Elliptic PDEs

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This is set of Notes on Elliptic PDEs collected by Mark Ma, PhD student at Columbia University, in preparation for his PhD qualifying exam.

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Chapter 1

Laplace Equation and Harmonic Functions

We define $u \in C^2(B_1)$ to be classically solution to Laplace's Equation if

$$\Delta u = 0 \qquad \text{in } B_1$$

Or more generally, consider $\Omega \subseteq \mathbb{R}^n$ open domain.

Definition 1.0.1 (Harmonic Function). *We say $u \in C^2(\Omega)$ solves the Laplace Equation in Ω classically if*

$$\Delta u = 0 \qquad \text{in } \Omega \tag{1.1}$$

In this case we call u a harmonic function in Ω. If $\Delta u \geq 0$ we call u subharmonic, and if $\Delta u \leq 0$ we call u superharmonic.

If $u \in C^2(\Omega)$ solves

$$\Delta u = f$$

for some $f \in C(\Omega)$, u is called solution to Poisson's Equation.

Change of Variables Before we start, let's introduce tools we occasionally use. We denote

$$\omega_n \equiv |B_1(0)| := \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \qquad \text{volume of unit sphere in } \mathbb{R}^n$$

Clearly $|B_r| = \omega_n r^n$ and $|\partial B_r| = n\omega_n r^{n-1}$. Notice applying

$$\Gamma(z+1) = z\Gamma(z), \qquad \Gamma(1) = 1$$

to $z = \frac{n}{2}$ one has

$$|\partial B_1(0)| = n\omega_n = \frac{n}{\Gamma(\frac{n}{2}+1)}\pi^{\frac{n}{2}} = \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})\frac{n}{2}} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \tag{1.2}$$

A change of variables formula. From sphere $\partial B_r(x)$ to $\partial B_1(0)$

$$\int_{\partial B_r(x)} u(y)\,dS(y) = \int_{\partial B_r(0)} u(x+y)dS(y) = \int_{\partial B_1(0)} u(x+ry)r^{n-1}dS(y)$$

$$\frac{1}{n\omega_n r^{n-1}}\int_{\partial B_r(x)} u(y)\,dS(y) =: \fint_{\partial B_r(x)} u(y)\,dS(y) = \fint_{\partial B_1(0)} u(x+ry)dS(y) \tag{1.3}$$

and from Ball $B_r(x)$ to unit ball $B_1(0)$

$$\int_{B_r(x)} u(y)dy = \int_{B_r(0)} u(x+y)dy = \int_{B_1(0)} u(x+ry)r^n dy$$

$$\frac{1}{\omega_n r^n}\int_{B_r(x)} u(y)\,dy =: \fint_{B_r(x)} u(y)dy = \fint_{B_1(0)} u(x+ry)dy$$

Laplacian Heuristically, what is a Laplacian? A useful way to think is that: up to a constant, the Laplacian is the only linear operator of second order which is translation invariant and rotation invariant. Indeed it can be seen as an operator that measures, infinitesimally, the difference between u at a given point x, and the average of u around x in the following sense ([FRRO22])

$$\Delta u(x) = \lim_{r\to 0} \frac{C_n}{r^2}\left(⨍_{B_r(x)} u(y) - u(x)dy\right)$$

Proof. For $u \in C^2$, apply Taylor Expansion around x so

$$u(y) = u(x) + \nabla u(x)\cdot(y-x) + \frac{1}{2}(y-x)^T D^2u(x)(y-x) + o(|y-x|)$$

Now

$$\begin{aligned} ⨍_{B_r(x)} u(y) - u(x)dy &= ⨍_{B_r(x)} \nabla u(x)\cdot(y-x) + ⨍_{B_r(x)} (y-x)^T\frac{1}{2}D^2u(x)(y-x)dy + ⨍_{B_r(x)} o(|y-x|^2) \\ &= ⨍_{B_r(0)} \nabla u(x)\cdot z + ⨍_{B_r(0)} z^T\frac{1}{2}D^2u(x)zdz + ⨍_{B_r(0)} o(|z|^2) \end{aligned}$$

Note

$$⨍_{B_r(0)} \nabla u(x)\cdot z = \nabla u(x)\cdot ⨍_{B_r(0)} z = 0 \qquad \text{due to symmetry of } B_r(0)$$

On the other hand

$$⨍_{B_r(0)} z^T\frac{1}{2}D^2u(x)zdz = \frac{1}{2}\partial_{ij}u(x) ⨍_{B_r(0)} z_iz_jdz$$

For $i \neq j$

$$⨍_{B_r(0)} z_iz_jdz = 0 \qquad \text{due to symmetry of } B_r(0)$$

For z_i^2, again using symmetry

$$\begin{aligned} ⨍_{B_r(0)} z_i^2dz &= \frac{1}{n}⨍_{B_r(0)} |z|^2dz = \frac{1}{n}\frac{1}{\omega_n r^n}\int_0^r \rho^{2+n-1}\int_{\partial B_1(0)} dSd\rho \\ &= \frac{1}{r^n}\int_0^r \rho^{n+1}d\rho = \frac{1}{n+2}r^2 \end{aligned}$$

Thus

$$⨍_{B_r(0)} z^T\frac{1}{2}D^2u(x)zdz = \frac{1}{2}\partial_{ii}u(x)\frac{1}{n+2}r^2 = \frac{r^2}{2(n+2)}\Delta u(x)$$

One conclude

$$\lim_{r\to 0}\frac{2(n+2)}{r^2}\left(⨍_{B_r(x)} u(y) - u(x)dy\right) = \Delta u(x)$$

□

Gauss-Green The Gauss Theorem and Green's Identities([Eva10] C.2). Here we need to assume Ω sufficiently regular, say C^1 boundary. Let $\nu = n$ denote outer unit normal to $\partial\Omega$.

1. For any vector field $w \in C^1(\overline{\Omega};\mathbb{R}^n)$, the Gauss-Green Theorem gives

$$\int_\Omega \operatorname{div}(w(x))\,dx = \int_{\partial\Omega} w(x)\cdot\nu(x)\,dS(x) \tag{1.4}$$

   where dS is $n-1$ dimensional surface measure.

2. For $u \in C^2(\overline{\Omega})$, let $w = Du \in C^1(\overline{\Omega};\mathbb{R}^n)$, and denote $\frac{\partial u}{\partial\nu} := Du\cdot\nu$, one obtain the Divergence Theorem

$$\int_\Omega \Delta u\,dx = \int_{\partial\Omega}\frac{\partial u}{\partial\nu}\,dS \tag{1.5}$$

3. For u, $v \in C^2(\overline{\Omega})$, let $w = vDu \in C^1(\overline{\Omega};\mathbb{R}^n)$, one has Green's first identity

$$\int_\Omega v\Delta u\,dx + \int_\Omega \nabla v\cdot\nabla u\,dx = \int_{\partial\Omega} v\frac{\partial u}{\partial\nu}\,dS \tag{1.6}$$

4. Interchange u, v in (1.6) and substract, one has Green's second identity

$$\int_\Omega (v\Delta u - u\Delta v)dx = \int_{\partial\Omega}\left(v\frac{\partial u}{\partial\nu} - u\frac{\partial v}{\partial\nu}\right)dS \tag{1.7}$$

Laplacian in Spherical Coordinates For $n = 2$ with $(r, \theta) \in (0, \infty) \times (0, 2\pi)$

$$\Delta u = \partial_{rr} u + \frac{1}{r} \partial_r u + \frac{1}{r^2} \partial_\theta^2 u$$

1.1 Mean Value Property

In this section we deduce all properties of the solution using Mean Value Property.

Definition 1.1.1 (Mean Value Property). *$u \in C(\Omega)$ satisfies mean value property (MVP) if*

$$u(x) = \fint_{\partial B_r(x)} u(y)\, dS(y) \qquad \forall\ B_r(x) \Subset \Omega \tag{1.8}$$

$$u(x) = \fint_{B_r(x)} u(y)\, dy \qquad \forall\ B_r(x) \Subset \Omega \tag{1.9}$$

Immediately notice that (1.8) and (1.9) are equivalent ([HL11] Remark 1.2).

Proof. $\Longrightarrow$ Let $B_r(x) \Subset \Omega$. For any $0 < \rho < r$, integrate from 0 to r

$$n\omega_n \rho^{n-1} u(x) = \int_{\partial B_\rho(x)} u(y)\, dS(y) \implies \omega_n r^n u(x) = \int_{B_r(x)} u(y)\, dy$$

$\Longleftarrow$ Let $B_r(x) \Subset \Omega$. One differentiate

$$\omega_n r^n u(x) = \int_{B_r(x)} u(y)\, dy = \int_0^r \left(\int_{\partial B_\rho(x)} u(y)\, dS(y) \right) d\rho \implies n\omega_n r^{n-1} u(x) = \int_{\partial B_r(x)} u(y)\, dS(y)$$

□

MVP and Harmonicity

1. Harmonic(sub/super) functions satisfy the Mean Value Property(inequalities).

   Theorem 1.1.1 ([GT01] Theorem 2.1; [HL11] Theorem 1.6; [Eva10] Theorem 2.2). *$u \in C^2(\Omega)$ s.t. $\Delta u = 0$ $(\geq, \leq 0)$ in Ω, then for any ball $B_r(y) \Subset \Omega$*

$$u(y) = (\leq, \geq) \fint_{B_r(y)} u\, dx \tag{1.10}$$

$$u(y) = (\leq, \geq) \fint_{\partial B_r(y)} u\, dS \tag{1.11}$$

   Proof. For any $0 < \rho < r$, one starts with change of variables on surface of the sphere. For any $x \in \partial B_\rho(y)$, $x = y + \rho\omega$ where $\omega \in \mathbb{S}^{n-1}$ so $u(x) = u(y + \rho\omega)$.

$$\begin{aligned}\int_{\partial B_\rho(y)} \frac{\partial u}{\partial \nu}(x)\, dS(x) &= \int_{\partial B_\rho(y)} \nabla u(x) \cdot \frac{x-y}{|x-y|}\, dS(x) \\ &= \rho^{n-1} \int_{\mathbb{S}^{n-1}} \nabla u(y+\rho\omega) \cdot \omega dS(\omega) = \rho^{n-1} \frac{\partial}{\partial_\rho} \left(\int_{\mathbb{S}^{n-1}} u(y+\rho\omega)\, dS(\omega) \right)\end{aligned}$$

   One may take the derivative out by DCT. Then one scales back

$$= \rho^{n-1} \frac{\partial}{\partial_\rho} \left(\frac{1}{\rho^{n-1}} \int_{\partial B_\rho(y)} u(x)\, dS(x) \right) \tag{1.12}$$

   Note on the other hand, one may apply the divergence theorem to LHS, and use the harmonicity assumption

$$\int_{\partial B_\rho(y)} \frac{\partial u}{\partial \nu}(x)\, dS(x) \overset{(1.5)}{=} \int_{B_\rho(y)} \Delta u(x) dx = (\geq, \leq)\ 0$$

   resulting in

$$\frac{\partial}{\partial_\rho} \left(\frac{1}{\rho^{n-1}} \int_{\partial B_\rho(y)} u(x)\, dS(x) \right) = (\geq, \leq)\ 0 \implies \frac{1}{\rho^{n-1}} \int_{\partial B_\rho(y)} u\, dS = (\leq, \geq)\ \frac{1}{r^{n-1}} \int_{\partial B_r(y)} u\, dS$$

   Apply Lebesgue Differentiation, and (1.11) follows

$$n\omega_n u(y) = \lim_{\rho \to 0} \frac{1}{\rho^{n-1}} \int_{\partial B_\rho(y)} u\, dS = (\leq, \geq)\ \frac{1}{r^{n-1}} \int_{\partial B_r(y)} u\, dS$$

To obtain solid MVT, rewrite for $0 < \rho < r$, integrate from 0 to r and (1.10) follows

$$\begin{aligned} n\omega_n \rho^{n-1} u(y) &= \ (\le, \ge) \ \int_{\partial B_\rho(y)} u\, dS \\ \omega_n r^n u(y) = \omega_n u(y) \int_0^r n\rho^{n-1} d\rho &= \ (\le, \ge) \ \int_0^r \int_{\partial B_\rho(y)} u\, dS = \int_{B_r(y)} u\, dx \end{aligned}$$

□

In fact one may simply compute via the following. let $u_r(x) = u(rx)$

$$\begin{aligned} \frac{d}{dr}\left(\fint_{\partial B_1} u_r \right) &= \fint_{\partial B_1} \nabla u(rx) \cdot x = \fint_{\partial B_r} \nabla u(y) \cdot \frac{y}{r} d\mathcal{H}^{n-1} = \frac{r}{n} \fint_{B_r} \Delta u \\ \frac{d}{dr}\left(\fint_{B_1} u_r \right) &= \fint_{B_r} \nabla u(y) \cdot \frac{y}{r} dy = \frac{1}{\omega_n r^n} \int_0^r \int_{\partial B_s} \nabla u(z) \cdot \frac{z}{r} d\mathcal{H}^{n-1}(z) ds \\ &= \frac{1}{\omega_n r^n} \int_0^r \frac{s}{r} \int_{\partial B_s} \nabla u(z) \cdot \frac{z}{s} d\mathcal{H}^{n-1}(z) ds \\ &= \frac{1}{\omega_n r^n} \int_0^r \frac{s}{r} \int_{B_s} \Delta u \, ds \\ &= \int_0^r (\frac{s}{r})^{n+1} \fint_{B_s} \Delta u ds \end{aligned}$$

2. In fact MVP implies Harmonicity, and moreover, smoothness of the solution.

Theorem 1.1.2 ([HL11] Theorem 1.8; [Eva10] Theorem 2.3, 2.6)**.** *If $u \in C(\Omega)$ s.t. mean value property holds in Ω. Then $u \in C^\infty(\Omega)$ and $\Delta u = 0$ in Ω.*

Proof. We first show u is in fact smooth. Let $\varphi \in C_0^\infty(B_1(0))$ s.t. $\int_{B_1} \varphi = 1$ and $\varphi(x) = \psi(|x|)$, i.e., φ is radial. One compute

$$\begin{aligned} \int_{B_1(0)} \varphi(x)\, dx &= \int_0^1 \int_{\partial B_r} \psi(r)\, dS(w) dr = \int_0^1 \int_{\partial B_1} \psi(r) r^{n-1}\, dS(w) dr \\ &= \int_0^1 n\omega_n r^{n-1} \psi(r) dr = 1 \end{aligned}$$

One define the mollifier $\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n}\varphi(\frac{x}{\varepsilon})$. To show u is smooth, one show u coincides with its mollification $u * \varphi_\varepsilon$. Let $x \in \Omega$ and choose $\varepsilon < \operatorname{dist}(x, \partial\Omega)$. Since φ is radial,

$$\begin{aligned} u * \varphi_\varepsilon(x) &= \int_{|x-y|<\varepsilon} \varphi_\varepsilon(x-y) u(y) dy = \int_{|x-y|<\varepsilon} \varphi_\varepsilon(y-x) u(y) dy = \int_{|y|<\varepsilon} \varphi_\varepsilon(y) u(y+x) dy \\ &= \frac{1}{\varepsilon^n} \int_{|y|<\varepsilon} u(x+y) \varphi(\frac{y}{\varepsilon}) dy = \int_{|y|<1} u(x+\varepsilon y) \varphi(y) dy \end{aligned}$$

One introduce coarea formula for polar coordinates

$$\begin{aligned} &= \int_0^1 \int_{\partial B_r(0)} u(x+\varepsilon y)\varphi(y)\, dS(y) dr = \int_0^1 \int_{\partial B_1(0)} u(x+\varepsilon r w)\psi(r) r^{n-1}\, dS(w) dr \\ &= \int_0^1 \psi(r) r^{n-1} \int_{|w|=1} u(x+\varepsilon r w)\, dS(w)\, dr \\ &= \int_0^1 \psi(r) r^{n-1} n\omega_n \frac{1}{n\omega_n} \int_{\partial B_1} u(x+\varepsilon r\omega) dS(\omega) dr = \int_0^1 \psi(r) r^{n-1} n\omega_n u(x) dr = u(x) \end{aligned}$$

Where the last line uses change of coordinates (1.3) and the MVP (1.9). Since $u * \varphi_\varepsilon$ is smooth in

$$\Omega_\varepsilon := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$$

u is smooth in Ω_ε. For any $x \in \Omega$ we can always squeeze in an ε, hence $u \in C^\infty(\Omega)$.

Now to show $\Delta u = 0$, for any $x \in \Omega$ and $B_r(x) \Subset \Omega$, one has divergence theorem and (1.12)

$$\begin{aligned} \int_{B_r(x)} \Delta u(y) dy &\overset{(1.5)}{=} \int_{\partial B_r(x)} \frac{\partial u}{\partial \nu}(y)\, dS(y) = \cdots \overset{(1.12)}{=} r^{n-1} \frac{\partial}{\partial r}\left(\frac{1}{r^{n-1}} \int_{\partial B_r(x)} u(y)\, dS(y) \right) \\ &\overset{(1.8)}{=} r^{n-1} \frac{\partial}{\partial r}(n\omega_n u(x)) = 0 \qquad \forall\ B_r(x) \Subset \Omega \end{aligned}$$

Hence $\Delta u(x) = 0$ for any $x \in \Omega$. □

MVP and Maximum Principle One has two sets of Maximum/Minimum Principle: Strong and Weak, both led by MVP.

1. Let Ω be connected, open subset of $\mathbb{R}^n$. Strong Maximum Principle says a subharmonic function cannot achieve interior supremum, unless itself is a constant.

   Theorem 1.1.3 ([GT01] Theorem 2.2; [HL11] Proposition 1.4; [Eva10] Theorem 2.4)**.** *Let $u \in C^2(\Omega)$. If $\Delta u \geq (\leq) 0$ in Ω, then u cannot achieve interior supremum (infimum) unless u is constant throughout Ω.*

   Proof. WLOG prove for $\Delta u \geq 0$. Consider the closed subset

   $$\Omega_M := \{x \in \Omega \mid u(x) = M\}$$

   Since we assumed Ω is connected, Ω_M as a subset cannot be both closed and open unless it is the whole domain $\Omega_M = \Omega$ or empty. Now assume there exists $y \in \Omega$ s.t. achieves interior supremum

   $$u(y) = \sup_{x \in \Omega} u(x) \equiv M$$

   hence $y \in \Omega_M$, and Ω_M is non-empty. To conclude $\Omega = \Omega_M$, i.e., u is constant throughout Ω, it suffices to prove Ω_M is open. Now for any $z \in \Omega_M$, we want to prove there exists $r > 0$ s.t. $B_r(z) \subseteq \Omega_M$. How to find such $r = r_z > 0$? We use Mean Value Property. Since $\Delta u \geq 0$ in Ω, there exists $r > 0$ s.t. $B_r(z) \subseteq \Omega$, and

   $$\fint_{B_r(z)} u(z) = u(z) \overset{(1.10)}{\leq} \fint_{B_r(z)} u$$

   $$\fint_{B_r(z)} u - u(z) \geq 0$$

   But $u - u(z) \leq 0$ in $B_r(z)$. Hence $u = u(z) = M$ in $B_r(z)$, and thus $B_r(z) \subseteq \Omega_M$. □

2. Furthermore, assume Ω (still connected) to be bounded. One has Weak Maximum Principle that says supremum must be achieved on the boundary.

   Theorem 1.1.4 ([GT01] Theorem 2.3; [HL11] Proposition 1.4; [Eva10] Theorem 2.4)**.** *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ for Ω bounded. If $\Delta u \geq (\leq) 0$ in Ω, then*

   $$\sup_{\Omega} u = \sup_{\partial\Omega} u \qquad (\inf_{\Omega} u = \inf_{\partial\Omega} u)$$

   In particular for harmonic function u over Ω

   $$\inf_{\partial\Omega} u \leq u(x) \leq \sup_{\partial\Omega} u \qquad \forall\ x \in \Omega$$

   Proof. WLOG assume $\Delta u \geq 0$. Note one direction

   $$\sup_{\partial\Omega} u \leq \sup_{\Omega} u$$

   is clear using continuous up to boundary. Now suppose there exists interior maximum, using Theorem 1.1.3 u is constant across Ω, thus

   $$\sup_{\partial\Omega} u = \sup_{\Omega} u$$

   If no interior maximum exists, using $\overline{\Omega}$ is compact, u must attain maximum somewhere on $\overline{\Omega}$, then maximum must be achieved on the boundary. Hence

   $$\sup_{\Omega} u \leq \sup_{\partial\Omega} u$$

   Both cases conclude the proof. □

One has two immediate corollaries of the maximum principle.

1. One has uniqueness of solutions to Dirichlet Problem over bounded domain (regardless of regularity)!

Corollary 1.1.1 ([GT01] Theorem 2.4; [HL11] Remark 1.9; [Eva10] Theorem 2.5)**.** *Let u, $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ for Ω bounded. Assume*

$$\begin{cases} \Delta u = \Delta v & \Omega \\ u = v & \partial\Omega \end{cases}$$

Then

$$u = v \qquad \Omega$$

Proof. Consider $w = u - v$ which solves

$$\begin{cases} \Delta w = 0 & \Omega \\ w = 0 & \partial\Omega \end{cases}$$

Apply Weak Maximum and Minimum Principle Theorem 1.1.4 so that

$$0 = \inf_{\partial\Omega} w \le w(x) = u(x) - v(x) \le \sup_{\partial\Omega} w = 0 \qquad \forall\, x \in \Omega$$

□

2. One has Comparison Principle as a slight-variant of the above.

Corollary 1.1.2 ([GT01] Section 2.2)**.** *Let u, $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ for Ω bounded. Assume*

$$\begin{cases} \Delta u = 0 & \Omega \\ \Delta v \ge (\le)\ 0 & \Omega \\ u = v & \partial\Omega \end{cases}$$

Then

$$u \ge (\le)\ v \qquad \Omega$$

Proof. Consider $w = u - v$ which solves

$$\begin{cases} \Delta w \le (\ge)\ 0 & \Omega \\ w = 0 & \partial\Omega \end{cases}$$

Apply one-sided Weak Minimum (Maximum) Principle Theorem 1.1.4 so that

$$0 = \inf_{\partial\Omega} w \le w(x) = u(x) - v(x) \qquad \forall\, x \in \Omega$$

or

$$u(x) - v(x) = w(x) \le \sup_{\partial\Omega} w = 0 \qquad \forall\, x \in \Omega$$

□

3. We remark that Corollary 1.1.1 fails for unbounded domain([HL11] Remark 1.9). Let u solve

$$\begin{cases} \Delta u = 0 & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

so $u = 0$ is a trivial solution. Now we make our choice of unbounded Ω.

(a) $\Omega = \{x \in \mathbb{R}^n \mid |x| > 1\}$. If $n = 2$, choose $u(x) = \log(|x|)$. Calculate

$$\Delta u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{1}{|x|}\left(\frac{x_i}{|x|}\right)\right) = \sum_{i=1}^{n}\left(\frac{-2}{|x|^3} \cdot \frac{x_i^2}{|x|} + \frac{1}{|x|^2}\right) = (n-2)\frac{1}{|x|^2} = 0$$

and indeed $u = 0$ on $\partial\Omega = \partial B_1(0)$. Note $u \to \infty$ as $|x| \to \infty$.

(b) $\Omega = \{x \in \mathbb{R}^n \mid |x| > 1\}$. If $n \ge 3$, choose $u(x) = \frac{1}{|x|^{n-2}} - 1$. Calculate

$$\Delta u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left((2-n)\frac{1}{|x|^{n-1}} \cdot \frac{x_i}{|x|}\right) = (2-n)\sum_{i=1}^{n}\left(-n\left(\frac{1}{|x|^{n+1}} \cdot \frac{x_i^2}{|x|}\right) + \frac{1}{|x|^n}\right) = 0$$

and indeed $u = 0$ on $\partial\Omega = \partial B_1(0)$. Note $u \to -1$ as $|x| \to \infty$. In this case uniqueness fails even if u is bounded over Ω.

(c) $\Omega = \{x \in \mathbb{R}^n \mid x_n > 0\}$. Choose

$$u(x) = x_n \tag{1.13}$$

Note $\Delta u = 0$ and $u = 0$ on $\partial\Omega = \{x \in \mathbb{R}^n \mid x_n = 0\}$. Here $u \to \infty$ as $x_n \to \infty$.

MVP and Harnack Inequality We prove Harnack using MVP.

Theorem 1.1.5 ([GT01] Theorem 2.5; [HL11] Theorem 1.15; [Eva10] Theorem 2.11)**.** *$u \in C^2(\Omega)$ s.t. $\Delta u = 0$ and $u \geq 0$. Then for any bounded subdomain $\Omega' \Subset \Omega$, there exists constant $C = C(n, \Omega, \Omega')$ s.t.*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u \tag{1.14}$$

Proof. We first prove Harnack on Balls. The trick is to cover balls using one another. For any $y \in \Omega$, there exists r s.t. $B_{4r}(y) \subseteq \Omega$. Take any $x_0, x_1 \in B_r(y)$, one may apply Theorem 1.1.1 to average $u(x_0)$ around $B_r(x_0)$ and $u(x_1)$ around $B_{3r}(x_1)$

$$\begin{aligned} u(x_0) &\overset{(1.9)}{=} \frac{1}{\omega_n r^n} \int_{B_r(x_0)} u\,dx \leq \frac{1}{\omega_n r^n} \int_{B_{2r}(y)} u\,dx \\ u(x_1) &\overset{(1.9)}{=} \frac{1}{\omega_n (3r)^n} \int_{B_{3r}(x_1)} u\,dx \geq \frac{1}{\omega_n (3r)^n} \int_{B_{2r}(y)} u\,dx \end{aligned}$$

where the inequalities use nonnegativity of u. One hence obtains

$$u(x_0) \leq 3^n u(x_1) \implies \sup_{B_r(y)} u \leq 3^n \inf_{B_r(y)} u$$

Then we connect 2 points in arbitrary bounded subdomains using balls. Let $\Omega' \Subset \Omega$ be bounded so $\overline{\Omega'}$ is compact. Choose arbitrary $x_0, x_1 \in \overline{\Omega'}$ and connect them using closed arc $\Gamma \subseteq \overline{\Omega'}$. Indeed Γ is compact. To cover Γ, one needs to choose the open balls with $4r < \text{dist}(\Gamma, \partial\Omega)$ so B_{4r} does not touch $\partial\Omega$. Now for the open cover $\Gamma \subseteq \bigcup_{z\in\Gamma} B_r(z)$, there exists finitely many, say N balls s.t. $\Gamma \subseteq \bigcup_1^N B_r(z_i)$. Note N depends on Ω' via x_0, x_1, Γ and depends on Ω via choice of r. Hence we apply Harnack on the sequence of balls $\{B_r(z_i)\}_1^N$ so

$$u(x_0) \leq 3^n u(z_1) \leq 3^{2n} u(z_2) \leq \cdots \leq 3^{Nn} u(z_n) \leq 3^{N(n+1)} u(x_1)$$

Choose $C = 3^{(N+1)n}$ so $\sup_{\Omega'} u \leq C \inf_{\Omega'} u$. □

MVP and Gradient Estimates We prove first order gradient bounds using MVP.

1. Bound $|\nabla u(0)|$ using $\frac{1}{R}\|u\|_{L^\infty(B_R)}$.

   Lemma 1.1.1 ([HL11] Lemma 1.10)**.** *Let $u \in C^2(B_R) \cap C^0(\overline{B_R})$ be harmonic. Then*

   $$|\nabla u(0)| \leq \frac{n}{R} \sup_{x\in\overline{B_R}} |u(x)| \tag{1.15}$$

   Proof. For any $i = 1, \cdots, n$, since MVP Theorem 1.1.2 yields $u \in C^\infty$, we may differentiate the equation so that u_{x_i} again satisfies Laplace Equation. In particular, u_{x_i} is again harmonic. Hence MVP applies

   $$\begin{aligned} u_{x_i}(0) &\overset{(1.9)}{=} \fint_{B_R} u_{x_i}(y)\,dy = \frac{1}{\omega_n R^n} \int_{B_R} u_{x_i}(y)\,dy \\ &\overset{(1.4)}{=} \frac{1}{\omega_n R^n} \int_{\partial B_R} u\nu_i\,dS(y) = \frac{n}{R} \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R} u\nu_i\,dS(y) \\ &= \frac{n}{R} \fint_{\partial B_R} u\nu_i\,dS(y) \\ |Du(0)| &\leq \frac{n}{R} \max_{x\in\overline{B_R}} |u(x)| \end{aligned}$$

   □

2. Bound $|\nabla u(0)|$ using $\frac{u(0)}{R}$ for non-negative u.

   Lemma 1.1.2 ([HL11] Lemma 1.11)**.** *Let $u \in C^2(B_R) \cap C^0(\overline{B_R})$ be non-negative and harmonic. Then*

   $$|\nabla u(0)| \leq \frac{n}{R} u(0) \tag{1.16}$$

   Proof.

   $$\begin{aligned} u_{x_i}(0) &\overset{(1.9),(1.4)}{=} \frac{n}{R} \fint_{\partial B_R} u\nu_i\,dS(y) \\ |Du(0)| &\leq \frac{n}{R} \fint_{\partial B_R} |u|\,dS(y) \overset{\text{non-negative}}{=} \frac{n}{R} \fint_{\partial B_R} u\,dS(y) \overset{(1.8)}{=} \frac{n}{R} u(0) \end{aligned}$$

   □

3. Bound $|\nabla u(0)|$ using $\frac{1}{R^{n+1}} \|u\|_{L^1(B_R)}$.

Lemma 1.1.3 ([Eva10] Theorem 2.7)**.** *Let $u \in C^2(B_R) \cap C^0(\overline{B_R})$ be harmonic. Then*

$$|\nabla u(0)| \leq \frac{n2^{n+1}}{\omega_n R^{n+1}} \|u\|_{L^1(B_R(0))} \tag{1.17}$$

Proof. First proceed as in (1.15) over $B_{R/2}$

$$u_{x_i}(0) \overset{(1.9),(1.4)}{=} \frac{2n}{R} \fint_{\partial B_{R/2}} u\nu_i \, dS(y) \leq \frac{2n}{R} \sup_{x \in \partial B_{R/2}} |u(x)|$$

Now for any $x \in \partial B_{R/2}$, notice $B_{R/2}(x) \subseteq B_R(0)$. But for such x

$$u(x) = \fint_{B_{R/2}(x)} u = \frac{2^n}{\omega_n R^n} \int_{B_{R/2}(x)} u$$
$$|u(x)| \leq \frac{2^n}{\omega_n R^n} \int_{B_R(0)} |u| = \frac{2^n}{\omega_n R^n} \|u\|_{L^1(B_R(0))}$$

Hence combining both yields

$$|u_{x_i}(0)| \leq \frac{2n}{R} \frac{2^n}{\omega_n R^n} \|u\|_{L^1(B_R(0))} = \frac{n2^{n+1}}{\omega_n R^{n+1}} \|u\|_{L^1(B_R(0))} \qquad \forall\, i = 1, \cdots, n$$

□

Liouville Theorem An immediate corollary of gradient estimate (1.15) is Liouville Theorem.

Corollary 1.1.3 ([HL11] Corollary 1.12; [Eva10] Theorem 2.8)**.** *Let $u \in C^2(\mathbb{R}^n)$ be harmonic. If u is bounded either from below or by above, then u is constant throughout $\mathbb{R}^n$.*

Proof. Fix any $x_0 \in \mathbb{R}^n$. If $u \in C^2(\mathbb{R}^n)$ is harmonic, then for any $B_R(x_0) \subset \mathbb{R}^n$, u is harmonic in $B_R(x_0)$. Then locally via translation, one has gradient estimate (1.15)

$$|Du(x_0)| \leq \frac{n}{R} \max_{x \in \overline{B}_R(x_0)} |u(x)| \leq \frac{C(n)}{R} \to 0 \qquad R \to \infty$$

Since LHS is independent of R, we conclude

$$|Du(x_0)| = 0 \qquad \forall\, x_0 \in \mathbb{R}^n$$

□

Corollary 1.1.4. *Let $u \in C^2(\mathbb{R}^n)$ be harmonic. If*

$$|u(x)| \leq C(1 + |x|^m)$$

Then u is a polynomial of degree m.

Proof. Apply gradient estimate to $|\alpha| = m + 1$ so that

$$|D^\alpha u(0)| \leq \frac{C}{R^{m+1}} \|u\|_{L^\infty(B_R)} \leq \frac{C}{R^{m+1}}(1 + R^m) \to 0 \qquad \text{as } R \to \infty$$

But 0 is generic. □

Higher Order Gradient Estimates One can obtain higher-order gradient estimates.

1. Bound $|\nabla^\alpha u(0)|$ using $\frac{1}{R^{|\alpha|}} \|u\|_{L^\infty(B_R)}$.

Lemma 1.1.4 ([HL11] Proposition 1.13; [GT01] Theorem 2.10)**.** *Let $u \in C^\infty(B_R) \cap C^0(\overline{B}_R)$ be harmonic. Then for $\alpha \in \mathbb{N}^n$ where $m = |\alpha| := \sum_{i=1}^n \alpha_i$*

$$|D^\alpha u(0)| \leq n^m e^{m-1} m! \cdot \frac{1}{R^m} \max_{x \in \overline{B}_R} |u(x)| \tag{1.18}$$

Proof. One proceed using induction. For $m = 1$ this holds via (1.15). Assume for m and we prove for $m+1$. Let's define some parameter r which gives us a little room to play with in B_R

$$r := (1-\theta)R \qquad 0 < \theta < 1$$

Then by assumption for m, for any $\alpha \in \mathbb{N}^n$ s.t. $|\alpha| = m+1$, let $\tilde{\alpha} := (\alpha_1 - 1, \alpha_2, \cdots, \alpha_n)$ WLOG so that $|\tilde{\alpha}| = m$. Apply (1.15) to the ball B_r so that

$$|D^\alpha u(0)| \le \frac{n}{r} \max_{x\in \overline{B}_r} |D^{\tilde{\alpha}} u(x)| = \frac{n}{(1-\theta)R} \max_{x\in \overline{B}_r} |D^{\tilde{\alpha}} u(x)|$$

But then for any $x \in B_r$, one apply inductive assumption so that on the ball $B_{R-r}(x) \subseteq B_R$

$$\begin{aligned} |D^{\tilde{\alpha}} u(x)| &\le \frac{n^m e^{m-1} m!}{(R-r)^m} \max_{y \in \overline{B}_{R-r}(x)} |u(y)| \\ &\le \frac{n^m e^{m-1} m!}{\theta^m R^m} \max_{y\in \overline{B}_R} |u(y)| \end{aligned}$$

Thus we add plug one to the other to obtain

$$|D^\alpha u(0)| \le \frac{n^{m+1} e^m (m+1)!}{R^{m+1}} \frac{1}{e(m+1)\theta^m (1-\theta)} \max_{y\in\overline{B}_R} |u(y)|$$

It suffices to notice upon taking

$$\theta = \frac{m}{m+1}$$

one has

$$\frac{1}{\theta^m (1-\theta)} = \frac{(m+1)^{m+1}}{m^m} < e(m+1)$$

□

2. Bound $|\nabla^\alpha u(0)|$ using $\frac{1}{R^{n+|\alpha|}} \|u\|_{L^1(B_R)}$.

Lemma 1.1.5 ([Eva10] Theorem 2.7)**.** *Let $u \in C^\infty(B_R) \cap C^0(\overline{B}_R)$ be harmonic. Then for $\alpha \in \mathbb{N}^n$ where $m = |\alpha|$*

$$|D^\alpha u(0)| \le \frac{(n2^{n+1}m)^m}{\omega_n R^{n+m}} \|u\|_{L^1(B_R(0))}$$

Proof. $m = 1$ holds via (1.17). Assume for $m-1$ and we prove for m. Fix $\alpha \in \mathbb{N}^n$ with $|\alpha| = m$. Then there exists $\beta \in \mathbb{N}^n$ with $|\beta| = m-1$ and $i \in \{1, \cdots, n\}$ s.t.

$$D^\alpha u = (D^\beta u)_{x_i}$$

We apply MVP and Gauss-Green as in (1.15) to $(D^\beta u)_{x_i}$ at 0 on $B_{R/m}$ so that

$$|D^\alpha u(0)| = |(D^\beta u)_{x_i}(0)| \overset{(1.9),(1.4)}{\le} \frac{nm}{R} \left\|D^\beta u\right\|_{L^\infty(B_{R/m}(0))}$$

Now notice for any $x \in B_{R/m}(0)$, $B_{R(m-1)/m}(x) \subseteq B_R(0)$. Hence for such x, using inductive hypothesis

$$\begin{aligned} |D^\beta u(x)| &\le \frac{(n2^{n+1}(m-1))^{m-1}}{\omega_n(\frac{(m-1)}{m}R)^{n+m-1}} \|u\|_{L^1(B_{R(m-1)/m}(0))} \le \frac{(n2^{n+1}(m-1))^{m-1}}{\omega_n(\frac{(m-1)}{m}R)^{n+m-1}} \|u\|_{L^1(B_R(0))} \\ |D^\alpha u(0)| &\le \frac{nm}{R} \cdot \frac{(n2^{n+1}(m-1))^{m-1}}{\omega_n(\frac{(m-1)}{m}R)^{n+m-1}} \|u\|_{L^1(B_R(0))} \\ &= \frac{n^m 2^{(n+1)m}}{\omega_n R^{n+m}} \frac{m(m-1)^{m-1} m^{n+m-1}}{2^{n+1}(m-1)^{n+m-1}} \|u\|_{L^1(B_R(0))} \end{aligned}$$

notice the constant simplifies to

$$\frac{m(m-1)^{m-1} m^{n+m-1}}{2^{n+1}(m-1)^{n+m-1}} = \frac{m^{m+n}}{2^{n+1}(m-1)^n} = m^m \frac{1}{2}(\frac{m}{2m-2})^n \le m^m \qquad \forall\, m \ge 2$$

□

Analyticity Using higher order gradient estimate with L^∞ norm, one improve smoothness to analyticity.

Corollary 1.1.5 ([HL11] Theorem 1.14; [Eva10] Theorem 2.10)**.** *Let* $u \in C^2(\Omega)$ *be harmonic. Then* u *is analytic in* Ω*.*

Proof. For any $x \in \Omega$, choose $R > 0$ s.t.

$$B_{2R}(x) \Subset \Omega$$

We wish to Taylor Expand u at the position $x + h$ for $|h| < R$ sufficiently small. To do so

$$u(x+h) = u(x) + \sum_{i=1}^{m-1} \frac{1}{i!} \left(\sum_{j=1}^{n} h_j \frac{\partial}{\partial x_j} \right)^i u(x) + R_m(h)$$

where the tail term writes for $x = (x_1, \cdots, x_n)$

$$R_m(h) = \frac{1}{m!} \left(\sum_{j=1}^{n} h_j \frac{\partial}{\partial x_j} \right)^m u(x_1 + \theta h_1, \cdots, x_n + \theta h_n)$$

for some $\theta \in (0,1)$. Now using (1.18) for some $|\alpha| = m$

$$|R_m(h)| \leq \frac{1}{m!} |h|^m n^m \|D^\alpha u\|_{L^\infty(B_R(x))} \leq \frac{(n^2|h|e)^m}{eR^m} \max_{x \in \overline{B}_{2R}} |u(x)| \to 0 \qquad m \to \infty$$

provided choosing h small s.t. $n^2|h|e \leq \frac{R}{2}$. One obtain analyticity since we sent order of Taylor Expansion to $m \to \infty$, and the error $|R_m(h)| \to 0$ for h small, independent of m. $\square$

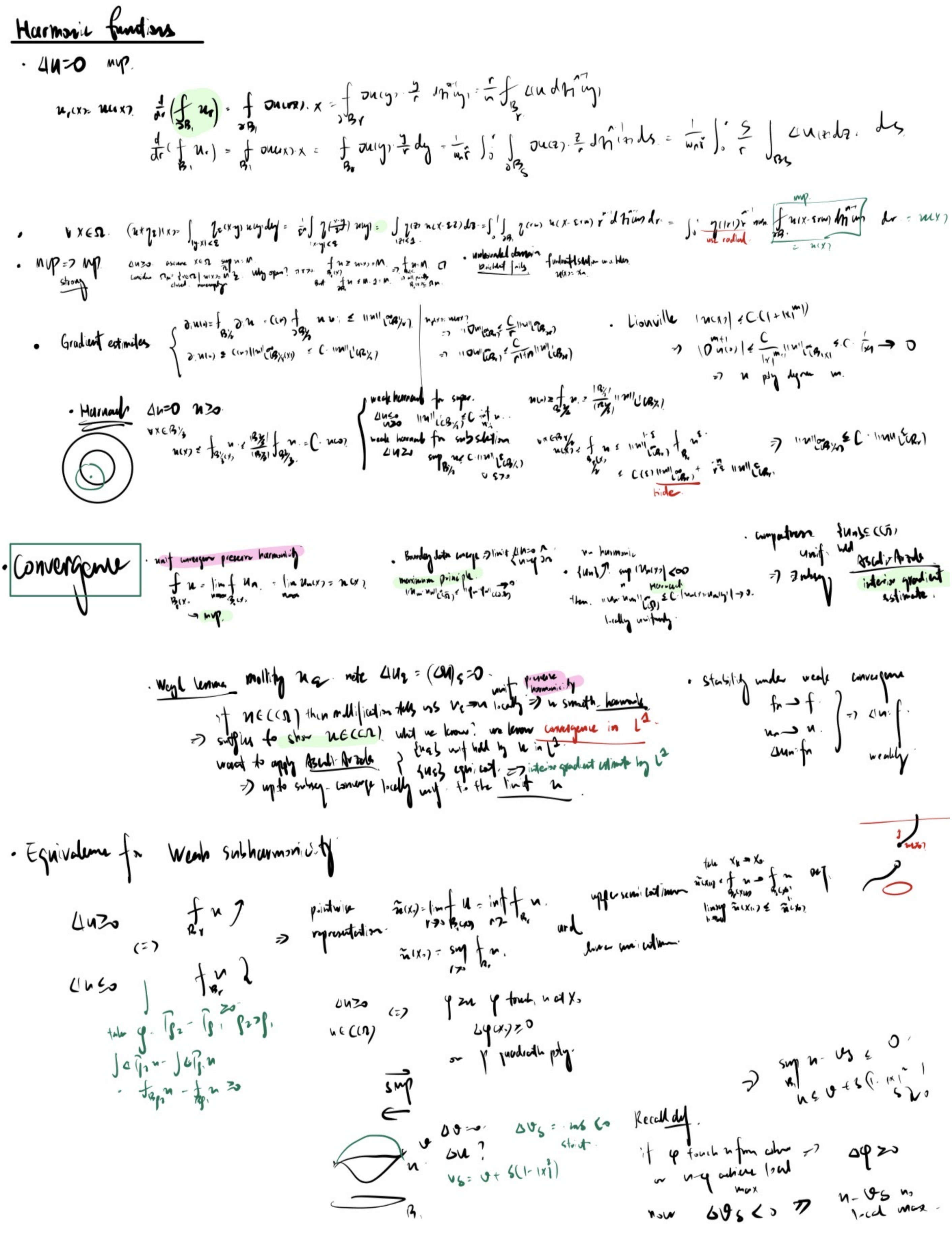


Figure 1.1: Harmonic Functions

1.2 Convergence Theorems

In this section we discuss the limit of sequence of harmonic functions.

Harmonicity and uniform convergence

1. One start by showing harmonicity is preserved under uniform convergence.

   Theorem 1.2.1 ([GT01] Theorem 2.8). *Let $\Omega \subseteq \mathbb{R}^n$ be bounded open subset. Let $u_n \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be sequence of harmonic functions defined on Ω, and assume*

   $$\sup_{x\in\overline{\Omega}}|u_n(x) - u(x)| \to 0 \qquad n \to \infty$$

   for some function u defined pointwise over $\overline{\Omega}$. Then $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, and u is harmonic over Ω.

   Proof. First, note $C^0(\overline{\Omega})$ is Banach space w.r.t. sup norm, necessarily $u \in C^0(\overline{\Omega})$. Now for any $x \in \Omega$ and $B_r(x) \Subset \Omega$, uniform convergence necessarily guarantees pointwise convergence, hence

   $$u(x) = \lim_{n\to\infty} u_n(x) \overset{(1.9)}{=} \lim_{n\to\infty} \fint_{B_r(x)} u_n(y)dy$$

   Now since we have uniform convergence

   $$|\fint_{B_r(x)} u_n(y)dy - \fint_{B_r(x)} u(y)dy| \le \fint_{B_r(x)} |u_n - u| \le \|u_n - u\|_{L^\infty(B_r(x))} \le \|u_n - u\|_{L^\infty(\Omega)} \to 0$$

   Thus

   $$u(x) = \lim_{n\to\infty} \fint_{B_r(x)} u_n(y)dy = \fint_{B_r(x)} u(y)dy$$

   But $x \in \Omega$ and $B_r(x) \Subset \Omega$ is arbitrary, hence $u \in C^0(\overline{\Omega})$ satisfies MVP. By Theorem 1.1.2, $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$. □

2. As immediate corollary, one has stability: If boundary values converge uniformly, the solution to Dirichlet Problems converges uniformly, and the limit solves the Dirichlet Problem with the limiting boundary data.

