(0) = 0 \in \mathbb{R}^N$, and $r^{-\beta}\left(\fint_{B_r}|U|^2\right)^{1/2} \leq C_H$ $\quad \forall\, 0<r<1/2$

or $U(0) \neq 0$, and $c_H|U(0)| \leq |U(x)| \leq C_H|U(0)|$ $\quad \forall\, |x| \leq r_H(|U(0)|)^{\frac{1}{\beta}}$



How does our Proposition help us?

Consider $N = 1$. one has equation $\Delta u = \frac{\gamma}{2}\omega^{\gamma} u^{\gamma-1} + \frac{1}{2}\partial_u G$ in $\{u>0\}$.

then $\forall\ x \in \{u=0\}$. $\left(⨍_{B_r} |u|^2\right)^{\frac{1}{2}} \leq C r^{\beta}$

$\forall\ x \in \{u>0\}$. take $B_\rho(x) \subsetneq \{u>0\}$ where $\rho = r_H\, u(x)^{\frac{1}{\beta}}$. now on $B_{\rho/2}(x)$. $\Delta u \in L^\infty$.

then $$\|\nabla u\|_{C^{0,\beta-1}(B_{\rho/2}(x))} \leq C \cdot \rho^{-\beta}\left(\|u\|_{L^\infty(B_\rho(x))} + \rho^2\left(\|u^{\gamma-1}\|_{L^\infty(B_\rho(x))} + K\right)\right)$$
$$\lesssim \rho^{-\beta}\left(u(x) + \rho^2\left(u(x)^{\gamma-1} + K\right)\right) \lesssim 1 + \rho^{2-\beta}\cdot\rho^{(\gamma-1)\beta} \lesssim 1.$$

Conclude using Campanato's approach

Consider $N \geq 2$

then $\forall\ x \in \{|U|=0\}$. $\left(⨍_{B_r} |U|^2\right)^{\frac{1}{2}} \leq C \cdot r^{\beta}$

$\forall\ x \in \{|U|>0\}$ on the ball $B_\rho(x) \subsetneq \{|U|>0\}$ $\rho \sim |U(x)|^{\frac{1}{\beta}}$

one has strict separation of two membranes $\Rightarrow$ induction.

Now we're left with two issues

① How to prove the Key Proposition (β-growth rate & Harnack Inequality) ?

② For more layers of membranes. how to run an induction argument ?

$\rightsquigarrow$ let's first look at the key Proposition.

- For scalar $N=1$, $G\equiv 0$ this is 1983 Phillips. 1986 Alt-Phillips.

- to avoid using the equation, we adopt method from 2020 De Silva-Savin

  the full argument for $N=1$ is established in 2026 M.

Idea : Dichotomy Argument

We assume $\left(\fint_{B_1} |U|^2\right)^{\frac{1}{2}} = a \geq a_*$ sufficiently large

then $\tilde{U} = \frac{1}{a} U$ is "perturbation" for minimizer to Dirichlet energy.

$$\tilde{U} \text{ minimizes } \int_{B_1} |\nabla \tilde{U}|^2 + a^{-1}\Big(\underbrace{a^{\gamma-1} W_N^{\gamma}(\tilde{U}) + a^{-1} G(x, a\tilde{U})}_{\leq C(1+|\tilde{U}|^2)}\Big)\,dx$$

treating $\delta = a^{-1}$. mimicking what we had for Hölder Regularity.

one obtain, for Φ as harmonic replacement of $\tilde{U}$ in B_1

$$\int_{B_1} |\tilde{U} - \Phi|^2 + |\nabla(\tilde{U} - \Phi)|^2 \le C \cdot \delta, \quad |\Phi(0)|^2 \le \fint_{B_1} |\Phi|^2 \le C_0^2$$

Now we distinguish into two cases.

① Assume $|\Phi(0)| \le c_0$ for certain c_0 small to choose.

then $\forall \phi_j$ $\fint_{B_\xi} |\phi_j - \phi_j(0)|^2 \lesssim \|\nabla \phi_j\|^2_{L^\infty(B_\xi)} \, \xi^2 \overset{\text{gradient estimate}}{\lesssim} \|\phi_j\|^2_{L^\infty(B_{2\xi})}$ this step uses $\sum_{i=1}^N u_i = 0$

$$\overset{\text{Harnack}}{\lesssim} |\phi_1(0) - \phi_N(0)|^2 \lesssim |\Phi(0)|^2 \lesssim c_0^2$$