   Corollary 1.2.1 ([GT01] Section 2.6). *Let $\Omega \subseteq \mathbb{R}^n$ be bounded open subset. Let $u_n \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be sequence of harmonic functions defined on Ω, and assume*

   $$\sup_{x\in\partial\Omega} |u_n(x) - \varphi(x)| \to 0$$

   for some function φ defined pointwise on $\partial\Omega$. Then the sequence $\{u_n\}$ uniformly converges to some $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ that solves

   $$\begin{cases} \Delta u = 0 & \Omega \\ u = \varphi & \partial\Omega \end{cases}$$

   Proof. First note $\partial\Omega$ is compact, and $C^0(\partial\Omega)$ is closed under supremum norm. Thus $\varphi \in C(\partial\Omega)$. Now we observe $\{u_n\}$ as a sequence of functions defined over Ω is Cauchy due to Weak Maximum Principle Theorem 1.1.4

   $$\sup_{x\in\Omega}|u_n(x) - u_m(x)| \le \sup_{x\in\partial\Omega} |u_n(x) - u_m(x)| \le \sup_{x\in\partial\Omega} |u_n(x) - \varphi(x)| + \sup_{x\in\partial\Omega} |u_m(x) - \varphi(x)| \to 0$$

   Thus using $C^0(\overline{\Omega})$ is Banach Space w.r.t. uniform convergenece, there exists $u \in C(\overline{\Omega})$ s.t.

   $$\sup_{x\in\Omega}|u_n(x) - u(x)| \to 0$$

   Thus using Theorem 1.2.1, u is harmonic in Ω. Also

   $$\sup_{x\in\partial\Omega} |u_n(x) - u(x)| \le \sup_{x\in\overline{\Omega}}|u_n(x) - u(x)| \to 0 \qquad \& \qquad \sup_{x\in\partial\Omega} |u_n(x) - \varphi(x)| \to 0$$

   by uniqueness of limits one obtain

   $$u(x) = \varphi(x) \qquad \partial\Omega$$

   □

3. In fact, one can relax uniform convergence to boundedness at one point using Harnack, if the sequence is monotonically increasing. This is known as Harnack's Convergence Theorem.

Theorem 1.2.2 ([GT01] Theorem 2.9)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open. Let $u_n \in C^2(\Omega)$ be monotonically increasing sequence of harmonic functions defined in Ω, and assume at some point $y \in \Omega$*

$$\sup_n |u_n(y)| \leq M < \infty$$

Then u_n converges locally uniformly to some harmonic function $u \in C(\Omega)$, i.e., $u_n \to u$ uniformly over any bounded subdomain $\Omega' \Subset \Omega$.

Proof. Since $\{u_n(y)\}$ is monotonically increasing sequence that is bounded, by MCT there exists some $w \in \mathbb{R}$ s.t.

$$|u_n(y) - w| \to 0 \qquad n \to \infty$$

Hence $\{u_n(y)\}$ is Cauchy sequence. Now consider any bounded subdomain Ω' that contains y. Using Harnack's Inequality Theorem 1.1.5, there exists $C = C(\Omega', n) > 0$ s.t.

$$\sup_{x\in\Omega'} |u_n(x) - u_m(x)| \leq C \inf_{x\in\Omega'} |u_n(x) - u_m(x)| \leq C|u_n(y) - u_m(y)| \to 0$$

as m, $n \to \infty$. Using $C^0(\overline{\Omega'})$ is closed under uniform norm, there exists $u' \in C^0(\overline{\Omega'})$ s.t.

$$\sup_{x\in\Omega'} |u_n(x) - u'(x)| \to 0$$

Using Theorem 1.2.1, we know u' is harmonic in Ω'. Now if we consider $\Omega'' \supsetneq \Omega'$ a large bounded subdomain, there exists u'' harmonic that u_n uniformly converges to over Ω''. But

$$\sup_{\Omega'} |u_n - u''| \leq \sup_{\Omega''} |u_n - u''| \to 0$$

So by uniqueness of limits

$$u''|_{\Omega'} = u'$$

Hence there exists one $u \in C(\Omega)$ that u_n converges locally uniformly to, that is harmonic in Ω. Notice for bounded subdomains not containing y, it suffices to pick a large subdomain that covers it and also contains y, and run our argument. □

Compactness One also record a Compactness Theorem for Harmonicitiy using L^∞ gradient bound and Arzela-Ascoli.

Theorem 1.2.3 ([GT01] Theorem 2.11)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open domain. Let $u_n \in C^2(\Omega)$ be uniformly bounded sequence of harmonic functions. Then there exists a subsequence u_{n_k} s.t.*

$$u_{n_k} \to u \qquad \textit{locally uniformly}$$

to some $u \in C(\Omega)$ harmonic.

Proof. For any $\Omega' \Subset \Omega$, using gradient bound (1.15)

$$\sup_{x\in\Omega'} |\nabla u_n(x)| \leq C(n) \|u_n\|_{L^\infty(\Omega)} \leq C(n)M < \infty \qquad \forall\ n$$

One can apply Arzela-Ascoli so that there exists $u \in C(\Omega')$ s.t.

$$u_{n_k} \to u \qquad \text{uniformly over } \Omega'$$

Now go to bigger subdomain, by uniqueness of limits the limit agrees on smaller subdomains. One can extract a subsequence by diagonalization that converges to u locally uniformly. Then using Theorem 1.2.1 u is harmonic over Ω. □

Weak Solution and Weyl's Lemma Uniform convergence that preserves harmonicity upgrades weak solution to classical solution.

We define a weak solution to Laplace equation. Let $\Omega \subseteq \mathbb{R}^n$ be open domain.

Definition 1.2.1 (Weakly Harmonic Function). *Given $f \in L^1_{loc}(\Omega)$, we say $u \in L^1_{loc}(\Omega)$ is weak solution to*

$$\Delta u = f$$

if

$$\int_\Omega u\Delta\varphi = \int_\Omega f\varphi \qquad \forall\ \varphi \in C_0^\infty(\Omega) \tag{1.19}$$

If $f = 0$, u is called weakly harmonic.

One has Weyl's Lemma that says 'Weakly Harmonic' already implies C^∞.

Lemma 1.2.1 (De Silva Analysis II 2025; [HL11] Theorem 1.16; [Wey40]). *If $u \in L^1_{loc}(\Omega)$ is weakly harmonic, $u \in C^\infty(\Omega)$ and is harmonic in Ω in the classical sense.*

Proof. Fix any $\varepsilon > 0$. Define

$$\Omega_\varepsilon := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$$

One want to prove $u_\varepsilon = u$ in Ω_ε where u_ε is mollification

$$u_\varepsilon(x) := \eta_\varepsilon * u(x)$$

1. We compute for any $x \in \Omega_\varepsilon$

$$\partial_{x_i}(u_\varepsilon)(x) = \partial_{x_i}\left(\int_\Omega \eta_\varepsilon(x-y)u(y)dy\right) = \int_\Omega (\eta_\varepsilon)_{x_i}(x-y)u(y)dy$$

$$\Delta u_\varepsilon(x) = \int_\Omega \Delta\eta_\varepsilon(x-y)u(y)dy \overset{u \text{ is weakly harmonic}}{=} 0$$

Since $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$ and we know u_ε is harmonic in Ω_ε, by Theorem 1.1.1 u_ε satisfies the MVP in Ω_ε.

2. Now apply MVP to u_ε. For any $x \in \Omega_\varepsilon$, we obtain for any $0 < r < \operatorname{dist}(x, \partial\Omega_\varepsilon)$

$$u_\varepsilon(x) = \fint_{\partial B_r(x)} u_\varepsilon(y)dS(y)$$

How lovely it would be if one can send $\varepsilon \to 0$! If we're able to prove that $u \in C(\Omega)$, then mollification tells us $u_\varepsilon \to u$ uniformly over any compact subset of Ω, hence using Theorem 1.2.1 over any compact subset, we know u is harmonic in Ω, and thus $u \in C^\infty(\Omega)$.

3. It suffices to show $u \in C(\Omega)$. To do so we wish to use Arzela-Ascoli to show $u_{k_j} \to u$ locally uniformly in Ω upon extracting a subsequence. Let's see why we can apply Arzela-Ascoli

(a) For any $\Omega' \Subset \Omega$, there exists $\varepsilon \le \varepsilon_0(\Omega')$ small enough so that

$$\Omega' \cup \{B_\varepsilon(x) \mid x \in \Omega'\} \Subset \Omega'' \Subset \Omega$$

and thus

$$\|u_\varepsilon\|_{L^\infty(\Omega')} = \sup_{x\in\Omega'}|u_\varepsilon(x)| = \sup_{x\in\Omega'}|\int_{B_\varepsilon(0)} u(x-y)\eta_\varepsilon(y)dy| \le \|u\|_{L^1(\Omega'')}$$

where $\|u\|_{L^1(\Omega'')} < \infty$ is due to $u \in L^1_{\text{loc}}(\Omega)$. Hence the sequence $\{u_\varepsilon\}$ is locally uniformly bounded in ε.

(b) To see u_ε is locally uniformly equi-continuous, for any $\Omega' \Subset \Omega'' \Subset \Omega$, for any $x, y \in \Omega'$, using u_ε are smooth one has bound

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le \|\nabla u_\varepsilon\|_{L^\infty(\Omega')}|x-y|$$

Hence it suffices to prove

$$\|\nabla u_\varepsilon\|_{L^\infty(\Omega')} \le C$$

for some constant C independent of ε. To do so we make use of gradient estimate in L^1 (1.17).

$$\|\nabla u_\varepsilon\|_{L^\infty(\Omega')} \le C(\Omega', \Omega'')\|u_\varepsilon\|_{L^1(\Omega'')}$$

Now we may take $\varepsilon \le \varepsilon_0(\Omega'')$ small enough so that

$$\Omega'' \cup \{B_\varepsilon(x) \mid x \in \Omega''\} \Subset \Omega''' \Subset \Omega$$

and thus

$$\|u_\varepsilon\|_{L^1(\Omega'')} = \int_{\Omega''}|\int_{B_\varepsilon(0)} u(x-y)\eta_\varepsilon(y)dy|dx \le \|u\|_{L^1(\Omega''')}|\Omega''|$$

Thus

$$\|\nabla u_\varepsilon\|_{L^\infty(\Omega')} \le C(\Omega', \Omega'')\|u\|_{L^1(\Omega''')}|\Omega''|$$

uniformly for ε small.

Hence Arzela-Ascoli applies. Using uniqueness of limit, we conclude $u \in C(\Omega)$.

□

Stability Under Weak Convergence One has stability w.r.t. weak convergence of the source f_k and their weak solutions. Weyl's Lemma says harmonicity is preserved even under the weak convergence.

Corollary 1.2.2 (De Silva Analysis II 2025)**.** *If $f_k \in L^1_{loc}(\Omega)$ and $f_k \rightharpoonup 0$ weakly. Let $u_k \in L^1_{loc}(\Omega)$ be weak solutions to*

$$\Delta u_k = f_k$$

and $u_k \rightharpoonup u$ weakly. Then $u \in C^\infty(\Omega)$ and is harmonic in Ω in the classical sense.

Proof. We directly compute for any $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega u\Delta\varphi = \int_\Omega (u - u_k + u_k)\Delta\varphi = \int_\Omega (u - u_k)\Delta\varphi + \int_\Omega f_k \Delta\varphi \to 0$$

where the two convergence follows directly by $u_k \rightharpoonup u$ and $f_k \rightharpoonup 0$. Thus $u \in L^1_{\text{loc}}(\Omega)$ is weakly harmonic, and using Weyl's Lemma 1.2.1 the result follows. □

1.3 Representation Formula

In this section we derive a representation formula for solutions to Laplace Equation over decent domains.

Fundamental Solutions We begin by seeking a radial solution, which we denote

$$v(|x|) = u(x)$$

where we expect u to solve Laplace Equation over $\mathbb{R}^n \setminus \{0\}$.

1. Let's see what ODE v solves ([Eva10] 2.2.1 a). Notice

$$|x| = \sqrt{x_1^2 + \cdots + x_n^2}$$

so that

$$\begin{aligned}
\frac{\partial |x|}{\partial x_i} &= \partial_{x_i}(\sqrt{x_1^2 + \cdots + x_n^2}) = \frac{1}{2}\frac{1}{|x|}\partial_{x_i}(x_1^2 + \cdots + x_n^2) = \frac{x_i}{|x|} \qquad \forall\, x \neq 0 \\
u_{x_i}(x) &= \partial_{x_i}(v(|x|)) = v'(|x|)\frac{\partial |x|}{\partial x_i} = v'(|x|)\frac{x_i}{|x|} \\
u_{x_i x_i}(x) &= \partial_{x_i}(v'(|x|)\frac{x_i}{|x|}) = v''(|x|)\frac{x_i^2}{|x|^2} + v'(|x|)(\frac{1}{|x|} - \frac{x_i^2}{|x|^3}) \\
\Delta u(x) &= v''(|x|) + \frac{n-1}{|x|}v'(|x|)
\end{aligned}$$

If one denote $r = |x|$ then $v(r) = v(|x|)$ solves the ODE

$$v''(r) + \frac{n-1}{r}v'(r) = 0 \qquad \forall\, r > 0$$

2. We proceed to obtain solution to the ODE. We denote $w(r) = v'(r)$ and view our ODE as

$$w'(r) + \frac{n-1}{r}w(r) = 0 \qquad \forall\, r > 0$$

If $w \neq 0$, then

$$\begin{aligned}
\frac{w'(r)}{w(r)} &= \frac{1-n}{r} \\
(\log(|w(r)|))' &= \frac{1-n}{r} \\
\log(|w(r)|) &= (1-n)\log(r) + C \\
|w(r)| &= e^C r^{1-n}
\end{aligned}$$

Hence unraveling $w(r) = v'(r)$ yields a general solution to the ODE.

$$v(r) = \begin{cases} C_1 \log(r) + C_2 & n = 2 \\ C_1 \frac{1}{2-n} r^{2-n} + C_2 & n \geq 3 \end{cases}$$

3. In this step we make smart choice of the constants C_1 and C_2, and therefore define the fundamental solution. We impose the condition that ([HL11] Section 1.3)

$$1 = \int_{\partial B_r(0)} v'(r)\, dS \overset{u(x)=v(|x|)}{=} \int_{\partial B_r(0)} \nabla u(x) \cdot \frac{x}{|x|}\, dS(x) \qquad \forall\, r > 0$$

If so, one can choose C_1 s.t.

$$\begin{aligned}
1 &= \int_{\partial B_r(0)} C_1 r^{1-n}\, dS = \int_{\partial B_1(0)} C_1 r^{1-n} r^{n-1}\, dS(\omega) = C_1 n\omega_n \\
C_1 &= \frac{1}{n\omega_n}
\end{aligned}$$

C_2 seems free, so for simplicity we take $C_2 = 0$.

Definition 1.3.1 (Fundamental Solution to Laplace Equation). *For any fixed $x \in \mathbb{R}^n$, the fundamental solution $\Gamma : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is defined as*

$$\Gamma(x-y) := \begin{cases} \frac{1}{2\pi}\log(|x-y|) & n=2 \\ \frac{1}{n(2-n)\omega_n}|x-y|^{2-n} & n \geq 3 \end{cases} \tag{1.20}$$

Abuse of notation, one denote

$$\Gamma(x-y) = \Gamma(|x-y|) = \Gamma(x,y) \qquad \forall\, x \neq y$$

Note we're treating x as the pole fixed in $\mathbb{R}^n$, and $y \in \mathbb{R}^n \setminus \{x\}$ as the variable. One has two immediate observations for the fundamental solution.

1. For any $x \in \mathbb{R}^n$ fixed

$$\Delta_y \Gamma(x-y) = 0 \qquad \forall\, y \neq x$$

   In particular $\Gamma(x-\cdot)$ has a singularity at $y = x$.

2. For any $x \in \mathbb{R}^n$ fixed, $\Gamma'(|x-y|)$ has precise formula. For any $n \geq 2$

$$\begin{aligned} \partial_{y_i}\Gamma(x-y) &= \frac{1}{n\omega_n}|x-y|^{1-n}\frac{y_i-x_i}{|x-y|} \\ \nabla_y \Gamma(x-y) &= \frac{1}{n\omega_n}\frac{y-x}{|x-y|^n} \end{aligned}$$

   Then for outward unit normal $\nu(y)$ on $\partial B_r(x)$, so that $\nu(y) = \frac{y-x}{|x-y|}$, one has

$$\begin{aligned} \frac{\partial\Gamma(x,y)}{\partial\nu(y)} &= \nabla_y\Gamma(x-y)\cdot\nu(y) = \frac{1}{n\omega_n}\frac{1}{|x-y|^{n-1}} \\ &= \Gamma'(|x-y|) \qquad \text{abuse of notation} \end{aligned}$$

   In particular, the choice of the constants are to make

$$\begin{aligned} \int_{\partial B_r(x)} \frac{\partial\Gamma(x,y)}{\partial\nu(y)}\,dS(y) &= \int_{\partial B_r(x)} \frac{1}{n\omega_n}\frac{1}{|x-y|^{n-1}}\,dS(y) \\ &= \int_{\partial B_1(0)} \frac{1}{n\omega_n}\frac{1}{r^{n-1}|z|^{n-1}}\cdot r^{n-1}dS(z) \qquad \text{change } y = x + rz \\ &= \int_{\partial B_1(0)} \frac{1}{n\omega_n}\frac{1}{r^{n-1}}\cdot r^{n-1}dS = \frac{1}{n\omega_n}|\partial B_1(0)| = 1 \qquad \forall\, r > 0 \end{aligned} \tag{1.21}$$

   Note Γ is not defined at $y = x$, hence $\Gamma(x-\cdot) \notin C^2(B_r(x))$. It does not violate Gauss-Green (1.4).

In particular if take $x = 0$, one has simple formulas

$$\int_{\partial B_r} \Gamma'(|y|) = 1 \qquad \forall\, r > 0$$

One could formally think of as (but not true)

$$\int_{\mathbb{R}^n} \Delta\Gamma = 1$$

Cutoff Γ_ρ Let's make rigorous the equation

$$\Delta\Gamma = \delta_0 \qquad \text{the Dirac Delta Measure}$$

In fact consider

$$\Gamma(x) := \begin{cases} C_n|x|^{2-n} & n \geq 3 \\ \frac{1}{2\pi}\log|x| & n = 2 \end{cases}$$

and

$$\Gamma_\rho(x) := \begin{cases} \Gamma(x) & |x| \geq \rho \\ P(x) & |x| < \rho \end{cases}$$

where P is the unique convex parabola that glues on $\partial B_\rho(0)$ in C^1 fashion s.t.

$$\Delta P(x) = \frac{1}{|B_\rho|}$$

In fact, let's build such parabola explicitly.

Lemma 1.3.1 (Savin Analysis II 2026)**.** *The cutoff* $\Gamma_\rho \in C^{1,1}(\mathbb{R}^n)$ *defined as*

$$\Gamma_\rho(x) = \begin{cases} \Gamma & |x| \geq \rho \\ \frac{1}{2n|B_\rho|}(|x|^2 - \rho^2) + \Gamma(\rho) & |x| \leq \rho \end{cases} \tag{1.22}$$

satisfies

$$\Delta\Gamma_\rho(x) = \frac{1}{|B_\rho|}\chi_{B_\rho} \qquad \textit{weakly}$$

and

$$\int_{\mathbb{R}^n} \Delta\Gamma_\rho = 1$$

Proof. Let $P(x) = a|x|^2 + b|x| + c$. Then match

$$\begin{aligned}
\Delta P(x) &= 2na = \frac{1}{|B_\rho|} \\
a &= \frac{1}{2n|B_\rho|} = \frac{1}{2n\omega_n\rho^n} \\
P'(|x|) &= 2a\rho + b = \Gamma'(\rho) = \frac{1}{n\omega_n}\rho^{1-n} \\
b &= 0 \\
P|_{\partial B_\rho} &= a\rho^2 + b\rho + c = \Gamma(\rho)
\end{aligned}$$

so that

$$P(x) = \frac{1}{2n|B_\rho|}(|x|^2 - \rho^2) + \Gamma(\rho) \qquad \forall\ x \in \overline{B_\rho}$$

Let's see $\Delta\Gamma_\rho = \frac{1}{|B_\rho|}\chi_{B_\rho}$. For any $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Gamma_\rho \Delta\varphi \\
&= \int_{|x|>\rho} \Gamma\Delta\varphi + \int_{|x|<\rho} P\Delta\varphi \\
&= \int_{|x|>\rho} \Delta\Gamma\varphi + \int_{\partial B_\rho} \frac{\partial\varphi}{\partial\nu}\Gamma - \int_{\partial B_\rho} \varphi\frac{\partial\Gamma}{\partial\nu} \qquad \text{here } \nu = -\frac{x}{|x|} \text{ is outwards w.r.t. } |x| > \rho \\
&+ \int_{|x|<\rho} \Delta P\varphi + \int_{\partial B_\rho} \frac{\partial\varphi}{\partial\nu}P - \int_{\partial B_\rho} \varphi\frac{\partial P}{\partial\nu} \qquad \text{here } \nu = \frac{x}{|x|} \text{ is outwards w.r.t. } |x| < \rho \\
&= \int_{|x|<\rho} \frac{1}{|B_\rho|}\varphi \qquad \text{boundary terms all cancel out due to } C^1 \text{ matching}
\end{aligned}$$

□

Lemma 1.3.2 (Convergence for Γ_ρ)**.** *Let* $n \geq 3$*. Then for any* $p < \frac{n}{n-2}$

$$\Gamma_\rho \to \Gamma \qquad \textit{in } L^p$$

Proof. Notice $\Gamma_\rho \to \Gamma$ a.e. $x \in \mathbb{R}^n$ (in fact only except the origin 0) for $\rho \to 0$. Hence in order to deduce

$$\lim_{\rho\to 0} \int_{\mathbb{R}^n} |\Gamma_\rho - \Gamma|^p = 0$$

It suffices to ensure DCT applies, i.e., there is a domination. But (for $n \geq 3$), for any fixed $\rho > 0$, the difference only happens in the region B_ρ, thus we look at

$$C_n^p \int_{B_\rho} |x|^{(2-n)p} dx = C(n,p) \int_0^\rho r^{(2-n)p+n-1} dr$$

This is integrable when

$$(2-n)p + n - 1 > -1 \iff p < \frac{n}{n-2}$$

□

Now the RHS converges weakly (star) to the Dirac measure δ_0 while LHS converges in L^p for any $1 \leq p < \frac{n}{n-2}$ (for $n \geq 3$) to Γ.

Using Stability Corollary 1.2.2 we know

$$\Delta\Gamma = \delta_0 \qquad \text{weakly}$$

Alternative Proof for Weyl In fact one has alternative proof for Weyl's Lemma.

Proof. Let

$$\varphi(x) := \Gamma_{\rho_2}(x) - \Gamma_{\rho_1}(x)$$

for $\rho_1 < \rho_2$ and convolve $\varphi_\varepsilon(x) := (\varphi * \eta_\varepsilon)(x)$.

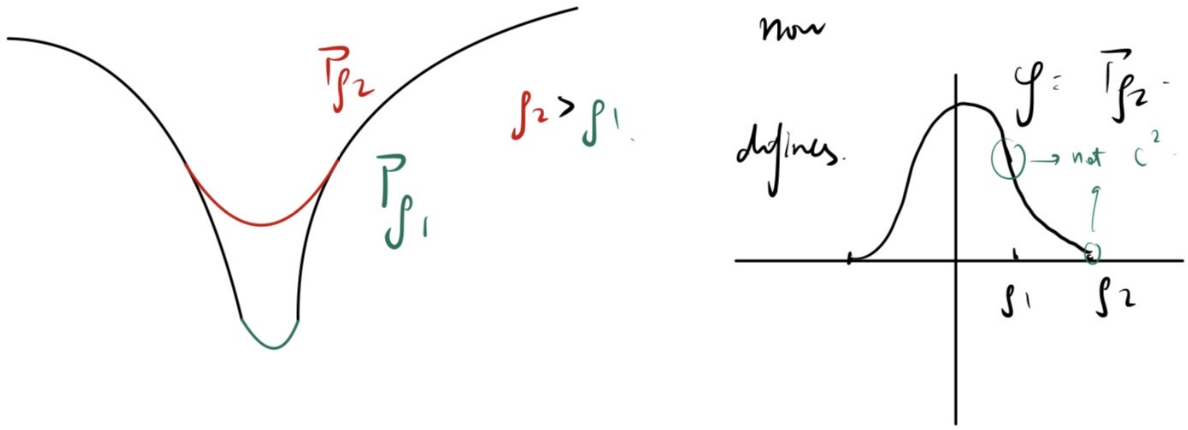


Figure 1.2: $\Gamma_{\rho_2} - \Gamma_{\rho_1}$

Notice

$$\begin{aligned} \varphi_\varepsilon(x) &= \Gamma_{\rho_2} * \eta_\varepsilon(x) - \Gamma_{\rho_1} * \eta_\varepsilon(x) \\ \partial_i \varphi_\varepsilon(x) &= (\partial_i \Gamma_{\rho_2})_\varepsilon - (\partial_i \Gamma_{\rho_1})_\varepsilon \\ \Delta\varphi_\varepsilon(x) &= (\Delta\Gamma_{\rho_2} - \Delta\Gamma_{\rho_1})_\varepsilon \end{aligned}$$

Now using u is weakly harmonic

$$0 = \int_\Omega u \Delta\varphi_\varepsilon$$

$$0 = \lim_{\varepsilon \to 0} \int_\Omega u\Delta\varphi_\varepsilon = \int_\Omega u\left(\frac{1}{|B_{\rho_2}|}\chi_{B_{\rho_2}} - \frac{1}{|B_{\rho_1}|}\chi_{B_{\rho_1}}\right)$$

Thus

$$⨍_{B_{\rho_2}(x)} u = ⨍_{B_{\rho_1}(x)} u \qquad \forall\ \rho_1 < \rho_2$$

This is almost MVP except that $u \in L^1_{\text{loc}}$ so the pointwise value does not make sense. But

1. On one hand, by Lebesgue Differentiation

   $$\lim_{\rho_1 \to 0} ⨍_{B_{\rho_1}(x)} u = u(x) \qquad \text{for a.e. } x \in \Omega, \text{ in particular, at the Lebesgue points}$$

   Thus sending $\rho_1 \to 0$ gives

   $$u(x) = ⨍_{B_{\rho_2}(x)} u \qquad \text{a.e. } x \in \Omega$$

   Where u is in fact defined pointwise a.e. by picking the representative that converges at the particular point.

2. On the other hand, $⨍_{B_{\rho_2}(x)} u$ is a continuous function in x. Thus u is continuous function and defined everywhere in Ω.

Therefore, since u satisfies MVP, it is harmonic. □

Global Solution to Poisson's Equation By convoluting the source f with our fundamental solution (1.20), one can define solutions to Poisson's equation over the whole $\mathbb{R}^n$.

Theorem 1.3.1 ([Eva10] Theorem 2.1)**.** *Let $f \in C_0^2(\mathbb{R}^n)$. Define u by*

$$u(x) := \int_{\mathbb{R}^n} \Gamma(x-y) f(y) dy \qquad \forall\ x \in \mathbb{R}^n \tag{1.23}$$

Then

1. *$u \in C^2(\mathbb{R}^n)$*
2. *u solves classically*

$$\Delta u(x) = f(x) \qquad \forall\ x \in \mathbb{R}^n$$

Proof. Making use of $y \in \mathbb{R}^n \mapsto x - y \in \mathbb{R}^n$ is bijection with Jacobian size 1, one can interchange

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y) dy = \int_{\mathbb{R}^n} \Gamma(y) f(x-y) dy$$

1. Let's first see why the function (1.23) is well-defined. Take $n \geq 3$ for simplicity. Since $f \in C_0^\infty(\mathbb{R}^n)$, for any $x \in \mathbb{R}^n$, there exists $R = R(x)$ sufficiently large s.t. $\text{supp}(f(x - \cdot)) \subseteq B_R$. Then

$$\begin{aligned} |u(x)| &\leq C(n) \int_{B_R} |y|^{2-n} |f(x-y)| dy \leq C(n) \|f\|_{L^\infty(\mathbb{R}^n)} \int_0^R \int_{\partial B_1} r^{2-n} r^{n-1} dS(\omega) dr \\ &= C(n) \|f\|_{L^\infty(\mathbb{R}^n)} R^2 < \infty \end{aligned}$$

   Hence $u(x)$ is well-defined pointwise over $\mathbb{R}^n$. In particular this shows for any $x \in \mathbb{R}^n$, the function

$$\Gamma(y) f(x-y) \in L^1_y(\mathbb{R}^n)$$

2. We first see why one can differentiate u. Now for any $i = 1, \cdots, n$

$$\frac{u(x + he_i) - u(x)}{|h|} = \int_{\mathbb{R}^n} \Gamma(y) \frac{f(x + he_i - y) - f(x-y)}{|h|} dy$$

   and since for $|h| \leq 1$ small

$$|\Gamma(y) \frac{f(x + he_i - y) - f(x-y)}{|h|}| \leq |\Gamma(y)| \sup_{B_{R+1}} |\nabla f| \in L^1_y(\mathbb{R}^n)$$

   By the dominated convergence theorem one can interchange differentiatio and integration, hence

$$u_{x_i}(x) = \int_{\mathbb{R}^n} \Gamma(y) f_{x_i}(x-y) dy$$

   Similarly

$$u_{x_i x_j}(x) = \int_{\mathbb{R}^n} \Gamma(y) f_{x_i x_j}(x-y) dy$$

   Since $\Gamma \in L^1_y(\mathbb{R}^n)$ and for each fixed $x \in \mathbb{R}^n$, $f_{x_i x_j}(x - \cdot)$ is uniformly continuous over its compact support, the function $u_{x_i x_j} \in C(\mathbb{R}^n)$. Thus $u \in C^2(\mathbb{R}^n)$.

3. Let's see why u solves the Poisson's Equation. For this, it suffices to check that

$$f(x) = \int_{\mathbb{R}^n} \Gamma(y) \Delta_x f(x-y) dy \qquad \forall\ x \in \mathbb{R}^n \tag{1.24}$$

   Since Γ has a singularity at $y = 0$, we divide the RHS into two portions

$$\int_{\mathbb{R}^n} \Gamma(y) \Delta_x f(x-y) dy = \int_{B_\varepsilon(0)} \Gamma(y) \Delta_x f(x-y) dy + \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Gamma(y) \Delta_x f(x-y) dy \tag{1.25}$$

   and study the limit as $\varepsilon \to 0$. For the following we fix $x \in \mathbb{R}^n$.
   For the first portion.

$$\begin{aligned} \int_{B_\varepsilon(0)} \Gamma(y) \Delta_x f(x-y) dy &\leq C(n) \left\|D^2 f\right\|_{L^\infty(B_\varepsilon(x))} \int_{B_\varepsilon(0)} |\Gamma(y)| dy = C(n) \left\|D^2 f\right\|_{L^\infty(B_\varepsilon(x))} \int_0^\varepsilon \Gamma(r) r^{n-1} dr \\ &\leq \begin{cases} C(n) \left\|D^2 f\right\|_{L^\infty(B_\varepsilon(x))} \varepsilon^2 |\log(\varepsilon)| & n = 2 \\ C(n) \left\|D^2 f\right\|_{L^\infty(B_\varepsilon(x))} \varepsilon^2 & n \geq 3 \end{cases} \end{aligned}$$

In any cases, the first portion goes to 0 as $\varepsilon \to 0$.

For the second portion. We need to study what happens in particular on $\partial B_\varepsilon(0)$. Take $R = R(x)$ large so that $f(x - \cdot)$ is supported within B_R. First notice $\Delta_y \Gamma(y) = 0$ away from the origin

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Gamma(y)\Delta_x f(x-y)dy &= \int_{B_R(0)\setminus B_\varepsilon(0)} \Gamma(y)\Delta_x f(x-y)dy \qquad \text{support of } D^2 f \\
&= \int_{B_R(0)\setminus B_\varepsilon(0)} \Gamma(y)\Delta_y f(x-y)dy \qquad \text{differentiating twice} \\
&= \int_{B_R(0)\setminus B_\varepsilon(0)} (\Gamma(y)\Delta_y f(x-y) - \Delta_y \Gamma(y) f(x-y))dy \qquad \text{use } \Delta_y\Gamma(y) = 0 \\
&= \int_{\partial(B_R(0)\setminus B_\varepsilon(0))} \left(\Gamma(y)\frac{\partial f}{\partial \nu_y}(x-y) - \frac{\partial \Gamma}{\partial \nu_y}(y) f(x-y)\right) dS(y) \qquad \text{Apply (1.7)} \\
&= -\int_{\partial B_\varepsilon(0)} \left(\Gamma(y)\frac{\partial f}{\partial \nu_y}(x-y) - \frac{\partial \Gamma}{\partial \nu_y}(y) f(x-y)\right) dS(y) \qquad \nu \text{ denotes outward normal}
\end{aligned}$$

Again the RHS consists of two sub-portions. For the first

$$\begin{aligned}
\left| -\int_{\partial B_\varepsilon(0)} \Gamma(y)\frac{\partial f}{\partial \nu_y}(x-y)dS(y)\right| &\le C(n)\left\|\nabla f\right\|_{L^\infty(B_\varepsilon(x))} \varepsilon^{n-1}|\Gamma(\varepsilon)| \\
&\le \begin{cases} C(n)\left\|\nabla f\right\|_{L^\infty(B_\varepsilon(x))} \varepsilon|\log(\varepsilon)| & n=2 \\ C(n)\left\|\nabla f\right\|_{L^\infty(B_\varepsilon(x))} \varepsilon & n \ge 3 \end{cases} \to 0 \qquad \varepsilon \to 0
\end{aligned}$$

For the second, interesting things happen.

$$\begin{aligned}
\int_{\partial B_\varepsilon(0)} \frac{\partial \Gamma}{\partial \nu_y}(y) f(x-y)dS(y) &= \frac{1}{n\omega_n}\int_{\partial B_1(0)} \varepsilon^{1-n}\varepsilon^{n-1} f(x-\varepsilon\omega)dS(\omega) \qquad \text{choice of (1.20)} \\
&\to \frac{1}{n\omega_n}\int_{\partial B_1(0)} f(x)dS(\omega) = f(x) \qquad \varepsilon \to 0 \qquad \text{using dominated convergence theorem}
\end{aligned}$$

Hence as $\varepsilon \to 0$ on RHS of (1.25)

$$\int_{\mathbb{R}^n} \Gamma(y)\Delta_x f(x-y)dy = \int_{B_\varepsilon(0)} \Gamma(y)\Delta_x f(x-y)dy + \int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \Gamma(y)\Delta_x f(x-y)dy \to f(x) \qquad \varepsilon \to 0$$

But LHS is independent of ε, hence (1.24) holds.

$\square$

1.3.1 Green's Representation

Intuitively, how to solve the Dirichlet problem for the Laplace operator?

$$\begin{cases} \Delta u = 0 & \Omega \\ u = \varphi & \partial\Omega \end{cases}$$

Recall the Green's formula

$$\begin{aligned}
\int_\Omega \Delta u v &= -\int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \\
&= \int_\Omega u \Delta v - \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v
\end{aligned}$$

Now if u is our solution, and v is some test function, in particular,

$$v = \Gamma_\rho * \eta_\varepsilon \qquad \text{for } \Gamma \text{ with fundamental solution at } x_0 \in \Omega$$

then the above writes

$$\begin{aligned}
0 &= \int_\Omega u\Delta\Gamma_\rho - \int_{\partial\Omega} u\frac{\partial \Gamma_\rho}{\partial \nu} + \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\Gamma_\rho \\
\int_\Omega u\Delta\Gamma_\rho &= \int_{\partial\Omega} u\frac{\partial \Gamma_\rho}{\partial \nu} - \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\Gamma_\rho \\
u(x_0) &= \int_{\partial\Omega} u\frac{\partial \Gamma}{\partial \nu} - \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\Gamma \qquad \text{upon passing } \rho, \varepsilon \to 0
\end{aligned}$$

The analysis are made below via the Green's Formula.

Green's Representation Formula In the following we mimic the proof for Theorem 1.3.1 and obtain the Green's Representation Formula over C^1 bounded domains. Notice the formula works for any function u sufficiently regular!

Theorem 1.3.2 ([HL11] Theorem 1.17; [Eva10] 2.2.4.a; [GT01] Section 2.4)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open bounded domain with C^1 regularity. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Then for any $x \in \Omega$*

$$u(x) = \int_{\partial\Omega} \left(u(y) \frac{\partial \Gamma}{\partial \nu_y}(x-y) - \frac{\partial u}{\partial \nu_y}(y)\Gamma(x-y) \right) dS(y) + \int_\Omega \Gamma(x-y)\Delta u(y) dy \tag{1.26}$$

Proof. For any $x \in \Omega$, take $B_r(x) \Subset \Omega$. Consider the quantity

$$\begin{aligned}\int_{\Omega \backslash B_r(x)} u(y)\Delta_y \Gamma(x-y) - \Delta u(y)\Gamma(x-y) dy &= -\int_{\Omega\backslash B_r(x)} \Delta u(y)\Gamma(x-y)dy \qquad \text{use } \Delta_y\Gamma(x-y) = 0 \text{ for any } y \neq x \\ &\to -\int_\Omega \Delta u(y)\Gamma(x-y)dy \qquad r \to 0 \end{aligned} \tag{1.27}$$

Using dominated convergence theorem, and the fact that $\Gamma(x-y)\Delta u(y) \in L^1_y(\Omega)$ as argued in Theorem 1.3.1. On the other hand, one can apply Green's second identity (1.7) to the LHS

$$\begin{aligned}&\int_{\Omega\backslash B_r(x)} u(y)\Delta_y\Gamma(x-y) - \Delta u(y)\Gamma(x-y)dy \\ &\overset{(1.7)}{=} \int_{\partial(\Omega\backslash B_r(x))} \left(u(y)\frac{\partial\Gamma}{\partial\nu_y}(x-y) - \frac{\partial u}{\partial \nu_y}(y)\Gamma(x-y)\right) dS(y) \\ &= \int_{\partial\Omega} \left(u(y)\frac{\partial\Gamma}{\partial\nu_y}(x-y) - \frac{\partial u}{\partial \nu_y}(y)\Gamma(x-y)\right) dS(y) - \int_{\partial B_r(x)} \left(u(y)\frac{\partial\Gamma}{\partial\nu_y}(x-y) - \frac{\partial u}{\partial \nu_y}(y)\Gamma(x-y)\right) dS(y) \qquad \nu \text{ outer normal}\end{aligned} \tag{1.28}$$

One want to study the limit of the second portion as $r \to 0$. Note this again consists of two sub-portions. We argue in the same way as Theorem 1.3.1. For the first

$$\begin{aligned}-\int_{\partial B_r(x)} u(y)\frac{\partial \Gamma}{\partial \nu_y}(x-y) &= -\frac{1}{n\omega_n}\int_{\partial B_1(0)} u(x+r\omega)r^{1-n}r^{n-1}dS(\omega) \qquad \text{using choice of fundamental solution (1.20)} \\ &\to -\frac{1}{n\omega_n}\int_{\partial B_1(0)} u(x)dS(\omega) = -u(x) \qquad r\to 0\end{aligned}$$

For the second

$$\begin{aligned}\left|\int_{\partial B_r(x)} \frac{\partial u}{\partial \nu_y}(y)\Gamma(x-y)dS(y)\right| &\le C(n)\left\|u\right\|_{C^1(B_r(x))} r^{n-1}|\Gamma(r)| \\ &\le \begin{cases} C(n)\left\|u\right\|_{C^1(B_r(x))} r|\log(r)| & n=2 \\ C(n)\left\|u\right\|_{C^1(B_r(x))} r & n \ge 3\end{cases} \to 0 \qquad r\to 0\end{aligned}$$

Rearranging the terms in (1.27), (1.28) and sending $r \to 0$ yields the result (1.26). □

Green's Function Assume the function $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is unknown. Then using the Representation formula (1.26), it suffices to use

$$u|_{\partial\Omega}, \qquad \left.\frac{\partial u}{\partial \nu_y}\right|_{\partial\Omega}, \qquad \Delta u|_\Omega$$

to determine the function value u everywhere in Ω. Now for a classical Dirichlet Boundary Value Problem for Laplace Equation, the outer-normal derivative term $\left.\frac{\partial u}{\partial \nu_y}\right|_{\partial\Omega}$ is usually unknown. But thanks to the clever observation of Green's, if we have enough information about the domain Ω, one can define 'Green's Function' to remove the middle term and fully determine a classical solution to the Dirichlet Boundary Value Problem.

Heuristically, starting from

$$u(x_0) = \int_{\partial\Omega} u\frac{\partial \Gamma_{x_0}}{\partial\nu} - \int_{\partial\Omega}\frac{\partial u}{\partial \nu}\Gamma_{x_0}$$

to cancel the second portion, let

$$G = \Gamma_{x_0} + \Phi$$

where the Φ solves

$$\begin{cases}\Delta\Phi = 0 & \Omega \\ \Phi = -\Gamma_{x_0} & \partial\Omega\end{cases}$$

Applying the Green's identity to G writes

$$\begin{aligned}
0 &= \int_{\Omega} u\Delta G - \int_{\partial\Omega} u\frac{\partial G}{\partial \nu} + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} G \\
u(x_0) &= \int_{\partial\Omega} u\frac{\partial G}{\partial \nu} - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} G \\
&= \int_{\partial\Omega} u\frac{\partial G}{\partial \nu} - \underbrace{\cancel{\int_{\partial\Omega} \frac{\partial u}{\partial \nu}(\Gamma_{x_0} + \Phi)}}_{\text{due to choice of } \Phi \text{ on } \partial\Omega}
\end{aligned}$$

Let' rigorously define Green's Function for a given domain Ω with C^1 regularity.

Definition 1.3.2 (Green's Function)**.** *If for any $x \in \Omega$, one can define a corrector function*

$$\Phi(x, \cdot) \in C^1(\overline{\Omega}) \cap C^2(\Omega)$$

that solves

$$\begin{cases} \Delta_y \Phi(x, y) = 0 & \forall\ y \in \Omega \\ \Phi(x, y) = -\Gamma(x - y) & \forall\ y \in \partial\Omega \end{cases} \tag{1.29}$$

Such $\Phi(x, \cdot)$ is the harmonic replacement of $-\Gamma(x - y)$ with pole at x. Whether such $\Phi(x, \cdot)$ exists depend on the domain.

Then one can define Green's Function

$$G(x, y) := \Phi(x, y) + \Gamma(x - y) \qquad \forall\ x \in \Omega,\ y \in \overline{\Omega},\ x \neq y \tag{1.30}$$

Notice Green's Function, by definition, necessarily solves

$$\begin{cases} \Delta_y G(x, y) = 0 & \forall\ y \neq x \in \Omega \\ G(x, y) = 0 & \forall\ y \in \partial\Omega \end{cases}$$

In particular, the Green's Function has boundary value 0, while keeps the singularity at x.

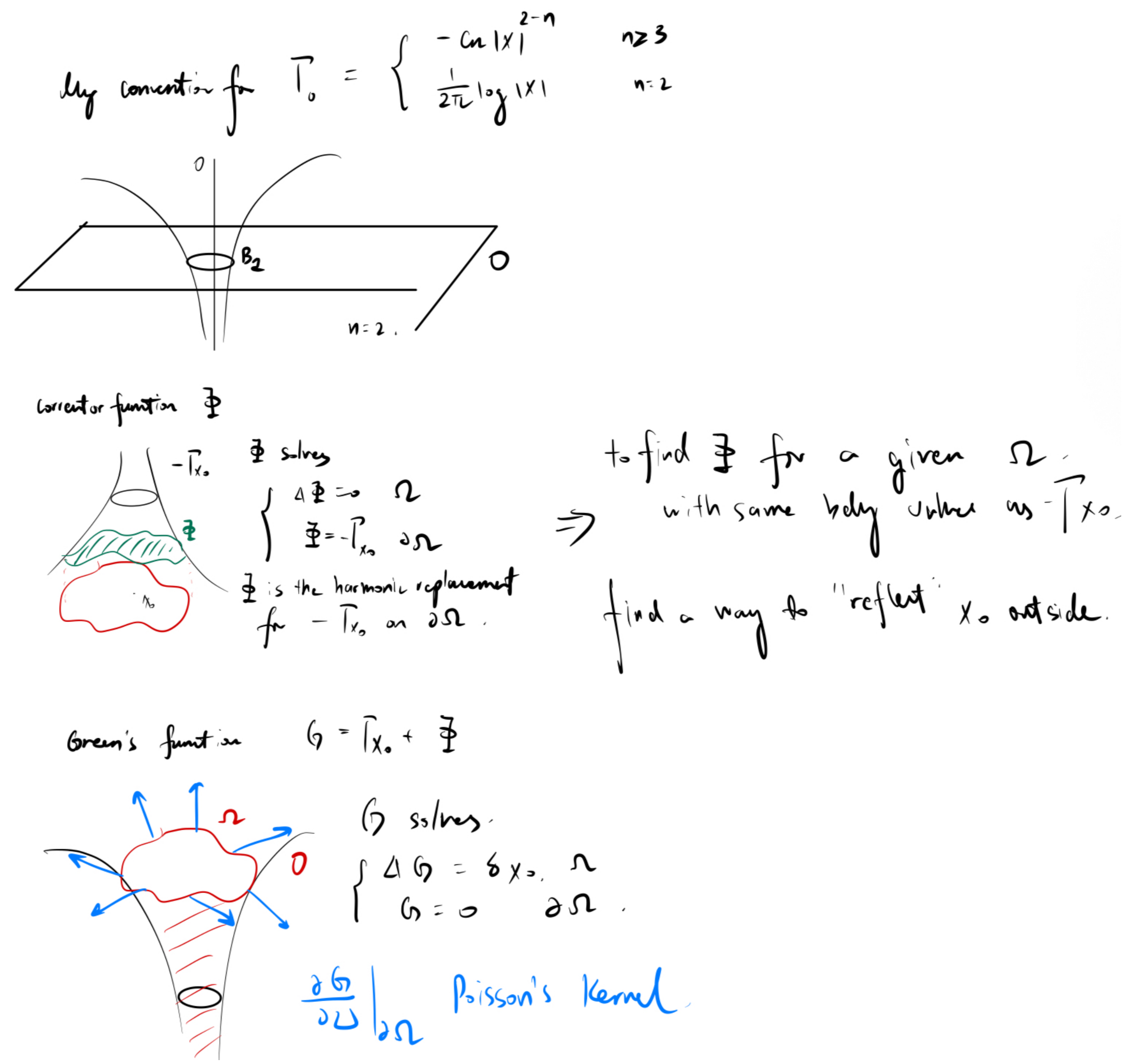


Figure 1.3: Fundamental Solution, Corrector Function, Green's Function

Why does such definition (1.30) help? One has the Green's Representation Formula improved, with G in place of Γ. And most importantly, we have the boundary gradient term removed.

Corollary 1.3.1 ([Eva10] 2.2.4.a; [HL11] Section 1.3; [GT01] Section 2.4)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open bounded domain with C^1 regularity, and let G be the associated Green's Function. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Then for any $x \in \Omega$*

$$u(x) = \int_{\partial\Omega} u(y)\frac{\partial G}{\partial \nu_y}(x,y)dS(y) + \int_{\Omega} G(x,y)\Delta u(y)dy \qquad \forall\ x \in \Omega \tag{1.31}$$

Proof. Apply Green's Second Identity (1.7) to u and $\Phi(x,\cdot)$ on Ω

$$\begin{aligned}\int_{\Omega} u(y)\Delta_y\Phi(x,y) - \Phi(x,y)\Delta u(y)\,dy &= \int_{\partial\Omega} u(y)\frac{\partial \Phi(x,y)}{\partial \nu_y} - \Phi(x,y)\frac{\partial u}{\partial \nu_y}\,dS(y)\\ -\int_{\Omega}\Phi(x,y)\Delta u(y)\,dy &= \int_{\partial\Omega} u(y)\frac{\partial \Phi(x,y)}{\partial \nu_y} - \Phi(x,y)\frac{\partial u}{\partial \nu_y}\,dS(y) \qquad \text{use } \Delta_y\Phi(x,y)=0\\ 0 &= \int_{\partial\Omega} u(y)\frac{\partial \Phi(x,y)}{\partial \nu_y} - \Phi(x,y)\frac{\partial u}{\partial \nu_y}\,dS(y) + \int_{\Omega}\Phi(x,y)\Delta u(y)\,dy\end{aligned}$$

Now we add this to our Representation Formula (1.26).

$$\begin{aligned} u(x) &= \int_{\partial\Omega} u(y)\frac{\partial G}{\partial \nu_y}(x,y)dS(y) + \int_{\Omega} G(x,y)\Delta u(y)dy - \int_{\partial\Omega} G(x,y)\frac{\partial u}{\partial \nu_y}dS(y) \\ &= \int_{\partial\Omega} u(y)\frac{\partial G}{\partial \nu_y}(x,y)dS(y) + \int_{\Omega} G(x,y)\Delta u(y)dy \qquad \text{using } G(x,y)=0 \text{ on } y\in\partial\Omega \end{aligned}$$

□

In particular, if we're given Dirichlet Boundary Value Problem where Green's Function G is well-defined over Ω

$$\begin{cases} \Delta u = f & \Omega \\ u = g & \partial\Omega \end{cases}$$

and assume the solution $u \in C^2(\Omega)\cap C^1(\overline{\Omega})$. One has unique solution from (1.31) ([Eva10] Theorem 2.12)

$$u(x) = \int_{\partial\Omega} g(x)\frac{\partial G}{\partial \nu_y}(x,y)dS(y) + \int_{\Omega} G(x,y)f(y)dy \qquad \forall\, x\in\Omega \tag{1.32}$$

Properties of Green's Function We discuss some properties of G (1.30).

1. One has uniqueness of G over bounded domains ([HL11] Section 1.3).

   Proof. By definition of Green's Function (1.30), it suffices to prove $\Phi(x,\cdot)$ is unique for each fixed $x\in\Omega$. But Φ solves the Dirichlet Boundary Value Problem (1.29) for each fixed $x\in\Omega$. If we know Ω is bounded, from the Weak Maximum Principe, one know from Corollary 1.1.1 that $\Phi(x,\cdot)$ is unique for each fixed $x\in\Omega$. Hence $G(x,y)$ is unique w.r.t. Ω. □

2. The Green's Function is Symmetric over $\Omega\times\Omega$.

   Property 1.3.1 ([Eva10] Theorem 1.13; [HL11] Proposition 1.20; [GT01] Problem 2.3(a))**.**

   $$G(x,y) = G(y,x) \qquad \forall\, x\neq y$$

   Proof. Fix any $x_1 \neq x_2 \in \Omega$. For $r>0$ small, apply Green's second identity (1.7) to

   $$G_1(y) := G(x_1,y) \qquad G_2(y) := G(x_2,y) \qquad \forall\, y\in\Omega\setminus(B_r(x_1)\cup B_r(x_2))$$

   on the domain $\Omega\setminus(B_r(x_1)\cup B_r(x_2))$. Then

   $$\begin{aligned} &\underbrace{\cancel{\int_{\Omega\setminus(B_r(x_1)\cup B_r(x_2))} G_1(y)\Delta G_2(y) - G_2(y)\Delta G_2(y)\,dy}}_{G(x,y)\text{ is harmonic away from } x=y\text{, i.e., } \Delta G_1(y)=\Delta G_2(y)=0 \text{ on } \Omega\setminus(B_r(x_1)\cup B_r(x_2)).} \\ \overset{(1.7)}{=} &\underbrace{\cancel{\int_{\partial\Omega}\left(G_1(y)\frac{\partial G_2(y)}{\partial\nu_y} - G_2\frac{\partial G_1(y)}{\partial \nu_y}\right)dS}}_{G(x,y)=0 \text{ for any } y\in\partial\Omega\text{. Hence } G_1=G_2=0 \text{ on } \partial\Omega.} \\ &- \int_{\partial B_r(x_1)}\left(G_1\frac{\partial G_2}{\partial\nu_y} - G_2\frac{\partial G_1}{\partial\nu_y}\right)dS - \int_{\partial B_r(x_2)}\left(G_1\frac{\partial G_2}{\partial\nu_y} - G_2\frac{\partial G_1}{\partial\nu_y}\right)dS \end{aligned}$$

   where ν always denotes the outer normal. Thus only the last two terms survive. We rewrite

   $$0 = \int_{\partial B_r(x_1)}\left(G_1\frac{\partial G_2}{\partial\nu_y} - G_2\frac{\partial G_1}{\partial\nu_y}\right)dS + \int_{\partial B_r(x_2)}\left(G_1\frac{\partial G_2}{\partial\nu_y} - G_2\frac{\partial G_1}{\partial\nu_y}\right)dS$$

   One wish to take $r\to 0$. For the integral on $\partial B_r(x_1)$, singularity occurs at $\frac{\partial G_1}{\partial \nu_y}$ as $y\to x_1$. This first term can be further divided into two sub-portions

   (a) For the first portion, we argue $G_1(y)$ alone does not create a problem. If $n\geq 3$

   $$\begin{aligned} \int_{\partial B_r(x_1)} |G_1\frac{\partial G_2}{\partial\nu_y}|\,dS &\leq \int_{\partial B_r(x_1)} |\Gamma(x_1,y)\frac{\partial G_2}{\partial\nu_y}|\,dS + \int_{\partial B_r(x_1)} |\Phi(x_1,y)\frac{\partial G_2}{\partial\nu_y}|\,dS \\ &\lesssim \|G_2\|_{C^1(B_r(x_1))}\, r^{2-n}r^{n-1} + \|\Phi(x_1,\cdot)\|_{L^\infty(\Omega)}\,\|G_2\|_{C^1(B_r(x_1))}\, r^{n-1} \overset{r\to 0}{\to} 0 \end{aligned}$$

   While for $n=2$

   $$\int_{\partial B_r(x_1)} |G_1\frac{\partial G_2}{\partial\nu_y}|\,dS \lesssim \|G_2\|_{C^1(B_r(x_1))}\,|\log(r)|r + \|\Phi(x_1,\cdot)\|_{L^\infty(\Omega)}\,\|G_2\|_{C^1(B_r(x_1))}\, r \overset{r\to 0}{\to} 0$$

(b) For the second portion, we argue $\frac{\partial G_1}{\partial \nu_y}$, which is more singular, gives $-G(x_2, x_1)$ as $r \to 0$.

$$-\int_{\partial B_r(x_1)} G_2 \frac{\partial G_1}{\partial \nu_y}\, dS(y) = -\int_{\partial B_r(x_1)} G_2 \frac{\partial \Gamma(x_1, y)}{\partial \nu_y}\, dS(y) - \int_{\partial B_r(x_1)} G_2 \frac{\partial \Phi(x_1, y)}{\partial \nu_y}\, dS(y)$$

Here the first term on RHS converges using Lebesgue Differentiation

$$\begin{aligned} -\int_{\partial B_r(x_1)} G_2 \frac{\partial \Gamma(x_1, y)}{\partial \nu_y}\, dS(y) &= -\frac{1}{n\omega_n} \int_{\partial B_r(x_1)} G_2 r^{1-n}\, dS(y) \\ &= -\frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(x_1)} G_2\, dS(y) \overset{r\to 0}{\to} -G_2(x_1) = -G(x_2, x_1) \end{aligned}$$

While the second term on RHS vanishes

$$\int_{\partial B_r(x_1)} |G_2 \frac{\partial \Phi(x_1, y)}{\partial \nu_y}|\, dS(y) \lesssim \|G_2\|_{C(B_r(x_1))} \|\Phi(x_1, \cdot)\|_{C^1(\Omega)} r^{n-1} \overset{r\to 0}{\to} 0$$

Hence

$$\int_{\partial B_r(x_1)} \left(G_1 \frac{\partial G_2}{\partial \nu_y} - G_2 \frac{\partial G_1}{\partial \nu_y} \right) dS \overset{r\to 0}{\to} -G(x_2, x_1)$$

For the integral on $\partial B_r(x_2)$, singularity occurs at $\frac{\partial G_2}{\partial \nu_y}(y)$ as $y \to x_2$. By a similar approach, one obtain

$$\int_{\partial B_r(x_2)} \left(G_1 \frac{\partial G_2}{\partial \nu_y} - G_2 \frac{\partial G_1}{\partial \nu_y} \right) dS \overset{r\to 0}{\to} G_1(x_2) = G(x_1, x_2)$$

Hence

$$G(x_1, x_2) = G(x_2, x_1) \qquad \forall\, x_1 \neq x_2 \in \Omega$$

□

3. The Green's Function is necessarily negative over bounded domain Ω.

Property 1.3.2 ([HL11] Proposition 1.21; [GT01] Problem 2.3(b)). *For any $x \neq y \in \Omega$*

$$\begin{aligned} &0 > G(x, y) > \Gamma(x - y) && n \geq 3 \\ &0 > G(x, y) > \Gamma(x - y) - \frac{1}{2\pi} \log(\text{diam}(\Omega)) && n = 2 \end{aligned}$$

Proof. Fix $x \in \Omega$. Observe that

(a) Since

$$\lim_{y \to x} G(x, y) = \lim_{y \to x} (\Gamma(x - y) + \Phi(x, y)) = -\infty$$

There exists $r > 0$ s.t.

$$G(x, y) < 0 \qquad \forall\, y \in \overline{B_r(x)}, \qquad y \neq x$$

(b) For the above $r = r(x) > 0$, $\Delta_y G(x, y) = 0$ on $\Omega \setminus B_r(x)$ and one has boundary data

$$G(x, y) = 0 \quad \forall\, y \in \partial\Omega \quad \text{and} \quad G(x, y) < 0 \quad \forall\, y \in \partial B_r(x)$$

Hence by Strong Maximum Principle Theorem 1.1.3 one has

$$G(x, y) < 0 \qquad \forall\, y \in \Omega \setminus B_r(x)$$

Now for any $x \neq y \in \Omega$, one can always find such $r > 0$ s.t. $y \in \Omega \setminus B_r(x)$ and therefore $G(x, y) < 0$. On the other hand,

(a) For $n \geq 3$

$$\Gamma(x - y) = \frac{1}{n\omega_n(2 - n)} |x - y|^{2-n} < 0 \qquad \forall\, x \neq y \in \mathbb{R}^n$$

and in particular, using (1.29)

$$\Gamma(x - y) < 0 \quad \forall\, y \in \partial\Omega \implies \Phi(x, y) = -\Gamma(x - y) > 0 \quad \forall\, y \in \partial\Omega$$

Since $\Delta_y \Phi(x, y) = 0$ in $y \in \Omega$, Strong Maximum Principle Theorem 1.1.3 yields

$$G(x, y) = \Gamma(x, y) + \Phi(x, y) > \Gamma(x, y) \qquad \forall\, y \neq x \in \Omega$$

(b) For $n = 2$

$$\Gamma(x-y) = \frac{1}{2\pi}\log|x-y| < \frac{1}{2\pi}\log(\text{diam}(\Omega)) \qquad \forall\, x \neq y \in \overline{\Omega}$$

and in particular,

$$\Gamma(x-y) < \frac{1}{2\pi}\log(\text{diam}(\Omega)) \qquad \forall\, y \in \partial\Omega \implies \Phi(x,y) = -\Gamma(x-y) > -\frac{1}{2\pi}\log(\text{diam}(\Omega)) \qquad \forall\, y \in \partial\Omega$$

Since $\Delta_y \Phi(x,y) = 0$ in $y \in \Omega$, again Strong Maximum Principle Theorem 1.1.3 yields

$$G(x,y) = \Gamma(x,y) + \Phi(x,y) > \Gamma(x,y) - \frac{1}{2\pi}\log(\text{diam}(\Omega)) \qquad \forall\, y \neq x \in \Omega$$

□

4. One has good convergence of the potential to the boundary $\partial\Omega$.

Property 1.3.3 ([GT01] Problem 2.3(c)). *Let $\Omega \subseteq \mathbb{R}^n$ be bounded open domain. Let $f \in L^1(\Omega) \cap L^\infty(\Omega)$. Then*

$$\int_\Omega G(x,y)f(y)dy \to 0 \qquad x \to \partial\Omega$$

Proof. First we check the integral is well-defined. Indeed by definition (1.30)

$$\int_\Omega G(x,y)f(y)dy = \int_\Omega \Gamma(x-y)f(y)dy + \int_\Omega \Phi(x,y)f(y)dy$$

$$|\int_\Omega G(x,y)f(y)dy| \leq |\int_{B_\varepsilon(x)} \Gamma(x-y)f(y)dy| + \|\Gamma(x-\cdot)\|_{C^0(\Omega\setminus B_\varepsilon(x))}\|f\|_{L^1(\Omega)} + \|\Phi(x,\cdot)\|_{C^0(\overline{\Omega})}\|f\|_{L^1(\Omega)}$$

$$|\int_{B_\varepsilon(x)} \Gamma(x-y)f(y)dy| \leq C(n)\|f\|_{L^\infty(\Omega)} \begin{cases} \int_0^\varepsilon r^{2-n}r^{n-1}dr & n \geq 3 \\ \int_0^\varepsilon |\log(r)|rdr & n = 2 \end{cases} < \infty$$

Thus the convolution is well-defined for any $x \in \Omega$. On the other hand, let's study the behavior as x tends to some point on $\partial\Omega$. For any $x_0 \in \partial\Omega$, and for any $\varepsilon > 0$, pick $x \in \Omega$ s.t. $|x_0 - x| < \varepsilon$. Then consider the pieces

$$\int_\Omega G(x,y)f(y)dy = \int_{B_{2\varepsilon}(x_0)\cap\Omega} G(x,y)f(y)dy + \int_{\Omega\setminus B_{2\varepsilon}(x_0)} G(x,y)f(y)dy$$

For the first piece

$$\begin{aligned}
|\int_{B_{2\varepsilon}(x_0)\cap\Omega} G(x,y)f(y)dy| &\leq \int_{B_{3\varepsilon}(x)\cap\Omega} |G(x,y)f(y)|dy \\
&\leq \int_{B_{3\varepsilon}(x)\cap\Omega} |\Gamma(x-y)f(y)|dy + C\|f\|_{L^1(B_{3\varepsilon}(x)\cap\Omega)} \qquad \text{using Property 1.3.2} \\
&\leq C(n)\|f\|_{L^\infty(\Omega)} \begin{cases} \int_0^{3\varepsilon} r^{2-n}r^{n-1}dr & n \geq 3 \\ \int^{3\varepsilon} |\log(r)|rdr & n = 2 \end{cases} + C(n)\|f\|_{L^\infty(\Omega)}\varepsilon^n \\
&\leq C(n)\|f\|_{L^\infty(\Omega)}\left(\varepsilon^2|\log(\varepsilon)| + \varepsilon^n\right) \qquad \text{uniformly in } x \in \Omega \text{ s.t. } |x - x_0| < \varepsilon
\end{aligned}$$

For the second piece

$$\begin{aligned}
|\int_{\Omega\setminus B_{2\varepsilon}(x_0)} G(x,y)f(y)dy| &\leq \int_{\Omega\setminus B_{2\varepsilon}(x_0)} |\Gamma(x-y)f(y)|dy + C(n)\|f\|_{L^1(\Omega)} \qquad \text{Property 1.3.2} \\
&\leq C(n,\varepsilon)\|f\|_{L^1(\Omega)} \qquad \text{uniformly in } x \in \Omega \text{ s.t. } |x - x_0| < \varepsilon
\end{aligned}$$

Thus Dominated Convergence Theorme applies

$$\begin{aligned}
\lim_{x\to x_0} \int_{\Omega\setminus B_{2\varepsilon}(x_0)} G(x,y)f(y)dy &= \int_{\Omega\setminus B_{2\varepsilon}(x_0)} \lim_{x\to x_0} G(x,y)f(y)dy \\
&= \int_{\Omega\setminus B_{2\varepsilon}(x_0)} G(x_0,y)f(y)dy \qquad \text{for } y \in \Omega \text{ fixed, } G(x_0,y) = \lim_{x\to x_0} G(x,y) \text{ for } x_0 \in \partial\Omega \\
&= \int_{\Omega\setminus B_{2\varepsilon}(x_0)} (\Gamma(x_0 - y) + \Phi(x_0,y))f(y)dy \qquad (1.30) \\
&= \int_{\Omega\setminus B_{2\varepsilon}(x_0)} (\Gamma(x_0 - y) - \Gamma(x_0 - y))f(y)dy = 0 \qquad (1.29)
\end{aligned}$$

Hence putting the two pieces together

$$|\limsup_{x\to x_0}\int_\Omega G(x,y)f(y)dy| \le C(n)\,\|f\|_{L^\infty(\Omega)}\left(\varepsilon^2|\log(\varepsilon)|+\varepsilon^n\right) \qquad \forall\,\varepsilon>0$$

Send $\varepsilon\to 0$ to conclude. □

1.3.2 Poisson's Integral Formula

Recall that, if the corrector function Φ exists over a domain, it suffices to compute

$$\frac{\partial G}{\partial\nu}(x,y)\Big|_{y\in\partial\Omega}$$

to determine the pointwise value of a harmonic function $u(x)$ via

$$u(x)=\int_{\partial\Omega}u(y)\frac{\partial G}{\partial\nu}(x,y)\,dS(y)$$

The outer normal derivative of G is known as the Poisson's Kernel.

We compute the Green's Representation Formula (1.31) precisely over domains of simple geometry. In particular, it suffices to compute the Green's Function. To do so we need to solve the equation for the corrector function (1.29), which involves clever geometric reflection tricks.

Notice the corrector function $\Phi(x,\cdot)$ solves (1.29) in the variable y with boundary data $-\Gamma(x-\cdot)$. Except for the singularity $y=x$, $\Gamma(x-\cdot)$ is smooth in Ω. One has a way of reflecting the singularity $y=x$ outside the domain, hence building an explicit solution using the radial-symmetry of Γ.

Properties of Poisson's Kernel Given domain $\Omega\subseteq\mathbb{R}^n$ bounded with C^1 boundary, and denote ν as the outward unit normal to $\partial\Omega$. The Poisson's kernel writes

$$K:\Omega\times\partial\Omega\to\mathbb{R}$$
$$(x,y)\mapsto\frac{\partial G(x,y)}{\partial\nu_y}$$

One has quite a few properties to say about this K.

1. First of all, plugging $u=1$ on $\partial\Omega$ into the expression yields

$$1=\int_{\partial\Omega}K(x,y)dS(y)$$

2. Using $x\neq y\in\partial\Omega$ separates strictly, one has

$$\Delta_x K(x,y)=0 \qquad \forall\,(x,y)\in\Omega\times\partial\Omega$$

3. $K(x,y)>0$

4. For any $\delta>0$ fixed, for any point $x_0\in\partial\Omega$ fixed

$$\int_{\partial\Omega\cap\{|y-x_0|>\delta\}}K(x,y)dS(y)\to 0 \qquad x\to x_0$$

Or equivalently, for $\delta>0$ fixed and $x_0\in\partial\Omega$ fixed, $K(x,y)\to 0$ as $x\to x_0$ uniformly in

$$\{y\in\partial\Omega\mid|y-x_0|\ge\delta\}$$

Poisson's Integral Formula over half space $\mathbb{R}^n_+$ Consider the unbounded domain ([Eva10] 2.2.4.b)

$$\Omega=\mathbb{R}^n_+$$

1. For any $x=(x_1,\cdots,x_{n-1},x_n)\in\mathbb{R}^n_+$ with $x_n>0$. Define its reflection point w.r.t. $\partial\mathbb{R}^n_+=\{x_n=0\}$ as

$$\tilde{x}:=(x_1,\cdots,x_{n-1},-x_n)\in\mathbb{R}^n_-$$

Now consider

$$\Phi(x,y):=-\Gamma(\tilde{x}-y) \qquad \forall\,y\in\mathbb{R}^n_+ \tag{1.33}$$

The good thing about this is that

(a) $\tilde{x} \notin \mathbb{R}^n_+$, hence $-\Gamma(\tilde{x} - \cdot)$ is smooth in $y \in \mathbb{R}^n_+$.

(b) The boundary values agree

$$\Phi(x, y) := -\Gamma(\tilde{x} - y) = -\Gamma(x - y) \qquad \text{on } \partial\mathbb{R}^n_+ = \{y_n = 0\}$$

because in this case

$$|\tilde{x} - y| = \sqrt{\sum_{i=1}^{n-1}(x_i - y_i)^2 + (-x_n - 0)^2} = \sqrt{\sum_{i=1}^{n-1}(x_i - y_i)^2 + (x_n)^2} = |x - y|$$

and $\Gamma(\cdot)$ is a radially-symmetric function.

(c) $\Phi(x, \cdot)$ indeed solves (1.29) in $y \in \mathbb{R}^n_+$.

$$\Delta_y \Phi(x, y) = -\Delta_y \Gamma(\tilde{x} - y) = 0 \qquad \forall\ y \in \mathbb{R}^n_+ \subseteq \{y \neq \tilde{x}\}$$

Hence we've checked (1.33) is indeed a corrector function.

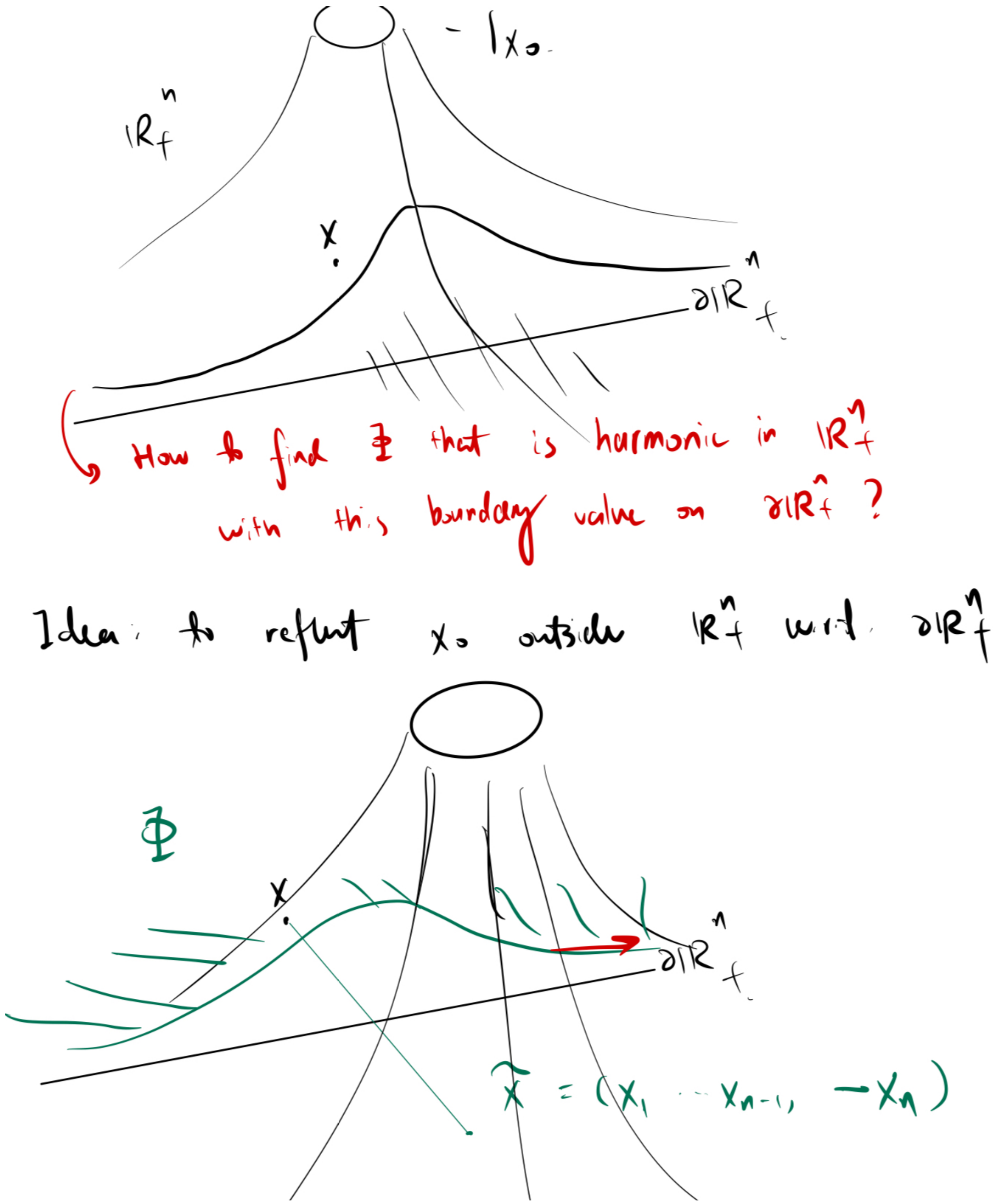


Figure 1.4: Reflection Point for Half Space

2. Now define the Green's Function as in (1.30)

$$G_{\mathbb{R}^n_+}(x,y) := -\Gamma(\tilde{x}-y) + \Gamma(x-y) \qquad \forall\, x \in \mathbb{R}^n_+,\ y \in \overline{\mathbb{R}^n_+},\ x \neq y \tag{1.34}$$

In view of Representation formula (1.32), one needs to calculate

$$\frac{\partial G}{\partial \nu_y}(x,y)\bigg|_{y\in\partial\mathbb{R}^n_+}$$

Notice our previously-derived Green's Representation formula works for bounded domains so the result does not apply directly. We need to prove this is indeed a solution later! First notice

$$\nu_{\partial\mathbb{R}^n_+}(y) = (0, \cdots, 0, -1) \in \mathbb{R}^n$$
$$\frac{\partial G}{\partial \nu_y}(x,y)\Big|_{y\in\partial\mathbb{R}^n_+} = \nabla_y G(x,y)\cdot \nu_{\partial\mathbb{R}^n_+}(y) = -\,G_{y_n}(x,y)|_{y\in\partial\mathbb{R}^n_+}$$

and

$$\begin{aligned} G_{y_n}(x,y) &= -\Gamma'(|\tilde{x}-y|)\frac{\partial|\tilde{x}-y|}{\partial y_n} + \Gamma'(|x-y|)\frac{\partial|x-y|}{\partial y_n} \\ &= -\frac{1}{n\omega_n}|x-y|^{1-n}\left(\frac{\tilde{x}-y}{|x-y|}\cdot(0,\cdots,0,-1) - \frac{x-y}{|x-y|}\cdot(0,\cdots,0,-1)\right) \qquad \text{using } |\tilde{x}-y| = |x-y| \text{ over } \partial\mathbb{R}^n_+ \\ &= -\frac{1}{n\omega_n}|x-y|^{1-n}\frac{1}{|x-y|}(y_n - \tilde{x}_n - y_n + x_n) \\ &= -\frac{2x_n}{n\omega_n}|x-y|^{-n} \qquad \text{using reflection point } \tilde{x}_n = -x_n \end{aligned}$$

Hence we define our Poisson's Kernel as

$$K_{\mathbb{R}^n_+}(x,y) := \frac{\partial G}{\partial \nu_y}(x,y)\Big|_{y\in\partial\mathbb{R}^n_+} = \frac{2x_n}{n\omega_n}\frac{1}{|x-y|^n} \qquad \forall\ x\in\mathbb{R}^n_+,\ y\in\partial\mathbb{R}^n_+ \tag{1.35}$$

3. Now suppose u solves the Dirichlet Boundary Value Problem

$$\begin{cases} \Delta u = 0 & \mathbb{R}^n_+ \\ u = g & \partial\mathbb{R}^n_+ \end{cases} \tag{1.36}$$

for an appropriate choice of g. One expect the solution defined by (1.32)

$$u(x) := \frac{2x_n}{n\omega_n}\int_{\partial\mathbb{R}^n_+}\frac{g(y)}{|x-y|^n}dy \qquad \forall\ x\in\mathbb{R}^n_+ \tag{1.37}$$

to be our solution that solves (1.36) classically.

Theorem 1.3.3 ([Eva10] Theorem 2.14)**.** *Let boundary data $g\in C(\mathbb{R}^{n-1})\cap L^\infty(\mathbb{R}^{n-1})$. Then the function u defined as (1.37) satisfies*

(a) $u\in C^\infty(\mathbb{R}^n_+)\cap L^\infty(\mathbb{R}^n_+)$ (interior regularity)

(b) $\Delta u = 0$ in $\mathbb{R}^n_+$ (interior equation)

(c) For any $x_0\in\partial\mathbb{R}^n_+$,

$$\lim_{x\to x_0} u(x) = g(x_0)$$

Hence $u\in C^0(\overline{\mathbb{R}^n_+})$ (continuously approach boundary data).

Proof. (a) We begin by verifying (Math Stacks Exchange)

$$\int_{\partial\mathbb{R}^n_+} K(x,y)dy = 1 \qquad \forall\ x\in\mathbb{R}^n_+ \tag{1.38}$$

For each $x\in\mathbb{R}^n_+$, denote $x=(x',x_n)$ where $x'\in\mathbb{R}^{n-1}$

$$\begin{aligned} \int_{\partial\mathbb{R}^n_+} K(x,y)dy &= \frac{2x_n}{n\omega_n}\int_{\partial\mathbb{R}^n_+}\frac{1}{|x-y|^n}dy = \frac{2x_n}{n\omega_n}\int_{\partial\mathbb{R}^n_+}\frac{1}{|x-x'+x'-y|^n}dy \\ &= \frac{2x_n}{n\omega_n}\int_{\mathbb{R}^{n-1}}\frac{1}{((x_n)^2+\sum_{i=1}^{n-1}(x_i-y_i)^2)^{\frac{n}{2}}}dy \\ &= \frac{2x_n}{n\omega_n}\int_{\mathbb{S}^{n-2}}\int_0^\infty\frac{1}{(x_n^2+r^2)^{\frac{n}{2}}}r^{n-2}dr dS(\omega) \qquad \text{change } r=|x'-y| \\ &= 2\frac{(n-1)\omega_{n-1}}{n\omega_n}x_n\int_0^\infty\frac{1}{(x_n^2+r^2)^{\frac{n}{2}}}r^{n-2}dr \\ &= 2\frac{(n-1)\omega_{n-1}}{n\omega_n}x_n^2\int_0^\infty\frac{(x_n u)^{n-2}}{(x_n^2+x_n^2u^2)^{\frac{n}{2}}}du \qquad \text{change } r = x_n u \\ &= 2\frac{(n-1)\omega_{n-1}}{n\omega_n}\int_0^\infty\frac{u^{n-2}}{(1+u^2)^{\frac{n}{2}}}du \overset{(1.2)}{=} \frac{2}{\sqrt{\pi}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\int_0^\infty\frac{u^{n-2}}{(1+u^2)^{\frac{n}{2}}}du \end{aligned}$$

We want to verify that

$$\int_0^\infty \frac{u^{n-2}}{(1+u^2)^{\frac{n}{2}}}du = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} \qquad \forall\, n \geq 2$$

i. For $n = 2$ indeed

$$\int_0^\infty \frac{1}{1+u^2}du = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan(\theta)^2}\sec^2(\theta)d\theta = \frac{\pi}{2} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{1}{2})}{\Gamma(1)} \qquad \text{using } \Gamma(\frac{1}{2}) = \sqrt{\pi}$$

ii. Assume for $n-1$ with $n \geq 3$. Then

$$\begin{aligned}\int_0^\infty \frac{u^{n-2}}{(1+u^2)^{\frac{n}{2}}}du &= \frac{1}{2-n}\int_0^\infty u^{n-3}d((1+u^2)^{\frac{2-n}{2}}) = \frac{n-3}{n-2}\int_0^\infty \frac{u^{n-4}}{(1+u^2)^{\frac{n-2}{2}}}du \qquad \text{IBP}\\ &= \frac{n-3}{n-2}\frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{n-3}{2})}{\Gamma(\frac{n-2}{2})} \qquad \text{Induction hypothesis}\\ &= \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} \qquad \text{using } \Gamma \text{ function } \Gamma(z+1) = \Gamma(z)z\end{aligned}$$

Making use of (1.38) one immediately obtain $u \in L^\infty(\mathbb{R}^n_+)$ via

$$|u(x)| \leq |\int_{\partial\mathbb{R}^n_+} K(x,y)g(y)dy| \leq \|g\|_{L^\infty(\partial\mathbb{R}^n_+)} \qquad \forall\, x \in \mathbb{R}^n_+$$

On the other hand, the function

$$K(\cdot,y) : x \in \mathbb{R}^n_+ \mapsto \frac{\partial G}{\partial \nu_y}(x,y)\Big|_{y\in\partial\mathbb{R}^n_+}$$

is harmonic in x for any $y \in \partial\mathbb{R}^n_+$ (notice this uses symmetry of G). In particular, the more derivatives one hit on $K(\cdot,y)$, the better integrability it inherits. In particular, using Dominated Convergence one can interchange derivative and integration so that

$$\begin{aligned}\partial_{x_i}u(x) &= \int_{\partial\mathbb{R}^n_+} \partial_{x_i}K(x,y)g(y)dy\\ \Delta_x u(x) &= \int_{\partial\mathbb{R}^n_+} \Delta_x K(x,y)g(y)dy = 0 \qquad \forall\, x \in \mathbb{R}^n_+\end{aligned}$$

This verifies $u \in C^\infty(\mathbb{R}^n_+)$ and that

$$\Delta u = 0 \qquad \mathbb{R}^n_+$$

(b) The crucial thing is whether u achieves boundary point continuously. For any $x_0 \in \partial\mathbb{R}^n_+$, using $g \in C(\mathbb{R}^{n-1})$, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, x_0) > 0$ s.t.

$$|y - x_0| \leq \delta \implies |g(y) - g(x_0)| < \frac{\varepsilon}{2}$$

One would hence like to understand the difference on two portions

$$\begin{aligned}|u(x) - g(x_0)| &\overset{(1.38)}{\leq} \int_{\partial\mathbb{R}^n_+} K(x,y)|g(y) - g(x_0)|dy\\ &= \underbrace{\int_{|y-x_0|\leq\delta} K(x,y)|g(y)-g(x_0)|dy}_{\text{use continuity of } g} + \underbrace{\int_{|y-x_0|>\delta} K(x,y)|g(y)-g(x_0)|dy}_{\text{use uniform convergence outside any neighborhood of size } \delta}\end{aligned}$$

The first portion is small due to continuity and unit integral

$$\int_{|y-x_0|\leq\delta} K(x,y)|g(y)-g(x_0)|dy \leq |g(y) - g(x_0)| < \frac{\varepsilon}{2}$$

The second portion is small due to the term $x_n \to 0$ as $x \to x_0$ built in Poisson Kernel (1.35). Notice we're sending $x \to x_0$, hence one can take

$$|x - x_0| < \frac{\delta}{2}$$

for our δ fixed above. Thus on the second portion

$$|y - x_0| < |x - y| + |x - x_0| \leq |x - y| + \frac{\delta}{2} \leq |x - y| + \frac{1}{2}|y - x_0| \implies \frac{1}{2}|y - x_0| < |x - y| \tag{1.39}$$

one can estimate

$$\begin{aligned}
\int_{|y-x_0|>\delta} K(x,y)|g(y) - g(x_0)|dy &\leq C(n)\|g\|_{L^\infty(\mathbb{R}^{n-1})} \int_{|y-x_0|>\delta} \frac{|x_n|}{|x-y|^n} dy \\
&\overset{(1.39)}{\leq} C(n)\|g\|_{L^\infty(\mathbb{R}^{n-1})} \int_{|y-x_0|>\delta} \frac{|x_n|}{|y-x_0|^n} dy \\
&\leq C(n)\|g\|_{L^\infty(\mathbb{R}^{n-1})} \delta^{-1}|x_n|
\end{aligned}$$

Now we take $\delta_2 = \delta_2(\varepsilon, n, x_0, \delta, g) < \frac{\delta}{2}$ s.t.

$$C(n)\|g\|_{L^\infty(\mathbb{R}^{n-1})} \delta^{-1}\delta_2 < \frac{\varepsilon}{2}$$

so that for any $|x - x_0| < \delta_2$ one has

$$\int_{|y-x_0|>\delta} K(x,y)|g(y) - g(x_0)|dy \leq \frac{\varepsilon}{2}$$

Hence for any $x_0 \in \partial\mathbb{R}^n_+$ and $\varepsilon > 0$, there exists $\delta_2 > 0$ s.t.

$$|u(x) - g(x_0)| \leq \varepsilon$$

□

Notice Uniqueness in general fails in view of (1.13).

Schwarz Reflection Principle Since we're dealing with half space model, let's discuss a useful technique to continuously reflect a harmonic function defined on one side to the other, known as the Schwarz Reflection.

Lemma 1.3.3 ([GT01] Problem 2.4)**.** *Let $\Omega^+ \subseteq \mathbb{R}^n_+$ and assume part of its boundary intersects the plane $\{x_n = 0\}$ with non-empty interior w.r.t. subspace topology ($\circ$ denotes interior w.r.t. topology in $\mathbb{R}^n$ intersect $\partial\mathbb{R}^n_+$)*

$$T := (\partial\Omega^+ \cap \partial\mathbb{R}^n_+)^\circ \neq \varnothing$$

Now given a harmonic function $u \in C^2(\Omega^+) \cap C^0(\Omega^+ \cup T)$ that continuously attains boundary T with value 0

$$\begin{cases} \Delta u = 0 & \Omega^+ \\ u = 0 & T \end{cases}$$

One may define its Schwarz Reflection

$$U(x_1, \cdots, x_n) := \begin{cases} u(x) = u(x_1, \cdots, x_n) & x_n \geq 0 \\ -u(\tilde{x}) = -u(x_1, \cdots, x_{n-1}, -x_n) & x_n < 0 \end{cases} \quad \forall\, x \in \Omega^+ \cup T \cup \Omega^- \tag{1.40}$$

where Ω^- is the reflection of Ω^+ w.r.t. $\{x_n = 0\}$

$$\Omega^- := \{x \in \mathbb{R}^n \mid \tilde{x} \in \Omega^+\}$$

Show that U is harmonic in $\Omega^+ \cup T \cup \Omega^-$.

Proof. For any $x \in \Omega^-$, compute

$$\Delta U(x) = -\sum_{i=1}^{n-1} \partial_{ii} u(\tilde{x}) - (-1)^2 \partial_{nn} u(\tilde{x}) = -\Delta u(\tilde{x}) = 0$$

It suffices to consider the points $x \in T$. But for any $r > 0$ s.t. $B_r(x) \subseteq \Omega^+ \cup T \cup \Omega^-$

$$\begin{aligned}
⨍_{B_r(x)} U(y)dy &= \frac{1}{\omega_n r^n} \int_{B_r(x)\cap\{y_n \geq 0\}} U(y)dy + \frac{1}{\omega_n r^n} \int_{B_r(x)\cap\{y_n<0\}} U(y)dy \\
&= \frac{1}{\omega_n r^n}\left(\int_{B_r(x)\cap\{y_n>0\}} u(y)dy - \frac{1}{\omega_n r^n} \int_{B_r(x)\cap\{y_n<0\}} u(\tilde{y})dy \right) \\
&= 0 = U(x) \qquad \forall\, x \in T
\end{aligned}$$

Thus U satisfies mean value property on T, and via Theorem 1.1.2 one obtain $\Delta U(x) = 0$ for any $x \in T$. Thus U is harmonic in the domain $\Omega^+ \cup T \cup \Omega^-$. □

Poisson's Integral Formula over ball $B_R(0)$ Consider the domain ([Eva10] 2.2.4.c., [GT01] Section 2.5)

$$\Omega = B_R(0)$$

1. For any $x \in B_R(0)$ with $|x| = R$, again we consider how to reflect $x \mapsto \tilde{x} \in \mathbb{R}^n \setminus B_R(0)$. The natural choice is

$$\tilde{x} := \begin{cases} \frac{R^2}{|x|^2}x & x \neq 0 \\ \infty & x = 0 \end{cases} \tag{1.41}$$

Why is so? Notice for $0 \neq x \in B_R$ fixed, and any $y \in \partial B_R$, consider the triangle formed by the three points

$$\Delta_{0xy}, \qquad \overline{0x} = |x|, \qquad \overline{0y} = R, \qquad \overline{xy} = |x-y|$$

We want to reflect in a way s.t. the triangles Δ_{0xy} and $\Delta_{0y\tilde{x}}$ are similar with the angle

$$\angle x0y = \angle y0\tilde{x} \qquad \text{preserved}$$

where

$$\Delta_{0y\tilde{x}}, \qquad \overline{0y} = R, \qquad \overline{0\tilde{x}} = |\tilde{x}|, \qquad \overline{y\tilde{x}} = |y - \tilde{x}|$$

To ensure similarity, one needs to ensure

$$\frac{\overline{0y}}{\overline{0x}} = \frac{\overline{0\tilde{x}}}{\overline{0y}} \implies \overline{0\tilde{x}} = |\tilde{x}| = \frac{R^2}{|x|}$$

To ensure $\tilde{x} \in \vec{0x}$, one take

$$\tilde{x} := \frac{R^2}{|x|^2}x$$

as defined. Such reflection is known as the *Kelvin Transformation* w.r.t. the sphere $\partial B_R(0)$, which maps 0 to ∞, interior to exterior, and vice versa. Notice under such definition, necessarily

$$\frac{|y-\tilde{x}|}{|x-y|} = \frac{R}{|x|} = \frac{|\tilde{x}|}{R} \qquad \forall\ x \in B_R,\ y \in \partial B_R \tag{1.42}$$

Now we define the corrector function

$$\Phi(x,y) := \begin{cases} -\Gamma(\frac{|x|}{R}(\tilde{x} - y)) & x \neq 0 \\ -\Gamma(R) & x = 0 \end{cases} \qquad \forall\ y \in B_R(0) \tag{1.43}$$

Intuitively, for $x = 0$, the pole is directly centered at the origin, and because the fundamental solution is radial, the corrector Φ is just constant across ∂B_R. For $x \neq 0$, for simplicity take $R = 1$

$$-|x|^{2-n}\Gamma(|y-\tilde{x}|) = -C_n|x|^{2-n}|y-\tilde{x}|^{2-n} \overset{(1.42)}{=} -C_n|y-x|^{2-n} \qquad \forall\ y \in \partial B_1$$

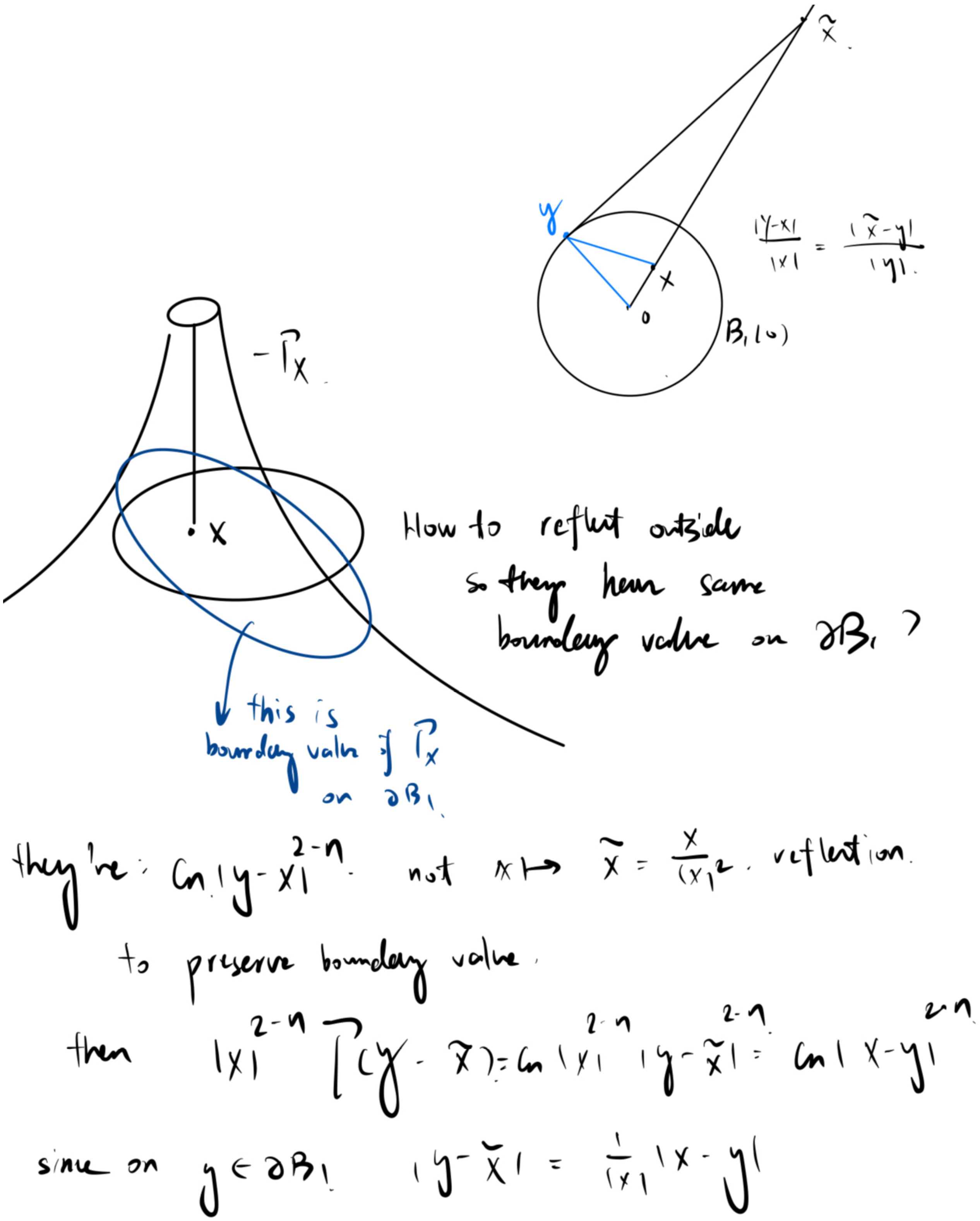


Figure 1.5: Reflection Point for Ball

We verify

(a) $\tilde{x} \in \mathbb{R}^n \setminus B_R(0)$ for any $0 \neq x \in B_R(0)$, hence $-\Gamma(\frac{|x|}{R}(\tilde{x} - \cdot))$ is well-defined, thus smooth in $y \in B_R(0)$. For $x = 0$, $\Phi(0, \cdot)$ is constant in y, hence smooth.