thus $$\fint_{B_\xi} |\tilde{U}|^2 \lesssim \fint_{B_\xi} |\tilde{U} - \Phi|^2 + |\Phi - \Phi(0)|^2 \lesssim \xi^{-d} \cdot \delta + c_0^2$$

$$\xi^{-2\beta} \fint_{B_\xi} |U|^2 \lesssim \xi^{-2\beta} \left(\xi^{-d} a^{-1} + c_0^2\right) a^2 \le \frac{1}{4} a^2$$

choose c_0 small. then choose a large

② Assume $c_0 \le |\Phi(0)| \le C_0$

Since Φ is harmonic, $\fint_{B_\eta} |\Phi - \Phi(0)|^2 \lesssim \eta^2$. Denote $M := a\cdot\Phi(0)$.

now $$\fint_{B_\eta} |U - M|^2 \lesssim \fint_{B_\eta} |U - a\Phi|^2 + |a\Phi - M|^2 \le \left(\eta^{-d}\cdot\delta + \eta^2\right) a^2$$

let $\varepsilon > 0$ be arbitrary. One may choose $\eta = \eta(\varepsilon) > 0$ small, then $a_* = a_*(\varepsilon, \xi)$ large

so that $$\left(\fint_{B_\eta} |U - M|^2\right)^{\frac{1}{2}} \le \varepsilon\cdot a$$

lemma (Dichotomy) $\forall \varepsilon > 0$, $\forall \xi \in (0, \frac{1}{2}]$, $\exists\, \eta(\varepsilon)$ small, $a_*(\varepsilon, \xi)$ large

s.t. for any U minimizer to $J^{\gamma, G}_{N, d}$, $G \in \mathcal{G}_K$, zero mean.

and that $\left(\fint_{B_1} |U|^2\right)^{\frac{1}{2}} = a \ge a_*$

Then either $\xi^{-\beta}\left(\fint_{B_\xi} |U|^2\right)^{1/2} \le \frac{1}{2} a$

or for $c_0 a \le |M| \le C_0 a$, $\left(\fint_{B_\eta} |U - M|^2\right)^{1/2} \le \varepsilon\cdot a$

In fact one may fix $\varepsilon_* > 0$ small so that using Hölder Regularity.

$$\frac{1}{4} C_0 a \le |U(x)| \le 4 C_0 a \text{ in } B_{\eta^*}, \quad \left(\fint_{B_r} |U - U(0)|^2 \right)^{1/2} \le \varepsilon_* a \cdot r^\alpha \quad \forall \ r \le \eta^*$$

Now we prove

> **Key Proposition** (β-growth rate & Harnack Inequality)
>
> There exists universal constants $c_H > 0$, $C_H \ge 1$, $r_H \in (0,1)$ depending only on d, N, γ, K, ω s.t. for U as zero mean minimizer to $J^{\gamma, G}_{N, d}$ with $G \in \mathcal{G}_K$, and $\fint_{B_1} |U|^2 \le 1$.
>
> either $U(0) = 0 \in \mathbb{R}^N$, and $r^{-\beta} \left(\fint_{B_r} |U|^2 \right)^{1/2} \le C_H \quad \forall \ 0 < r < 1/2$
>
> or $U(0) \ne 0$, and $c_H |U(0)| \le |U(x)| \le C_H |U(0)| \quad \forall \ |x| \le r_H (|U(0)|)^{\frac{1}{\beta}}$

Proof Let $\xi = 1/2$. We work with sequence $a_k := (2^{-k})^{-\beta} \left(\fint_{B_{2^{-k}}} |U|^2 \right)^{\frac{1}{2}} = \left(\fint_{B_1} |U_{2^{-k}}|^2 \right)^{\frac{1}{2}}$

Notice $a_{k+1} \le \bar{C} a_k \ \forall k$ for $\bar{C} = \bar{C}(d, \gamma)$.

Consider the set $\mathcal{K} := \{ k \in \mathbb{N} \mid a_{k+1} \le \bar{C} A + 2^{-k} \}$ for A to be determined

1) $0 \in \mathcal{K}$

2) if $\mathcal{K} = \mathbb{N}$, then a_k is uniformly bounded. thus $U(0) = 0$ and $\left(\fint_{B_r} |U|^2 \right)^{\frac{1}{2}} \lesssim r^\beta$.