(b) The boundary values agree. For any $x \neq 0$

$$\Phi(x,y) := -\Gamma(\frac{|x|}{R}(\tilde{x} - y)) = -\Gamma(\frac{|x|}{R}|\tilde{x} - y|) \overset{(1.42)}{=} -\Gamma(\frac{|x|}{R}\frac{R}{|x|}|x - y|) = -\Gamma(x - y) \qquad \forall\ y \in \partial B_R(0)$$

while for $x = 0$

$$\Phi(0, y) = -\Gamma(R) = -\Gamma(-y) \qquad \forall\ y \in \partial B_R(0)$$

(c) For any $x \neq 0$, $\Phi(x, \cdot)$ solves (1.29) in $y \in B_R(0)$

$$\Delta_y \Phi(x,y) = -\frac{|x|^2}{R^2}\Delta_y \Gamma(\frac{|x|}{R}(\tilde{x}-y)) = 0 \qquad \forall\, y \in B_R(0)$$

For $x = 0$,

$$\Delta_y \Phi(0,y) = 0 \qquad \forall\, y \in B_R(0)$$

Hence (1.43) is a corrector function for any $x \in B_R(0)$.

2. Now we define the Green's Function as in (1.30) ([GT01] (2.23))

$$G_{B_R(0)}(x,y) := \begin{cases} -\Gamma(\frac{|x|}{R}(\tilde{x}-y)) + \Gamma(x-y) & x \in B_R(0)\setminus\{0\} \\ -\Gamma(R) + \Gamma(y) & x = 0 \end{cases} \qquad \forall\, y \in \overline{B_R(0)}, \qquad y \neq x$$
$$= -\Gamma(|\frac{x\cdot y}{R} - R|) + \Gamma(|x-y|) \qquad \forall\, x \in B_R, \qquad y \in \overline{B_R}, \qquad x \neq y \tag{1.44}$$

Indeed, the equality holds. For any $x \neq 0$

$$-\Gamma(\frac{|x|}{R}(\tilde{x}-y)) = -\Gamma(\frac{R}{|x|}x - \frac{|x|}{R}y) = -\Gamma(|\frac{R}{|x|}x - \frac{|x|}{R}y|) = -\Gamma(\sqrt{R^2 - 2x\cdot y + \frac{|x|^2|y|^2}{R^2}})$$
$$= -\Gamma(|\frac{x\cdot y}{R} - R|) \qquad \text{which has no singularity at } x=0\text{, and coincides with } x=0 \text{ case}$$

Again, in view of Representation formula (1.32), we need to compute ([HL11] Corollary 1.23)

$$\frac{\partial G}{\partial \nu_y}(x,y)\Big|_{y\in\partial B_R}$$

First note outer unit-normal takes the form

$$\nu_{\partial B_R(0)}(y) = \frac{y}{|y|}$$

Then we compute $\nabla_y G(x,y)$ for $y \in \partial B_R$.

$$\nabla_y G(x,y) = -\Gamma'(|\frac{x\cdot y}{R} - R|)\left(\frac{2\frac{|x|^2}{R^2}|y|\frac{y}{|y|} - 2x}{2|\frac{x\cdot y}{R} - R|}\right) + \Gamma'(|x-y|)\frac{y-x}{|x-y|}$$
$$= -\Gamma'(|\frac{x\cdot y}{R} - R|)\frac{\frac{|x|^2}{R^2}y - x}{|\frac{x\cdot y}{R} - R|} + \Gamma'(|x-y|)\frac{y-x}{|x-y|}$$
$$\nabla_y G(x,y)\cdot\nu(y) = -\Gamma'(|\frac{x\cdot y}{R} - R|)\frac{\frac{|x|^2}{R} - \frac{x\cdot y}{R}}{|\frac{x\cdot y}{R} - R|} + \Gamma'(|x-y|)\frac{R - \frac{x\cdot y}{R}}{|x-y|} \qquad \forall\, y \in \partial B_R$$
$$= -\frac{1}{n\omega_n}|\frac{x\cdot y}{R} - R|^{-n}(\frac{|x|^2}{R} - \frac{x\cdot y}{R}) + \frac{1}{n\omega_n}|x-y|^{-n}(R - \frac{x\cdot y}{R})$$

We need to distinguish the cases where $x = 0$ or not. If $x = 0$ then the above writes

$$\nabla_y G(0,y)\cdot\nu(y) = \frac{1}{n\omega_n R^n}R = \frac{1}{n\omega_n R^{n-1}}$$

If $x \neq 0$ then one can use similarity of triangles in reflection

$$\nabla_y G(x,y)\cdot\nu(y) \overset{(1.42)}{=} -\frac{1}{n\omega_n}|x-y|^{-n}(\frac{|x|^2}{R} - \frac{x\cdot y}{R}) + \frac{1}{n\omega_n}|x-y|^{-n}(R - \frac{x\cdot y}{R})$$
$$= \frac{1}{n\omega_n|x-y|^n}\left(R - \frac{x\cdot y}{R} + \frac{x\cdot y}{R} - \frac{|x|^2}{R}\right) = \frac{R^2 - |x|^2}{n\omega_n R|x-y|^n}$$

Hence we define our *Poisson's Kernel* as (which works for both $x = 0$ and $x \neq 0$)

$$K_{B_R(0)}(x,y) := \frac{\partial G}{\partial \nu_y}(x,y)\Big|_{y\in\partial B_R} = \frac{R^2 - |x|^2}{n\omega_n R|x-y|^n} \qquad \forall\, x \in B_R(0), \qquad y \in \partial B_R(0) \tag{1.45}$$

3. Now suppose u solves the Dirichlet Boundary Value Problem

$$\begin{cases} \Delta u = 0 & B_R(0) \\ u = g & \partial B_R(0) \end{cases} \tag{1.46}$$

for an appropriate choice of g. One expect the solution defined by (1.32)

$$u(x) := \frac{R^2 - |x|^2}{n\omega_n R} \int_{\partial B_R(0)} \frac{1}{|x-y|^n} g(y) dS(y) \qquad \forall\, x \in B_R(0) \tag{1.47}$$

to be our solution that solves (1.46) classically.

Theorem 1.3.4 ([Eva10] Theorem 2.15; [HL11] Theorem 1.24; [GT01] Theorem 2.6)**.** *Let boundary data $g \in C(\partial B_R(0))$. Then the function u defined as (1.47) satisfies*

(a) $u \in C^\infty(B_R(0))$ (interior regularity)

(b) $\Delta u = 0$ in $B_R(0)$ (interior equation)

(c) For any $x_0 \in \partial B_R(0)$,

$$\lim_{x \to x_0} u(x) = g(x_0)$$

Hence $u \in C^0(\overline{B_R(0)})$ (continuously approach the boundary data).

Proof. (a) Note $G(x,y)$, and thus $\frac{\partial G}{\partial \nu_y} = K(\cdot, y)$ is harmonic in x if $y \in \partial B_R(0)$. Hitting ∂_{x_i} on $K(x,y)$ remains smooth, thus L^1 over bounded domains. Hence by Dominated Convergence Theorem one interchange

$$\partial_{x_i} u(x) = \int_{\partial B_R} \partial_{x_i} K(x,y) g(y) dS(y)$$
$$\Delta_x u(x) = \int_{\partial B_R} \Delta_x K(x,y) g(y) dS(y) = 0 \qquad \forall\, x \in B_R(0)$$

and hence harmonicity implies smoothness via Theorem 1.1.2.

(b) We verify

$$\int_{\partial B_R} K(x,y) dS(y) = 1$$

Indeed, since now we're working with bounded domain, Green's Representation applies. In particular, we apply (1.32) to the function $u = 1 \in C^2(B_R) \cap C^1(\overline{\Omega})$ so that

$$1 = \int_{\partial B_R} \frac{\partial G}{\partial \nu_y}(x,y) dS(y) = \int_{\partial B_R} K(x,y) dS(y)$$

Now to show u continuously approach boundary data, pick any $x_0 \in \partial B_R(0)$. Using $g \in C(\partial B_R)$, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, x_0) > 0$ s.t.

$$|y - x_0| < \delta \implies |g(y) - g(x_0)| < \frac{\varepsilon}{2}$$

One estimate

$$\begin{aligned} |u(x) - g(x_0)| &= \Big|\int_{\partial B_R} K(x,y)(g(y) - g(x_0)) dS(y)\Big| \\ &\le \underbrace{\int_{|y-x_0|<\delta} K(x,y)|g(y) - g(x_0)| dS(y)}_{\text{use continuity of } g} + \underbrace{\int_{|y-x_0|\ge\delta} K(x,y)|g(y) - g(x_0)| dS(y)}_{\text{use uniform convergence outside any neighborhood of size } \delta} \end{aligned}$$

For the first portion

$$\int_{|y-x_0|<\delta} K(x,y)|g(y) - g(x_0)| dS(y) \le |g(y) - g(x_0)| < \frac{\varepsilon}{2}$$

For the second portion, arguing as before

$$\begin{aligned} \int_{|y-x_0|\ge\delta} K(x,y)|g(y) - g(x_0)| dS(y) &\le C(n,R) \|g\|_{C(\partial B_R)} \int_{|y-x_0|\ge\delta} \frac{R^2 - |x|^2}{|x-y|^n} dS(y) \\ &\overset{(1.39)}{\le} C(n,R) \|g\|_{C(\partial B_R)} \int_{|y-x_0|\ge\delta} \frac{R^2 - |x|^2}{|x_0 - y|^n} dS(y) \\ &\le C(n,R) \|g\|_{C(\partial B_R)} \delta^{-n} \cdot (R^2 - |x|^2) \end{aligned}$$

It suffices to choose $\delta_2 = \delta_2(\varepsilon, x_0, \delta, g, n, R) < \delta$ s.t.

$$|x_0 - x| < \delta_2 \implies C(n,R)\,\|g\|_{C(\partial B_R)}\,\delta^{-n} \cdot (R^2 - |x|^2) < \frac{\varepsilon}{2}$$

Hence for any $\varepsilon > 0$ there exists δ_2 s.t.

$$|u(x) - g(x_0)| < \varepsilon \qquad \forall\ |x - x_0| < \delta_2$$

□

Kelvin Transform Since we're dealing with inversion w.r.t. spheres, we introduce Kelvin Transform. The inversion w.r.t. unit ball as in (1.41) extends to

$$x \in \mathbb{R}^n \setminus \{0\} \to x^* := \frac{x}{|x|^2} \in \mathbb{R}^n \setminus \{0\} \tag{1.48}$$

1. Immediately observe $x \mapsto x^*$ is smooth. Also, for any $x \in \mathbb{R}^n \setminus \{0\}$

$$(x^*)^* = \frac{x^*}{|x^*|^2} = \frac{\frac{x}{|x|^2}}{\frac{|x|^2}{|x|^4}} = x$$

yields $x \mapsto x^*$ is surjective. On the other hand, suppose $x_1,\, x_2 \in \mathbb{R}^n \setminus \{0\}$ satisfy

$$\begin{aligned} \frac{x_1}{|x_1|^2} = x_1^* = x_2^* &= \frac{x_2}{|x_2|^2} \\ x_1|x_2|^2 - x_2|x_1|^2 &= 0 \\ |x_1| &= |x_2| \\ x_1 &= x_2 \end{aligned}$$

Thus $x \mapsto x^*$ is injective. Now $x \mapsto x^*$ is smooth diffeomorphism from $\mathbb{R}^n \setminus \{0\}$ onto itself whose inverse agrees with itself.

2. We define the Kelvin Transform of a function $u : \Omega \subseteq \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ as

$$u^*(x) := |x|^{2-n} u(x^*) = |x|^{2-n} u(\frac{x}{|x|^2}) \qquad \forall\ x \in \Omega^* := \{x \in \Omega \mid x^* \in \Omega\} \tag{1.49}$$

Now assume u solves the Poisson's Equation

$$\Delta u = f \qquad \Omega \subseteq \mathbb{R}^n \setminus \{0\}$$

Then its Kelvin Transform (1.49) solves ([GT01] Problem 4.7)

$$\Delta u^*(x) = |x|^{-n-2} f(\frac{x}{|x|^2}) \qquad \forall\ x \in \Omega^* \tag{1.50}$$

Proof. We compute using brute force

$$\begin{aligned}
\partial_i u^*(x) &= \partial_i(|x|^{2-n}u(\frac{x}{|x|^2})) = (2-n)|x|^{1-n}\frac{x_i}{|x|}u(\frac{x}{|x|^2}) + |x|^{2-n}\partial_j u(\frac{x}{|x|^2})\partial_i\left(\frac{x_j}{|x|^2}\right) \\
&= (2-n)|x|^{-n}x_i u(\frac{x}{|x|^2}) + |x|^{2-n}\partial_j u(\frac{x}{|x|^2})\left(\frac{\delta_{ij}}{|x|^2} - 2\frac{x_i x_j}{|x|^4}\right) \\
\partial_{ii}u^*(x) &= -n(2-n)|x|^{-n-2}x_i^2 u(\frac{x}{|x|^2}) + (2-n)|x|^{-n}u(\frac{x}{|x|^2}) + (2-n)|x|^{-n}x_i\partial_j u(\frac{x}{|x|^2})\left(\frac{\delta_{ij}}{|x|^2} - 2\frac{x_i x_j}{|x|^4}\right) \\
&+ (2-n)|x|^{-n}x_i\partial_j u(\frac{x}{|x|^2})\left(\frac{\delta_{ij}}{|x|^2} - 2\frac{x_i x_j}{|x|^4}\right) + |x|^{2-n}\partial_{jk}u(\frac{x}{|x|^2})\left(\frac{\delta_{ik}}{|x|^2} - 2\frac{x_i x_k}{|x|^4}\right)\left(\frac{\delta_{ij}}{|x|^2} - 2\frac{x_i x_j}{|x|^4}\right) \\
&+ |x|^{2-n}\partial_j u(\frac{x}{|x|^2})\left(-2\frac{\delta_{ij}}{|x|^4}x_i - 2\frac{1}{|x|^4}(x_j + x_i\delta_{ij}) + 8\frac{x_i^2 x_j}{|x|^6}\right) \\
\Delta u^*(x) &= -n(2-n)|x|^{-n}u(\frac{x}{|x|^2}) + n(2-n)|x|^{-n}u(\frac{x}{|x|^2}) + 2(2-n)|x|^{-n}\partial_j u(\frac{x}{|x|^2})\left(\frac{\sum_i x_i\delta_{ij}}{|x|^2} - 2\frac{x_j}{|x|^2}\right) \\
&+ |x|^{2-n}\partial_{jk}u(\frac{x}{|x|^2})\left(\frac{\sum_i \delta_{ik}\delta_{ij}}{|x|^4} - 2\frac{x_j x_k}{|x|^4}\frac{\delta_{ij}}{|x|^2} - 2\frac{x_i x_j}{|x|^4}\frac{\delta_{ik}}{|x|^2} + 4\frac{x_j x_k}{|x|^6}\right) \\
&+ |x|^{2-n}\partial_j u(\frac{x}{|x|^2})\left(-2\frac{\sum_i \delta_{ij}x_i}{|x|^4} - 2\frac{n x_j}{|x|^4} - 2\frac{\sum_i x_i\delta_{ij}}{|x|^4} + 8\frac{x_j}{|x|^4}\right) \\
&= |x|^{2-n}\partial_j u(\frac{x}{|x|^2})\left(2(2-n)\frac{\sum_i x_i\delta_{ij}}{|x|^4} - 4(2-n)\frac{x_j}{|x|^4} - 4\frac{\sum_i \delta_{ij}x_i}{|x|^4} - 2\frac{n x_j}{|x|^4} + 8\frac{x_j}{|x|^4}\right) \\
&+ |x|^{2-n}\partial_{jk}u(\frac{x}{|x|^2})\frac{\sum_i \delta_{ik}\delta_{ij}}{|x|^4} \\
&= |x|^{2-n}\partial_j u(\frac{x}{|x|^2})\left(-2n\frac{\sum_i x_i\delta_{ij}}{|x|^4} + 2n\frac{x_j}{|x|^4}\right) + |x|^{2-n}\partial_{jk}u(\frac{x}{|x|^2})\frac{\sum_i \delta_{ik}\delta_{ij}}{|x|^4} \\
&= |x|^{-n-2}\Delta u(\frac{x}{|x|^2}) = |x|^{-n-2}f(\frac{x}{|x|^2})
\end{aligned}$$

□

This amounts to say that the Laplacian Operator is invariant under Kelvin Transformation.

Poisson's Integral Formula and its Consequences In this paragraph we deduce properties of harmonic function using Poisson's Integral Formula, instead of MVP.

1. In fact, one can first deduce MVP from Poisson's Integral Formula.

   Corollary 1.3.2 ([HL11] Remark 1.25)**.** *Let $B_R \subseteq \mathbb{R}^n$. Let u solve (1.46). Then*

$$u(0) = \frac{1}{n\omega_n R^{n-1}}\int_{\partial B_R} g(y)\,dS(y) = \fint_{\partial B_R(0)} g dS \tag{1.51}$$

   Proof. Let $x = 0$ in Poisson's Integral Formula (1.47). Then

$$u(0) = \frac{R^2}{n\omega_n R}\int_{\partial B_R}\frac{g(y)}{|y|^n}\,dS(y) = \frac{1}{n\omega_n R^{n-1}}\int_{\partial B_R} g(y)\,dS(y)$$

   □

2. One has a more quantitative version of Harnack's Inequality from Poisson's Integral Formula.

   Corollary 1.3.3 ([HL11] Lemma 1.26)**.** *Let u be harmonic and $u \geq 0$ in $B_R(x_0)$. Then for any $x \in B_R(x_0)$*

$$\frac{R-|x-x_0|}{R+|x-x_0|}\left(\frac{R}{R+|x-x_0|}\right)^{n-2}u(x_0) \leq u(x) \leq \frac{R+|x-x_0|}{R-|x-x_0|}\left(\frac{R}{R-|x-x_0|}\right)^{n-2}u(x_0) \tag{1.52}$$

   Notice as $|x-x_0| \to R$ the estimate fails as the denominator $R-|x-x_0| \to 0$. In analog with the classical Harnack's Inequality, one shall take $B_{r/2}(x_0)$ to estimate sup *against* inf*.*

   Proof. WLOG assume $x_0 = 0$ and $u \in C(\overline{B_R})$. By Poisson's Integral Formula (1.47), for any $x \in B_R$

$$u(x) = \frac{R^2-|x|^2}{n\omega_n R}\int_{\partial B_R}\frac{u(y)}{|x-y|^n}\,dS(y)$$

Using

$$R-|x| \leq |x-y| \leq R+|x| \iff \frac{1}{(R+|x|)^n} \leq \frac{1}{|x-y|^n} \leq \frac{1}{(R-|x|)^n} \qquad \forall\, y \in \partial B_R,\ x \in B_R$$

one obtain using Mean Value Property

$$\begin{aligned}
u(x) &\geq \frac{1}{n\omega_n R}\frac{R^2-|x|^2}{(R+|x|)^n}\int_{\partial B_R} u(y)\,dS(y) = \frac{1}{n\omega_n R}\frac{R-|x|}{(R+|x|)^{n-1}}\int_{\partial B_R} u(y)\,dS(y)\\
&= \frac{R-|x|}{(R+|x|)^{n-1}}\frac{R^{n-2}}{n\omega_n R^{n-1}}\int_{\partial B_R} u(y)\,dS(y) \overset{(1.51)}{=} \frac{R-|x|}{R+|x|}\left(\frac{R}{R+|x|}\right)^{n-2} u(0)\\
u(x) &\leq \frac{1}{n\omega_n R}\frac{R^2-|x|^2}{(R-|x|)^n}\int_{\partial B_R} u(y)\,dS(y) = \frac{1}{n\omega_n R}\frac{R+|x|}{(R-|x|)^{n-1}}\int_{\partial B_R} u(y)\,dS(y)\\
&= \frac{R+|x|}{(R-|x|)^{n-1}}\frac{R^{n-2}}{n\omega_n R^{n-1}}\int_{\partial B_R} u(y)\,dS(y) \overset{(1.51)}{=} \frac{R+|x|}{R-|x|}\left(\frac{R}{R-|x|}\right)^{n-2} u(0)
\end{aligned}$$

Hence translating back yields (1.52). □

Instead of pointwise estimate, of course we can formulate the Harnack's Inequality over balls, in a cleaner form.

Corollary 1.3.4 ([FRRO22] Theorem 2.1)**.** *Let u be harmonic and $u \geq 0$ in $B_1(0)$. Then there exists $C = C(n) > 0$ s.t. for any $\rho \in (0,1)$*

$$\sup_{x\in B_\rho} u(x) \leq \frac{C}{(1-\rho)^n}\inf_{x\in B_\rho} u(x) \tag{1.53}$$

Proof. Again we make use of Poisson's Integral Formula (1.47)

$$u(x) = (1-|x|^2)\fint_{\partial B_1}\frac{u(y)}{|x-y|^n}dS(y) \qquad \forall\, x \in B_1$$

For any $y \in \partial B_1$ on the boundary and $x \in B_1$ in the interior, one has

$$1-|x| \leq |x-y| \leq 1+|x| \implies (1+|x|)^{-n} \leq \frac{1}{|x-y|^n} \leq (1-|x|)^{-n} \qquad \forall\, x\in B_1, \qquad y \in \partial B_1$$

Using $u \geq 0$, the trivial bound follows

$$\begin{aligned}
u(x) &= (1-|x|^2)\fint_{\partial B_1}\frac{u(y)}{|x-y|^n}dS(y) \geq \frac{1-|x|^2}{(1+|x|)^n}\fint_{\partial B_1} u\,dS(y)\\
&= \frac{1-|x|}{(1+|x|)^{n-1}}\fint_{\partial B_1} u\,dS(y)\\
u(x) &\leq \frac{1-|x|^2}{(1-|x|)^n}\fint_{\partial B_1} u\,dS(y) = \frac{1+|x|}{(1-|x|)^{n-1}}\fint_{\partial B_1} u\,dS(y)
\end{aligned}$$

Fix $\rho \in (0,1)$. Now for any $x_1,\, x_2 \in B_\rho$ one may compare

$$\begin{aligned}
u(x_1) &\leq \frac{1+|x_1|}{(1-|x_1|)^{n-1}}\fint_{\partial B_1} u\,dS(y) \leq \frac{1+|x_1|}{(1-|x_1|)^{n-1}}\frac{(1+|x_2|)^{n-1}}{1-|x_2|}u(x_2)\\
&\leq \frac{(1+\rho)^n}{(1-\rho)^n}u(x_2) \leq \frac{2^n}{(1-\rho)^n}u(x_2) \qquad \forall\, x_1,\, x_2 \in B_\rho
\end{aligned}$$

Taking sup on LHS and inf on RHS, (1.53) follows. □

3. One has Liouville Theorem immediately from Harnack derived above.

   Corollary 1.3.5 ([HL11] Corollary 1.27)**.** *Let u be harmonic function in $\mathbb{R}^n$ bounded from either above or below. Then u is constant.*

   Proof. (a) If u is bounded from below, it suffices to consider $u \geq 0$. This is done by taking $u + C$ for C sufficiently large s.t. $u + C \geq 0$ in $\mathbb{R}^n$. Now using Harnack (1.52), for any $R > 0$

   $$\frac{R-|x|}{R+|x|}\left(\frac{R}{R+|x|}\right)^{n-2}u(0) \leq u(x) \leq \frac{R+|x|}{R-|x|}\left(\frac{R}{R-|x|}\right)^{n-2}u(0)$$

   Let $R \to \infty$ on both sides to arrive at $u(0) = u(x)$ for any $x \in \mathbb{R}^n$.

(b) If on the other hand u is bounded from above, it suffices to consider $u \geq 0$ as well. This is done by taking $-u + C$ for C sufficiently large s.t. $-u + C \geq 0$ in $\mathbb{R}^n$. Flipping the signs in (1.52) yields the same result $u(0) = u(x)$ as $R \to \infty$.

□

1.4 Maximum Principle

In this section we derive estimates of harmonic function using Maximum Principle. This technique translates to more general elliptic operators. We first prove Theorem 1.4.1, the Weak Maximum Principle for subharmonic function (Comparison Principle), then in the sequel we apply this to different test function carefully chosen. In particular, we deduce

1. Interior gradient estimates using Bochner's Technique (1.56)
2. Hopf's Lemma (Theorem 1.4.4)
3. Global Hölder estimates inherited from Hölder Boundary data (1.65).

Weak Maximum Principle via Touching We first prove the Weak Maximum Principle for subharmonic functions, avoid using MVP! In the following we demonstrate two strategies

1. A proof by introducing small perturbation to the solution
2. A proof by constructing a barrier and touching the solution.

Theorem 1.4.1 ([HL11] Theorem 1.29)**.** *Let $u \in C^2(B_1) \cap C^0(\overline{B}_1)$ be subharmonic $\Delta u \geq 0$ in B_1. Then*

$$\sup_{x \in B_1} u(x) \leq \sup_{x \in \partial B_1} u(x) \tag{1.54}$$

Proof following [HL11] *Theorem 1.29.* The only problem that stops us is the semi-positive definite of $D^2 u(x)$ as implied by $\Delta u \geq 0$ instead of strict inequality. We give us some room by lifting u. For any $\varepsilon > 0$. Define

$$u_\varepsilon(x) := u(x) + \varepsilon |x|^2 \tag{1.55}$$

Then one immediately observe

$$\Delta u_\varepsilon(x) = \Delta u(x) + \varepsilon (\sum_i \partial_i (2|x| \frac{x_i}{|x|})) = \Delta u(x) + 2n\varepsilon > 0$$

$$\sup_{x \in B_1} u(x) \leq \sup_{x \in B_1} u_\varepsilon(x) \qquad \forall\, \varepsilon > 0$$

1. Suppose there's interior supremum of u_ε, i.e., $x_0 \in B_1$ s.t. $u_\varepsilon(x_0) = \sup_{x \in B_1} u_\varepsilon(x)$. Then necessarily

$$\nabla u_\varepsilon(x_0) = 0, \qquad D^2 u_\varepsilon(x_0) \leq 0$$

Hence taking trace of Hessian one has

$$\Delta u_\varepsilon(x_0) \leq 0$$

which contradicts $\Delta u_\varepsilon(x) > 0$ for any $x \in B_1$. Thus supremum of u_ε must be achieved on the boundary

$$\sup_{x \in B_1} u_\varepsilon(x) = \sup_{x \in \partial B_1} u_\varepsilon(x)$$

2. Hence

$$\sup_{x \in B_1} u(x) \leq \sup_{x \in B_1} u_\varepsilon(x) = \sup_{x \in \partial B_1} u_\varepsilon(x) \leq \sup_{x \in \partial B_1} u(x) + \varepsilon \qquad \forall\, \varepsilon > 0$$

Take $\varepsilon \to 0$ to conclude.

□

Proof following De Silva Analysis II 2025. WLOG up to subtracting $\sup_{x \in \partial B_1} u$ from u, it suffices to assume

$$\sup_{x \in \partial B_1} u = 0$$

We wish to prove

$$u(x) \leq 0 = \sup_{x \in \partial B_1} u \qquad \forall\, x \in B_1$$

1. We give us some room by lowering a concave parabola on top of u. We define a family of concave parabola as

$$\phi_t(x) := t(1 - |x|^2)$$

so that

$$\begin{aligned}
\partial_{x_i}\phi_t(x) &= -2tx_i \\
\Delta\phi_t(x) &= -2nt < 0 \qquad \forall\ t > 0 \\
\phi_t(x) &= 0 \qquad \forall\ x \in \partial B_1
\end{aligned}$$

Since u is continuous up to the boundary, and $u = \phi$ at ∂B_1, one can start with lowering t from t large enough so that

$$\phi_t(x) \geq u(x) \qquad \forall\ x \in \overline{B_1}$$

2. Assume that there exists $x_0 \in B_1$ in the interior s.t.

$$\sup_{\overline{B_1}} u = u(x_0) > 0$$

Strict positivity of $u(x_0)$ ensures there must exist some t s.t. ϕ_t can touch u at x_0. So we consider the largest $\bar{t}$ s.t.

$$\phi_{\bar{t}}(x_0) = u(x_0), \qquad \phi_{\bar{t}} \geq u \qquad \text{in a neighborhood of } x_0$$

Thus x_0 is a local minimum for their difference $\phi_{\bar{t}} - u$, which is always non-negative. **Since x_0 is interior**, necessarily

$$\Delta(\phi_{\bar{t}} - u)(x_0) \geq 0$$

But notice their difference satisfies

$$\Delta(\phi_t - u)(x) < 0 \qquad \text{using choice of } \phi \text{ and } u \text{ is subharmonic in } B_1$$

Hence we reach a contradiction at $\bar{t}$ and x_0.

□

Bochner's Technique (Interior Gradient Estimate & Harnack's Inequality) In $\mathbb{R}^n$, Bochner's Formula involves hitting Δ on $|\nabla u|^2$. We demonstrate how to use such technique in proving estimates.

1. If one assume for u harmonic, then Bochner tells us

$$\begin{aligned}
\Delta(|\nabla u|^2) &= \sum_{i=1}^n \partial_{x_ix_i}(\sum_{j=1}^n(\partial_{x_j}u)^2) = \sum_{i=1}^n \partial_{x_i}(\sum_{j=1}^n 2(\partial_{x_j}u)\partial_{x_j}\partial_{x_i}u) \\
&= 2\sum_{i=1}^n\sum_{j=1}^n \left((\partial_i\partial_j u)^2 + \partial_j u \partial_{jii}u\right) \\
&= 2|D^2u|^2 \qquad \text{using } \Delta u = 0
\end{aligned}$$

That the RHS has a sign usually helps us by simply throwing the term away. Notice in particular $|\nabla u|^2$ is subharmonic. Also notice Ric does not occur due to flatness of $\mathbb{R}^n$.

We use Bernstein-Bochner's Technique to deduce interior gradient estimate.

Theorem 1.4.2 ([HL11] Proposition 1.31)**.** *Let $u \in C^2(B_1) \cap C^0(\overline{B_1})$ be harmonic in $B_1(0)$. Then there exists $C = C(n)$ s.t.*

$$\sup_{B_{1/2}} |\nabla u| \leq C(n) \sup_{\partial B_1} u \tag{1.56}$$

Proof. Let $\eta \in C_0^\infty(B_1)$ s.t. $\eta = 1$ on $B_{1/2}$. Since we want to capture interior behavior of $|\nabla u|$, we calculate (with Einstein summation)

$$\begin{aligned}
\Delta(\eta^2|\nabla u|^2) &= \nabla\cdot\left(2\eta|\nabla u|^2\nabla\eta + 2\eta^2 D^2u\nabla u\right) = \partial_i\left(2\eta|\nabla u|^2\partial_i\eta + 2\eta^2\partial_{ij}u\partial_j u\right) \\
&= 2(\partial_i\eta)^2|\nabla u|^2 + 4\eta\partial_i\eta\partial_{ij}u\partial_j u + 2\eta|\nabla u|^2\partial_{ii}\eta + 4\eta\partial_i\eta\partial_{ij}u\partial_j u + \cancel{2\eta^2\partial_{iij}u\partial_j u} + 2\eta^2(\partial_{ij}u)^2 \qquad \text{harmonic} \\
&= \left(2(\partial_i\eta)^2 + 2\eta\partial_{ii}\eta\right)|\nabla u|^2 + 8\eta\partial_i\eta\partial_{ij}u\partial_j u + 2\eta^2(\partial_{ij}u)^2 \\
&\geq \left(2(\partial_i\eta)^2 + 2\eta\partial_{ii}\eta\right)|\nabla u|^2 - C|\nabla\eta|^2|\nabla u|^2 - \eta^2(\partial_{ij}u)^2 + 2\eta^2(\partial_{ij}u)^2 \qquad \text{Young's Inequality} \\
&\geq (-C|\nabla\eta|^2 + 2\eta\Delta\eta)|\nabla u|^2 + \eta^2|D^2u|^2 \qquad \text{combing terms} \\
&\geq (-C|\nabla\eta|^2 + 2\eta\Delta\eta)|\nabla u|^2 \qquad \text{simply throwing away the last term due to non-negative} \\
&\geq -C|\nabla u|^2 \qquad \text{up to some constant that depends on choice of } \eta
\end{aligned}$$

Now the second clever point is that, one can reinterpret $|\nabla u|^2$ using harmonicity

$$\Delta(u^2) = \partial_i(2u\partial_i u) = 2|\nabla u|^2 + 2u\Delta u \overset{u \text{ harmonic}}{=} 2|\nabla u|^2$$

Thus plugging this interpretation into the above yields

$$\Delta(\eta^2|\nabla u|^2 + Cu^2) \geq 0 \qquad \text{for } C \text{ large enough constant}$$

Thus using Weak Maximum Principle for subharmonic functions Theorem 1.4.1

$$\sup_{x\in B_1}(\eta^2|\nabla u|^2 + Cu^2) \leq \sup_{x\in\partial B_1}(\eta^2|\nabla u|^2 + Cu^2) = C\sup_{x\in\partial B_1} u^2$$
$$\sup_{x\in B_{1/2}}|\nabla u|^2 \leq C\sup_{x\in\partial B_1} u^2 \qquad \text{using choice of cutoff}$$

which yields (1.56) upon taking square root. □

In particular, (1.56) yields interior Hölder estimate([HL11] Proposition 1.31). For any $\alpha \in [0,1]$

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \leq \frac{|u(x)-u(y)|}{|x-y|} \leq C\sup_{\partial B_1} u \qquad \forall\ x,\ y \in B_{1/2},\ x \neq y \tag{1.57}$$

In other words

$$\|u\|_{C^\alpha(B_{1/2})} \leq C(n)\sup_{\partial B_1} u \qquad \forall\ \alpha \in [0,1]$$

One can also do Hölder-rescaling. Apply (1.57) to

$$u_\rho(x) := \frac{u(\rho x)}{\rho^\alpha} \qquad \forall\ x \in B_1$$

for any ρ small of your choice, one get

$$\frac{|u(\rho x)-u(\rho y)|}{\rho^\alpha|x-y|^\alpha} = \frac{|u_\rho(x)-u_\rho(y)|}{|x-y|^\alpha} \overset{(1.57)}{\leq} C\sup_{\partial B_1} u_\rho = C\frac{1}{\rho^\alpha}{}_{x\in\partial B_1} u(\rho x)$$
$$[u]_{C^\alpha(\overline{B_{\rho/2}})} = [u_\rho]_{C^\alpha(\overline{B_{1/2}})} \leq C\frac{1}{\rho^\alpha}\|u\|_{C(\overline{B_\rho})} \tag{1.58}$$

This is the rescaled-version of interior Hölder Estimate for Harmonic Function.

2. If one take $v = \log u$ for $u > 0$ harmonic, one has additional structure that (Einstein summation)

$$\Delta v = \partial_{ii}(\log(u)) = \partial_i(\frac{1}{u}\partial_i u) = -\frac{1}{u^2}\sum_i(\partial_i u)^2 + \frac{1}{u}\Delta u \overset{\Delta u=0}{=} -\frac{1}{u^2}\sum_i(\partial_i u)^2$$
$$= -\sum_i(\frac{1}{u}\partial_i u)^2 = -\sum_i(\partial_i v)^2 = -|\nabla v|^2 \tag{1.59}$$

Then hitting Δ on $w = |\nabla v|^2$ gives us some different equation. Following similar computation as above

$$\Delta w = \Delta(|\nabla v|^2) = \partial_{ii}(\sum_j(\partial_j v)^2) = \partial_i(2\partial_j v\partial_{ji}v) = 2(\partial_{ij}v)^2 + 2\partial_j v\partial_j(\Delta v)$$
$$\overset{(1.59)}{=} 2(\partial_{ij}v)^2 - 2\partial_j v\partial_j(|\nabla v|^2)$$
$$\Delta w + 2\nabla v\cdot\nabla w = 2|D^2 v|^2$$

Now how can one manage to use $|D^2 v|$? We note

$$|D^2 v|^2 = \sum_{i,j}(\partial_{ij}v)^2 \geq \sum_i(\partial_{ii}v)^2 \overset{\text{AM-GM}}{\geq} \frac{1}{n}(\sum_i\partial_{ii}v)^2 = \frac{(\Delta v)^2}{n}$$
$$\overset{(1.59)}{=} \frac{|\nabla v|^4}{n} = \frac{w^2}{n} \tag{1.60}$$

and one obtain

$$\Delta w + 2\nabla v\cdot\nabla w \geq \frac{1}{n}w^2 \tag{1.61}$$

Now at point x where w reaches interior extrema, one can use

$$\nabla w(x) = 0 \qquad \Delta w(x) \leq 0$$

so that necessarily

$$w(x) = 0$$

On the other hand, if RHS of (1.61) writes $\frac{w^2}{n} - C$ for some positive constant, then one has pointwise estimate

$$w(x) \leq C$$

Let's see how the above Log-Bochner's technique help us to deduce Harnack Inequality.

Theorem 1.4.3 ([HL11] Lemma 1.32, Corollary 1.33)**.** *Let $u \in C^2(B_1) \cap C^0(\overline{B_1})$, $u \geq 0$ and harmonic in B_1. Then there exists $C = C(n) > 0$ s.t. for any $x_1, x_2 \in B_{1/2}$,*

$$u(x_1) \leq Cu(x_2)$$

Proof. First of all notice it suffices to assume $u > 0$ in B_1. If not, u achieves interior minimum and thus $u \equiv 0$ throughout B_1 and we're done. Now for any $x_1, x_2 \in B_{1/2}$, look at the estimate from FTC

$$\log(\frac{u(x_1)}{u(x_2)}) = \log u(x_1) - \log u(x_2) \leq |x_1 - x_2| \int_0^1 |\nabla(\log u)(tx_1 + (1-t)x_2)| dt$$

If one can obtain universal log estimate

$$\sup_{x \in B_{1/2}} |\nabla(\log u)(x)| \leq C(n) \tag{1.62}$$

Then the above writes

$$\log(\frac{u(x_1)}{u(x_2)}) \overset{(1.62)}{\leq} C(n)|x_1 - x_2| \leq C(n)$$
$$u(x_1) \leq e^{C(n)} u(x_2) \qquad \forall\, x_1, x_2 \in B_{1/2}$$

Hence it suffices to prove (1.62).

Now we define

$$v := \log u \qquad w := |\nabla v|^2$$

indicated as above. We want to obtain uniform gradient estimate on

$$\sup_{x \in B_{1/2}} |\nabla v(x)| \leq C(n)$$

The difference with (1.56) is that v here is not harmonic, and RHS needs to be independent of v. In our language, we seek for $C = C(n) > 0$ s.t.

$$w(x) \leq C \qquad \forall\, x \in B_{1/2} \tag{1.63}$$

then taking square root on both sides yields the result. Hence it suffices to prove for (1.63).

(a) We try for test function $\varphi \in C_0^\infty(B_1)$. As before we compute

$$\begin{aligned}\Delta(\varphi w) &= \partial_{ii}(\varphi \sum_j (\partial_j v)^2) = \partial_i(\partial_i \varphi w + 2\varphi \partial_j v \partial_{ji} v) \\ &= \Delta\varphi w + \nabla\varphi \cdot \nabla w + 2\nabla\varphi \cdot D^2 v \nabla v + 2\varphi |D^2 v|^2 + 2\varphi \partial_j v \partial_{jii} v \\ &\overset{(1.59)}{=} \Delta\varphi w + \nabla\varphi \cdot \nabla w + 2\nabla\varphi \cdot D^2 v \nabla v + 2\varphi |D^2 v|^2 - 2\varphi \nabla v \cdot \nabla w\end{aligned}$$

Let's expand the term

$$\nabla\varphi \cdot \nabla w = \partial_i \varphi \partial_i (\sum_j (\partial_j v)^2) = 2\partial_i \varphi (\partial_j v \partial_{ji} v) = 2\nabla\varphi \cdot D^2 v \nabla v$$

Thus one has

$$\begin{aligned}\Delta(\varphi w) + 2\varphi \nabla v \cdot \nabla w &= \Delta\varphi |\nabla v|^2 + 4\nabla\varphi \cdot D^2 v \nabla v + 2\varphi |D^2 v|^2 \\ \Delta(\varphi w) + 2\nabla v \cdot \nabla(\varphi w) &= \Delta\varphi |\nabla v|^2 + 4\nabla\varphi \cdot D^2 v \nabla v + 2\nabla v \cdot \nabla\varphi |\nabla v|^2 + 2\varphi |D^2 v|^2 \qquad \text{squeeze } \varphi \text{ into } \nabla\end{aligned}$$

We look at RHS. One want to group the terms with $|D^2v|^2$, hence we rewrite and apply Young's Inequality

$$\nabla\varphi \cdot D^2v\nabla v = \sqrt{\varphi}\frac{1}{\sqrt{\varphi}}\nabla\varphi \cdot D^2v\nabla v \geq -\frac{1}{4}\varphi|D^2v|^2 - C\frac{|\nabla\varphi|^2}{\varphi}|\nabla v|^2$$

so that

$$\begin{aligned}\Delta(\varphi w) + 2\nabla v \cdot \nabla(\varphi w) &\geq \varphi|D^2v|^2 + \left(\Delta\varphi - C\frac{|\nabla\varphi|^2}{\varphi}\right)|\nabla v|^2 - 2|\nabla\varphi||\nabla v|^3 \\ &\overset{(1.60)}{\geq} \frac{\varphi}{n}w^2 + \left(\Delta\varphi - C\frac{|\nabla\varphi|^2}{\varphi}\right)|\nabla v|^2 - 2|\nabla\varphi||\nabla v|^3\end{aligned}$$

(b) Now one needs clever choice of cutoff so that $\frac{|\nabla\varphi|^2}{\varphi}$ is bounded in $B_{1/2}$. In fact one can take $\varphi = \eta^4$ for some $\eta \in C_0^\infty(B_1)$ with $\eta|_{B_{1/2}} = 1$. Then

$$\begin{aligned}\frac{|\nabla\varphi|^2}{\varphi} &= 16\frac{\eta^6|\nabla\eta|^2}{\eta^4} = 16\eta^2|\nabla\eta|^2 \\ \Delta\varphi &= \partial_i(4\eta^3\partial_i\eta) = 12\eta^2|\nabla\eta|^2 + 4\eta^3\Delta\eta\end{aligned}$$

We rewrite

$$\begin{aligned}\Delta(\eta^4 w) + 2\nabla v \cdot \nabla(\eta^4 w) &\geq \frac{\eta^4}{n}w^2 + \left(4\eta^3\Delta\eta - C\eta^2|\nabla\eta|^2\right)|\nabla v|^2 - 8\eta^3|\nabla\eta||\nabla v|^3 \\ &\geq \frac{\eta^4}{n}w^2 - C_1(\eta,n)|\nabla v|^2 - C_2(\eta,n)|\nabla v|^3 \\ &= \frac{\eta^4}{n}w^2 - C_1(\eta,n)w - C_2(\eta,n)w^{\frac{3}{2}} \\ &\geq \frac{\eta^4}{2n}w^2 - C(\eta,n) \qquad \text{Young's Inequality}\end{aligned}$$

Notice $\eta^4 w$ is non-negative, continuous and compactly supported in B_1. Using cutoff, $\eta^4 w|_{\partial B_1} = 0$. Assume at $x_0 \in \operatorname{supp}(\eta)$ the function $\eta^4 w(x_0)$ achieves interior maximum. To show for (1.63), it suffices to obtain the estimate at

$$w(x_0) \leq C(\eta, n)$$

To do so, we use

$$\nabla(\eta^4 w)(x_0) = 0 \qquad \Delta(\eta^4 w)(x_0) \leq 0$$

so that

$$\frac{1}{2n}\eta^4(x)w^2(x) \leq \frac{\eta^4(x_0)}{2n}w^2(x_0) \leq C(\eta,n) \qquad \forall\, x \in \operatorname{supp}(\eta)$$

Restrict to $B_{1/2}$ and fix choice of η to recover (1.63).

□

Hopf's Lemma In this paragraph we prove the Laplace baby version of Hopf's Lemma, which says the outward normal derivative at the maximum point on the boundary of a ball is strictly positive.

Theorem 1.4.4 ([HL11] Proposition 1.34; De Silva Analysis II 2025)**.** *Let $u \in C^2(B_1) \cap C^1(\overline{B_1})$ be subharmonic in B_1. Assume $x_0 \in \partial B_1$ s.t.*

$$u(x) < u(x_0) \qquad \forall\, x \in \overline{B_1}, \qquad x \neq x_0$$

Then

$$\frac{\partial u}{\partial \nu}(x_0) > 0$$

Proof. Let's first see why this is non-trivial. Using $u(x) < u(x_0)$ for any $x \neq x_0$, one can only obtain

$$\frac{\partial u}{\partial \nu}(x_0) = \lim_{t\to 0}\frac{u(x_0) - u(x_0 - t\nu)}{t} \geq 0$$

since we're taking the limit as $t \to 0$. The key to prove is 'strictness'.

1. We add a small perturbation. For $\alpha > 0$ to be chosen, define

$$v(x) := e^{-\alpha|x|^2} - e^{-\alpha} \qquad \forall\ x \in \overline{B_1} \tag{1.64}$$

so that

$$\begin{aligned} \partial_i v(x) &= e^{-\alpha|x|^2}(-2\alpha x_i) \\ \Delta v(x) &= (-2n\alpha + 4\alpha^2|x|^2)e^{-\alpha|x|^2} \end{aligned}$$

We look at the region $\frac{1}{2} \le |x| \le 1$. Then one can choose α sufficiently large s.t.

$$\begin{aligned} -2n\alpha + 4\alpha^2|x|^2 > 0 \qquad &\forall\ \frac{1}{2} \le |x| \le 1 \\ \alpha > 2n \qquad &\text{suffices} \end{aligned}$$

Hence

$$\Delta v(x) > 0 \qquad \forall\ \frac{1}{2} \le |x| \le 1$$

Also notice

$$\begin{aligned} v(x) \ge 0 \qquad &\forall\ \frac{1}{2} \le |x| \le 1 \\ v(x) = 0 \qquad &|x| = 1 \end{aligned}$$

2. Now for any $\varepsilon > 0$, define perturbation as

$$u_\varepsilon(x) := u(x) - u(x_0) + \varepsilon v(x)$$

so that

$$\begin{aligned} u_\varepsilon(x) \le 0 \qquad &|x| = 1 \\ u_\varepsilon(x) < 0 \qquad &|x| = \frac{1}{2} \qquad \text{for } \varepsilon > 0 \text{ chosen small enough} \\ \Delta u_\varepsilon(x) = \Delta u + \varepsilon \Delta v > 0 \qquad &\text{using } u \text{ subharmonic and } \Delta v > 0 \end{aligned}$$

Applying Weak Maximum Principle Theorem 1.4.1 for Subharmonic function u_ε on $B_1 \setminus B_{1/2}$ therefore yields

$$u_\varepsilon(x) < 0 \qquad \forall\ \frac{1}{2} < |x| < 1$$

In other words, using $v(x_0) = 0$

$$\begin{aligned} u(x_0) - u(x) > \varepsilon v(x) \qquad &\forall\ \frac{1}{2} < |x| < 1 \\ \frac{1}{t}\left(u(x_0) - u(x_0 - t\nu)\right) > -\varepsilon \frac{1}{t}(v(x_0) - v(x_0 - t\nu)) \qquad &\forall\ 0 < t < \frac{1}{2} \end{aligned}$$

sending $t \to 0$ yields

$$\frac{\partial u}{\partial \nu}(x_0) \ge -\varepsilon \frac{\partial v}{\partial \nu}(x_0)$$

But we can compute the RHS

$$\frac{\partial v}{\partial \nu}(x_0) = v'(|x_0|) = -2\alpha|x_0|e^{-\alpha|x_0|^2} = -2\alpha e^{-\alpha} < 0$$

so that

$$\frac{\partial u}{\partial \nu}(x_0) \ge -\varepsilon \frac{\partial v}{\partial \nu}(x_0) = 2\varepsilon\alpha e^{-\alpha} > 0$$

□

Interior Cone Condition We claim that one can also do for interior cone condition. It suffices to build the cone barrier. Let's define in $\mathbb{R}^2$

$$u(r,\theta) = e^{-\alpha r^2}\cos(\theta)$$

Then

$$\begin{aligned}\partial_r u &= -2\alpha r e^{-\alpha r^2}\cos(\theta)\\ \partial_{rr} u &= -2\alpha u + 4\alpha^2 r^2 u\end{aligned}$$

Hence

$$\begin{aligned}\Delta u &= \partial_{rr} u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u\\ &= \left(-4\alpha + 4\alpha^2 r^2 - \frac{1}{r^2}\right) u\\ &= \left(\left(2\alpha r^2 - 1\right)^2 - 2\right)\frac{u}{r^2}\end{aligned}$$

Now this has a negative sign if we choose $r \le r_0$ small universal, and then choose α small depending on r_0.

Global Hölder Regularity from Hölder Boundary data Notice up until now we're doing mainly interior estimates. From Poisson's Integral Formula we know certain solutions defined by Green's Representation achieves boundary data continuously (given continuous boundary data).

But in general, given boundary data with higher regularity, does the solution to Laplace Equation inherit higher regularity up to the boundary as well?

Theorem 1.4.5 ([HL11] Lemma 1.35)**.** *Let $u \in C^2(B_1) \cap C^0(\overline{B_1})$ be harmonic function with boundary data $\varphi \in C^\alpha(\partial B_1)$ for $\alpha \in (0,1)$. Then $u \in C^{\alpha/2}(\overline{B_1})$ and*

$$[u]_{C^{\alpha/2}(\overline{B_1})} \le C(n,\alpha)[\varphi]_{C^\alpha(\partial B_1)}$$

Proof. 1. We first claim it suffices to prove for any $x_0 \in \partial B_1$

$$\sup_{x\in B_1}\frac{|u(x)-u(x_0)|}{|x-x_0|^{\alpha/2}} \le C(\alpha)\sup_{x\in\partial B_1}\frac{|\varphi(x)-\varphi(x_0)|}{|x-x_0|^\alpha} \tag{1.65}$$

In philosophy, to obtain global estimates, it suffices to first do for one fixed point on the boundary. To see why if suffices, we denote

$$d_x := \operatorname{dist}(x,\partial B_1) \qquad \forall\, x \in B_1$$

We want to use (1.65) to deduce

$$\frac{|u(x)-u(y)|}{|x-y|^{\frac{\alpha}{2}}} \le C[\varphi]_{C^\alpha(\partial B_1)} \qquad \forall\, x,\, y \in B_1 \qquad x \ne y$$

For any such x, y, pick x_0, $y_0 \in \partial B_1$ s.t. the distance are realized

$$d_x = |x-x_0| \qquad d_y = |y-y_0|$$

WLOG assume $0 < d_y \le d_x$, i.e., y is closer to boundary ∂B_1.

(a) If x, y are sufficiently close, one can simply reduce to interior estimates. In particular if

$$|x-y| \le \frac{1}{2}d_x = \frac{1}{2}|x-x_0|$$

One may apply rescaled-interior Hölder estimate (1.57) to the function $u-u(x_0)$ in the ball $B_{d_x}(x)$. Thus

$$\begin{aligned}\frac{|u(x)-u(y)|}{|x-y|^{\frac{\alpha}{2}}} &\le C\frac{1}{d_x^{\frac{\alpha}{2}}}\sup_{B_{d_x}(x)}|u-u(x_0)|\\ &\overset{(1.65)}{\le} C(\alpha)\sup_{x\in\partial B_1}\frac{|\varphi(x)-\varphi(x_0)|}{|x-x_0|^\alpha} \le C(\alpha)[\varphi]_{C^\alpha(\partial B_1)}\end{aligned}$$

(b) If x, y are not close, i.e.

$$|x-y| > \frac{1}{2}d_x = \frac{1}{2}|x-x_0|$$

One controls the distance to the boundary via

$$|x_0-y_0| \leq |x_0-x| + |x-y| + |y-y_0| \leq 5|x-y|$$

so that

$$\begin{aligned}|u(x)-u(y)| &\leq |u(x)-u(x_0)| + |u(x_0)-u(y_0)| + |u(y_0)-u(y)| \\ &\overset{(1.65)}{\leq} C\left(d_x^{\alpha/2} + |x_0-y_0| + d_y^{\alpha/2}\right)[\varphi]_{C^\alpha(\partial B_1)} \\ &\leq C(\alpha)|x-y|^{\frac{\alpha}{2}}[\varphi]_{C^\alpha(\partial B_1)}\end{aligned}$$

Since both cases are dealt with, one has global Hölder Estimate.

2. Now we prove (1.65). Notice the claim is invariant under translation and rotation, hence it suffices to do for the ball

$$B_1((1,0,\cdots,0)) = B_1(e_1)$$

and for the point

$$x_0 = 0 = (0,\cdots,0)$$

Why is this ball a smart choice? On the this ball, the points satisfy

$$\begin{aligned}\sqrt{(x_1-1)^2 + x_2^2 + \cdots + x_n^2} &= 1 \\ x_1^2 + \cdots + x_n^2 &= 2x_1 \\ |x|^2 &= 2x_1 \qquad \forall\, x \in \partial B_1(e_1)\end{aligned} \tag{1.66}$$

and this relation simplifies the computation a lot! Also, WLOG assume

$$\varphi(x_0) = \varphi(0) = u(0) = 0$$

We want to build a test function v defined on $B_1(e_1)$ s.t.

$$u = \varphi \leq v \qquad \partial B_1(e_1) \tag{1.67}$$

$$\Delta v \leq 0 \qquad B_1(e_1) \tag{1.68}$$

Then since $\Delta u = 0$ in $B_1(e_1)$ is harmonic, one has by the comparison principle

$$u \leq v \qquad B_1(e_1)$$

Now we build this v. Let

$$v(x) := 2^{\frac{\alpha}{2}} \sup_{x\in B_1(e_1)} \frac{|\varphi(x)|}{|x|^\alpha} x_1^{\frac{\alpha}{2}} \qquad \forall\ B_1(e_1)$$

We can ensure (1.67) (This is why we need Hölder Regularity of φ !)

$$\begin{aligned}|\varphi(x)| = \frac{|\varphi(x)|}{|x|^\alpha}|x|^\alpha &\leq \left(\sup_{x\in B_1(e_1)} \frac{|\varphi(x)|}{|x|^\alpha}\right)|x|^\alpha \\ &\overset{(1.66)}{=} \left(\sup_{x\in B_1(e_1)} \frac{|\varphi(x)|}{|x|^\alpha}\right) 2^{\frac{\alpha}{2}} x_1^{\frac{\alpha}{2}} = v(x)\end{aligned}$$

We can also ensure (1.68) by the simple computation

$$\Delta v = 2^{\frac{\alpha}{2}} \sup_{x\in B_1(e_1)} \frac{|\varphi(x)|}{|x|^\alpha} \frac{\alpha}{2}(\frac{\alpha}{2}-1)x_1^{\frac{\alpha}{2}-2} < 0 \qquad \text{using } \alpha < 2$$

Thus using the comparison principle

$$\begin{aligned}|u(x)| &\leq 2^{\frac{\alpha}{2}} \sup_{x\in B_1(e_1)} \frac{|\varphi(x)|}{|x|^\alpha} x_1^{\frac{\alpha}{2}} \\ &\leq 2^{\frac{\alpha}{2}} \sup_{x\in B_1(e_1)} \frac{|\varphi(x)|}{|x|^\alpha}|x|^{\frac{\alpha}{2}} \\ \sup_{x\in B_1(e_1)} \frac{|u(x)|}{|x|^{\frac{\alpha}{2}}} &\leq 2^{\frac{\alpha}{2}} \sup_{x\in B_1(e_1)} \frac{|\varphi(x)|}{|x|^\alpha}\end{aligned}$$

and this gives (1.65) upon translation and rotation.

□

Touch Let $|\Delta u| \le K$ in B_1 and $u = 0$ on ∂B_1. Then

$$\|u\|_{L^\infty(B_1)} \le \frac{K}{2n}$$

Proof. Denote $\phi = \frac{1}{2n}(|x|^2 - 1)$. Then

$$\begin{aligned} \Delta\phi &= 1 \\ \phi &= 0 \qquad \partial B_1 \end{aligned}$$

Now $u + K\phi$ satisfies

$$0 \le \Delta(u + K\phi)$$

hence is subharmonic. Using Maximum Principle for subharmonic functions yields

$$\max_{B_1}(u + K\phi) \le 0 \implies u \le -K\phi \le \frac{K}{2n} \qquad B_1$$

On the other hand $u - K\phi$ satisfies

$$\Delta(u - K\phi) \le 0$$

hence is superharmonic. Using Minimum Principle yields

$$\inf_{B_1}(u - K\phi) \ge \inf_{\partial B_1}(u - K\phi) = 0 \implies u \ge K\phi \ge -\frac{K}{2n} \qquad B_1$$

□

1.5 Perron's Method

Up until now one only has method to solve Laplace's Equation over certain domains via Green's Representation and Poisson's Integral Formula. For general bounded open domain, can one also build a solution to Laplace's Equation?

Perron's Solution resolves the problem for a large family of general domains. In the following, let $\Omega \subseteq \mathbb{R}^n$ be bounded open domain.

Weakly-Subharmonic We begin by defining a weaker notion than subharmonic functions, which allows for only continuity, but makes use of one-sidedness.

Definition 1.5.1 (Weakly Sub-Harmonic; [HL11] Definition 6.1)**.** *$u \in C(\Omega)$ is weakly subharmonic if*

$$u \leq v \quad in\ B, \qquad \forall\ B\ ball \Subset \Omega \quad and \quad v \in C^2(B) \cap C(\overline{B})\ harmonic \quad s.t. \quad u = v \quad \partial B$$

$u \in C(\Omega)$ is weakly superharmonic if $u \geq v$ in B for any B and v as above.

One observe that a weakly-subharmonic function inherits the Comparison Principle of an actual subharmonic function.

1. First and most importantly, a weak subharmonic function also has Strong Maximum Principle.

   Property 1.5.1 ([HL11] Lemma 6.2; [GT01] Section 2.8(i))**.** *Let $\Omega \subseteq \mathbb{R}^n$ be bounded, open and connected. Let $u \in C(\overline{\Omega})$ be weakly-subharmonic function. Then u cannot achieve interior supremum unless u is constant throughout Ω.*

   Proof. (a) Let $M := \sup_{x\in\Omega} u(x)$. Consider the closed subset

   $$\Omega_M := \{x \in \Omega \mid u(x) = M\}$$

   Since Ω is connected, Ω_M as a subset cannot be both open and closed unless it is either the whole domain $\Omega = \Omega_M$ or $\Omega = \varnothing$. Now assume there exists $y \in \Omega$ that achieves interior supremum

   $$u(y) = \sup_{x\in\Omega} u(x) = M$$

   hence $\Omega_M \neq \varnothing$. If we're able to prove Ω_M is open, necessarily $\Omega = \Omega_M$, i.e., $u \equiv M$ throughout Ω.

   (b) Now for any $x_0 \in \Omega_M$, consider $B = B_r(x_0) \Subset \Omega$. One can solve for $v \in C^2(B) \cap C^0(\overline{B})$ s.t.

   $$\begin{cases} \Delta v = 0 & B \\ v = u & \partial B \end{cases}$$

   using the Poisson's Integral Formula (1.47). Since u is weakly subharmonic, necessarily

   $$u \leq v \qquad B$$

   One on hand, since $v \in C^2(B) \cap C^0(\overline{B})$ is harmonic, by Weak Maximum Principle Theorem 1.1.4

   $$v(x) \leq \sup_{\partial B} v(x) = \sup_{\partial B} u \leq \sup_{x\in\Omega} u(x) = M \qquad \forall\ x \in B$$

   On the other hand, using weak subharmonicity of u at x_0

   $$M = u(x_0) \leq v(x_0) \leq M$$

   Thus v achieves interior supremum at the point x_0. Using Strong Maximum Principle Theorem 1.1.3

   $$v \equiv M \qquad \text{throught } B$$

   In particular

   $$M = \sup_{\partial B} v(x) = \sup_{\partial B} u$$

   But B around x_0 is arbitrary. By choosing any ball of smaller radius around x_0 and applying the above, one conclude

   $$B_r(x_0) \subseteq \Omega_M$$

   Thus we've fit an open ball around x_0 into Ω_M, so Ω_M is open.

   ∎

2. As an immediate corollary, one has the comparison principle.

   Corollary 1.5.1 ([HL11] Lemma 6.2). *Let $\Omega \subseteq \mathbb{R}^n$ be bounded, open and connected. Let $u \in C(\overline{\Omega})$ be weakly-subharmonic, and $v \in C(\overline{\Omega})$ be weakly-superharmonic s.t.*

$$u \leq v \qquad \partial\Omega$$

   Then either

$$u < v \qquad \Omega$$

   or

$$u \equiv v \qquad \overline{\Omega}$$

As a technical lemma, one notice weak-subharmonicity is preserved under taking finite maximum.

Property 1.5.2 ([GT01] Section 2.8(iii)). *Let $u_1, \cdots, u_N$ be weakly subharmonic in Ω, then $\max\{u_1, \cdots, u_N\}$ is weakly subharmonic in Ω.*

Proof. Take any $B \Subset \Omega$ and $v \in C^2(B) \cap C(\overline{B})$ harmonic s.t.

$$\max\{u_1, \cdots, u_N\} = v \qquad \partial B$$

There exists $v_i \in C^2(B) \cap C(\overline{B})$ harmonic for $i = 1, \cdots, N$ s.t.

$$u_i = v_i \leq v \qquad \partial B$$

Using each u_i is weakly subharmonic,

$$u_i \leq v_i \qquad B$$

But both v_i and v are harmonic in B with boundary data $v_i \leq v$. Using comparison principle

$$u_i \leq v_i \leq v \qquad B \qquad \forall\, i = 1, \cdots, N$$

Thus one get

$$\max\{u_1, \cdots, u_N\} \leq v \qquad B$$

□

We remark that, if one a prior showed equivalence between weak subharmonicity and subharmonic in the viscosity sense, then 'subharmonicity' is closed under taking maximum is trivial.

Harmonic Lifting Now given a weakly subharmonic function u, for any ball B fixed, one can construct a function w that lifts u to a harmonic function on B.

Definition 1.5.2 (Harmonic Lifting(Replacement); [GT01] Section 2.8(ii)). *Let $u \in C(\Omega)$ be weakly-subharmonic function, and let $B \Subset \Omega$. The harmonic lifting of u on B is $w \in C(\Omega)$ that satisfies*

$$\begin{cases} \Delta w = 0 & B \\ w = u & \Omega \setminus B \end{cases}$$

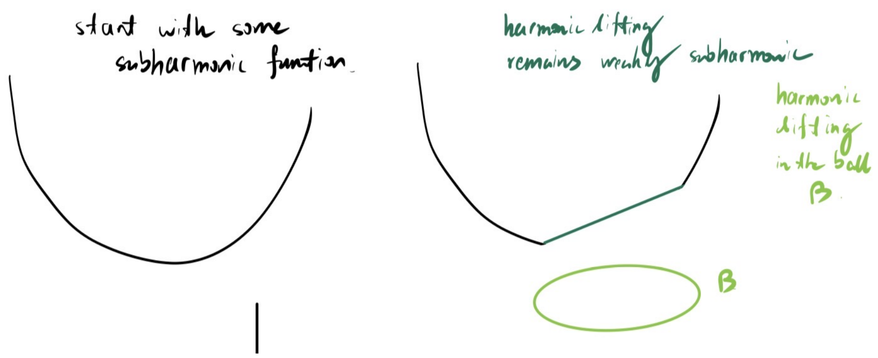


Figure 1.6: Harmonic Lifting

One immediately observe that

1. For any $B \Subset \Omega$, due to Uniqueness Corollary 1.1.1, the harmonic lifting w of u on B is uniquely determined.
2. Since u is weakly-subharmonic and w is harmonic in B, and they agree on $\Omega \setminus B$

$$u \leq w \qquad \Omega$$

3. The nontrivial property of harmonic lifting is that, it remains weakly subharmonic upon lifting the weakly subharmonic function.

Property 1.5.3 ([HL11] Lemma 6.3)**.** *Let $u \in C(\Omega)$ be weakly subharmonic, and $B \Subset \Omega$. Then the harmonic lifting $w \in C(\Omega)$ of u in B is weakly subharmonic.*

Proof. We aim to prove that for any $B' \Subset \Omega$ and $v \in C^2(B') \cap C(\overline{B'})$ s.t.

$$\Delta v = 0 \qquad B' \qquad v = w \qquad \partial B'$$

one can infer that

$$w \leq v \qquad B'$$

One think about what B' consists of. We need to make use of u and B. Indeed one can partition

$$B' = (B' \cap B) \cup (B' \setminus B)$$

and we treat the two portions separately.

(a) On $B' \setminus B$, by definition of w as harmonic lifting of u in B

$$w = u \qquad B' \setminus B \subseteq \Omega \setminus B$$

now since v is harmonic in $B' \setminus B$ and u is weakly subharmonic over Ω, in particular

$$u \leq v \qquad B'$$

Thus one obtain

$$w = u \leq v \qquad B' \setminus B$$

(b) On $B' \cap B$ (WLOG assume this is non-empty), both w and v are harmonic. Using Weak Maximum Principle Theorem 1.1.4

$$w - v \leq \sup_{\partial(B' \cap B)} w - v$$

Now what is the supremum on the boundary? Notice

$$\partial(B' \cap B) = (\partial B' \cap B) \cup (B' \cap \partial B) \cup (\partial B' \cap \partial B)$$

Hence using choice of $v = w$ on $\partial B'$

$$w - v = 0 \qquad \partial B' \cap B$$

using $w = u$ on ∂B as harmonic lifting, and u weakly subharmonic, v harmonic, hence $u \leq v$ in B'

$$w - v = u - v \leq 0 \qquad B' \cap \partial B$$

using again $v = w$ on $\partial B'$

$$w - v = 0 \qquad \partial B' \cap \partial B$$

Thus we conclude

$$w - v \leq \sup_{\partial(B' \cap B)} w - v = 0 \qquad B' \setminus B$$

□

Perron's Solution Given $\Omega \subseteq \mathbb{R}^n$ bounded connected open domain. Let $\varphi \in C(\partial\Omega)$ be continuous function defined on the boundary. Perron constructs a function defined using boundary data s.t. it is harmonic within Ω.

Let the admissible set be

$$S_\varphi := \{v \in C(\overline{\Omega}) \mid v \text{ weakly subharmonic in } \Omega \text{ and } v \leq \varphi \ \partial\Omega\} \tag{1.69}$$

and we define our Perron's solution to be

$$u(x) := \sup_{v \in S_\varphi} v(x) \qquad \forall\, x \in \Omega \tag{1.70}$$

Proposition 1.5.1 ([HL11] Lemma 6.4; [GT01] Theorem 2.12)**.** *Given $\varphi \in C(\partial\Omega)$. u defined as in (1.70) is harmonic in Ω*

$$\Delta u(x) = 0 \qquad \forall\, x \in \Omega$$

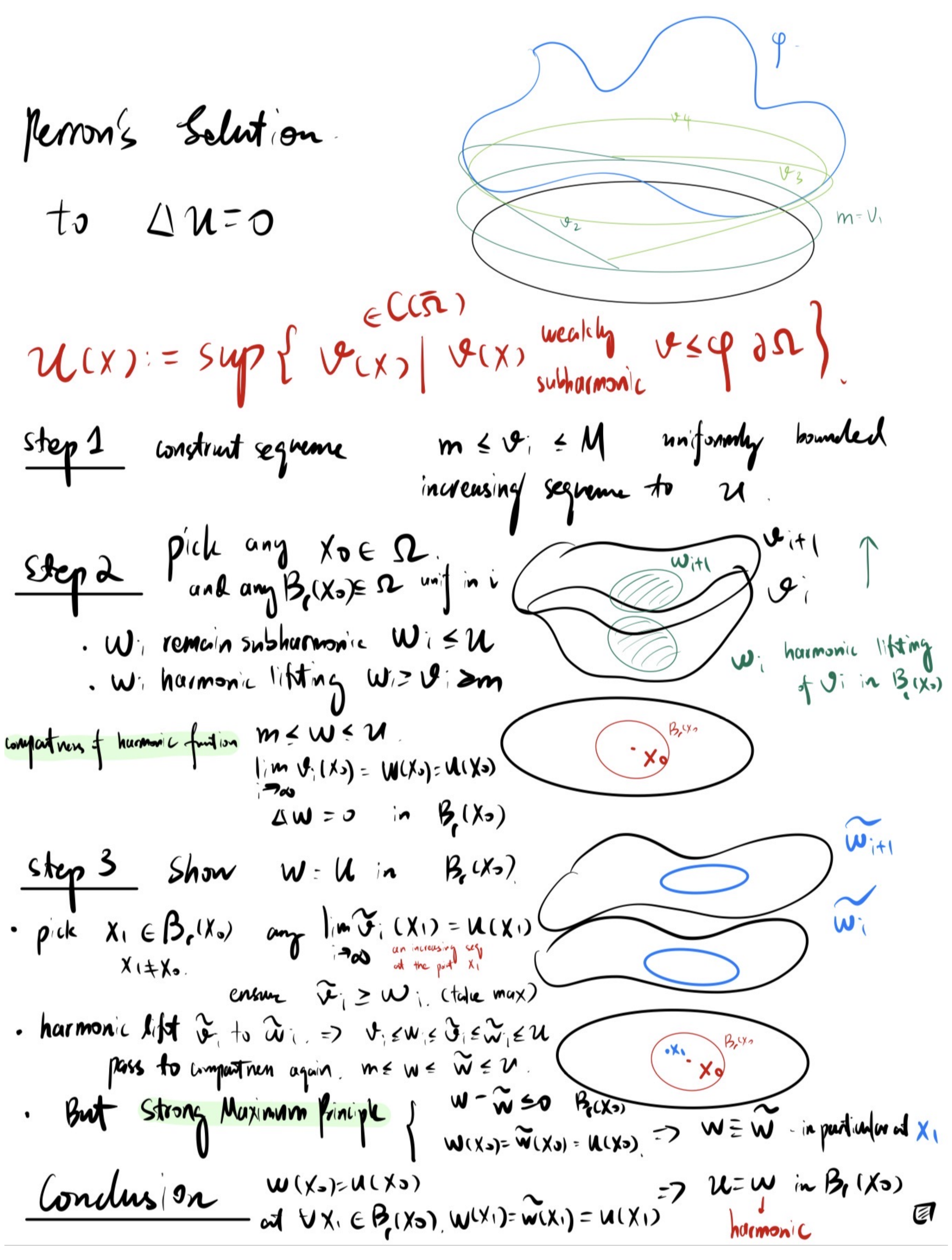


Figure 1.7: Perron's Solution

Proof. The idea is as follows. Using definition of (1.70) as supremum, one can construct an increasing sequence of functions v_i in S_φ that approaches u. Then for each v_i we consider a harmonic lifting w_i. We show that the sequence w_i converges to some $w \in C(\overline{\Omega})$ that is harmonic. Finally we demonstrate that in fact $u = w$ in Ω. In the following we setup step by step. Denote

$$m := \inf_{x \in \partial\Omega} \varphi(x) \qquad M := \sup_{x \in \partial\Omega} \varphi(x)$$

1. First we need a reason to construct the v_i. Note $S_\varphi \neq \varnothing$ because constants are weakly-subharmonic, hence
$$m \in S_\varphi$$
and therefore it's eligible to take an increasing sequence. Next we want v_i to have uniform upper bound. Notice M as constant is harmonic and $\varphi \leq M$ on $\partial\Omega$. Thus
$$v \leq M \qquad \forall\, v \in S_\varphi \implies u(x) \leq M \qquad \forall\, x \in \overline{\Omega}$$
and we have a uniform upper bound
$$v_i \leq M$$
We also want to ensure $v_i \geq m$ has uniform lower bound in our sequence. To do so we need to prove the admissible set S_φ is closed under taking finite maximum. Take $v_1, \cdots, v_N \in S_\varphi$. Using Property 1.5.2, we know
$$\max\{v_1, \cdots, v_N\} \qquad \text{remains weakly subharmonic in } \Omega$$
Then notice $v_i \leq \varphi$ on $\partial\Omega$ for any i indeed implies $\max\{v_1, \cdots, v_N\} \leq \varphi$ on $\partial\Omega$, and therefore the finite maximum belongs to S_φ. Now, since $m \in S_\varphi$ and $v_i \in S_\varphi$ for any i, one can redefine v_i as
$$\max\{m, v_i\} \in S_\varphi$$
and we have a uniform lower bound for v_i.

2. Now fix a point $x_0 \in \Omega$ and construct $m \leq v_i \in S_\varphi$ as above s.t.
$$u(x_0) = \lim_{i\to\infty} v_i(x_0)$$
Take $B_r(x_0) \Subset \Omega$, and construct w_i as harmonic lifting of v_i on $B_r(x_0)$, i.e.
$$\begin{cases} \Delta w_i = 0 & B_r(x_0) \\ w_i = v_i & \Omega \setminus B_r(x_0) \end{cases}$$
Why does w_i have a limit w, and why w remains harmonic in $B_r(x_0)$? This is due to usage of Compactness Theorem 1.2.3, which asserts a uniformly bounded sequence of harmonic functions converge locally uniformly to a harmonic function. To apply the Theorem one need uniform boundedness of w_i. On one hand, from Property 1.5.3, we know $w_i \in C(\Omega)$ are weakly subharmonic. Also
$$w_i = v_i \leq \varphi \qquad \partial\Omega \qquad \forall\, i$$
Thus $w_i \in S_\varphi$ for any i. In particular
$$w_i \leq u \leq M \qquad \overline{\Omega} \qquad \forall\, i$$
On the other hand, using w_i as harmonic lifting
$$m \leq v_i \leq w_i \qquad \overline{\Omega} \qquad \forall\, i$$
Hence w_i is uniformly bounded over Ω by
$$m \leq w_i \leq M \qquad \forall\, i$$
Now apply Theorem 1.2.3 one has $w \in C(B_r(x_0))$ s.t. $w_i \to w$ locally uniformly (up to subsequence) in $B_r(x_0)$ and $\Delta w = 0$ in $B_r(x_0)$. We know
$$m \leq w \leq u \qquad B_r(x_0)$$
and in particular at the point x_0
$$u(x_0) = \lim_{i\to\infty} v_i(x_0) \leq \lim_{i\to\infty} w_i(x_0) = w(x_0) \leq u(x_0) \implies w(x_0) = u(x_0) \tag{1.71}$$

3. In the third step we show
$$w = u \qquad B_r(x_0)$$
Take any other point $x_1 \in B_r(x_0)$. Again, one can construct a sequence $\tilde{v}_i \in S_\varphi$ s.t.
$$u(x_1) = \lim_{i\to\infty} \tilde{v}_i(x_1)$$
$$\tilde{v}_i \geq w_i \qquad \forall\, i \qquad \text{using Property 1.5.2}$$

where $w_i \in S_\varphi$ as are in previous step. Again we construct $\tilde{w}_i$ as harmonic lifting of $\tilde{v}_i$ on $B_r(x_0)$

$$\begin{cases} \Delta \tilde{w}_i = 0 & B_r(x_0) \\ \tilde{w}_i = \tilde{v}_i & \Omega \setminus B_r(x_0) \end{cases}$$

For similar reasons as above, since

$$m \le v_i \le w_i \le \tilde{v}_i \le \tilde{w}_i \le u \le M \qquad \forall\ i$$

Again $\tilde{w}_i$ is uniformly bounded sequence of harmonic functions over $B_r(x_0)$. Apply Theorem 1.2.3, we know $\tilde{w}_i \to \tilde{w}$ locally uniformly to harmonic function $\tilde{w} \in C(B_r(x_0))$. Moreover

$$m \le w \le \tilde{w} \le u \le M \qquad B_r(x_0)$$

Let's pause and see what we aim at. In principle, since this x_1 is arbitrary, it suffices to prove

$$w(x_1) = u(x_1)$$

Our trick is to transit using $\tilde{w}$.

(a) It is immediate $u(x_1) = \tilde{w}(x_1)$ due to construction

$$u(x_1) = \lim_{i\to\infty} \tilde{v}_i(x_1) \le \lim_{i\to\infty} \tilde{w}_i(x_1) = \tilde{w}(x_1) \le u(x_1)$$

(b) To show $w(x_1) = \tilde{w}(x_1)$, we want to make use of Maximum Principe. First notice

$$w - \tilde{w} \le 0 \qquad B_r(x_0)$$

and at the point x_0

$$u(x_0) \overset{(1.71)}{=} w(x_0) \le \tilde{w}(x_0) \le u(x_0)$$

hence $w - \tilde{w}$ achieves interior maximum at x_0. Now using $w - \tilde{w}$ is harmonic in $B_r(x_0)$, by Strong Maximum Principle Theorem 1.1.3

$$w - \tilde{w} \equiv 0 \qquad B_r(x_0)$$

In particular equality holds true at x_1.

The proof demonstrates how compactness of harmonic functions lies behind existence of solutions. □

Perron's Solution Approaching Boundary Data Notice up until now one has not discussed how the Perron's solution u (1.70) approach boundary data φ. In fact, whether u can continuously approach boundary data relies on the boundary of our domain. Let $\Omega \subseteq \mathbb{R}^n$ be bounded open connected domain.

Barrier Function and Regular Points Let $x_0 \in \partial\Omega$. How can one possibly show that as $x \to x_0$ for $x \in \Omega$, one get $|u(x) - \varphi(x_0)| \to 0$? The idea is to use some $w_{x_0}(x)$ s.t. $\lim_{x\to x_0} w_{x_0}(x) = 0$ and bound

$$|u(x) - \varphi(x_0)| \lesssim w_{x_0}(x)$$

We define precisely what we mean by w_{x_0}. Notice this definition is purely dependent on the domain Ω.

Definition 1.5.3 ([GT01] Section 2.8)**.** *Let $x_0 \in \partial\Omega$. We call $w_{x_0} \in C(\overline{\Omega})$ a barrier function at x_0 if*

1. *w_{x_0} is weakly superharmonic in Ω.*
2. *$w_{x_0} > 0$ over $\Omega \setminus \{x_0\}$ and $w_{x_0}(x_0) = 0$.*

In particular, we call $x_0 \in \partial\Omega$ a regular point if there exists a barrier function as in Definition 1.5.3 at x_0.

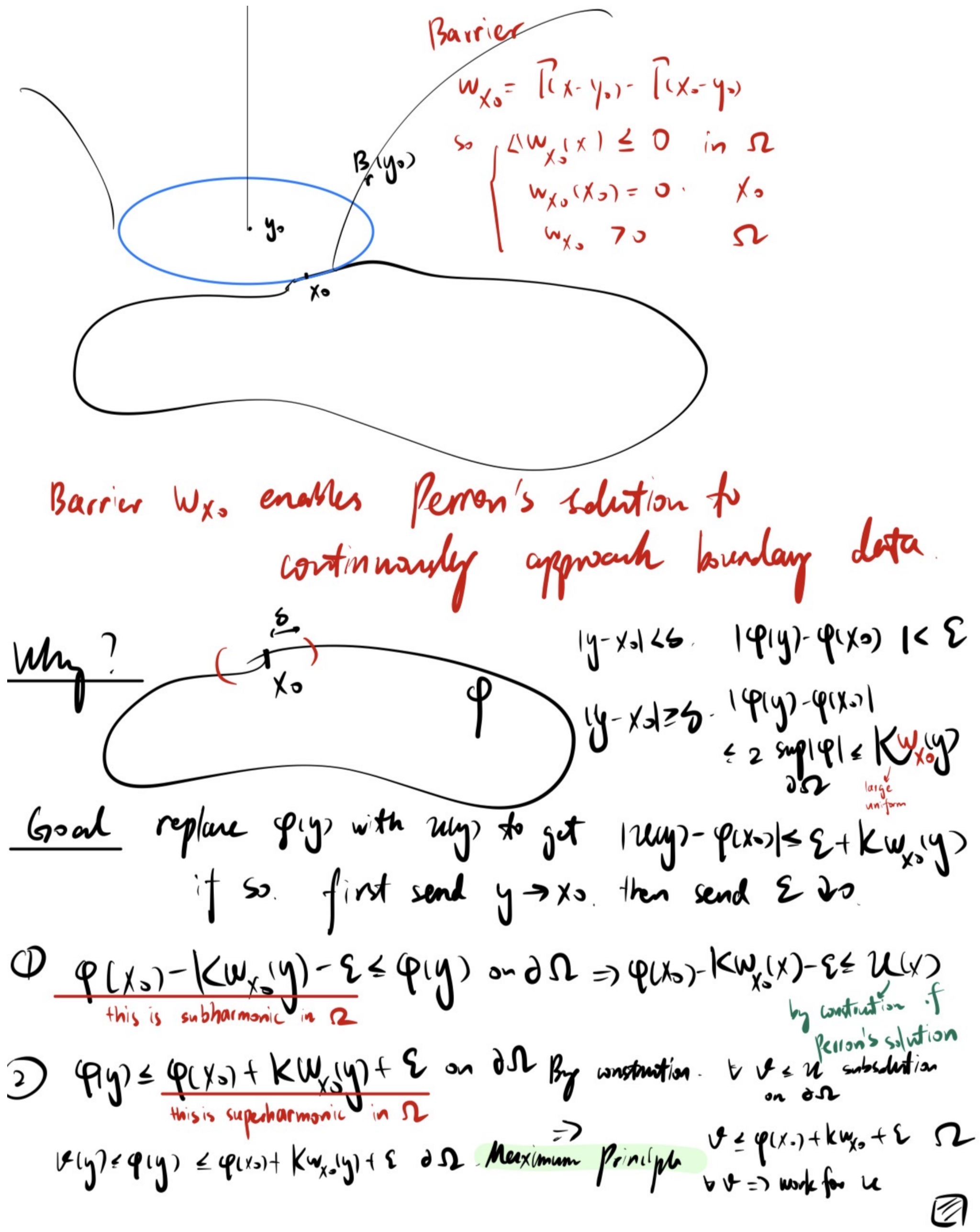


Figure 1.8: Barrier Function and Exterior Ball Condition

Let's see how the existence of a barrier function at a boundary point implies continuity up to the point.

Lemma 1.5.1 ([GT01] Lemma 2.13; [HL11] Lemma 6.5)**.** *Let $x_0 \in \partial\Omega$ be a regular point and w_{x_0} the barrier function at x_0. For $u \in C(\Omega)$ Perron's Solution defined as in (1.70) with $\varphi \in C(\partial\Omega)$ boundary data, one has*

$$\lim_{x \to x_0} |u(x) - \varphi(x_0)| = 0$$

Proof. Let's see what tools we have. First, since $\varphi \in C(\partial\Omega)$, for any $\varepsilon > 0$, there exists $\delta = \delta(x_0, \varepsilon) > 0$ s.t.

$$|y - x_0| < \delta \implies |\varphi(y) - \varphi(x_0)| < \varepsilon$$

Now for $|y - x_0| \geq \delta$ a positive distance away from x_0, we use $w_{x_0} > 0$ to ensure there exists $K = K(w_{x_0}, \delta) > 0$ large s.t.

$$|y - x_0| \geq \delta \implies K w_{x_0}(y) \geq 2 \sup_{\partial\Omega} |\varphi|$$

Thus globally over $\partial\Omega$

$$|\varphi(y) - \varphi(x_0)| \leq \varepsilon + K w_{x_0}(y) \qquad \forall\ y \in \partial\Omega$$

rewriting above yields

$$\varphi(x_0) - \varepsilon - K w_{x_0}(y) \leq \varphi(y) \leq \varphi(x_0) + \varepsilon + K w_{x_0}(y) \qquad \forall\ y \in \partial\Omega$$

1. Since $w_{x_0}(x)$ is weakly-superharmonic, $\varphi(x_0)-\varepsilon-Kw_{x_0}(x)$ is weakly-subharmonic in Ω. Moreover, using definition of S_φ (1.69)

$$\varphi(x_0)-\varepsilon-Kw_{x_0}(y)\leq\varphi(y) \qquad \text{on the boundary } y\in\partial\Omega$$

Thus

$$\varphi(x_0)-\varepsilon-Kw_{x_0}\in S_\varphi \qquad \text{is admissible}$$

Using definition of Perron's Solution (1.70)

$$\varphi(x_0)-\varepsilon-Kw_{x_0}(x)\leq u(x) \qquad \forall\ x\in\Omega$$

2. On the other hand $\varphi(x_0)+\varepsilon+Kw_{x_0}(x)\in C(\overline{\Omega})$ is weakly-superharmonic in Ω. For any $v\in S_\varphi$

$$v(y)\leq\varphi\leq\varphi(x_0)+\varepsilon+Kw_{x_0}(y) \qquad \forall\ y\in\partial\Omega$$

Then using weak version of Comparison Principle for Weakly Subharmonic and Weakly Superharmonic function Corollary 1.5.1 one has

$$v(x)\leq\varphi(x_0)+\varepsilon+Kw_{x_0}(x) \qquad \forall\ x\in\Omega$$

Notice this works uniformly for all $v\in S_\varphi$ thus

$$u(x)\leq\varphi(x_0)+\varepsilon+Kw_{x_0}(x) \qquad \forall\ x\in\Omega$$

Combining both sides, one obtain

$$|u(x)-\varphi(x_0)|\leq\varepsilon+Kw_{x_0}(x)$$

Letting $x\to x_0$ on both sides yields

$$\lim_{x\to x_0}|u(x)-\varphi(x_0)|\leq\varepsilon$$

Notice this works for any $\varepsilon>0$ hence the result follows. □

Solvability of Dirichlet Boundary Value Problem over Regular Domain What's more powerful is that, all boundary points being regular is equivalent to all BVP problems being solvable over the domain Ω.

Theorem 1.5.1 ([GT01] Theorem 2.14)**.** *Let $\Omega\subseteq\mathbb{R}^n$ be open bounded connected domain. The boundary value problem*

$$\begin{cases}\Delta u=0 & \Omega\\ u=\varphi & \partial\Omega\end{cases}$$

is solvable (i.e., $u\in C^\infty(\Omega)\cap C^0(\overline{\Omega})$ solution exists) for all boundary data $\varphi\in C(\Omega)$ iff all points of $\partial\Omega$ are regular.

Proof. Assume all points are regular. Using Proposition 1.5.1 u is harmonic in Ω, and using Lemma 1.5.1, u approaches the boundary data continuously at all boundary points. Thus the BVP is solable for any boundary data $\varphi\in C(\partial\Omega)$. On the other hand, assume BVP is solvable for all continuous boundary data φ. Then for any $x_0\in\partial\Omega$, one take

$$\varphi(x):=|x-x_0| \qquad \forall\ x\in\overline{\Omega}$$

Indeed

1. One solve the Boundary Value problem with boundary data φ, then the solution $u_\varphi\in C^\infty(\Omega)\cap C^0(\overline{\Omega})$ is harmonic in Ω, in particular, $u\in C^0(\overline{\Omega})$ is weakly-superharmonic.
2. The boundary data $\varphi(x)>0$ for any $x\in\overline{\Omega}\setminus\{x_0\}$ and $\varphi(x_0)=0$.

Hence u_φ is a barrier at the point x_0, and by definition, x_0 is regular. □

Exterior Ball Condition It suffices to determine what domains have regular boundary. We give a sufficient condition that builds up a barrier for free: the exterior ball condition.

Definition 1.5.4. *A bounded open connected domain $\Omega \subseteq \mathbb{R}^n$ satisfies the exterior ball condition if for any $x_0 \in \partial\Omega$, there exists exterior ball $B_r(y_0) \subseteq \mathbb{R}^n$ s.t.*

$$\Omega \cap B_r(y_0) = \varnothing \qquad \overline{\Omega} \cap \overline{B_r(y_0)} = \{x_0\}$$

Now given a domain with exterior ball condition, and given $x_0 \in \partial\Omega$, $B_r(y_0)$ as above, we claim the function

$$w_{x_0}(x) := \Gamma(x - y_0) - \Gamma(x_0 - y_0) \qquad \forall\, x \in \overline{\Omega}$$

is a barrier function at x_0. Indeed

1. $\Delta w_{x_0}(x) = \Delta_x \Gamma(x - y_0) = 0$ for any $x \neq y_0$. Since $y \notin \overline{\Omega}$, this is sufficient, and thus $w_{x_0} \in C(\overline{\Omega})$ is weakly superharmonic.
2. Notice for any $x \in \overline{\Omega} \setminus \{x_0\}$

$$\Gamma(x - y_0) = \Gamma(|x - y_0|) > \Gamma(|x_0 - y_0|) = \Gamma(x_0 - y_0) \iff w_{x_0}(x) > 0$$

   and

$$w_{x_0}(x_0) = \Gamma(x_0 - y_0) - \Gamma(x_0 - y_0) = 0$$

Hence domains with exterior ball condition necessarily satisfies Theorem 1.5.1.

C^2 Domains satisfy Exterior Ball Condition In particular, all C^2 boundaries are domains with exterior ball condition.

Remark 1.5.1 ([GT01] Problem 2.11)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be bounded, open, connected domain with $\partial\Omega$ C^2, i.e., for any $x_0 \in \partial\Omega$ there exists $r > 0$ and $g \in C^2(\mathbb{R}^{n-1})$ s.t. upon relabelling and reorienting*

$$\Omega \cap B_r(x_0) = \{x \in B_r(x_0) \mid x_n > g(x_1, \cdots, x_{n-1})\}$$

Then Ω satisfies the exterior ball condition.