3) let $k \in \mathcal{K}$ be the first such that $k+1 \notin \mathcal{K}$

then claim: $a_k > A$ (if not $a_{k+1} \le \bar{C} a_k \le \bar{C} A$ contradict $k+1 \notin \mathcal{K}$)

now choose $A \ge a_*$ from Dichotomy lemma

so that either $a_{k+1} \le \frac{1}{2} a_k$

or $\frac{1}{4} C_0 a_k \le |U_{2^{-k}}(x)| = (2^{-k})^{-\beta} |U(2^{-k}x)| \le 4 C_0 a_k \quad \forall x \in B_{\eta_*}$

claim the first alternative cannot happen (if so, $a_{k+1} \le \frac{1}{2} a_k \le \frac{1}{2}(\bar{C}A + 2^{-k})$ contradict $k+1 \notin \mathcal{K}$)

thus plug in $x=0$ $\quad 2^{-k\beta} \lesssim |U(0)| \le 4 C_0 a_k (2^{-k})^{\beta} \lesssim 2^{-k\beta}$

so that $\quad |U(0)|^{1/\beta} \lesssim 2^{-k}$.

choose r_H so small s.t. $r_H |U(0)|^{1/\beta} \le 2^{-k} \eta_*$

thus $\quad c_H |U(0)| \le |U(x)| \le C_H |U(0)| \quad \forall x \in B_{r_H |U(0)|^{1/\beta}}$ $\square$

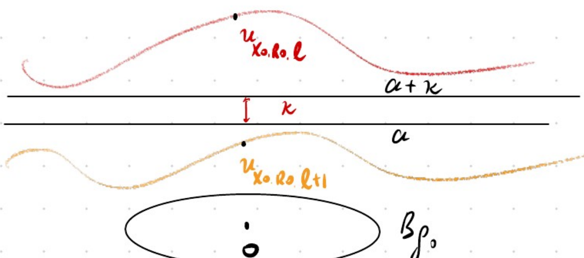
What's the remaining issue?

take U minimizer to $J^{\gamma,\sigma}_{N,d}$ zero mean and $\left(⨍_{B_1}|U|^2\right)^{\frac{1}{2}} \le 1$.

- for $x_0 \in \{|U|=0\} \cap B_{1/2}$ full contact set. $\left(⨍_{B_r}|U|^2\right)^{\frac{1}{2}} \le C \cdot r^{\beta}$
- for $x_0 \in \{|U|>0\} \cap B_{1/2}$. $\exists R_0 \sim |U(x_0)|^{\frac{1}{\beta}}$ s.t. $|U_{x_0,R_0}(y)| = R_0^{-\beta}|U(x_0+R_0 y)| \sim 1$ in B_1

there exists $\ell \in \{1, \dots, N-1\}$ such that $u_{x_0,R_0,\ell}(0) - u_{x_0,R_0,\ell+1}(0) \gtrsim 1/2$

now using $U \in C^{0,\alpha}$ Hölder Regularity.



for $\rho_0 \in (0,1)$ universal. for some $a \in \mathbb{R}$

$$u_{x_0,R_0,\ell} \ge a + \kappa \ge a > u_{x_0,R_0,\ell+1} \quad \text{in } B_{\rho_0}.$$

two membranes separate!

In the endpoint cases $\gamma=0$ or $\gamma=1$. the upper block $U^+ = (u_1, \dots, u_\ell)$ and lower block $U^- = (u_{\ell+1}, \dots, u_N)$ minimize respectively J_ℓ and $J_{N-\ell}$

But. this is not necessarily the case for general $\gamma \in (0,1)$.

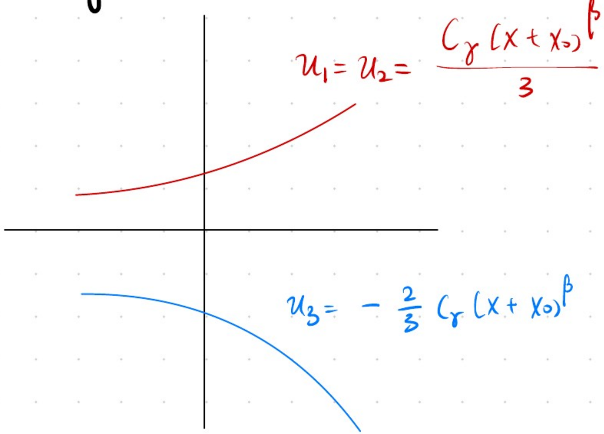
Why is this an issue?

Say for $\gamma = 0$. if $u_\ell > u_{\ell+1}$ in B_1. assume V^+ minimize J_ℓ^0 (instead of U^+)

let V^- minimize $J^0_{N-\ell}$. then using $\Delta v_\ell \le 0 \le \Delta v_{\ell+1} \Rightarrow v_\ell > v_{\ell+1}$ so $\int_{B_1} \chi_{\{v_\ell > v_{\ell+1}\}} = |B_1|$

now $J_N^0(V) = J_\ell^0(V^+) + J_{N-\ell}^0(V^-) + |B_1| < J_\ell^0(U^+) + J_{N-\ell}^0(U^-) + |B_1| = J_N^0(U)$. contradiction.