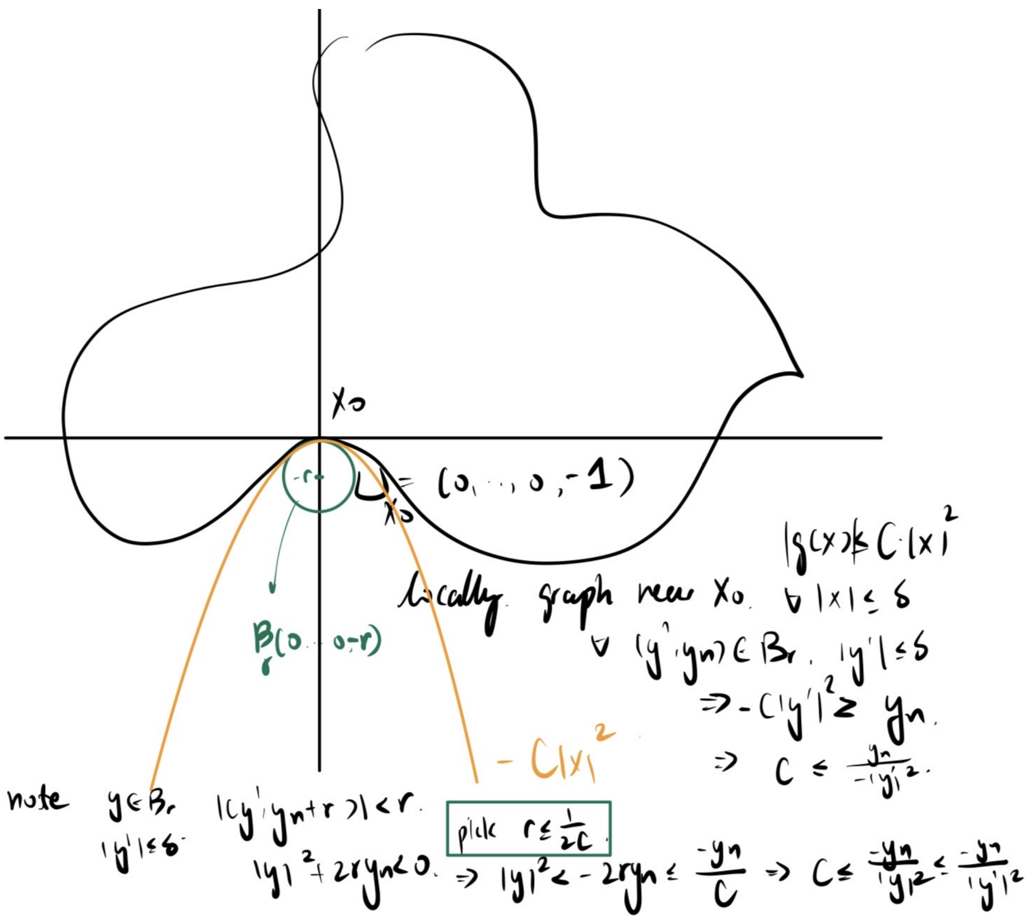


Figure 1.9: C^2 domain satisfies Exterior Ball Condition

Proof. Upon translation we take $x_0 = (0, \cdots, 0) = (x_0', 0)$ origin so that $g(x_0') = 0$. Since boundary is C^2, one has outer-unit normal defined $\nu \in C^1(\partial\Omega; \mathbb{R}^n)$. Upon orienting, take $\nu(x_0) = (0, \cdots, 0, -1)$. In particular, using explicit expression of ν one has

$$(0, \cdots, 0, -1) = \nu(x_0) = \frac{1}{\sqrt{|\nabla g|^2 + 1}}(\partial_1 g, \cdots, \partial_{n-1} g, -1)\Big|_{x = x_0'} \implies \nabla g(x_0') = 0 \in \mathbb{R}^{n-1}$$

Now since $g \in C^2$, the Hessian of g at x_0 is also continuous. In particular using Taylor, for any $x \in \mathbb{R}^{n-1}$ s.t. $|x - x_0'| = |x| \leq \delta$ with $\delta > 0$ small

$$g(x) = g(x_0') + x \cdot \nabla g(x_0') + \frac{1}{2} x^T D^2 g(x_0') x + o(\delta^2)$$

With our cboice of $g(x_0') = 0$ and $\nabla g(x_0') = 0$ one has

$$|g(x)| \leq \frac{1}{2} |D^2 g(x_0')||x|^2 + o(|x|^2)$$

Since $g \in C^2$, all second derivatives commute, hence the hessian matrix

$$D^2 g(x_0')$$

is symmetric. A symmetric matrix is diagonalizable, hence there exists real eigenvalues $\lambda_1, \cdots, \lambda_{n-1} \in \mathbb{R}$ of $D^2 g(x_0')$. Now one can choose $C > \max\{\lambda_1, \cdots, \lambda_{n-1}\}$ so that

$$|D^2 g(x_0')| \leq \max\{\lambda_1, \cdots, \lambda_{n-1}\} < C$$

Thus there exists $\delta = \delta(C) > 0$ small so that for any $|x| \leq \delta$

$$|g(x)| \leq C|x|^2$$

Why do we want a control on $|g|$? This is because we want to pick a $r > 0$ s.t. the ball

$$B_r((0, \cdots, 0, -r)) = B_r$$

satisfies $B_r \cap \{0\} = \{0\}$ while $B_r \subseteq \Omega^c$.

Now for the three objects

$$\text{graph of } g \text{ over } |x| \leq \delta, \qquad \{-C|x|^2 \mid |x| \leq \delta\}, \qquad B_r = B_r(0, \cdots, 0, -r)$$

How do we ensure the parabola lies on top of the ball? We want to ensure that for any $y = (y', y_n) \in B_r$ with $|y'| \leq \delta$, necessarily

$$-C|y'|^2 \geq y_n \tag{1.72}$$

Notice $y \in B_r$ with $|y'| \leq \delta$ yields

$$|(y', y_n + r)| < r$$
$$|y'|^2 + y_n^2 + 2ry_n < 0$$

Ah! We pick

$$r \leq \frac{1}{2C}$$

so that the above implies

$$\begin{aligned} |y'|^2 + y_n^2 &< -2ry_n \\ &\leq -\frac{1}{C} y_n \\ C|y'|^2 \leq C|y|^2 &\leq -y_n \end{aligned}$$

Hence such choice of r ensures (1.72) is satisfied. Thus for

$$r < \min\{\frac{1}{2C}, \delta\}$$

The ball $B_r((0, \cdots, 0, -r))$ touches $\partial\Omega$ only at the point $x_0 = 0 \in \mathbb{R}^n$ and $B_r \cap \Omega = \varnothing$. □

1.6 H^1 Theory for Laplace Equation

In previous sections, we defined the Laplace operator in the classical sense, i.e., for $u \in C^2$, we defined

$$\Delta u = \sum_{i=1}^{n} \partial_{ii} u$$

Yet one can understand the Laplace Operator via the weak sense, which naturally arises from the Calculus of Variations. A ground rule is that, PDE comes from Variational Problems, and solutions to PDEs are critical points for the energies.

1.6.1 Dirichlet Problem in H^1

1. Let $\Omega \subseteq \mathbb{R}^n$ be any bounded open domain (no assumption on regularity at the moment), and consider any function g defined on $\partial\Omega$. Let's think about the problem

$$\text{Minimize} \qquad \mathcal{E}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx \qquad \text{among functions } u \text{ s.t. } u = g \text{ on } \partial\Omega$$

Notice to make sense of the Minimization Problem above, one only need to specify order-one differentiability of u, and moreover, the derivative suffices to be defined in an integral sense s.t. ∇u is L^2. This allows us to choose

$$u \in H^1(\Omega) = W^{1,2}(\Omega) \qquad \text{as our function space to work with}$$

On the other hand, one can interpret $u = g$ in the trace sense, hence it allows us to pick $g \in L^2(\partial\Omega)$. Remark that the Trace theory requires $\partial\Omega$ with certain regularity. C^1 is preferable, but usually $\partial\Omega$ Lipschitz suffices.

Thus one can write formally our Minimization problem (1.74). Given $\Omega \subseteq \mathbb{R}^n$ open, bounded domain with $\partial\Omega$ Lipschitz. Given $g \in L^2(\partial\Omega)$. We may define our admissible set (the set among which we minimize our energy)

$$\mathcal{A} := \{u \in H^1(\Omega) \mid u = g \text{ in the trace sense}\} \tag{1.73}$$

The minimization problem seeks for

$$u_0 \in \mathcal{A} \qquad \text{s.t.} \qquad \mathcal{E}(u_0) = \inf\{\mathcal{E}(u) \mid u \in \mathcal{A}\} \tag{1.74}$$

2. What are some necessary conditions for a minimizer u to (1.74)?

For any $v \in C_0^\infty(\Omega)$ compactly supported within Ω, consider $u + \varepsilon v$ for $\varepsilon > 0$ small. Since v is compactly support, this does not touch boundary data g, thus $u + \varepsilon v$ is valid competitor. Using that u is minimizer, one shall obtain

$$\begin{aligned} \mathcal{E}(u) &\le \mathcal{E}(u + \varepsilon v) \\ \int_\Omega |\nabla u|^2 &\le \int_\Omega |\nabla(u + \varepsilon v)|^2 = \int_\Omega |\nabla u|^2 + 2\varepsilon \nabla u \cdot \nabla v + \varepsilon^2 |\nabla v|^2 \\ -\frac{\varepsilon}{2}\int_\Omega |\nabla v|^2 &\le \int_\Omega \nabla u \cdot \nabla v \end{aligned}$$

Taking $\varepsilon \to 0$, this amounts to say

$$0 \le \int_\Omega \nabla u \cdot \nabla v$$

On the other hand, choosing $u - \varepsilon v$ a downward perturbation yields the other side, i.e.

$$0 \ge \int_\Omega \nabla u \cdot \nabla v$$

Thus a minimizer u to (1.74) necessarily satisfies

$$\int_\Omega \nabla u \cdot \nabla v = 0 \qquad \forall\, v \in C_0^\infty(\Omega) \tag{1.75}$$

Assume for now $u \in C^2(\Omega)$. Then by Green's First identity (1.6) (essentially Integration by Parts)

$$0 = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} dS - \int_\Omega v\Delta u = -\int_\Omega v \Delta u \qquad \text{using } v \in C_0^\infty \text{ compactly supported}$$

Hence assuming for u enough regularity, u as a minimizer to (1.74) necessarily solves

$$\int_\Omega v\Delta u = 0 \qquad \forall\, v \in C_0^\infty(\Omega) \tag{1.76}$$

which amounts to say $\Delta u = 0$ in Ω a.e. via the fundamental theorem of Calculus of Variations ([FRRO22] Corollary 1.2).

3. In view of (1.76), we understand Minimizers to (1.74) as 'weak solutions' to some Dirichlet Boundary Value Problem.

   Notice $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, which is the dual of $H^1(\Omega)$ for Ω bounded. Thus one is allowed to pass to the limit and relax (1.75) to $H_0^1(\Omega)$.

   Definition 1.6.1 (Weak Solution to Laplace BVP; [FRRO22] Definition 1.9)**.** *Given $\Omega \subseteq \mathbb{R}^n$ open, bounded domain with $\partial\Omega$ Lipschitz. Given $g \in L^2(\partial\Omega)$.*

   We say u is weak solution to the Dirichlet Boundary Value Problem

$$\begin{cases} \Delta u = 0 & \Omega \\ u = g & \partial\Omega \end{cases}$$

   if $u \in \mathcal{A}$ as in (1.73), and u satisfies

$$\int_\Omega \nabla u \cdot \nabla v = 0 \qquad \forall\, v \in H_0^1(\Omega) \tag{1.77}$$

4. More generally, one may consider a forced Minimization problem. Given $\Omega \subseteq \mathbb{R}^n$ bounded, open, Lipschitz domain, and $g \in L^2(\partial\Omega)$ as before. Moreover, let $f \in L^2(\Omega)$.

$$\text{Minimize} \qquad \mathcal{E}_f(u) := \int_\Omega \frac{1}{2}|\nabla u|^2 - fu\,dx \qquad \text{among functions } u \in \mathcal{A} \tag{1.78}$$

   Following a similar procedure, assume u as minimizer to (1.78), compute for any $v \in C_0^\infty(\Omega)$

$$\begin{aligned} \lim_{\varepsilon\to 0} \frac{\mathcal{E}_f(u+\varepsilon v) - \mathcal{E}_f(u)}{\varepsilon} &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\int_\Omega \varepsilon\nabla u \cdot \nabla v + \frac{\varepsilon^2}{2}|\nabla v|^2 - \varepsilon v f\right) \\ &= \int_\Omega \nabla u \cdot \nabla v - vf \end{aligned}$$

   Thus u minimizer to (1.78) necessarily satisfies

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega vf \qquad \forall\, v \in C_0^\infty(\Omega)$$

   Again, assuming $u \in C^2$ and doing Integration by parts for the LHS, one obtain

$$-\int_\Omega \Delta u v = \int_\Omega fv \qquad \forall\, v \in C_0^\infty(\Omega)$$

   which amounts to say $-\Delta u = f$ in Ω a.e.

   Definition 1.6.2 (Weak Solution to Poisson BVP; [FRRO22] Definition 1.9)**.** *Given $\Omega \subseteq \mathbb{R}^n$ open, bounded domain with $\partial\Omega$ Lipschitz. Given $g \in L^2(\partial\Omega)$. Given $f \in L^2(\Omega)$.*

   We say u is weak solution to the Dirichlet Boundary Value Problem

$$\begin{cases} -\Delta u = f & \Omega \\ u = g & \partial\Omega \end{cases}$$

   if $u \in \mathcal{A}$ as in (1.73), and u satisfies

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv \qquad \forall\, v \in H_0^1(\Omega) \tag{1.79}$$

   Notice for convention, we write $-\Delta u = f$ because then in the weak form (1.79) we have positive signs on both sides.

Existence and Uniqueness (Calculus of Variations) As a first thing, to ensure it makes sense to discuss weak solutions (1.77) and (1.79), one need to show existence of minimizers to the Minimization problems (1.74), (1.78), which are indeed 'weak solutions' as we've discussed. We demonstrate the proof via Calculus of Variations.

We do for the general case with the forced energy.

Theorem 1.6.1 ([FRRO22] Theorem 1.10)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be bounded, open, and Lipschitz. Let $g \in L^2(\partial\Omega)$ and $f \in L^2(\Omega)$. Also assume $\mathcal{A} \neq \varnothing$. Then there exists a unique minimizer $u_0 \in \mathcal{A}$ as in (1.73) s.t.*

$$\mathcal{E}_f(u_0) = \inf \left\{ \int_\Omega \frac{1}{2}|\nabla u|^2 - fu\, dx \mid u \in \mathcal{A} \right\}$$

Proof. 1. First we show existence. Since we've assumed $\mathcal{A} \neq \varnothing$, pick any $w \in \mathcal{A}$. Then we compute

$$\begin{aligned} \mathcal{E}_f(w) &= \int_\Omega \frac{1}{2}|\nabla w|^2 - fw \\ |\mathcal{E}_f(w)| &\leq \int_\Omega \frac{1}{2}|\nabla w|^2 + \|f\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} < \infty \end{aligned}$$

Thus

$$\inf \left\{ \int_\Omega \frac{1}{2}|\nabla u|^2 - fu\, dx \mid u \in \mathcal{A} \right\} \leq \mathcal{E}_f(w) < \infty$$

and hence we're allowed to take minimizing sequences using the definition of infimum. We take

$$\{u_k\} \subseteq \mathcal{A} \qquad \mathcal{E}_f(u_k) \leq \mathcal{E}_f(w) \qquad \forall\ k, \qquad \lim_{k\to\infty} \mathcal{E}_f(u_k) = \inf \left\{ \int_\Omega \frac{1}{2}|\nabla u|^2 - fu\, dx \mid u \in \mathcal{A} \right\}$$

A common procedure in showing the limit exists is by passing to a weak limit via compactness. We need uniform bounds of $\{u_k\}$ in H^1. First observe via Poincaré and Trace Inequality

$$\begin{aligned} \int_\Omega |u_k|^2 dx &\leq C(n,\Omega) \left(\int_\Omega \frac{1}{2}|\nabla u_k|^2 + \int_{\partial\Omega} |g|^2 d\mathcal{H}^{n-1} \right) \\ &\leq C(n,\Omega) \left(\int_\Omega \frac{1}{2}|\nabla u_k|^2 - u_k f + \int_\Omega u_k f + \|g\|^2_{L^2(\partial\Omega)} \right) \\ &\leq C(n,\Omega) \left(\mathcal{E}_f(w) + \frac{\varepsilon}{2} \int_\Omega |u_k|^2 + \frac{1}{2\varepsilon} \int_\Omega |f|^2 + \|g\|^2_{L^2(\partial\Omega)} \right) \qquad \text{using Young's} \\ \int_\Omega |u_k|^2 dx &\leq C(n,\Omega) \left(\mathcal{E}_f(w) + \|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\partial\Omega)} \right) \qquad \text{choosing } \varepsilon = \varepsilon(n,\Omega) \text{ small and absorb} \end{aligned} \tag{1.80}$$

On the other hand

$$\begin{aligned} \int_\Omega \frac{1}{2}|\nabla u_k|^2 &= \int_\Omega \frac{1}{2}|\nabla u_k|^2 - u_k f + \int_\Omega u_k f \\ &\leq \mathcal{E}_f(w) + \|u_k\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)} \\ &\overset{(1.80)}{\leq} C(n,\Omega) \left(\mathcal{E}_f(w) + \|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\partial\Omega)} \right) \end{aligned}$$

Thus we have the sequence u_k uniformly bounded in H^1.

(a) Using Banach Alaoglu, unit balls in weak* topology are compact, hence there exists $u_0 \in H^1(\Omega)$ s.t. $u_k \to u_0$ weakly in H^1 up to subsequence. Due to the nature of weak limit

$$\|u_0\|_{H^1(\Omega)} \leq \liminf_{k\to\infty} \|u_k\|_{H^1(\Omega)}$$

(b) Using Rellich, $W^{1,2} \Subset L^2$, passing to another subsequence $u_k \to u_0$ strongly in L^2.

Now using Trace Inequality

$$\begin{aligned} \|u_k - u_0\|_{L^2(\partial\Omega)} C(n,\Omega) &\left(\|u_k - u_0\|_{L^2(\Omega)} + \|\nabla u_k - \nabla u_0\|_{L^2(\Omega)} \right) \\ &\leq C(n,\Omega,f,g,\mathcal{E}_f(w)) \qquad \forall\ k \end{aligned}$$

Since $\{u_k - u_0\}$ is uniformly bounded sequence in $L^2(\partial\Omega)$, by Banach Alaoglu again, up to another subsequence, $u_k \to u_0$ weakly in $L^2(\partial\Omega)$. But $u_k = g$ on $\partial\Omega$ for any k since $u_k \in \mathcal{A}$, one obtain

$$u_0 = g \qquad \partial\Omega$$

And thus $u_0 \in \mathcal{A}$ is admissible. Now

$$\begin{aligned}\mathcal{E}_f(u_0) &\le \liminf_{k\to\infty} \int_\Omega \frac{1}{2}|\nabla u_k|^2 - \lim_{k\to\infty}\int_\Omega f u_k \qquad \text{using weak limit in } H^1\text{, and strong limit in } L^2 \\ &\le \lim_{k\to\infty} \mathcal{E}_f(u_k) = \inf\left\{\int_\Omega \frac{1}{2}|\nabla u|^2 - f u\,dx \mid u \in \mathcal{A}\right\}\end{aligned}$$

Thus u_0 is minimizer to (1.78).

2. We then show Uniqueness. For any $v \in H_0^1(\Omega)$, consider $u + v$

$$\begin{aligned}\mathcal{E}_f(u_0 + v) &= \int_\Omega \frac{1}{2}|\nabla u_0|^2 + \nabla u_0 \cdot \nabla v + \frac{1}{2}|\nabla v|^2 - f u_0 - f v \\ &= \mathcal{E}_f(u_0) + \int_\Omega \frac{1}{2}|\nabla v|^2 + \int_\Omega \nabla u_0 \cdot \nabla v - f v \\ &\overset{(1.79)}{=} \mathcal{E}_f(u_0) + \int_\Omega \frac{1}{2}|\nabla v|^2\end{aligned}$$

If $v \not\equiv 0$, $\int |\nabla v|^2 \ne 0$ due to $v \in H_0^1$ (we identify a.e.), thus

$$\mathcal{E}_f(u_0 + v) > \mathcal{E}_f(u_0)$$

gives strict inequality so $u_0 + v$ is not minimizer. Consequently u_0 is unique.

□

One shall remark that, given $g \in L^2(\partial\Omega)$, the condition $\mathcal{A} \ne \varnothing$ is not guaranteed.

Remark 1.6.1 ([FRRO22] Remark 1.11)**.** *Given $\Omega \subseteq \mathbb{R}^n$ bounded open, and Lipschitz, $\mathcal{A} \ne \varnothing$ iff*

$$\int_{\partial\Omega}\int_{\partial\Omega} \frac{|g(x) - g(y)|^2}{|x - y|^{n+1}}\,dxdy < \infty$$

Existence and Uniqueness (Functional Analysis) On the other hand, one is allowed to forget totally about the underlying Minimization problem and merely look at the weak solution defined as in Definition 1.6.2. We demonstrate an alternative proof using Functional Analysis, in particular, Riesz Representation, which works for zero boundary data.

Theorem 1.6.2 (Weinstein Analytic Methods for PDEs 2024)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be bounded, open and Lipschitz. Let $f \in L^2(\Omega)$. Then there exists unique weak solution $u_0 \in H_0^1(\Omega)$ to the Dirichlet Boundary Value Problem*

$$\begin{cases} -\Delta u = f & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

as in Definition 1.6.2, i.e.

$$\int_\Omega \nabla u_0 \cdot \nabla v = \int_\Omega f v \qquad \forall\ v \in H_0^1(\Omega)$$

Proof. 1. Recall H^1 with the classic inner product

$$(u, v)_{H^1} := \int_\Omega uv + \nabla u \cdot \nabla v$$

defines a Hilbert space. $H_0^1(\Omega) \subseteq H^1(\Omega)$ as a subspace indeed inherits the same inner product, and due to $H_0^1(\Omega)$ is closed w.r.t. norm induced by $(\cdot,\cdot)_{H^1}$, $(H_0^1(\Omega), (\cdot,\cdot)_{H^1})$ again defines a Hilbert space.

The point in this proof is that, one can define another inner product

$$(u, v)_{H_0^1} := \int_\Omega \nabla u \cdot \nabla v \qquad \forall\ u,\, v \in H_0^1(\Omega)$$

on H_0^1, which turns out to be equivalent to the classic one. In the first step we see why they induce equivalent norms, i.e.

$$(H_0^1(\Omega), \|\cdot\|_{H^1}) \cong (H_0^1(\Omega), \|\cdot\|_{H_0^1})$$

To do so, we simply use Poincaré. For any $u \in H_0^1(\Omega)$

$$\|u\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla u|^2 \le \int_\Omega |u|^2 + |\nabla u|^2 = \|u\|^2_{H^1(\Omega)}$$
$$\|u\|^2_{H^1(\Omega)} = \int_\Omega |u|^2 + |\nabla u|^2 \le C(n,\Omega) \int_\Omega |\nabla u|^2 \le C(n,\Omega) \|u\|^2_{H_0^1(\Omega)}$$

Hence $(H_0^1(\Omega), \|\cdot\|_{H_0^1})$ has the same convergence in norm as a Hilbert space, which is to say, it is a Hilbert space itself.

2. The reason why one wish $(H_0^1(\Omega), \|\cdot\|_{H_0^1})$ defines a Hilbert space is to apply Riesz Representation. Consider the map

$$T_f : (H_0^1(\Omega), \|\cdot\|_{H_0^1}) \to \mathbb{R}$$
$$v \mapsto \int_\Omega fv$$

T_f is indeed linear. To see T_f is bounded, one check

$$\begin{aligned} |T_f(v)| &= |\int_\Omega fv| \le \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\le C(n,\Omega) \|f\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} && \text{using Poincaré again} \\ &= C(n,\Omega) \|f\|_{L^2(\Omega)} \|v\|_{H_0^1(\Omega)} && \text{by definition of our new inner product} \end{aligned}$$

Thus the operator norm is bounded for any $f \in L^2(\Omega)$

$$\|T_f\| = \sup_{v \in H_0^1(\Omega), \|v\|_{H_0^1(\Omega)}=1} |T_f(v)| \le C(n,\Omega) \|f\|_{L^2(\Omega)}$$

and one obtain a bounded linear operator.

Now using Riesz Representation, there exists a unique element $u_0 \in H_0^1(\Omega)$ s.t.

$$T_f(v) = (u_0, v)_{H_0^1} \qquad \forall\ v \in H_0^1(\Omega)$$

i.e. (1.79) is satisfied

$$\int_\Omega fv = \int_\Omega \nabla u_0 \cdot \nabla v \qquad \forall\ v \in H_0^1(\Omega)$$

□

Solvability of Dirichlet Problem in H^1

Theorem 1.6.3. *There exists a unique $u \in H^1(B_1)$ for any given $\varphi \in H^1(B_1)$ as boundary data*

$$\Delta u = f + \partial_i g_i \qquad B_1, \qquad u - \varphi \in H_0^1(B_1) \tag{1.81}$$

and $f, g_i \in L^2(B_1)$.

Moreover

$$\|u\|_{H^1} \le C(n) \left(\|f\|_{L^2} + \sum_i \|g_i\|_{L^2} + \|\varphi\|_{H^1} \right)$$

By definition of weak solutions, u solves (1.81) if

$$\int_{B_1} \nabla u \cdot \nabla v + \int_{B_1} fv - \int_{B_1} g \cdot \nabla v = 0 \qquad \forall\ v \in H_0^1(B_1)$$

We let the RHS be in H^1 and they act on H_0^1 as

$$\langle f + \text{div} g, v\rangle := \int fv - g \cdot \nabla v \qquad \forall\ v \in H_0^1$$

Immediately notice $f + \text{div} g$ is a bounded linear functional on H_0^1. So $H^{-1} = (H_0^1)^*$ and

$$\|f + \text{div} g\|_{H^{-1}} \le \|f\|_{L^2} + \|g\|_{L^2}$$

Fredholm Alternative We've seen that

$$\begin{aligned} \Delta : H_0^1 &\to H^{-1} \\ u &\mapsto \Delta u \end{aligned}$$

is a linear isomorphism. Thus the inverse

$$\begin{aligned} \Delta^{-1} : H^{-1} &\to H_0^1 \\ f + \mathrm{div} g &\mapsto \Delta^{-1}(f + \mathrm{div} g) \end{aligned}$$

is bounded.

Moreover, Since H_0^1 is compactly embedded in L^2

$$H_0^1 \Subset L^2 \subsetneq H^{-1}$$

We see that

$$\Delta^{-1} : H^{-1} \to H^{-1}$$

and the restrictions of Δ^{-1}

$$\Delta^{-1} : L^2 \to L^2$$

are compact linear operators.

But what happens when we have a zero order term?

Now consider the eigenvalue problem for $f \in L^2$

$$\Delta u + \lambda u = f, \qquad u \in H_0^1$$

If we denote the compact linear self-adjoint operator

$$T = \Delta^{-1} : L^2 \to L^2$$

Then applying T to both sides of the equation yields

$$u + \lambda T u = T f$$

We check T is self-adjoint.

Proof. For any f, $g \in L^2$, there exists unique u_f, $u_g \in H_0^1$ s.t.

$$\begin{aligned} \int \nabla u_f \cdot \nabla v &= \int f v \qquad \forall\ v \in H_0^1 \\ \int \nabla u_g \cdot \nabla v &= \int g v \qquad \forall\ v \in H_0^1 \end{aligned}$$

Now

$$\begin{aligned} \langle \Delta^{-1} f, g \rangle &= \int u_f g = \int \nabla u_f \cdot \nabla u_g \\ &= \int u_g f = \langle f, \Delta^{-1} g \rangle \end{aligned}$$

□

This is solvable iff

$$T f \in \mathrm{Range}(I + \lambda T) = \ker(I + \lambda T)^{\perp}$$

i.e.

$$\int T f v dx = 0 \qquad \forall\ v \in V_\lambda$$

where

$$V_\lambda := \{ v \in H_0^1 \mid \Delta v + \lambda v = 0 \}$$

The eigenspace V_λ is finite dimensional and is nontrivial only for a sequence of $\lambda \to \infty$.

1.6.2 Caccioppoli

Theorem 1.6.4 (De Silva Analysis Ii Spring 2025). *For*

$$\Delta u = f + \text{div} g \qquad B_1$$

One has

$$\|\nabla u\|_{L^2(B_{1/2})} \leq C\left(\|u\|_{L^2(B_1)} + \|f\|_{L^2(B_1)} + \|g\|_{L^2(B_1)}\right)$$

Proof. When we deal with weak solutions, we always come up with smart test functions. (In the classical solutions we build barriers).

$$-\int_{B_1} \nabla u \cdot \nabla \varphi = \int_{B_1} f\varphi - g \cdot \nabla \varphi$$

$$\varphi := \eta^2 u \qquad \text{for } \eta \in C_0^\infty(B_1) \text{ s.t. } \eta = 1 \text{ on } B_{1/2}$$

We plug in and obtain

$$-\int_{B_1} \nabla u \cdot \nabla(\eta^2 u) = \int f\eta^2 u - g \cdot \nabla(\eta^2 u)$$

$$-\int |\nabla u|^2\eta^2 - 2\int \eta u \nabla u \cdot \nabla \eta = \int f\eta^2 u - \int (\eta g \cdot \nabla(\eta u) + \eta u g \cdot \nabla \eta)$$

On the other hand notice by product rule

$$\int |\nabla(\eta u)|^2 = \int |(\nabla \eta)u + \eta \nabla u|^2 = \int u^2|\nabla \eta|^2 + \eta^2|\nabla u|^2 + 2u\eta\nabla u \cdot \nabla \eta$$

Thus one can rewrite the LHS and equate with the RHS

$$\begin{aligned} -\int |\nabla u|^2\eta^2 - 2\int \eta u \nabla u \cdot \nabla \eta &= -\int |\nabla(\eta u)|^2 + \int u^2|\nabla \eta|^2 \\ &= \int f\eta^2 u - \int (\eta g \cdot \nabla(\eta u) + \eta u g \cdot \nabla \eta) \end{aligned}$$

Then

$$\begin{aligned} \int |\nabla(\eta u)|^2 &= \int u^2|\nabla \eta|^2 - \int f\eta^2 u - \eta g \cdot \nabla(\eta u) - \eta u g \cdot \nabla \eta \\ &\leq \int u^2|\nabla \eta|^2 + C\left(\int f^2\eta^2 + \int \eta^2 u^2\right) \\ &\quad + \left(\frac{C}{\varepsilon}\int \eta^2 g^2 + \varepsilon \int |\nabla(\eta u)|^2\right) + C\left(\int \eta^2 u^2 + \int g^2|\nabla \eta|^2\right) \end{aligned}$$

$$\int_{B_1} |\nabla(\eta u)|^2 \leq C\left(\|u\|_{L^2}^2 + \|f\|_{L^2}^2 + \|g\|_{L^2}^2\right)$$

□

One also has Caccioppoli for boundary version.

Theorem 1.6.5. *For*

$$\Delta u = f + \text{div} g \qquad B_1^+$$

with

$$u = 0 \qquad \{x_n = 0\}$$

One has

$$\|\nabla u\|_{L^2(B_{1/2}^+)} \leq C\left(\|u\|_{L^2(B_1^+)} + \|f\|_{L^2(B_1^+)} + \|g\|_{L^2(B_1^+)}\right)$$

1.6.3 Higher Regularity

Theorem 1.6.6. *Let u solve weakly for $f \in L^2$*

$$\Delta u = f \qquad B_1$$

Then $u \in H^2_{loc}(B_1)$ and

$$\|u\|_{H^2(B_{1/2})} \leq C\left(\|u\|_{L^2(B_1)} + \|f\|_{L^2(B_1)}\right)$$

If moreover $f \in H^k$ then $u \in H^{k+2}_{loc}$ and

$$\|u\|_{H^{k+2}(B_{1/2})} \leq C\left(\|u\|_{L^2(B_1)} + \|f\|_{H^k(B_1)}\right)$$

Proof for H^2. **Discrete Integration by Parts**. We define

$$\nabla_k^h u(x) := \frac{1}{h}(u(x+he_k) - u(x))$$

Thus we show for any $\phi \in H^1(\Omega')$ with $\Omega' \Subset B_1$

$$\begin{aligned}\int_{\Omega'} \nabla_k^h u(x)\phi(x) &= \frac{1}{h}\int u(x+he_k)\phi(x) - \frac{1}{h}\int u(x)\phi(x) \\ &= \frac{1}{h}\int u(x)\phi(x-he_k) - \frac{1}{h}\int u(x)\phi(x) \\ &= \int u(x)\frac{1}{h}(\phi(x-he_k) - \phi(x)) = -\int_{\Omega'} u(x)\nabla_k^{-h}\phi(x)\end{aligned}$$

Work for $\nabla_k^{-h} v$ test function. Assume for any $v \in H_0^1$

$$\begin{aligned}\int_{\Omega'} \nabla u \cdot \nabla\nabla_h^{-k} v + \int_{\Omega'} fu = 0 \\ -\int_{\Omega'} \nabla\nabla_h^k u \cdot \nabla v + \int_{\Omega'} fu = 0\end{aligned}$$

Pick $v = \eta^2\nabla_k^h u$. Let $\eta \in C_0^\infty$ with $\eta = 1$ in $B_{1/2}$ but $\eta = 0$ outside $B_{3/4}$, and $0 \le \eta \le 1$. Compute

$$\begin{aligned}\int_{\Omega'} \nabla\nabla_h^k u \cdot \nabla v &= \int_{\Omega'} \nabla\nabla_h^k u \cdot \nabla(\eta^2\nabla_k^h u) \\ &= \int 2\eta\nabla\nabla_h^k u \cdot \nabla\eta\nabla_k^h u + \eta^2|\nabla\nabla_k^h u|^2\end{aligned}$$

Plugging back yields

$$\begin{aligned}0 &= -\int 2\eta\nabla\nabla_h^k u \cdot \nabla\eta\nabla_k^h u - \int \eta^2|\nabla\nabla_k^h u|^2 + \int_{\Omega'} fu \\ \int \eta^2|\nabla\nabla_k^h u|^2 &= \int 2\eta\nabla\nabla_h^k u \cdot \nabla\eta\nabla_k^h u - \int_{\Omega'} fu \\ &\le \varepsilon\int \eta^2|\nabla\nabla_k^h u|^2 + C(\varepsilon)\int |\nabla\eta|^2|\nabla_k^h u|^2 + \int f^2 + \int u^2 \\ \int \eta^2|\nabla\nabla_k^h u|^2 &\le C\left(\int_{B_{3/4}} |\nabla_k^h u|^2 + \int f^2 + \int u^2\right) \\ &\le C\left(\int u^2 + \int f^2\right)\end{aligned}$$

where the last step uses Caccioppoli. Now send $h \to 0$. □

Proof for Higher Order Derivatives. Assume the result holds for $k \in \mathbb{N}$. Then we already have interior H^{k+2} regularity of u. Then we observe what $\partial_i^{k+1} u$ solves. Note for any $\phi \in H_0^1$

$$\begin{aligned}\int \nabla\partial_i^{k+1} u \cdot \nabla\phi &= (-1)^{k+1}\int \nabla u \cdot \nabla\partial_i^{k+1}\phi \\ &= (-1)^{k+2}\int f\partial_i^{k+1}\phi \\ &= -\int \partial_i^{k+1} f\phi\end{aligned}$$

Thus the equation is weakly satisfied

$$\Delta\partial_i^{k+1} u = \partial_i^{k+1} f$$

The result follows by taking for any $j = 1, \cdots, n$

$$v := \eta^2\nabla_j^h\partial_i^{k+1} u$$

□

In particular if

$$\Delta u = 0$$

weakly, then $u \in C^\infty$ by Sobolev Embedding.

1.7 Weak Sub/Superharmonicity

Besides 'weak solutions', one is of course welcome to define 'weak sub/supersolutions'. The notion of weakness here is of course in the energy sense, i.e., different from Definition 1.5.1 for the Perron's Solution.

Weak Subharmonic and Weak Superharmonic Functions Recall for any $v \in H_0^1$

$$\int_\Omega \Delta u v = -\int_\Omega \nabla u \cdot \nabla v$$

Now ensuring test functions v are signed, one may compare Δu with f via the following

Definition 1.7.1 ([FRRO22] Definition 1.9). *Let $\Omega \subseteq \mathbb{R}^n$ be open domain. Let $f \in L^2(\Omega)$ and $u \in H^1(\Omega)$. We say*

$$\Delta u \geq f \qquad \text{weakly in } \Omega$$

if

$$-\int_\Omega \nabla u \cdot \nabla v \geq \int_\Omega vf \qquad \forall\ v \in H_0^1(\Omega), \qquad v \geq 0$$

We say

$$\Delta u \leq f \qquad \text{weakly in } \Omega$$

if

$$-\int_\Omega \nabla u \cdot \nabla v \leq \int_\Omega vf \qquad \forall\ v \in H_0^1(\Omega), \qquad v \geq 0$$

In particular we call u weakly subharmonic if

$$\Delta u \geq 0 \qquad \text{weakly in } \Omega \qquad \Longleftrightarrow \qquad \int_\Omega \nabla u \cdot \nabla v \leq 0 \qquad \forall\ v \in H_0^1(\Omega), \qquad v \geq 0$$

and we call u weakly superharmonic if

$$\Delta u \leq 0 \qquad \text{weakly in } \Omega \qquad \Longleftrightarrow \qquad \int_\Omega \nabla u \cdot \nabla v \geq 0 \qquad \forall\ v \in H_0^1(\Omega), \qquad v \geq 0$$

In fact, one may even reduce the above definition to $u \in L^1_{\text{loc}}(\Omega)$ via

$$\Delta u \geq 0 \iff \int_\Omega u \Delta \varphi \geq 0 \qquad \forall\ \varphi \geq 0$$

and

$$\Delta u \leq 0 \iff \int_\Omega u \Delta \varphi \leq 0 \qquad \forall\ \varphi \geq 0$$

1.7.1 Pointwise Representation

Increasing/Decreasing Averages The important thing about weak subharmonicity (resp. superharmonicity) is that, one has increasing (resp. decreasing) spherical (ball) averages.

Lemma 1.7.1 ([FRRO22] Chapter 1; [Caf98] Lemma 2). *If $u \in L^1_{loc}(\Omega)$ is weakly subharmonic, then the average is an increasing function of r*

$$r \mapsto \fint_{B_r(x_0)} u \qquad \nearrow \qquad \forall\ B_r(x_0) \Subset \Omega \tag{1.82}$$

If u is weakly superharmonic, then the average is an decreasing function in r

$$r \mapsto \fint_{B_r(x_0)} u \qquad \searrow \qquad \forall\ B_r(x_0) \Subset \Omega \tag{1.83}$$

Proof. WLOG do for $n \geq 3$. Recall the fundamental solution and its cutoff

$$\Gamma(x) = C_n |x|^{2-n}, \qquad \Gamma_\rho(x) = \begin{cases} \Gamma(x) & |x| \geq \rho \\ P(x) & |x| \leq \rho \end{cases}$$

where P are convex paraboloid that glue in C^1 fashion and

$$\Delta P = \frac{1}{|B_\rho|}$$

Now take test function

$$\varphi_\varepsilon = \varphi * \eta_\varepsilon, \qquad \varphi = \Gamma_{\rho_2} - \Gamma_{\rho_1} \qquad \rho_1 \leq \rho_2$$

so that

$$0 \leq \lim_{\varepsilon \to 0} \int_\Omega u \Delta \varphi_\varepsilon = \int_\Omega u \left(\frac{1}{|B_{\rho_2}|} \chi_{B_{\rho_2}} - \frac{1}{|B_{\rho_1}|} \chi_{B_{\rho_1}} \right)$$

Thus the ball average is increasing. □

One has an immediate compactness result.

Lemma 1.7.2. *Let $\Omega \subseteq \mathbb{R}^n$ be open domain. Let $w_n \in L^1_{loc}(\Omega)$ be a sequence of locally uniformly bounded functions satisfying (1.82) (alternatively (1.83)). Assume*

$$w_n \to w_\infty : \Omega \to \mathbb{R} \qquad \textit{pointwise}$$

Then necessarily $w_\infty \in L^1_{loc}(\Omega)$ satisfies (1.82) (alternatively (1.83)).

Proof. Take any $x \in \Omega$ and $B_r(x) \Subset \Omega$. Since w_n is locally uniformly bounded in n and converges pointwise to w_∞

$$\lim_{n \to \infty} \int_{B_r(x)} w_n = \int_{B_r(x)} w_\infty$$

is ensured by DCT, where domination comes from

$$\sup_{B_r(x)} |w_n| \leq C_{x,r} \qquad \forall\ n$$

This gives $w_\infty \in L^1_{\text{loc}}(\Omega)$.

Now for any $0 < r_1 < r_2 < \text{dist}(x, \partial\Omega)$, for any $n \in \mathbb{N}$, using w_n satisfies (1.82) WLOG, one obtain

$$⨍_{B_{r_1}(x)} w_n \leq ⨍_{B_{r_2}(x)} w_n \qquad \forall\ n$$

Passing to the limit via DCT gives

$$⨍_{B_{r_1}(x)} w_\infty \leq ⨍_{B_{r_2}(x)} w_\infty$$

Hence w_∞ satisfies (1.82). The exact same argument applies to (1.83). □

Pointwise Representation and Upper/Lower semi-continuity The following is a most important feature of weakly subharmonic/superharmonic functions: they have a pointwise representation!

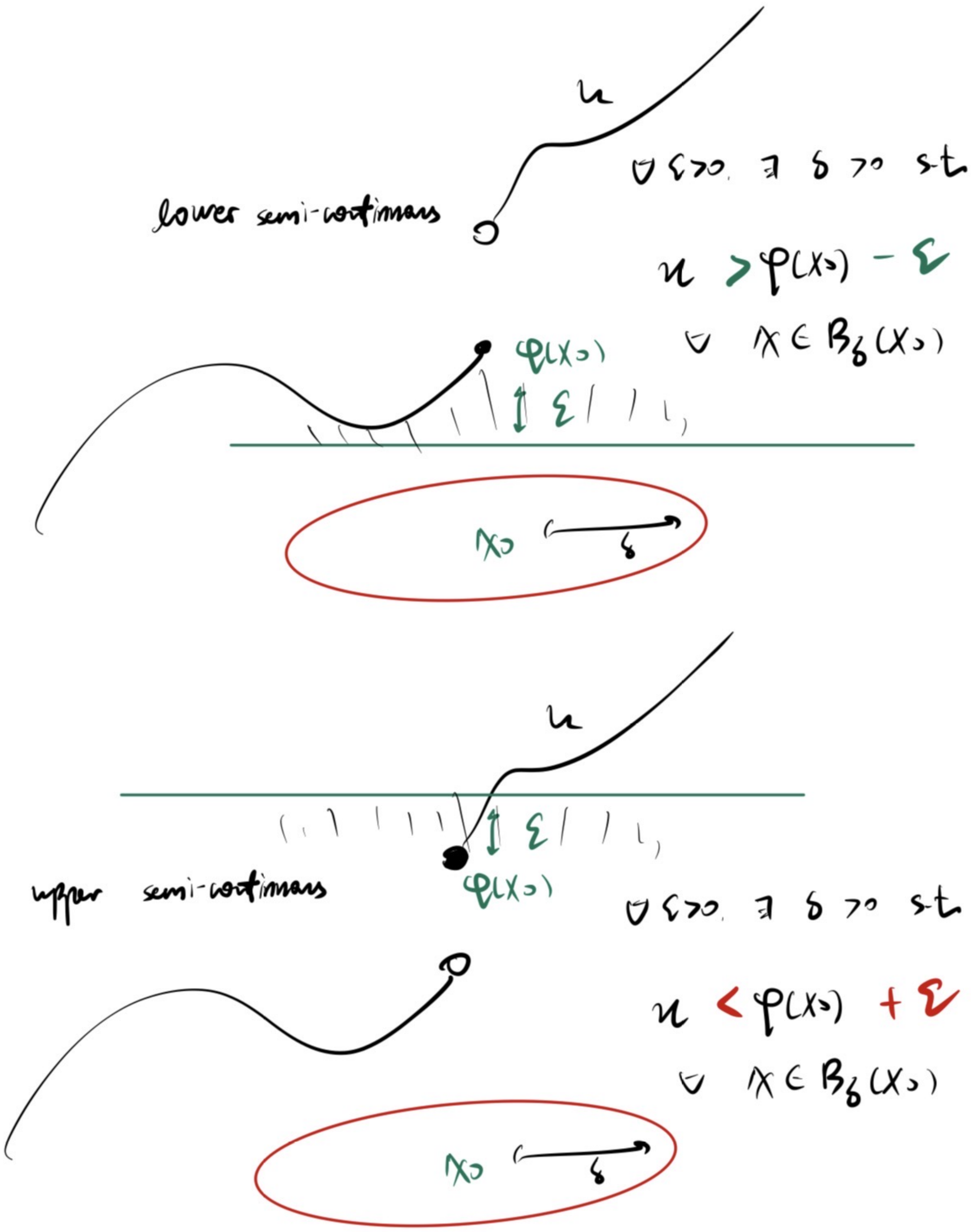


Figure 1.10: Upper and Lower Semicontinuity

Lemma 1.7.3 ([FRRO22] Lemma 1.17)**.** *Assume $u \in L^1_{loc}(\Omega)$.*

1. *If u satisfies (1.82), then one may redefine u*

$$\tilde{u}(x_0) := \lim_{r\to 0} \fint_{B_r(x_0)} u = \inf_{r>0} \fint_{B_r(x_0)} u \in [-\infty, \infty) \qquad \forall\ x_0 \in \Omega \tag{1.84}$$

s.t. $\tilde{u} = u$ a.e. $x \in \Omega$. Moreover, such $\tilde{u}$ is upper semi-continuous, i.e., for any $x_0 \in \Omega$ and for any $\varepsilon > 0$, there exists $U \subseteq \Omega$ open neighborhood around x_0 s.t.

$$\tilde{u}(x) < \tilde{u}(x_0) + \varepsilon \qquad \forall\ x \in U$$

or equivalently

$$\limsup_{x\to x_0} \tilde{u}(x) \leq \tilde{u}(x_0) \qquad \forall\ x_0 \in \Omega$$

2. *If u satisfies (1.83), then one may redefine u*

$$\tilde{u}(x_0) := \lim_{r\to 0} \fint_{B_r(x_0)} u = \sup_{r>0} \fint_{B_r(x_0)} u \in (-\infty, \infty] \qquad \forall\ x_0 \in \Omega \tag{1.85}$$

s.t. $\tilde{u} = u$ *a.e.* $x \in \Omega$*. Moreover, such* $\tilde{u}$ *is lower semi-continuous, i.e., for any* $x_0 \in \Omega$ *and for any* $\varepsilon > 0$*, there exists* $U \subseteq \Omega$ *open neighborhood around* x_0 *s.t.*

$$\tilde{u}(x) > \tilde{u}(x_0) - \varepsilon \qquad \forall\ x \in U$$

or equivalently

$$\liminf_{x \to x_0} \tilde{u}(x) \geq \tilde{u}(x_0) \qquad \forall\ x_0 \in \Omega$$

Proof. 1. Note the definition (1.84) is well-defined if u has increasing average, and (1.85) is well-defined if u has decreasing average, then $\tilde{u}(x_0) = u(x_0)$ if x_0 is a Lebesgue point, hence $\tilde{u} = u$ a.e. in Ω.

2. Consider any $x_0 \in \Omega$ and a sequence $x_k \to x_0$. Using dominated convergence theorem to characteristic functions

$$⨍_{B_r(x_0)} u = \lim_{k\to\infty} ⨍_{B_r(x_k)} u$$

Now if u satisfies (1.82)

$$⨍_{B_r(x_0)} u = \lim_{k\to\infty} ⨍_{B_r(x_k)} u \geq \limsup_{k\to\infty} \tilde{u}(x_k)$$

Letting $r \to 0$ yields

$$\tilde{u}(x_0) = \lim_{r\to 0} ⨍_{B_r(x_0)} u \geq \limsup_{k\to\infty} \tilde{u}(x_k)$$

On the other hand if u satisfies (1.83)

$$⨍_{B_r(x_0)} u = \lim_{k\to\infty} ⨍_{B_r(x_k)} u \leq \liminf_{k\to\infty} \tilde{u}(x_k)$$

Letting $r \to 0$ yields

$$\tilde{u}(x_0) = \lim_{r\to 0} ⨍_{B_r(x_0)} u \leq \liminf_{k\to\infty} \tilde{u}(x_k)$$

□

1.7.2 Equivalence for Weakly Subharmonic Functions

In this paragraph we collect the Equivalence characterisations for a function being subharmonic. Some of the statements were proven earlier or later. But no harm doing it again.

Proposition 1.7.1 (Strong Subharmonicity)**.** *Let* $u \in C^2(\Omega)$*. Then the following are equivalent*

(a) $\Delta u \geq 0$ *pointwise classically.*

(b) u *satisfies the spherical Mean Value Inequality, i.e.*

$$⨍_{\partial B_t(x)} u \leq ⨍_{\partial B_s(x)} u \qquad \forall\ B_t(x) \subseteq B_s(x) \Subset \Omega$$

(c) u *satisfies the Mean Value Inequality*

$$⨍_{B_t(x)} u \leq ⨍_{B_s(x)} u \qquad \forall\ B_t(x) \subseteq B_s(x) \Subset \Omega$$

Proof. 1. (a) iff (b). WLOG we do at 0. For any $r > 0$, we employ rescaling

$$\begin{aligned}\frac{\partial}{\partial r}\left(⨍_{\partial B_r} u\right) &= \frac{\partial}{\partial r}\left(⨍_{\partial B_1} u(rx)dS(x)\right) = ⨍_{\partial B_1} \nabla u(rx)\cdot x\, dS(x)\\ &= ⨍_{\partial B_r} \frac{\partial u}{\partial \nu} dS = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r} \frac{\partial u}{\partial \nu}\, dS = \frac{r}{n} ⨍_{B_r} \Delta u \geq 0\end{aligned}$$

Thus $⨍_{\partial B_r} u$ is increasing in r.

2. (a) iff (c). WLOG we do at 0.

$$\begin{aligned}\frac{\partial}{\partial r}\left(\fint_{B_r} u\right) &= \frac{\partial}{\partial r}\left(\fint_{B_1} u(rx)\right) = \fint_{B_1} \nabla u(rx)\cdot x\,dx \\ &= \fint_{B_r} \nabla u(y)\cdot \frac{y}{r}dy = \frac{1}{\omega_n r^n}\int_0^r \frac{s}{r}\int_{\partial B_s}\nabla u(z)\cdot\frac{z}{s}\,dS(z)ds \\ &= \frac{1}{\omega_n r^n}\int_0^r \frac{s}{r}\int_{B_s}\Delta u(z)\,dzds \geq 0\end{aligned}$$

Thus $\fint_{B_r} u$ is increasing in r.

□

Proposition 1.7.2 (Savin Analysis II 2026)**.** *Let $u \in L^1_{loc}(\Omega)$. Then the following are equivalent*

(a) $\Delta u \geq 0$ weakly subharmonic, i.e.

$$\int_\Omega u\Delta\varphi \geq 0 \qquad \forall\ \varphi \in C_0^\infty(\Omega), \qquad \varphi \geq 0$$

(b) u satisfies the Mean Value Inequality

$$r \mapsto \fint_{B_r(x)} u \qquad \textit{increasing in } r \qquad B_r(x) \Subset \Omega$$

In particular, the Mean Value Inequality implies that u is upper semi-continuous.

Proof. 1. (a) implies (b). Let $\Gamma_\rho \in C^{1,1}(\mathbb{R}^n)$ denote (1.22)

$$\Gamma_\rho(x) := \begin{cases}\Gamma(x) & |x| \geq \rho \\ P(x) & |x| \leq \rho\end{cases}$$

where P is convex parabola s.t.

$$\Delta P = \frac{1}{|B_\rho|}\chi_{B_\rho}$$

Now for any $\rho_1 < \rho_2$, take the function

$$\varphi(x) = \Gamma_{\rho_2}(x) - \Gamma_{\rho_1}(x)$$

and mollify it φ_ε. Now

$$\int_\Omega u\Delta\varphi_\varepsilon \geq 0$$

while the LHS writes (using $\varphi \in C^{1,1}$ and a.e. convergence and DCT)

$$\lim_{\varepsilon\to 0}\int_\Omega u\Delta\varphi_\varepsilon = \int_\Omega u\Delta\varphi = \int_\Omega u(\frac{1}{|B_{\rho_2}|}\chi_{B_{\rho_2}} - \frac{1}{|B_{\rho_1}|}\chi_{B_{\rho_1}})$$

Thus

$$\fint_{B_{\rho_2}} u \geq \fint_{B_{\rho_1}} u \qquad \forall\ \rho_1 < \rho_2$$

2. (b) implies upper semi-continuity.

Building on the previous argument, since the Mean Value Inequality holds, one may define $\tilde{u}$ pointwise via

$$\tilde{u}(x) := \lim_{r\to 0}\fint_{B_r(x)} u$$

By Lebesgue Differentiation, $u = \tilde{u}$ a.e. at the Lebesgue points.

Now using $u \in L^1_{\text{loc}}$, $\fint_{B_r(x)} u$ are continuous in x (this is done by doing translation and then DCT applied to characteristic functions). Thus $\tilde{u}$ as infimum of continuous functions is *upper semi-continuous* via: choose $x_k \to x_0$, then for any $r > 0$

$$\fint_{B_r(x_0)} u = \lim_{x_k\to x_0}\fint_{B_r(x_k)} u \geq \limsup_{k\to\infty}\tilde{u}(x_k)$$

On the other hand letting $r \to 0$ gives

$$\lim_{r\to 0}\fint_{B_r(x_0)} u = \tilde{u}(x_0) \geq \limsup_{k\to\infty}\tilde{u}(x_k)$$

3. (b) implies (a). For u not smooth, mean value inequality is preserved for u_ε. But u_ε are smooth, thus using the previous characterisation
$$\Delta(u_\varepsilon) \geq 0$$
Now $u_\varepsilon \to u$ in L^1_{loc}, and thus Weak Stability Corollary 1.2.2 preserves the inequality
$$\Delta u \geq 0 \qquad \text{weakly}$$

□

Subharmonic in Viscosity sense

Definition 1.7.2 ([FRRO22] Definition 1.20). *We say $u \in C(\overline{\Omega})$ is subharmonic (resp. superharmonic) in the viscosity sense if for any $\phi \in C^2(\Omega)$ that touches u from above (resp. below) at $x_0 \in \Omega$,*
$$\phi \geq (\textit{resp.} \leq)\ u \qquad u(x_0) = \phi(x_0)$$
One has
$$\Delta\phi(x_0) \geq 0\ (\textit{resp.} \leq 0)$$

Note in this case, $|x|$ are subharmonic and $-|x|$ are superharmonic.

Lemma 1.7.4 ([FRRO22] Proposition 1.21). *Let u_1, u_2 be two subharmonic functions, then $\max\{u_1, u_2\}$ remains subharmonic. For two superharmonic functions, $\min\{u_1, u_2\}$ remains superharmonic.*

Proposition 1.7.3 (Savin Analysis II 2026). *Let $u \in L^1_{loc}(\Omega)$ and upper semi-continuous. Then the following are equivalent.*

(a) $\Delta u \geq 0$ weakly subharmonic.

(b) u is subharmonic in the viscosity sense, i.e., for any $\varphi \in C^2(\Omega)$ that touches u from above at $x_0 \in \Omega$, necessarily
$$\Delta\varphi(x_0) \geq 0$$
Equivalently, this is to say a strict superharmonic C^2 function cannot touch u from above in the interior.

(c) For any P quadratic polynomial that touches u from above at $x_0 \in \Omega$, necessarily
$$\Delta P(x_0) \geq 0$$

Proof. 1. (a) implies (b). A weakly subharmonic function u satisfies strong maximum principle, i.e., assume $\Delta\varphi < 0$ in Ω and that $u \leq \varphi$, then
$$\begin{cases} \Delta(u - \varphi) \geq 0 & \Omega \\ u - \varphi \leq 0 & \partial\Omega \end{cases}$$
Now the strong maximum principle says if there is $x_0 \in \Omega$ s.t. $u(x_0) - \varphi(x_0) = 0$, then $u - \varphi \equiv 0$. But this contradicts.

2. (b) implies (c) is trivial since P itself is C^2. For (c) implies (b), let $x_0 \in \Omega$ and let φ touch u from above at x_0. Then consider the polynomial
$$P = u(x_0) + \nabla\varphi(x_0) \cdot (x - x_0) + \frac{1}{2}(x - x_0)^T D^2\varphi(x_0)(x - x_0) + \frac{\varepsilon}{2}|x - x_0|^2$$
For $|x - x_0|$ sufficiently small, P touches u from above at x_0. Hence
$$0 \leq \Delta P(x_0) = \Delta\varphi(x_0) + \varepsilon n$$
Take $\varepsilon \to 0$ to deduce
$$\Delta\varphi(x_0) \geq 0$$

3. (b) implies (a). We show that u lies below its harmonic replacement in any ball. Consider B_1, and let v be its harmonic replacement with same boundary data $v = u$ on ∂B_1. Then we perturb v upwards via
$$v_\delta(x) = v(x) + \delta(1 - |x|^2)$$
so that
$$\Delta v_\delta(x) = \Delta v(x) - 2n\delta < 0$$
is strictly superharmonic. Then $u - v_\delta$ cannot achieve local maximum in B_1. Thus
$$\sup_{B_1} u - v_\delta \leq \sup_{\partial B_1} u - v_\delta = 0$$
and we arrive at
$$u(x) \leq v(x) + \delta(1 - |x|^2) \qquad \forall\ \delta > 0$$
Take $\delta \to 0$.

□

1.7.3 Maximum Principle

One obtain a weak version of maximum principle as an immediate consequence.

Lemma 1.7.5 ([FRRO22] Proposition 1.13)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be bounded open set. Let $u \in H^1(\Omega)$. Assume*

$$\begin{cases} \Delta u \leq 0 & \Omega \text{ weakly} \\ u \geq 0 & \partial\Omega \text{ trace sense} \end{cases}$$

Then $u \geq 0$ in Ω.

Proof. By definition

$$\int_\Omega \nabla u \cdot \nabla v \geq 0 \qquad \forall\, v \in H_0^1(\Omega), \qquad v \geq 0$$

What functions can we test against? Note one can write

$$u = u^+ - u^-$$

where both u^+, $u^- \geq 0$. Recall $u \in H^1(\Omega)$ iff both u^+, $u^- \in H^1(\Omega)$. Since $u \geq 0$ on $\partial\Omega$, this is to say $u^- = 0$ on $\partial\Omega$. Thus $u^- \in H_0^1(\Omega)$ which is non-negative, is a valid competitor. We obtain

$$\int_\Omega \nabla u \cdot \nabla u^- \geq 0 \tag{1.86}$$

Recall our target is to show

$$u \geq 0 \quad \Omega \qquad \Longleftrightarrow \qquad u^- = 0 \quad \Omega$$

Since one already specified trace $u^-|_{\partial\Omega} = 0$, it suffices to prove

$$\int_\Omega |\nabla u^-|^2 = 0$$

But this is

$$\begin{aligned} \int_\Omega |\nabla u^-|^2 &= \int_\Omega \nabla u^- \cdot \nabla u^- \\ &= \int_\Omega \nabla u^- \cdot (\nabla u^- - \nabla u^+) \qquad \text{using } u^- u^+ = 0 \text{ in } \Omega \\ &= -\int_\Omega \nabla u^- \cdot \nabla u \overset{(1.86)}{\leq} 0 \end{aligned}$$

Thus $u \geq 0$ in Ω. □

L^∞ Estimate Making use of the weak comparison principle, one obtain a trivial L^∞ estimate for weak solution to Definition 1.6.2. In some sense, this is to say variational structure recovers the classical theory.

Corollary 1.7.1 ([FRRO22] Lemma 1.14)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded domain, and let $u \in H^1(\Omega)$ be weak solution in Definition 1.6.2 to*

$$\begin{cases} -\Delta u = f & \Omega \\ u = g & \partial\Omega \end{cases}$$

Then one has estimate

$$\|u\|_{L^\infty(\Omega)} \leq C(n, \Omega) \left(\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\partial\Omega)} \right) \tag{1.87}$$

In particular, if f, $g \in L^\infty$ then $u \in L^\infty(\Omega)$.

Proof. If either $\|f\|_\infty$ or $\|g\|_\infty = \infty$ this is trivial. Otherwise redefine

$$\begin{aligned} \tilde{u}(x) &:= \frac{1}{\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\partial\Omega)}} u(x) \\ \tilde{f} &:= \frac{1}{\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\partial\Omega)}} f \\ \tilde{g} &= \frac{1}{\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\partial\Omega)}} g \end{aligned}$$

So that $\tilde{u} \in H^1(\Omega)$ solves weakly

$$\begin{cases} -\Delta \tilde{u} = \tilde{f} & \Omega \\ \tilde{u} = \tilde{g} & \partial\Omega \end{cases}$$

with

$$\left\|\tilde{f}\right\|_{L^\infty}, \ \|\tilde{g}\|_{L^\infty} \leq 1$$

It suffices to bound $\|\tilde{u}\|_\infty \leq C$.

One wish to build a barrier w and compare with $\tilde{u}$. Take R large so $\Omega \subseteq B_R(0)$. Define

$$w(x) := \frac{R^2 - |x|^2}{M} + 1$$

so that

$$\partial_i w = -\frac{2x_i}{M}$$
$$\Delta w = -\frac{2n}{M}$$

Pick $M = 2n$ so that

$$\begin{cases} \Delta w = -1 & \Omega \\ w \geq 1 & \partial\Omega \end{cases}$$

Now let's evaluate the difference $w - \tilde{u}$. Compute

$$\Delta(w - \tilde{u}) = -1 + \tilde{f} \leq 0 \qquad \Omega$$
$$w - \tilde{u} \geq 1 - \tilde{g} \geq 0 \qquad \partial\Omega$$

Thus apply the Maximum Principle Lemma 1.7.5 to $w - \tilde{u}$ so that

$$w - \tilde{u} \geq 0 \qquad \Omega$$

And the bound follows

$$\tilde{u}(x) \leq \|w\|_{L^\infty(\Omega)} \leq \frac{R^2}{2n} + 1 = C(n, \Omega) \qquad \text{a.e.} \quad x \in \Omega$$

One still need to prove for the other side. To do so consider

$$\tilde{w} := -w$$

so that

$$\begin{cases} \Delta \tilde{w} = 1 & \Omega \\ \tilde{w} \leq -1 & \partial\Omega \end{cases}$$

Evaluate the difference $\tilde{u} - \tilde{w}$.

$$\Delta(\tilde{u} - \tilde{w}) = -\tilde{f} - 1 \leq 0 \qquad \Omega$$
$$\tilde{u} - \tilde{w} \geq \tilde{g} + 1 \geq 0 \qquad \partial\Omega$$

Thus apply Lemma 1.7.5 to $\tilde{u} - \tilde{w}$ yields

$$\tilde{u} \geq \tilde{w} \qquad \text{a.e. } \Omega \iff -\tilde{u}(x) \leq C(n, \Omega) \qquad \text{a.e. } \Omega$$

Hence one has the bound

$$\|\tilde{u}\|_{L^\infty} \leq C(n, \Omega)$$
$$\|u\|_{L^\infty(\Omega)} \leq C(n, \Omega) \left(\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\partial\Omega)} \right)$$

□

On the other hand, one has a rescaled variant of the estimate (1.87) in balls $B_r(0)$.

Corollary 1.7.2 (Rescaled Version of [FRRO22] Lemma 2.14)**.** *Let $B_r = B_r(0) \subseteq \mathbb{R}^n$. Let $u \in H^1(\Omega)$ be weak solution to*

$$\begin{cases} -\Delta u = f & B_r \\ u = g & \partial B_r \end{cases}$$

Then one has estimate

$$\|u\|_{L^\infty(B_r)} \leq C(n) \left(r^2 \|f\|_{L^\infty(B_r)} + \|g\|_{L^\infty(\partial B_r)} \right) \tag{1.88}$$

Proof. Let's do it directly without dividing by large constant so we see more clearly where the terms comes from. Define barrier function

$$w(x) := \frac{r^2 - |x|^2}{2n} \|f\|_{L^\infty(B_r)} + \|g\|_{L^\infty(\partial B_r)}$$

Then w solves

$$\begin{cases} \Delta w = -\|f\|_{L^\infty(B_r)} & B_r \\ w \geq \|g\|_{L^\infty(\partial B_r)} & \partial B_r \end{cases}$$

Then the difference between w and u solves

$$\begin{aligned} \Delta(w-u) = -\|f\|_{L^\infty(B_r)} + f \leq 0 \qquad & B_r \\ w - u = \|g\|_{L^\infty(\partial B_r)} - g \geq 0 \qquad & \partial B_r \end{aligned}$$

Thus apply Maximum Principle Lemma 1.7.5 so that

$$\begin{aligned} w - u \geq 0 \qquad & B_r \\ u(x) \leq w(x) = \frac{r^2 - |x|^2}{2n} \|f\|_{L^\infty(B_r)} + \|g\|_{L^\infty(\partial B_r)} & \\ \leq C(n)\left(r^2 \|f\|_{L^\infty(B_r)} + \|g\|_{L^\infty(\partial B_r)}\right) & \qquad \forall\, x \in B_r \end{aligned}$$

Now consider the other side

$$\begin{aligned} \Delta(u+w) = -f - \|f\|_{L^\infty(B_r)} \leq 0 \qquad & B_r \\ u + w \geq g + \|g\|_{L^\infty(\partial B_r)} \geq 0 \qquad & \partial B_r \end{aligned}$$

Apply Maximum Principle Lemma 1.7.5 so that

$$u(x) \geq -w(x) \geq -C(n)\left(r^2 \|f\|_{L^\infty(B_r)} + \|g\|_{L^\infty(\partial B_r)}\right) \qquad \forall\, x \in B_r$$

□

1.8 Harnack's Inequality

Recall in the previous sections, we derived Hölder Estimates (1.57), (1.58) from Interior Gradient Estimates, where the latter uses either Mean Value Property (1.15), or Bochner's Technique (1.56).

In this section, we recover Hölder, and higher regularity using *Harnack's Inequality*. This will turn out to be useful in more general elliptic type operators.

Recall that, we have already derived Harnack's Inequality for harmonic functions in various means. The ground rule is to assume $u \geq 0$ harmonic in an open set.

1. Assuming Mean Value Property, we derived Harnack's as in Theorem 1.1.5. $u \geq 0$ allows us to compare directly via MVP.

2. Assuming Poisson's Integration Formula

$$u(x) = (1 - |x|^2) \fint_{\partial B_1(0)} \frac{u(y)}{|x-y|^n} dS(y) \qquad x \in B_1$$

   we derived Harnack's Inequality (1.52), and (1.53). $u \geq 0$ allows us to compare directly via the formula.

3. We also derive Harnack's Inequality Theorem 1.4.3 via Bochner's Technique, leveraging (1.59), (1.60), (1.61). Assuming harmonicity makes sense of the key formula (1.59), and $u \geq 0$ allows us to take $v = \log(u)$.

In the following we demonstrate the sequence

$$\text{Harnack Inequality} \implies \text{Oscillation Decay} \implies \text{Hölder Regularity}$$

This is essentially the logic to gain regularity for more general elliptic operators.

Also, in the following we use Weak Solutions defined as in Definition 1.6.2. Although we showed Harnack's Inequality for at least continuous functions, in the sequel we never differentiate the solution nor the equation. Since we work on balls B_1, using Corollary 1.7.1, $u \in L^\infty(B_1)$ automatically follows from $u \in H^1(B_1)$ if we impose L^∞ boundary data. Hence one allow for u weakly solving the equation upon identifying u equivalent modulo a.e.

1.8.1 Harnack to Hölder for Harmonic Function

Oscillation Decay We assume Harnack's Inequality of the form (which we'll use)

$$\begin{cases} \Delta u = 0 & \text{weakly harmonic in } B_1 \\ u \geq 0 & \text{non-negative in } B_1 \end{cases} \implies \sup_{B_{1/2}} u \leq C \inf_{B_{1/2}} u \qquad \text{for some } C(n) > 0 \tag{1.89}$$

Given $\Omega \subseteq \mathbb{R}^n$ open domain, and a function $u \in L^\infty(\Omega)$, one define its oscillation in Ω as

$$\operatorname{osc}_{\Omega} u \leq \sup_{\Omega} u - \inf_{\Omega} u$$

where inf and sup are understood as essential inf and sup.

Lemma 1.8.1 ([FRRO22] Corollary 2.3)**.** *Let $u \in H^1(B_1)$ be harmonic. There exists $\theta = \theta(n) \in (0,1)$ s.t.*

$$\operatorname{osc}_{B_{1/2}} u \leq (1-\theta) \operatorname{osc}_{B_1} u \tag{1.90}$$

Proof. If either $\sup_{B_1} u$ or $\inf_{B_1} u$ is ∞ there is nothing to prove. Otherwise we work with $u \in L^\infty(B_1)$.

Define a function

$$w(x) := u(x) - \inf_{B_1} u$$

Since u satisfies $\Delta u = 0$, indeed $\Delta w = 0$. Also by our definition, $w \geq 0$ in B_1. If $w = 0$ somewhere in the interior, then $u \equiv \inf_{B_1} u$ constant and (1.90) follows trivially. Hence we assume $w > 0$ in B_1.

One may apply Harnack's Inequality to w as in (1.89). One obtain

$$\sup_{B_{1/2}} w \leq C \inf_{B_{1/2}} w \iff -\frac{1}{C} \sup_{B_{1/2}} w \geq -\inf_{B_{1/2}} w \tag{1.91}$$

In particular, we consider $0 < \frac{1}{C} < 1$.

1. First we see $1 \leq C$. If $\inf_{B_{1/2}} w > 0$ this trivially follows via

$$\inf_{B_{1/2}} w \leq \sup_{B_{1/2}} w \leq C \inf_{B_{1/2}} w \implies 1 \leq C$$

   Otherwise from Harnack's Inequality (1.91)

$$\sup_{B_{1/2}} w \leq C \inf_{B_{1/2}} w = 0 \implies \sup_{B_{1/2}} w = \inf_{B_{1/2}} w = 0$$

   Thus $w \equiv 0$ in $B_{1/2}$. But this contradicts our assumption $w > 0$.

2. Then we rule out $C = 1$. If so, again

$$\sup_{B_{1/2}} w = \inf_{B_{1/2}} w$$

   Using definition of w this gives

$$\sup_{B_{1/2}} u = \inf_{B_{1/2}} u \implies \operatorname{osc}_{B_{1/2}} u = 0$$

   Hence (1.90) holds trivially, so WLOG one may assume $C > 1$.

Thus one do estimate

$$\begin{aligned}
\operatorname{osc}_{B_{1/2}} u &= \sup_{B_{1/2}} u - \inf_{B_{1/2}} u \\
&= (\sup_{B_{1/2}} u - \inf_{B_1} u) - (\inf_{B_{1/2}} u - \inf_{B_1} u) = \sup_{B_{1/2}} w - \inf_{B_{1/2}} w = \operatorname{osc}_{B_{1/2}} w \\
&\overset{(1.91)}{\leq} \left(1 - \frac{1}{C}\right) \sup_{B_{1/2}} w = \left(1 - \frac{1}{C}\right) (\sup_{B_{1/2}} u - \inf_{B_1} u) \leq \left(1 - \frac{1}{C}\right) \operatorname{osc}_{B_1} u
\end{aligned}$$

(1.90) follows upon defining $\theta := \frac{1}{C}$. □

Hölder Regularity To obtain Hölder Regularity, one first need to use linearity of the Equation, and rescale ([FRRO22] Corollary 2.6). Assume $u \in H^1(B_r(x_0))$ is harmonic. Then $\tilde{u}(x) := u(x_0 + rx)$ is harmonic in B_1 as

$$\Delta \tilde{u}(x) = r^2 \Delta u(x_0 + rx) = 0$$

Now for θ as in (1.90), one write the rescaled version of

$$\begin{aligned}
\operatorname{osc}_{B_{r/2}(x_0)} u &= \operatorname{osc}_{x \in B_{1/2}} \tilde{u}(x) \overset{(1.90)}{\leq} (1-\theta) \operatorname{osc}_{x \in B_1} \tilde{u}(x) \\
&= (1-\theta) \operatorname{osc}_{x \in B_1} u(x_0 + rx) = (1-\theta) \operatorname{osc}_{B_r(x_0)} u
\end{aligned} \tag{1.92}$$

One make use of (1.92) to obtain Hölder Regularity for some $\alpha > 0$.

Lemma 1.8.2 ([FRRO22] Corollary 2.7)**.** *Let $u \in H^1(B_2) \cap L^\infty(B_2)$ be harmonic in B_2. Then there exists $\alpha = \alpha(n) > 0$ and $C = C(n) > 0$ s.t.*

$$[u]_{C^{0,\alpha}(\overline{B_{1/2}})} \leq C \|u\|_{L^\infty(B_2)} \tag{1.93}$$

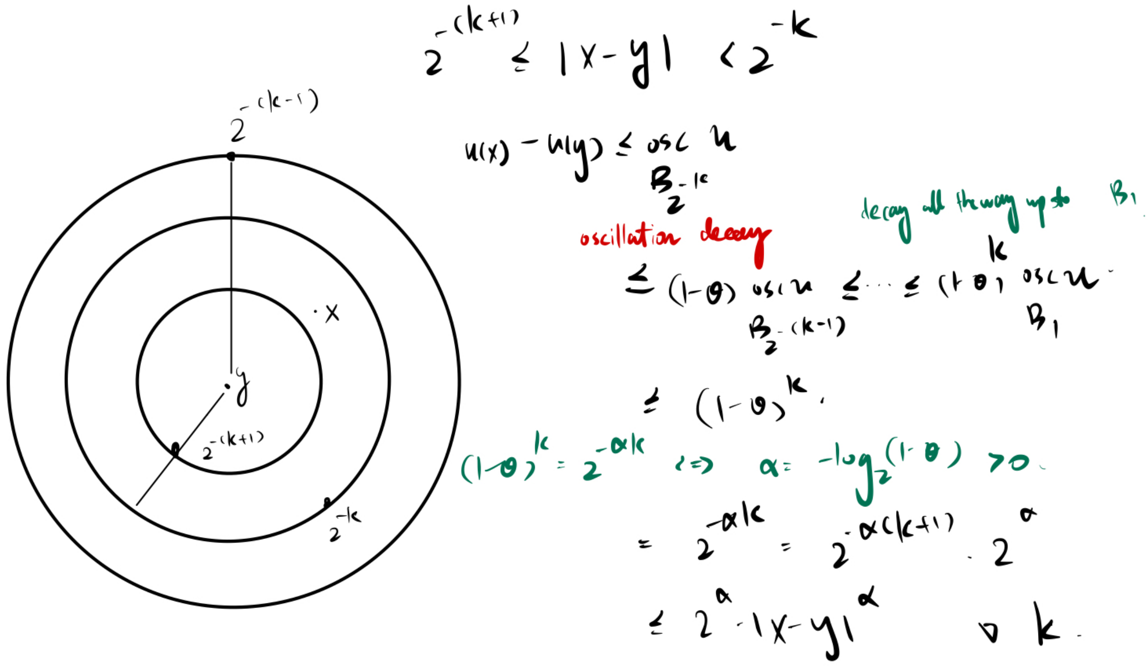


Figure 1.11: Oscillation Decay implies Hölder Regularity for Laplace

Proof. A first thing to do is rescaling the size of u

$$\tilde{u} := \frac{1}{2\,\|u\|_{L^\infty(B_2)}} u$$

so that

$$\|\tilde{u}\|_{L^\infty(B_2)} \leq \frac{1}{2}$$

Why $\frac{1}{2}$? Since it looks better

$$\underset{B_2}{\operatorname{osc}} \tilde{u} \leq 2\,\|\tilde{u}\|_{L^\infty(B_2)} \leq 1$$

Now let's estimate. For any x, $y \in B_{1/2}$ with $x \neq y$, one may pick $k \in \mathbb{N}$ sufficiently large s.t.

$$x \in B_{2^{-k}}(y) \setminus B_{2^{-(k+1)}}(y) \iff 2^{-(k+1)} \leq |x-y| < 2^{-k} \tag{1.94}$$

Let's control the function values at the points x, y

$$\begin{aligned} |\tilde{u}(x) - \tilde{u}(y)| &\leq \underset{B_{2^{-k}}(y)}{\operatorname{osc}} \tilde{u} \overset{(1.92)}{\leq} (1-\theta) \underset{B_{2^{-(k-1)}}(y)}{\operatorname{osc}} \tilde{u} \\ &\leq \cdots \leq (1-\theta)^k \underset{B_1(y)}{\operatorname{osc}} \tilde{u} \qquad \text{applying (1.92) } k \text{ times} \\ &\leq (1-\theta)^k \underset{B_2}{\operatorname{osc}} \tilde{u} \qquad \text{using } B_1(y) \subseteq B_2 \text{ for } y \in B_{1/2} \\ &\leq (1-\theta)^k \end{aligned}$$

To obtain some power, we rewrite

$$(1-\theta)^k = 2^{-k\alpha} \iff \alpha := -\log_2(1-\theta) > 0$$

Thus one obtain

$$\begin{aligned} |\tilde{u}(x) - \tilde{u}(y)| &\leq 2^{-k\alpha} = 2^{\alpha} 2^{-(k+1)\alpha} \\ &\overset{(1.94)}{\leq} 2^{\alpha} |x-y|^{\alpha} \qquad \text{using } x \text{ stays positive distance away from } y \end{aligned}$$

Dividing to LHS and taking supremum in x, y yields

$$\sup_{y\in B_{1/2}} \sup_{x\in B_{2^{-k}}(y)\setminus B_{2^{-(k+1)}}(y)} \frac{|\tilde{u}(x)-\tilde{u}(y)|}{|x-y|^\alpha} \le 2^\alpha \qquad \forall\, k\in\mathbb{N}$$

$$[\tilde{u}]_{C^{0,\alpha}(\overline{B_{1/2}})} \le 2^\alpha$$

Unravelling $\tilde{u}$ yields (1.93). □

1.8.2 Harnack to Hölder for Poisson's Equation

One shall see that Harnack's Inequality remains true for Poisson's Equation, i.e., with RHS

$$\Delta u = f$$

Let's first state a version of Harnack's Inequality for Poisson's Equation.

Theorem 1.8.1 ([FRRO22] Theorem 2.9). *Let $u\in H^1(B_1)$ and $f\in L^\infty(B_1)$ solve weakly*

$$\begin{cases}\Delta u = f & B_1\\ u\ge 0 & \partial B_1\end{cases}$$

Then there exists $C=C(n)>0$ s.t.

$$\sup_{B_{1/2}} u \le C\left(\inf_{B_{1/2}} u + \|f\|_{L^\infty(B_1)}\right) \tag{1.95}$$

Proof. Let's write

$$u = v + w$$

where v solves weakly

$$\begin{cases}\Delta v = 0 & B_1\\ v = u\ge 0 & \partial B_1\end{cases}$$

and w solves weakly

$$\begin{cases}\Delta w = f & B_1\\ w = 0 & \partial B_1\end{cases}$$

For v we apply the classical Harnack's Inequality for harmonic functions (1.89) so that

$$\sup_{B_{1/2}} v \le C\inf_{B_{1/2}} v \tag{1.96}$$

while for w we apply L^∞ estimate (1.87)

$$\sup_{B_1} w \le C\,\|f\|_{L^\infty(B_1)} \tag{1.97}$$

Thus one obtain

$$\begin{aligned}\sup_{B_{1/2}} u &\le \sup_{B_{1/2}} v + \sup_{B_{1/2}} w\\ &\overset{(1.96),(1.97)}{\le} C\left(\inf_{B_{1/2}} v + \|f\|_{L^\infty(B_1)}\right)\\ &\le C\left(\inf_{B_{1/2}} u + \sup_{B_{1/2}} w + \|f\|_{L^\infty(B_1)}\right) \overset{(1.97)}{\le} C(\inf_{B_{1/2}} u + \|f\|_{L^\infty(B_1)})\end{aligned}$$

□

Oscillation Decay Similar as in (1.90), one obtain an oscillation decay, but with error introduced by f.

Lemma 1.8.3 ([FRRO22] Corollary 2.10). *Let $u\in H^1(B_1)$ and $f\in L^\infty(B_1)$ solve weakly*

$$\Delta u = f \qquad B_1$$

There exists $\theta=\theta(n)\in(0,1)$ s.t.

$$\operatorname*{osc}_{B_{1/2}} u \le (1-\theta)\operatorname*{osc}_{B_1} u + \|f\|_{L^\infty(B_1)} \tag{1.98}$$

Proof. WLOG assume $u \in L^\infty(B_1)$. Define

$$w(x) := u(x) - \inf_{B_1} u$$

Then w solves weakly the Poisson's Equation with non-negative boundary data

$$\begin{cases} \Delta w = f & B_1 \\ w \geq 0 & \partial B_1 \end{cases}$$

Hence we apply (1.95) to w

$$\sup_{B_{1/2}} w \leq C(\inf_{B_{1/2}} w + \|f\|_{L^\infty(B_1)})$$
$$-\inf_{B_{1/2}} w \leq -\frac{1}{C}\sup_{B_{1/2}} w + \|f\|_{L^\infty(B_1)}$$

WLOG one may take $C > 1$. To use the above, note

$$\begin{aligned} \operatorname*{osc}_{B_{1/2}} u = \operatorname*{osc}_{B_{1/2}} w &= \sup_{B_{1/2}} w - \inf_{B_{1/2}} w \\ &\leq \left(1 - \frac{1}{C}\right) \sup_{B_{1/2}} w + \|f\|_{L^\infty(B_1)} \\ &= \left(1 - \frac{1}{C}\right) \left(\sup_{B_{1/2}} u - \inf_{B_1} u\right) + \|f\|_{L^\infty(B_1)} \\ &\leq \left(1 - \frac{1}{C}\right) \operatorname*{osc}_{B_1} u + \|f\|_{L^\infty(B_1)} \end{aligned}$$

Define $\theta := \frac{1}{C} \in (0, 1)$. □

Hölder Regularity To obtain Hölder Regularity, as before one need to use linearity of

$$\Delta u = f$$

and rescale ([FRRO22] Remark 2.11). This time, however, due to our forcing, Harnack's Inequality is not invariant under rescaling.

Assume

$$\Delta u = f \qquad B_r(x_0)$$

Then defining

$$\begin{aligned} \tilde{u}(x) &:= u(x_0 + rx) \qquad \forall\ x \in B_1 \\ \tilde{f}(x) &:= r^2 f(x_0 + rx) \qquad \forall\ x \in B_1 \\ \Delta \tilde{u}(x) &= r^2 \Delta u(x_0 + rx) = r^2 f(x_0 + rx) = \tilde{f}(x) \qquad \forall\ x \in B_1 \end{aligned}$$

As one zoom in, the force term gets smaller and smaller. For θ as in (1.98)

$$\begin{aligned} \operatorname*{osc}_{B_{r/2}(x_0)} u = \operatorname*{osc}_{B_{1/2}} \tilde{u} &\overset{(1.98)}{\leq} (1-\theta) \operatorname*{osc}_{B_1} \tilde{u} + \left\|\tilde{f}\right\|_{L^\infty(B_1)} \\ &= (1-\theta) \operatorname*{osc}_{B_r(x_0)} u + r^2 \|f\|_{L^\infty(B_r(x_0))} \end{aligned} \tag{1.99}$$

Again one make use of (1.99) to obtain Hölder Regularity for some $\alpha > 0$. In this proof, we need to decide how close we take our points x, y to be, and allow the oscillation term to eat the forcing term via an inductive procedure, which is delicate analysis.

However, note this is essentially (8.92).

Lemma 1.8.4 ([FRRO22] Corollary 2.12)**.** *Let $u \in H^1(B_2) \cap L^\infty(B_2)$ and $f \in L^\infty(B_2)$ solve weakly*

$$\Delta u = f \qquad B_2$$

Then there exists $\alpha = \alpha(n)$, $C = C(n) > 0$ s.t.

$$[u]_{C^{0,\alpha}(\overline{B_{1/2}})} \leq C\left(\|u\|_{L^\infty(B_2)} + \|f\|_{L^\infty(B_2)}\right) \tag{1.100}$$

Proof. A first thing to do is to normalize

$$\tilde{u} := \frac{1}{2\|u\|_{L^\infty(B_2)} + \|f\|_{L^\infty(B_2)}} u$$
$$\tilde{f} := \frac{1}{2\|u\|_{L^\infty(B_2)} + \|f\|_{L^\infty(B_2)}} f$$

so that

$$\|\tilde{u}\|_{L^\infty(B_2)} \le \frac{1}{2}$$
$$\operatorname*{osc}_{B_2} \tilde{u} \le 1$$
$$\left\|\tilde{f}\right\|_{L^\infty(B_2)} \le 1$$

Now take any x, $y \in B_{1/2}$ with $x \neq y$. There must exists $k \in \mathbb{N}$ s.t.

$$x \in B_{2^{-k}}(y) \setminus B_{2^{-(k+1)}}(y) \iff 2^{-(k+1)} \le |x-y| < 2^{-k} \tag{1.101}$$

Let's pick some $k_0 \in \mathbb{N}$ fixed to be chosen later, and divide into two cases: $k > k_0$ or $k \le k_0$.

1. In the case where $k \le k_0$, x and y are sufficiently far away from each other. Then for any $\alpha > 0$ the following holds

$$\begin{aligned} |\tilde{u}(x) - \tilde{u}(y)| &\le \operatorname*{osc}_{B_{1/2}} \tilde{u} \le 1 = 2^{\alpha(k+1)} 2^{-\alpha(k+1)} && \forall\ \alpha > 0 \\ &\le 2^{\alpha(k+1)} |x-y|^\alpha && \text{using } \alpha > 0 \text{ and lower bound in (1.101)} \\ &\le 2^{\alpha(k_0+1)} |x-y|^\alpha && \text{using } \alpha > 0 \text{ and our assumption } k \le k_0 \end{aligned}$$

This deals with

$$\sup_{y \in B_{1/2}} \sup_{x \in B_{2^{-k}}(y) \setminus B_{2^{-(k+1)}}(y)} \frac{|\tilde{u}(x) - \tilde{u}(y)|}{|x-y|^\alpha} \le 2^{\alpha(k_0+1)} \qquad \forall\ k \le k_0, \qquad \forall\ \alpha > 0 \tag{1.102}$$

We remark that, in this step, one has two free parameters to choose: k_0 and α.

2. Now we pick $k_0 = k_0(n)$, $\alpha = \alpha(n)$ and deal with the case $k > k_0$, i.e., when x and y are sufficiently close to each other.

Let's first try to run our argument using (1.99)

$$\begin{aligned} |\tilde{u}(x) - \tilde{u}(y)| \le \operatorname*{osc}_{B_{2^{-k}}(y)} \tilde{u} &\overset{(1.99)}{\le} (1-\theta) \operatorname*{osc}_{B_{2^{-(k-1)}}(y)} \tilde{u} + 4^{-(k-1)} \left\|\tilde{f}\right\|_{L^\infty(B_{2^{-(k-1)}}(y))} \\ &\le (1-\theta) \operatorname*{osc}_{B_{2^{-(k-1)}}(y)} \tilde{u} + 4^{-(k-1)} \\ &= (1-\theta) \operatorname*{osc}_{B_{2^{-(k-1)}}(y)} \tilde{u} + 4^{-k_0} 4^{k_0-(k-1)} \end{aligned}$$

Let's choose $k_0 = k_0(\theta) = k_0(n)$ sufficiently large so that

$$4^{1-k_0} \le \frac{1}{2}\theta$$

Then one obtain an iterative relation

$$\operatorname*{osc}_{B_{2^{-(k+1)}}(y)} \tilde{u} \le (1-\theta) \operatorname*{osc}_{B_{2^{-k}}(y)} \tilde{u} + \frac{1}{2}\theta 4^{k_0-k} \qquad \forall\ k \ge k_0 \tag{1.103}$$

This seems like an induction. So indeed, one can pick α to ensure some induction holds. Let's aim for the induction as such: for some $\tilde{C} = \tilde{C}(n) > 0$ and $\alpha = \alpha(n) > 0$

$$\operatorname*{osc}_{B_{2^{-k}}(y)} \tilde{u} \le \tilde{C} 2^{-k\alpha} \qquad \forall\ k \ge k_0 \tag{1.104}$$

(a) For the base step $k = k_0$, we simply let

$$\operatorname*{osc}_{B_{2^{-(k_0+1)}}(y)} \tilde{u} \le \operatorname*{osc}_{B_2} \tilde{u} \le 1 = 2^{k_0\alpha} 2^{-k_0\alpha}$$

So we pick our $\tilde{C} = \tilde{C}(n)$ fixed to be

$$\tilde{C} := 2^{k_0\alpha}$$

where we save $\alpha(n)$ for later to pick.

(b) Now for the induction step, assume (1.104) holds for k. One wish to prove for $k+1$. Thus, plugging into our iteration (1.103) one obtain

$$\begin{aligned}\operatorname*{osc}_{B_{2^{-(k+1)}}(y)} \tilde{u} &\leq (1-\theta)\operatorname*{osc}_{B_{2^{-k}}(y)} \tilde{u} + \frac{1}{2}\theta 4^{k_0-k} \\ &\leq (1-\theta)\tilde{C}2^{-k\alpha} + \frac{1}{2}\theta 4^{k_0-k} \qquad \text{using inductive hypothesis}\end{aligned}$$

To ensure the induction works for step $k+1$, one seek for $\alpha = \alpha(n) > 0$ s.t. the following holds

$$\begin{aligned}(1-\theta)\tilde{C}2^{-k\alpha} + \frac{1}{2}\theta 4^{k_0-k} &= (1-\theta)2^{(k_0-k)\alpha} + \frac{1}{2}\theta 2^{2(k_0-k)} \\ &= 2^{k_0\alpha}\left((1-\theta)2^{-k\alpha} + \frac{1}{2}\theta 2^{2(k_0-k)-k_0\alpha}\right) \\ &\leq 2^{k_0\alpha}2^{-(k+1)\alpha} \\ \iff (1-\theta)2^{-k\alpha} + \frac{1}{2}\theta 2^{2(k_0-k)-k_0\alpha} &\leq 2^{-(k+1)\alpha}\end{aligned}$$

Since $k \geq k_0$, picking $\alpha \leq 2$ yields

$$2^{2(k_0-k)-k_0\alpha} \leq 2^{\alpha(k_0-k)-k_0\alpha} = 2^{-\alpha k}$$

Thus the above simplifies to ensuring

$$\begin{aligned}\left((1-\theta)+\frac{1}{2}\theta\right)2^{-k\alpha} &\leq 2^{-(k+1)\alpha} \\ 1-\frac{1}{2}\theta &\leq 2^{-\alpha} \\ \alpha &\leq -\log_2(1-\frac{1}{2}\theta)\end{aligned}$$

Defining

$$\alpha = \alpha(n) := \min\{2, -\log_2(1-\frac{1}{2}\theta)\} > 0$$

yields

$$\operatorname*{osc}_{B_{2^{-(k+1)}}(y)} \tilde{u} \leq \tilde{C}2^{-(k+1)\alpha}$$

Now going back to the original estimate, for $k > k_0$ and x, y s.t. (1.101), one has

$$\begin{aligned}|\tilde{u}(x)-\tilde{u}(y)| &\leq \operatorname*{osc}_{B_{2^{-k}}(y)} \tilde{u} \overset{(1.104)}{\leq} \tilde{C}2^{-k\alpha} \\ &= \tilde{C}2^{\alpha}2^{-(k+1)\alpha} \leq \tilde{C}2^{\alpha}|x-y|^{\alpha} \qquad \text{using } \alpha > 0 \text{ and lower bound in (1.101)}\end{aligned}$$

And this deals with

$$\sup_{y\in B_{1/2}}\sup_{x\in B_{2^{-k}}(y)\setminus B_{2^{-(k+1)}}(y)} \frac{|\tilde{u}(x)-\tilde{u}(y)|}{|x-y|^{\alpha}} \leq \tilde{C}2^{\alpha} \qquad \forall\, k > k_0 \tag{1.105}$$

Thus combining both (1.102) and (1.105), and defining

$$C = C(n) := \max\{2^{\alpha(k_0+1)}, \tilde{C}2^{\alpha}\}$$

one conclude (1.100) via unraveling $\tilde{u}$. □

1.8.3 Weak Harnack for Sub/Superharmonic Functions

In this section we develop the Weak Harnack Inequalities for Subharmonic and Superharmonic Functions, which will turn out to be useful when one only have one-sided operators.