But for general $\gamma \in (0,1)$. we can construct a counter example.



note $U^+ = (u_1, u_2)$ does not minimize J_2^γ

because $J_2^\gamma(U^+) = \int |\nabla u_1|^2 + |\nabla u_2|^2 + \cancel{(u_1 - u_2)_+^\gamma}$

a minimizer should be harmonic!

This is why we introduce $G \in \mathcal{G}_K$

take any minimizer U to $J^{\gamma, G}_{N,d}$

then immediately $U^+ = (u_1, \dots, u_\ell)$ minimize $J^{\gamma, G^+_\ell}_{\ell, d}$

where $G^+_\ell(x, V) := w^\gamma_\ell (v_\ell - u_{\ell+1})^\gamma_+ + G(x, V, U^-) - w^\gamma_\ell (-u_{\ell+1})^\gamma_+ - G(x, 0, U^-)$

among $A^+ := \{ V = (v_1, \dots, v_\ell) \mid v_1 \geq \dots \geq v_\ell \geq u_{\ell+1} \}$

But there's two constraints!

① A^+ is restricted to $v_\ell \geq u_{\ell+1}$

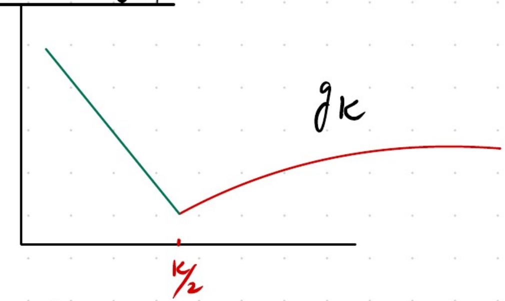
② $G^+_\ell \notin \mathcal{G}_K$

Even if we assume $u_\ell \geq u_{\ell+1} + K$, so that $\gamma (u_\ell - u_{\ell+1})^{\gamma-1} \leq \gamma K^{\gamma-1}$ bounded.

Q Can we redesign our functional so that the G-term belongs to $\mathcal{G}_K$ and we minimize among A ?

<u>Consider huge Lipschitz Penalization for v_ℓ close to $u_{\ell+1}$</u>

$$g_K(s) := \begin{cases} s^\gamma & s \ge K/2 \\ |s - K/2| + (K/2)^\gamma & s < K/2 \end{cases}$$



define $G^+_{\ell,K}(x,V) := \omega_\ell^\gamma g_K(v_\ell - u_{\ell+1}) + G(x,V,U^-) - \omega_\ell^\gamma g_K(-u_{\ell+1}) - G(x,0,U^-)$

so that $\quad G^+_{\ell,K} \in \mathcal{G}_{K + \omega_\ell^\gamma \|g_K\|_{C^{0,1}}}$

<u>lemma</u> let V^* be minimizer to

$$J_{\ell,d}^{\gamma, G^+_{\ell,K}}(V) := \int_{B_1} \sum_i^{\ell} |\nabla v_i|^2 + \sum_i^{\ell-1} \omega_i^\gamma (v_i - v_{i+1})^\gamma + G^+_{\ell,K}(x,V) \quad \text{among } \mathcal{A}$$

and assume that $\quad v^*_\ell \ge u_{\ell+1} + K/2 \quad$ in $\quad B_1$ $\longrightarrow$ this is all that's left to justify

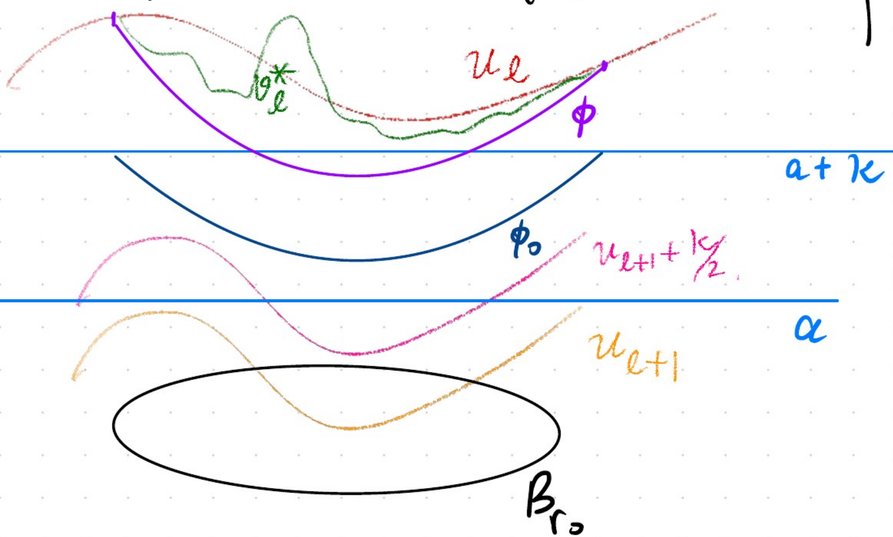
then $\quad U^+ \quad$ minimizes $\quad J_{\ell,d}^{\gamma, G^+_{\ell,K}} \quad$ among $\mathcal{A}$

Lemma Let U minimize $J^{r,G}_{N,d}$ and $u_\ell \ge a+k > a \ge u_{\ell+1}$ in B_1

then there exists $r_0 = r_0(k) > 0$ s.t. any V^* minimizer to $J^{r,G^+_{\ell,k}}_{\ell,d}$ among $\mathcal{A}$ in B_{r_0}

satisfies $V^*_\ell \ge u_{\ell+1} + k/2$ B_{r_0}

Proof Let $L_k := k + \omega_\ell^r \|g_k\|_{C^{0,1}}$ solve $\begin{cases} \Delta \phi = L_k \\ \phi = u_\ell & \partial B_{r_0} \end{cases}$ $\begin{cases} \Delta \phi_0 = L_k \\ \phi_0 = a+k & \partial B_{r_0} \end{cases}$



By comparison principle.

$$\phi \ge \phi_0$$

note $\phi_0(x) = a + k + \frac{L_k}{2d}\left(|x|^2 - r_0^2\right)$

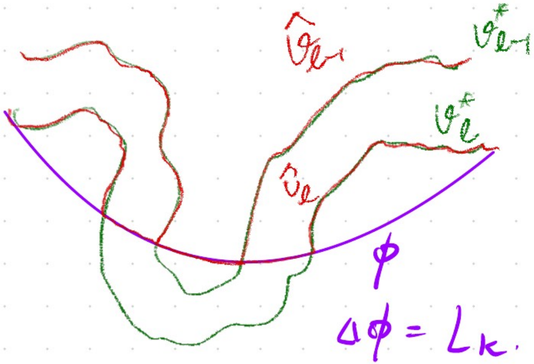
$$\ge a + k - \frac{L_k}{2d} r_0^2$$

$$> a + k/2 \ge u_{\ell+1} + k/2$$

By choosing r_0 small.

Hence it suffices to show $V^*_\ell \ge \phi$.

We use minimality of V^*. Consider competitor $\hat{V} := \left(\max\{ v_i^*, \phi \} \right)_{1 \le i \le l}$



write $v_i^* = \hat{v}_i - (\phi - v_i^*)_+$

$$\int_{B_{r_0}} \sum_{i=1}^{l} \left(|\nabla \hat{v}_i|^2 - |\nabla v_i^*|^2 \right) = \sum_i^l \int 2 \nabla \hat{v}_i \cdot \nabla(\phi - v_i^*)_+ - |\nabla(\phi - v_i^*)_+|^2$$

$$\le \sum_i^l \int_{\{\phi > v_i^*\}} 2 \nabla \phi \cdot \nabla(\phi - v_i^*) = -2 \sum_i^l \int_{\{\phi > v_i^*\}} L_k (\phi - v_i^*)$$

and $\quad 0 \le \hat{v}_i - \hat{v}_{i+1} \le v_i^* - v_{i+1}^*$

and $\quad G^+_{l,k}(x, \hat{V}) - G^+_{l,k}(x, V^*) \le L_k |\hat{V} - V^*| \le \sum_i^l L_k (\phi - v_i^*)$

therefore $\quad J_{l,d}^{r, G^+_{l,k}}(\hat{V}) - J_{l,d}^{r, G^+_{l,k}}(V^*) \le - \sum_i^l \int_{\{\phi > v_i^*\}} L_k (\phi - v_i^*)$

$$\Rightarrow \quad v_i^* \ge \phi \quad \text{a.e. in } B_{r_0} \qquad \square$$

future directions :

- blow up analysis
- classification of 1-D global minimizing cones
- classification of 2-D nontrivial extension minimizing cones
- Improvement of flatness.

Thank you