Weak Harnack for Supersolution

Theorem 1.8.2 ([FRRO22] Lemma B.4)**.** *Let $u \in C(\overline{B_1}) \cap H^1(B_1)$ solve*

$$\begin{cases} \Delta u \le 0 & B_1 \\ u \ge 0 & B_1 \end{cases}$$

Then there exists $C = C(n) > 0$ s.t.

$$\|u\|_{L^1(B_{1/2})} \le C \inf_{B_{1/2}} u$$

Proof. For any $x \in B_{1/3}$, via Mean Value Property

$$u(x) \ge \fint_{B_{2/3}(x)} u \ge \frac{1}{|B_{2/3}|} \int_{B_{1/3}} u$$

Now for any $x \in \partial B_{1/3}$, consider the ball $B_{1/6}(x)$

$$\begin{aligned} \inf_{B_{1/6}(x)} u &\ge c \|u\|_{L^1(B_{1/6}(x))} \\ &\ge c \int_{B_{1/6}(x) \cap B_{1/3}} u \ge c|B_{1/3} \setminus B_{1/6}| \inf_{B_{1/6}(x) \cap B_{1/3}} u \\ &\ge \tilde{c} \inf_{B_{1/3}} u \end{aligned}$$

Thus

$$\inf_{B_{1/2}} u \ge c \inf_{B_{1/3}} u \ge c \|u\|_{L^1(B_{1/3})} \ge c \|u\|_{L^1(B_{1/2})}$$

where in the last step we used decreasing average of superharmonic function. □

Weak Harnack for Subsolution

Theorem 1.8.3 ([FRRO22] Lemma B.5)**.** *Let $u \in C(\overline{B_1}) \cap H^1(B_1)$ solve*

$$\Delta u \ge 0 \qquad B_1$$

Then for any $\varepsilon > 0$, there exists $C_\varepsilon = C(n, \varepsilon) > 0$ s.t.

$$\sup_{B_{1/2}} u \le C_\varepsilon \|u\|_{L^\varepsilon(B_1)}$$

Proof. By mean value property, for any $r > 0$

$$\|u\|_{L^\infty(B_{r/2})} \le C \fint_{B_r} u \le C \|u\|^{1-\varepsilon}_{L^\infty(B_r)} \fint_{B_r} u^\varepsilon$$

for any $\varepsilon > 0$. Now by Young's Inequality, for any $\delta > 0$

$$\begin{aligned} \|u\|^{1-\varepsilon}_{L^\infty(B_r)} \fint_{B_r} u^\varepsilon &\le \delta \|u\|_{L^\infty(B_r)} + C_\delta \left(\fint_{B_r} u^\varepsilon \right)^{\frac{1}{\varepsilon}} \\ &= \delta \|u\|_{L^\infty(B_r)} + C_\delta r^{-\frac{n}{\varepsilon}} \|u\|_{L^\varepsilon(B_r)} \\ \|u\|_{L^\infty(B_{r/2})} &\le \delta \|u\|_{L^\infty(B_r)} + C_\delta r^{-\frac{n}{\varepsilon}} \|u\|_{L^\varepsilon(B_r)} \end{aligned}$$

To conclude, use Lemma 1.8.5 below. □

Absorption Trick To deal with the above issue, we introduce a clever trick.

Lemma 1.8.5 ([FRRO22] Lemma 2.27)**.** *Let S be a sub-additive function defined on balls of B_1.*
Let $\alpha \in \mathbb{R}$, $\gamma > 0$. Then there exists $\delta = \delta(\alpha, n) > 0$ s.t. if

$$S(B_{r/2}(x_0)) \le \delta \cdot S(B_r(x_0)) + r^{-\alpha} \cdot \gamma \qquad \forall\ B_r(x_0) \subseteq B_1 \tag{1.106}$$

Then there exists certain $C = C(n, \alpha) > 0$ s.t.

$$S(B_{1/2}) \le C \cdot \gamma$$

Proof. Given (1.106), we rewrite

$$r^\alpha S(B_{r/2}(x_0)) \le \delta \cdot r^\alpha S(B_r(x_0)) + \gamma \qquad \forall\ B_r(x_0) \subseteq B_1$$

Let's in fact stay a bit from the boundary and write

$$(\frac{r}{2})^\alpha S(B_{r/4}(x_0)) \le \delta \cdot (\frac{r}{2})^\alpha S(B_{r/2}(x_0)) + \gamma \qquad \forall\ B_r(x_0) \subseteq B_1$$

Denote

$$\tilde{Q} := \sup_{B_r(x_0)\subseteq B_1} (\frac{r}{2})^\alpha S(B_{r/4}(x_0)), \qquad Q := \sup_{B_r(x_0)\subseteq B_1} r^\alpha S(B_{r/2}(x_0))$$

The above immediately gives

$$\tilde{Q} \le \delta Q + \gamma$$

Now we claim it suffices to prove there exists $C = C(n, \alpha) > 0$ s.t.

$$CQ \le \tilde{Q} \tag{1.107}$$

If so, one conclude via choosing δ small universal depending on $C = C(n, \alpha)$

$$\begin{aligned} CQ &\le \tilde{Q} \le \delta Q + \gamma \\ (C - \delta)Q &\le \gamma \\ Q &\le \frac{1}{C - \delta}\gamma \end{aligned}$$

Take $x_0 = 0$ and $r = 1$ to conclude.

We prove (1.107). Take any $B_r(x_0) \subseteq B_1$, one may cover $B_{r/2}(x_0)$ with finitely many balls $B_{r/8}(z_j)$ for $z_j \in B_{r/2}(x_0)$, and $j = 1, \cdots, N$ for $N = N(n)$ open depending on the dimension.

Thus for any $B_r(x_0) \subseteq B_1$

$$\begin{aligned} r^\alpha S(B_{r/2}(x_0)) &\le r^\alpha S(\bigcup_{j=1}^N B_{r/8}(z_j)) \le r^\alpha \sum_{j=1}^N S(B_{r/8}(z_j)) \\ &\le N4^\alpha \tilde{Q} \\ Q &\le N4^\alpha \tilde{Q} \end{aligned}$$

□

1.9 Miscellaneous

These were mainly question taken from Ovidiu's lecture notes.

Lemma 1.9.1. *Let $u \in C^2(B_1^+) \cap C^0(\overline{B_1^+})$ be harmonic in the half ball. Then*

1. *If $u = 0$ on the flat portion of the boundary where $x_n = 0$, then its odd reflection*

$$\overline{u}(x_1, \cdots, x_{n-1}, x_n) := \begin{cases} u(x_1, \cdots, x_{n-1}, x_n) & x_n \geq 0 \\ -u(x_1, \cdots, x_{n-1}, -x_n) & x_n < 0 \end{cases}$$

is harmonic in B_1.

Proof. It suffices to check MVP on $x_0 \in \{x_n = 0\} \cap B_1$. For $r > 0$ small so $B_r(x_0) \Subset B_1$

$$\fint_{B_r(x_0)} \overline{u} = \frac{1}{\omega_n r^n} \left(\int_{B_r(x_0)^+} u(x_1, \cdots, x_{n-1}, x_n) - \int_{B_r(x_0)^-} u(x_1, \cdots, x_{n-1}, -x_n) \right) = 0 = \overline{u}(x_0)$$

where the last step is simply achieved by a reparametrization. □

2. *If $\partial_{x_n} u = 0$ on $x_n = 0$ then its even reflection*

$$\overline{u}(x_1, \cdots, x_{n-1}, x_n) := \begin{cases} u(x_1, \cdots, x_{n-1}, x_n) & x_n \geq 0 \\ u(x_1, \cdots, x_{n-1}, -x_n) & x_n < 0 \end{cases}$$

is harmonic in B_1.

Proof. Take any $x_0 \in \{x_n = 0\} \cap B_1$. For $r > 0$ small so $B_r(x_0) \Subset B_1$

$$\begin{aligned} \fint_{B_r(x_0)} \partial_{x_n} \overline{u} &= \frac{1}{\omega_n r^n} \left(\int_{B_r(x_0)^+} \partial_{x_n} u(x_1, \cdots, x_{n-1}, x_n) + \int_{B_r(x_0)^-} -\partial_{x_n} u(x_1, \cdots, x_{n-1}, -x_n) \right) \\ &= 0 \end{aligned}$$

Thus we know $\partial_{x_n} \overline{u}$ is harmonic in B_1. But all other derivatives are continuous across $\{x_n = 0\}$ by the trivial extension. But one can repeat the continuity across for higher derivatives, and thus one can switch

$$\Delta(\partial_{x_n} \overline{u}) = \partial_{x_n}(\Delta \overline{u}) = 0$$

This means $\Delta \overline{u}$ is constant along x_n direction. But in the interior B_1^+ or B_1^- we know $\Delta u = 0$. Thus $\Delta \overline{u} = 0$ in B_1. □

Lemma 1.9.2. *If*

$$\Delta u = f \qquad \Omega$$

in weak sense. Then

$$\Delta(u * g_\varepsilon) = f * g_\varepsilon \qquad \Omega_\varepsilon = \{\text{dist}(x, \partial\Omega) > \varepsilon\}$$

in weak sense for $g_\varepsilon \in L^1$ compactly supported in B_ε.

Proof. For any $\varphi \in C_0^2(\Omega_\varepsilon)$

$$\begin{aligned} \int_{\Omega_\varepsilon} \int_{B_\varepsilon(x)} u(y) g_\varepsilon(x-y) dy \Delta\varphi(x) dx &= \int_{\Omega_\varepsilon} \int_{B_\varepsilon} u(x-y) g_\varepsilon(y) dy \Delta\varphi(x) dx \\ &= \int_{B_\varepsilon} \int_{\Omega_\varepsilon} u(x-y) \Delta\varphi(x) dx g_\varepsilon(y) dy \\ &= \int_{B_\varepsilon} \int_{\Omega_\varepsilon} f(x-y) \varphi(x) dx g_\varepsilon(y) dy \\ &= \int_{\Omega_\varepsilon} \int_{B_\varepsilon} f(x-y) g_\varepsilon(y) dy \varphi(x) dx \\ &= \int_{\Omega_\varepsilon} \int_{B_\varepsilon(x)} f(y) g_\varepsilon(x-y) dy \cdot \varphi dx \end{aligned}$$

□

Lemma 1.9.3. *1. Assume $u \in C^2(\overline{\Omega})$ with $u = 0$ on $\partial\Omega$, where $\partial\Omega \in C^1$. Let $\tilde{u}$ denote the extension of u by 0 outside Ω.*

Show that

$$\Delta\tilde{u} = \Delta u\chi_\Omega - \partial_\nu u\, d\mathcal{H}^{n-1}\big|_{\partial\Omega}$$

where χ_Ω is the characteristic function of Ω and $d\mathcal{H}^{n-1}\big|_{\partial\Omega}$ denotes surface measure of $\partial\Omega$.

Proof. Let's make sense of the above in the weak sense. For any $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\begin{aligned}\int_{\mathbb{R}^n} \tilde{u}\Delta\varphi = \int_\Omega u\Delta\varphi &= -\int_\Omega \nabla u\cdot\nabla\varphi + \cancel{\int_{\partial\Omega} u\frac{\partial\varphi}{\partial\nu}} \\ &= \int_\Omega \Delta u\varphi - \int_{\partial\Omega}\frac{\partial u}{\partial\nu}\varphi \\ &= \int_{\mathbb{R}^n}\Delta u\chi_\Omega\varphi - \int_{\mathbb{R}^n}\partial_\nu u\varphi\, d\mathcal{H}^{n-1}\llcorner\partial\Omega\end{aligned}$$

□

2. *If $\Delta u = 0$ in Ω and $u = \partial_\nu u = 0$ on a an open smooth portion $\Gamma \subseteq \partial\Omega$ of the boundary, then u is identically 0 in Ω ([GT01] Exercise 2.2).*

Proof. Take $x_0 \in \Gamma \subseteq \partial\Omega$ and consider a small enough ball $B_r(x_0)$ s.t. $\partial B_r(x_0)\cap\Gamma$ does not exceed the smooth boundary portion on which $u = \partial_\nu u = 0$. Now extend u by zero outside Ω.

For any $\varphi \in C_0^\infty(B_r(x_0))$

$$\begin{aligned}\int_{B_r(x_0)} u\Delta\varphi = \int_{B_r(x_0)\cap\Omega} u\Delta\varphi &= -\int_{B_r(x_0)\cap\Omega}\nabla u\cdot\nabla\varphi + \cancel{\int_{\Gamma\cap B_r(x_0)} u\frac{\partial\varphi}{\partial\nu}} \\ &= \int_{B_r(x_0)\cap\Omega}\Delta u\varphi - \cancel{\int_{B_r(x_0)\cap\Gamma}\frac{\partial u}{\partial\nu}}\varphi = 0\end{aligned}$$

Thus u (extended by zero outside) is weakly harmonic in $B_r(x_0)$. By Weyl's lemma, we know u is classically harmonic in $B_r(x_0)$.

But then there is an open set on which u vanishes. Thus all derivatives of u vanish there. Now using analyticity of u as harmonic function, we know at all points in the domain, u is 0. □

Lemma 1.9.4 ([GT01] Exercise 2.14)**.** *Assume that $u \geq 0$ and $\Delta u \leq 0$ in $\mathbb{R}^n$. Show that*

1. *if $n = 2$, u is constant.*

Proof. Denote

$$m(\rho) := \min_{|x|=\rho} u(x)$$

Fix $0 < r < R$ and define

$$H(x) = m(r)\frac{\log(R)-\log(|x|)}{\log(R)-\log(r)} + m(R)\frac{\log(|x|)-\log(r)}{\log(R)-\log(r)}$$

Such H is harmonic in the annulus $\{r \leq |x| \leq R\}$. Now

$$\begin{aligned}H(x) = m(r) = \min_{|x|=r} u(x) \leq u(x) \qquad \forall\, |x| = r \\ H(x) = m(R) = \min_{|x|=R} u(x) \leq u(x) \qquad \forall\, |x| = R\end{aligned}$$

Thus by the infimum principle

$$H(x) \leq u(x) \qquad \forall\, r \leq |x| \leq R$$

Now for each fixed $r \leq \rho \leq R$, taking infimum in $|x| = \rho$ yields

$$H(x) \leq m(\rho) \qquad \forall\, |x| = \rho$$

Note for fixed $|x| = \rho$

$$H(x) = m(r)\frac{\log(R) - \log(\rho)}{\log(R) - \log(r)} + m(R)\frac{\log(\rho) - \log(r)}{\log(R) - \log(r)} \leq m(\rho)$$
$$m(r)\frac{\log(R) - \log(\rho)}{\log(R) - \log(r)} \leq m(\rho)$$
$$m(R)\frac{\log(\rho) - \log(r)}{\log(R) - \log(r)} \leq m(\rho)$$

For r fixed and send $R \to \infty$ gives

$$m(r) \leq m(\rho) \qquad \forall\, \rho \geq r$$

For R fixed and send $r \to 0$ gives

$$m(R) \leq m(\rho) \qquad \forall\, \rho \leq R$$

combining both gives

$$m(t) \leq m(\rho) \qquad \forall\, t \in [0, \infty)$$

so m achieves maximum at ρ. But ρ is arbitrary. Thus $m = m(1)$ has to be a constant. But on the other hand, since ∂B_1 is compact, u is continuous, and most importantly, $u \geq 0$ is bounded from above, the infimum of u must be achieved somewhere on ∂B_1. The strong infimum principle therefore forces the superharmonic function to be constant. □

2. *if $n \geq 3$, then u is not necessarily a constant.*

Proof. If superharmonicity is in the weak sense, then consider $|x|^{2-n}$ but on ∂B_1 we conduct a harmonic replacement in B_1. Thus this gives a non-negative superharmonic function, and nontrivial. It is superharmonic because we're taking minimum among two superharmonic functions.

If want C^2 over whole $\mathbb{R}^n$ consider

$$u(x) = \frac{1}{(1 + |x|^2)^{\frac{n-2}{2}}}$$

Then compute

$$\begin{aligned}
\partial_i u(x) &= \frac{2-n}{2}(1 + |x|^2)^{-\frac{n}{2}} 2x_i = (2 - n)(1 + |x|^2)^{-\frac{n}{2}} x_i \\
\partial_{ii} u(x) &= (n - 2)n(1 + |x|^2)^{-\frac{n+2}{2}} x_i^2 + (2 - n)(1 + |x|^2)^{-\frac{n}{2}} \\
\Delta u &= (n - 2)n(1 + |x|^2)^{-\frac{n+2}{2}} |x|^2 + n(2 - n)(1 + |x|^2)^{-\frac{n}{2}} \\
&= n(n - 2)\left((1 + |x|^2)^{-\frac{n+2}{2}} - (1 + |x|^2)^{-\frac{n}{2}}\right) \leq 0
\end{aligned}$$

While also $u \geq 0$. □

Lemma 1.9.5. *Assume that $u \in C^2 \cap L^\infty$ is harmonic function defined in $B_1 \setminus \{0\}$. If u vanishes continuously on ∂B_1, then $u \equiv 0$.*

Proof. Since $u \in L^\infty$, assume WLOG that $u(0) = a > 0$ is a boundary condition. Then one may define a sequence of functions

$$u_\varepsilon(x) := \begin{cases} \frac{a}{\varepsilon^{2-n} - 1}\left(|x|^{2-n} - 1\right) & |x| > \varepsilon \\ a & |x| \leq \varepsilon \end{cases}$$

Now the family of functions satisfy

$$u_\varepsilon|_{\partial B_1} = 0, \qquad u_\varepsilon|_{\partial B_\varepsilon} = a \qquad \Delta u_\varepsilon = 0 \qquad B_1 \setminus B_\varepsilon$$

Using Weak maximum principle for u the harmonic function defined on $B_1 \setminus \{0\}$ with boundary values $u = 0$ on ∂B_1 and $u = a$ on $\{0\}$, we know $0 \leq u \leq a$ for any $x \in B_1 \setminus \{0\}$. In particular

$$\begin{cases} u \leq u_\varepsilon & \partial B_1 \cup \partial B_\varepsilon \\ \Delta u = \Delta u_\varepsilon = 0 & B_1 \setminus B_\varepsilon \end{cases}$$

Now the maximum principle (comparison principle) says for any $x \in B_1 \setminus B_\varepsilon$

$$0 \leq u(x) \leq u_\varepsilon(x) = \frac{a}{\varepsilon^{2-n} - 1}\left(|x|^{2-n} - 1\right)$$

But this value on RHS converges to 0 as $\varepsilon \to 0$. Since $u(x)$ does not depend on ε, pass $\varepsilon \to 0$ on RHS to see $u(x) = 0$. Note for any $x \in B_1 \setminus \{0\}$, one can always find $\varepsilon > 0$ small so that $x \in B_1 \setminus B_\varepsilon(0)$. Thus $u \equiv 0$ on $B_1 \setminus \{0\}$.

□

Lemma 1.9.6 (Rellich Identity)**.** *Let $u \in C^2(B_r) \cap C^1(\overline{B_r})$ be a harmonic function. Compute*

$$\operatorname{div}(x|\nabla u|^2 - 2(x \cdot \nabla u)\nabla u)$$

and deduce that

$$(n-2)\int_{B_r} |\nabla u|^2 = r\int_{\partial B_r} |\nabla u|^2 - 2(\partial_\nu u)^2 d\sigma$$

Proof. Compute

$$\begin{aligned}
\operatorname{div}(x|\nabla u|^2 - 2(x \cdot \nabla u)\nabla u) &= \partial_i(x_i\partial_j u^2 - 2x_j\partial_j u\partial_i u)\\
&= n\partial_j u^2 + 2x_i\partial_j u\partial_{ij} u - 2\delta_{ij}\partial_j u\partial_i u - 2x_j\partial_{ij} u\partial_i u - 2x_j\partial_j u\partial_{ii} u\\
&= n|\nabla u|^2 - 2|\nabla u|^2 - 2(x\cdot\nabla u)\Delta u\\
&= (n-2)|\nabla u|^2 - 2(x\cdot\nabla u)\Delta u = (n-2)|\nabla u|^2 \qquad \text{using harmonicity}
\end{aligned}$$

Now using divergence theorem

$$\begin{aligned}
\int_{B_r}\operatorname{div}(x|\nabla u|^2 - 2(x \cdot \nabla u)\nabla u) &= \int_{\partial B_r} x\cdot\frac{x}{|x|}|\nabla u|^2 - 2(x\cdot\nabla u)\nabla u\cdot\frac{x}{|x|}\\
&= \int_{\partial B_r} r|\nabla u|^2 - 2r(\partial_\nu u)^2\, d\sigma
\end{aligned}$$

□

Lemma 1.9.7. *Find the radial solutions to the biharmonic equation*

$$\Delta^2 u = 0$$

and compute its fundamental solution.

Proof. Assume $u(x) = v(|x|)$. Then compute

$$\begin{aligned}
\partial_i u(x) &= v'(|x|)\frac{x_i}{|x|}\\
\partial_{ii} u(x) &= v''(|x|)\frac{x_i^2}{|x|^2} + v'(|x|)\frac{1}{|x|} - v'(|x|)\frac{x_i^2}{|x|^3}\\
\Delta u &= v''(|x|) + \frac{n-1}{|x|}v'(|x|)\\
\partial_i\Delta u &= v'''(|x|)\frac{x_i}{|x|} - \frac{n-1}{|x|^3}x_i v'(|x|) + \frac{n-1}{|x|^2}x_i v''(|x|)\\
\partial_{ii}\Delta u &= v^{(4)}(|x|)\frac{x_i^2}{|x|^2} + v^{(3)}(|x|)\frac{1}{|x|} - v^{(3)}(|x|)\frac{x_i^2}{|x|^3}\\
&\quad + 3\frac{n-1}{|x|^5}x_i^2 v'(|x|) - \frac{n-1}{|x|^3}v'(|x|) - \frac{n-1}{|x|^4}x_i^2 v''(|x|)\\
&\quad - 2\frac{n-1}{|x|^4}x_i^2 v''(|x|) + \frac{n-1}{|x|^2}v''(|x|) + \frac{n-1}{|x|^3}x_i^2 v^{(3)}(|x|)\\
\Delta^2 u(x) &= v^{(4)}(|x|) + v^{(3)}(|x|)\frac{n-1}{|x|}\\
&\quad + (3-n)\frac{n-1}{|x|^3}v'(|x|) - \frac{n-1}{|x|^2}v''(|x|)\\
&\quad + (n-2)\frac{n-1}{|x|^2}v''(|x|) + \frac{n-1}{|x|}v^{(3)}(|x|)\\
&= v^{(4)}(|x|) + \frac{2(n-1)}{|x|}v^{(3)}(|x|) + \frac{(n-3)(n-1)}{|x|^2}v''(|x|) - \frac{(n-3)(n-1)}{|x|^3}v'(|x|)
\end{aligned}$$

Thus the radial solutions solve

$$0 = v^{(4)}(r) + \frac{2(n-1)}{r}v^{(3)}(r) + \frac{(n-3)(n-1)}{r^2}v''(r) - \frac{(n-3)(n-1)}{r^3}v'(r)$$

To solve this, let

$$w(r) := v''(r) + \frac{n-1}{r}v'(r)$$

Thus $\Delta^2 u = 0$ corresponds to $\Delta w = 0$, which further gives

$$\begin{aligned} w(r) &= Ar^{2-n} + B \qquad n \geq 3 \\ w(r) &= A\log(r) + B \qquad n = 2 \end{aligned}$$

For $n \geq 3$. Now the trick is to write $p(r) = v'(r)$ and writes

$$\begin{aligned} p'(r) + \frac{n-1}{r}p(r) &= Ar^{2-n} + B \\ (r^{n-1}p(r))' &= Ar + Br^{n-1} \qquad \text{multiply by } r^{n-1} \text{ on both sides} \\ r^{n-1}p(r) &= \frac{A}{2}r^2 + \frac{B}{n}r^n + C \\ v'(r) &= \frac{A}{2}r^{3-n} + \frac{B}{n}r + Cr^{1-n} \end{aligned}$$

Now to proceed, if $n \neq 4$

$$\begin{aligned} v(r) &= \frac{A}{8-2n}r^{4-n} + \frac{B}{2n}r^2 + \frac{C}{2-n}r^{2-n} + D \\ &= c_1 r^{4-n} + c_2 r^{2-n} + c_3 r^2 + c_4 \end{aligned}$$

If $n = 4$

$$\begin{aligned} v'(r) &= \frac{A}{2}r^{-1} + \frac{B}{n}r + Cr^{-3} \\ v(r) &= c_1 \log r + c_2 r^{-2} + c_3 r^2 + c_4 \end{aligned}$$

For $n = 2$. Again write $p(r) = v'(r)$ so

$$\begin{aligned} p'(r) + \frac{1}{r}p(r) &= A\log(r) + B \\ (rp(r))' &= Ar\log(r) + Br \qquad \text{multiply by } r \text{ on both sides} \\ rp(r) &= A\left(\frac{1}{2}r^2\log(r) - \frac{1}{4}r^2\right) + \frac{B}{2}r^2 + C \\ p(r) &= \frac{A}{2}r\log(r) + (\frac{B}{2} - \frac{A}{4})r + Cr^{-1} \\ v(r) &= \frac{A}{2}\left(\frac{1}{2}r^2\log(r) - \frac{1}{4}r^2\right) + (\frac{B}{4} - \frac{A}{8})r^2 + C\log(r) + D \\ &= c_1 r^2\log(r) + c_2\log(r) + c_3 r^2 + c_4 \end{aligned}$$

Now the philosophy is, one want to match coefficients so that $\Delta v = \Gamma$ the fundamental solution

$$\Gamma(r) = \begin{cases} \frac{1}{(2-n)n\omega_n} r^{2-n} & n \geq 3 \\ \frac{1}{2\pi}\log(r) & n = 2 \end{cases}$$

Imagine hitting two derivatives on v, then one want to pick out the singular term that gives precisely the singular term characterizing the fundamental solution.

For $n \geq 5$, choose r^{4-n} as singular term

$$\begin{aligned} \partial_i(r^{4-n}) &= (4-n)|x|^{3-n}\frac{x_i}{|x|} \\ \partial_{ii}(r^{4-n}) &= (4-n)(3-n)|x|^{2-n}\frac{x_i^2}{|x|^2} + (4-n)|x|^{3-n}\frac{1}{|x|} - (4-n)|x|^{-n}x_i^2 \\ \Delta(r^{4-n}) &= (4-n)(3-n)|x|^{2-n} + n(4-n)|x|^{2-n} - (4-n)|x|^{2-n} = 2(4-n)|x|^{2-n} \end{aligned}$$

so

$$\Phi(x) = \frac{1}{2(4-n)(2-n)n\omega_n}|x|^{4-n}$$

For $n = 4$, choose $\log(r)$ as singular term so

$$\partial_i(\log(r)) = \frac{x_i}{|x|^2}$$
$$\partial_{ii}\log(r) = \frac{1}{|x|^2} - 2\frac{x_i^2}{|x|^4}$$
$$\Delta(\log(r)) = \frac{n-2}{|x|^2} = \frac{2}{|x|^2}$$

so

$$\Phi(x) = \frac{1}{2(2-n)n\omega_n}\log(r) = -\frac{1}{8\pi^2}\log(r)$$

For $n = 3$, choose r as singular term so

$$\partial_i r = \frac{x_i}{|x|}$$
$$\partial_{ii} r = \frac{1}{|x|} - \frac{x_i^2}{|x|^3}$$
$$\Delta r = \frac{n-1}{|x|} = \frac{2}{|x|}$$

so

$$\Phi(x) = \frac{1}{2(2-n)n\omega_n} r = -\frac{1}{8\pi}|x|$$

For $n = 2$, choose $r^2\log(r)$ as singular term so

$$\partial_i(r^2\log(r)) = 2x_i\log(|x|) + x_i$$
$$\partial_{ii}(r^2\log(r)) = 2\log(|x|) + 2\frac{x_i^2}{|x|^2} + 1$$
$$\Delta(r^2\log(r)) = 4\log(|x|) + 4$$

so (we drop 4)

$$\Phi(x) = \frac{1}{8\pi}|x|^2\log(|x|)$$

□

Chapter 2

Poisson's Equation and $C^{2,\alpha}$ Estimates

In this section we obtain sharp regularity estimates for Poisson's Equation given certain RHS forcing. We establish

1. $C^{2,\alpha}$ Estimates via an Integral Representation, i.e., Potential Theory.
2. Schauder Estimates via Comparison Principle approach, including Wang's Method [Wan06] and Campanato's Method.
3. Calderón-Zygmund $W^{2,p}$ Estimates

2.1 Potential Theory

In view of Green's Representation (1.32), the study of solution to Poisson's Equation should be closely related to the study of Newtonian Potential. Recall the fundamental solution Γ writes (1.20)

$$\Gamma(x) := \begin{cases} \frac{1}{2\pi}\log(|x|) & n = 2 \\ \frac{1}{n(2-n)\omega_n}\frac{1}{|x|^{n-2}} & n \geq 3 \end{cases} \qquad \forall\, x \neq 0$$

Let $\Omega \subseteq \mathbb{R}^n$ be open domain (not necessarily bounded). Define their convolution as (if well-defined)

$$w(x) := \int_\Omega \Gamma(x-y)f(y)dy \tag{2.1}$$

2.1.1 Potential Theory Basics

Let's see for now $n \geq 3$. When is $\int_{B_\varepsilon(0)} \Gamma^p$ convergent?

$$\int_{B_\varepsilon(0)} \Gamma^p = C(n)\int_0^\varepsilon \frac{1}{r^{p(n-2)}}r^{n-1}dr$$

This is convergent if

$$p(2-n) + n - 1 > -1 \iff p < \frac{n}{n-2}$$

Compute

$$\frac{1}{p'} + \frac{1}{\frac{n}{n-2}} = 1 \implies p' = \frac{n}{2}$$

When is $\int_{B_\varepsilon(0)} |\nabla\Gamma|^p$ convergent? For any $n \geq 2$

$$\int_{B_\varepsilon(0)} |\nabla\Gamma|^p = C(n)\int_0^\varepsilon \frac{1}{r^{p(n-1)}}r^{n-1}dr$$

is convergent if

$$p(1-n) + n - 1 > -1 \iff p < \frac{n}{n-1}$$

Compute

$$\frac{1}{p'} + \frac{1}{\frac{n}{n-1}} = 1 \implies p' = n$$

But what about $|D^2\Gamma|$? Similar computation requires

$$p < \frac{n}{n} = 1$$

so we know $\int_{B_\varepsilon(0)} |D^2\Gamma|$ is not integrable for any $\varepsilon > 0$. This is the key enemy that we deal with.

2.1.1.1 Newtonian Potential C^0 Regularity

Let's begin by studying: When is (2.1) well-defined? Due to the singularity at 0, it is natural to study

$$w(x) = \int_{\Omega \setminus B_\varepsilon(x)} \Gamma(x-y)f(y)dy + \int_{B_\varepsilon(x)} \Gamma(x-y)f(y)dy$$

for some $\varepsilon > 0$ fixed. Let's discuss the case for Ω not necessarily bounded.

Well-definedness of w for $n \geq 3$ For $n \geq 3$, $\Gamma(x-y) \sim \frac{1}{|x-y|^{n-2}}$. What conditions do we need on f ? On one hand, one need $f \in L^1(\Omega)$ so that

$$\int_{\Omega \setminus B_\varepsilon(x)} \Gamma(x-y)f(y)dy \lesssim \frac{1}{\varepsilon^{n-2}} \int_{\Omega \setminus B_\varepsilon(x)} |f| \tag{2.2}$$

is bounded for any $\varepsilon > 0$ fixed. On the other hand

$$\begin{aligned}\int_{B_\varepsilon(x)} \Gamma(x-y)f(y)dy &\leq C(n) \int_{B_\varepsilon(x)} \frac{1}{|x-y|^{n-2}}|f(y)|dy \\ &\leq C(n) \left(\int_{B_\varepsilon(x)} \frac{1}{|x-y|^{q(n-2)}} \right)^{\frac{1}{q}} \left(\int_{B_\varepsilon(x)} |f|^p \right)^{\frac{1}{p}} \qquad \frac{1}{p} + \frac{1}{q} = 1\end{aligned}$$

Notice

$$\int_{B_\varepsilon(x)} \frac{1}{|x-y|^{q(n-2)}} = C(n) \int_0^\varepsilon \frac{1}{r^{q(n-2)}} r^{n-1} dr = C(n) \int_0^\varepsilon r^{n-1-q(n-2)} dr < \infty$$
$$\iff n - q(n-2) > 0 \iff \frac{1}{q} > \frac{n-2}{n} \iff p > \frac{n}{2}$$

Hence picking $f \in L^p(\Omega)$ for $p > \frac{n}{2}$ deals with the second term (Math Stack Exchange). In particular one may pick $f \in L^\infty(\Omega)$ in a more straightforward sense

$$\int_{B_\varepsilon(x)} \Gamma(x-y)f(y)dy \leq C(n) \|f\|_{L^\infty(\Omega)} \int_{B_\varepsilon(x)} \frac{1}{|x-y|^{n-2}}|dy \leq C(n) \|f\|_{L^\infty(\Omega)} \varepsilon^2$$

We collect the conditions for $n \geq 3$

1. $f \in L^1(\Omega) \cap L^p(\Omega)$ for $p > \frac{n}{2} > 1$
2. or $f \in L^1(\Omega) \cap L^\infty(\Omega)$ suffices.
3. otherwise Ω bounded, and $f \in L^p(\Omega)$ for any $p \in (\frac{n}{2}, \infty]$ suffices.

In general, the philosophy is that one need L^1 integrability far away, and L^p for $p > \frac{n}{2}$ high integrability near the origin.

From here one obtain the estimate for any f compactly supported.

$$\|w\|_{L^\infty} \leq C \|f\|_{L^\infty}$$

Well-definedness of w for $n = 2$ For $n = 2$, $\Gamma(x-y) \sim |\log(|x-y|)|$, this is the logarithmic potential. Due to growth of size $\log|x|$ at ∞, one need more rapid decay of f at ∞. In particular one may choose $f \in L^1(\Omega)$ with compact support so that

$$\int_{\Omega \setminus B_\varepsilon(x)} \Gamma(x-y)f(y)dy \lesssim \sup_{z \in x - \text{supp}(f) \setminus B_\varepsilon(0)} |\log(|z|)| \int_{\text{supp}(f)} |f|$$

is bounded for any $\varepsilon > 0$. On the other hand, near the singularity, one could take $f \in L^\infty(\Omega)$ so that

$$\int_{B_\varepsilon(x)} \Gamma(x-y)f(y)dy \leq C(n) \|f\|_{L^\infty(\Omega)} \int_0^\varepsilon |\log(r)| r dr \leq C(n) \|f\|_{L^\infty(\Omega)} \varepsilon^2 \log(\varepsilon)$$

Let's also try using Hölder

$$\begin{aligned}\int_{B_\varepsilon(x)} \Gamma(x-y)f(y)dy &\leq C(n) \int_{B_\varepsilon(x)} |\log(|x-y|)||f(y)|dy \\ &\leq C(n) \left(\int_{B_\varepsilon(x)} |\log(x-y)|^q \right)^{\frac{1}{q}} \left(\int_{B_\varepsilon(x)} |f|^p \right)^{\frac{1}{p}} \qquad \frac{1}{p} + \frac{1}{q} = 1\end{aligned}$$

Note (WLOG assume $\varepsilon < 1$)

$$\int_{B_\varepsilon(x)} |\log(x-y)|^q dy = C(n)\int_0^\varepsilon |\log(r)|^q r dy = C(n)(-1)^q \left(\log(\varepsilon)\varepsilon^2 - q\int_0^\varepsilon r\log(r)^{q-1}dr\right)$$
$$= \cdots \leq C(n,q)|\log(\varepsilon)|\varepsilon^2 \qquad \forall\, q \in \mathbb{N}$$

Hence any $1 < p \leq \infty$ is valid. But even $p = 1$ itself is valid. Notice

$$C(n)\int_{B_\varepsilon(x)} |\log(|x-y|)||f(y)|dy \leq C(n)\int_{\partial B_1(0)}\int_0^\varepsilon |\log(r)|r|f(x+r\omega)|drdS(\omega) \leq C(n,\varepsilon)\int_{B_\varepsilon(x)} |f(y)|dy$$

Hence for $n = 2$ it is valid to take

1. $f \in L^1(\Omega)$ with compact support.
2. otherwise Ω bounded, and $f \in L^1(\Omega)$ suffices.

In general, the philosophy is that one need better integrability $\int_{\Omega\setminus B_\varepsilon(x)} |\log(|x-y|)f(y)|dy < \infty$ far away, and only need L^1 integrability near the origin.

Continuity Let any $x \in \mathbb{R}^n$ and $h > 0$

$$w(x+h) - w(x) = \int_{\mathbb{R}^n} \Gamma(y)\left(f(x+h-y) - f(x-y)\right)dy$$

which is well-defined once we know $f \in L^p$ for $p > \frac{n}{2}$ (and hopefully compactly supported, and WLOG $n \geq 3$). Now to sending $h \to 0$, the limit passes under the integral due to DCT. Thus w is continuous pointwise $w \in C(\mathbb{R}^n)$.

2.1.1.2 Newtonian Potential C^1 Regularity

Well-definedness of $\partial_i w$ On the other hand, if we study the derivatives

$$\int_\Omega \partial_{x_i}\Gamma(x-y)f(y)dy$$

The convolution works out better. Notice for any $n \geq 2$

$$\partial_{x_i}\Gamma(x) = \frac{1}{n\omega_n}\frac{1}{|x|^{n-1}}\frac{x_i}{|x|} \implies |\nabla\Gamma(x)| \lesssim \frac{1}{|x|^{n-1}}$$

Thus the far-away portion works the same as (2.2) provided $f \in L^1(\Omega)$

$$\int_{\Omega\setminus B_\varepsilon(x)} \partial_{x_i}\Gamma(x-y)f(y)dy \lesssim \frac{1}{\varepsilon^{n-1}}\int_{\Omega\setminus B_\varepsilon(x)} |f|$$

On the other hand, if we assume for $f \in L^\infty(\Omega)$

$$\int_{B_\varepsilon(x)} \partial_{x_i}\Gamma(x-y)f(y)dy \leq C(n)\|f\|_{L^\infty(\Omega)}\int_0^\varepsilon r^{1-n}r^{n-1}dr \leq C(n)\|f\|_{L^\infty(\Omega)}\varepsilon$$

Let's also try Hölder.

$$\int_{B_\varepsilon(x)} \partial_{x_i}\Gamma(x-y)f(y)dy \leq C(n)\left(\int_{B_\varepsilon(x)} \frac{1}{|x-y|^{q(n-1)}}\right)^{\frac{1}{q}}\left(\int_{B_\varepsilon(x)} |f|^p\right)^{\frac{1}{p}} \qquad \frac{1}{p}+\frac{1}{q} = 1$$
$$\int_{B_\varepsilon(x)} \frac{1}{|x-y|^{q(n-1)}} = C(n)\int_0^\varepsilon r^{-q(n-1)+n-1}dr < \infty$$
$$\iff n - q(n-1) > 0 \iff p > n$$

Hence for any $n \geq 2$ it is valid to take

1. $f \in L^1(\Omega) \cap L^p(\Omega)$ for $p > n$
2. or $f \in L^1(\Omega) \cap L^\infty(\Omega)$ suffices.
3. Otherwise Ω bounded, and $f \in L^p$ for any $p \in (n, \infty]$ is valid.

In general one need L^1 far away, and L^p for $p > n$ integrability near the origin.

From here one again obtain the estimate for any f compactly supported.

$$\|\nabla w\|_{L^\infty} \leq C\|f\|_{L^\infty}$$

Continuity of the first derivative $w \in C^1$ We further study: when is (2.1) C^1? To do so we construct C^1 functions converging locally uniformly to w and $\partial_i w$.

Lemma 2.1.1 ([GT01] Lemma 4.1)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open domain. Let $f \in L^1(\Omega) \cap L^\infty(\Omega)$ (for $n = 2$, one need more integrability at ∞ so that (2.1) is well-defined). Then the Newtonian potential $w \in C^1(\mathbb{R}^n)$ and for any $i = 1, \cdots, n$*

$$\partial_i w(x) = \int_\Omega \partial_{x_i}\Gamma(x-y)f(y)dy \qquad \forall\ x \in \mathbb{R}^n$$

Proof. Choose $\eta \in C^1(\mathbb{R})$ s.t.

$$0 \le \eta \le 1, \qquad |\eta'| \le 2, \qquad \eta(t) = 0 \qquad \forall\ t \le 1, \qquad \eta(t) = 1 \qquad \forall\ t \ge 2 \tag{2.3}$$

and define our sequence of functions as

$$w_\varepsilon(x) := \int_\Omega \Gamma(x-y)\eta(\frac{|x-y|}{\varepsilon})f(y)dy$$

First of all w_ε is well-defined due to integrability of f to ensure well-definedness of w. For the derivative we compute

$$\partial_i w_\varepsilon(x) = \int_\Omega \left(\partial_{x_i}\Gamma(x-y)\eta(\frac{|x-y|}{\varepsilon}) + \Gamma(x-y)\eta'(\frac{|x-y|}{\varepsilon})\frac{1}{\varepsilon}\frac{x_i - y_i}{|x-y|}\right) f(y)dy$$

Here $w_\varepsilon \in C^1(\mathbb{R}^n)$ since $\eta(\frac{|x-y|}{\varepsilon})$ cuts off singularity at origin smoothly, and $\int |\nabla\Gamma(x-y)f(y)|dy < \infty$ due to $f \in L^1 \cap L^\infty$. The second portion is due to well-definedness of w.

Now why does w_ε converge to w, and $\partial_i w_\varepsilon$ converge to $v(x) := \int_\Omega \partial_{x_i}\Gamma(x-y)f(y)dy$? We compute

$$\begin{aligned}
|w(x) - w_\varepsilon(x)| &\le \int_\Omega |\Gamma(x-y)(1 - \eta(\frac{|x-y|}{\varepsilon}))f(y)|dy \\
&\le \int_{|x-y|\le 2\varepsilon} |\Gamma(x-y)f(y)|dy \le \begin{cases} C(n)\,\|f\|_\infty\,\varepsilon^2 & n \ge 3 \\ C(n)\,\|f\|_\infty\,\varepsilon^2|\log(\varepsilon)| & n = 2 \end{cases} \\
|v(x) - \partial_i w_\varepsilon(x)| &\le \int_\Omega |\partial_{x_i}\Gamma(x-y)(1 - \eta(\frac{|x-y|}{\varepsilon})) - \Gamma(x-y)\frac{1}{\varepsilon}\eta'(\frac{|x-y|}{\varepsilon})||f(y)|dy \\
&\le \int_{|x-y|\le 2\varepsilon} (|\nabla\Gamma(x-y)| + \frac{2}{\varepsilon}|\Gamma(x-y)|)|f(y)|dy \\
&\le \begin{cases} C(n)\,\|f\|_\infty\,\varepsilon & n \ge 3 \\ C(n)\,\|f\|_\infty\,(\varepsilon + \varepsilon|\log(\varepsilon)|) & n = 2 \end{cases}
\end{aligned}$$

Thus $w_\varepsilon \to w$ and $\partial_i w_\varepsilon \to v$ locally uniformly in x as $\varepsilon \to 0$. Due to $C^1(\overline{B_r})$ is Banach space for any $r > 0$, the locally uniform convergence gives $w \in C^1(\mathbb{R}^n)$ and $\partial_i w = v$ in $\mathbb{R}^n$. □

2.1.2 Newtonian Potential C^2 Regularity

We further study: when is (2.1) C^2? The answer here is not as straightforward as Lemma 2.1.1 due to the term

$$\int_{|x-y|\le 2\varepsilon} (|\partial_{x_j}\partial_{x_i}\Gamma(x-y)| + \frac{2}{\varepsilon}|\partial_{x_i}\Gamma(x-y)|)|f(y)|dy$$

as one shall possibly expect. But here

$$\int_{|x-y|\le 2\varepsilon} \frac{1}{\varepsilon}|\partial_{x_i}\Gamma(x-y)| \sim \int_0^\varepsilon \frac{1}{\varepsilon}r^{1-n}r^{n-1}dr = O(1)$$

$$\int_{|x-y|\le 2\varepsilon} |\partial_{x_j}\partial_{x_i}\Gamma(x-y)| \sim \int_0^\varepsilon r^{-n}r^{n-1}dr = \infty$$

There is no decay and this is in fact ∞! The clever way to introduce decay here is to make use of f. Notice even a tiny bit of ε^α power should make this work, hence choosing $f \in C^\alpha$ shall help us.

We remark two things

1. First, (2.4) holds for $x \in \Omega$ instead of $x \in \mathbb{R}^n$ since we need to use $f \in C^\alpha$ which is defined only in Ω.
2. Second, choosing Ω bounded simplifies a lot by not violating divergence theorem and not considering far away integrability.

Lemma 2.1.2 ([GT01] Lemma 4.2). *Let $\Omega \subseteq \mathbb{R}^n$ be bounded open domain. Let $f \in L^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$ for $\alpha \in (0,1]$ locally Hölder continuous. Then the Newtonian Potential $w \in C^2(\Omega)$, and for any $i, j = 1, \cdots, n$*

$$\partial_{ij} w(x) = \int_{\Omega_0} \partial_{x_i x_j} \Gamma(x-y)(f(y) - f(x))dy + f(x) \int_{\partial\Omega_0} \partial_{x_i} \Gamma(x-y)\nu_j(y)dS(y) \qquad \forall\ x \in \Omega \tag{2.4}$$

where Ω_0 is any bounded domain containing Ω_0 s.t. divergence theorem holds, and $f \equiv 0$ outside Ω. In particular

$$\Delta w(x) = f(x) \qquad \forall\ x \in \Omega \tag{2.5}$$

Proof. Let's first study why the RHS of (2.4) is well-defined.

1. For the second part, notice $y \in \partial\Omega_0$ is far from the singularity. Since Ω_0 is bounded, $|f(x)| < \infty$ due to $f \in C(\Omega)$, the integral is finite.
2. For the first part though, one need to make use of the structure $f(y) - f(x)$ and the Hölder regularity. Again, using Ω_0 is bounded domain, and f extended to 0 outside Ω, the integral is finite far away from x. One is left to check the quantity for $\varepsilon > 0$

$$\begin{aligned} |\int_{B_\varepsilon(x)} \partial_{x_i x_j} \Gamma(x-y)(f(y) - f(x))dy| &\le C(n)[f]_{C^{0,\alpha}(B_\varepsilon(x))} \int_0^\varepsilon r^{-n} r^{n-1} r^\alpha dr \\ &\le C(n,\alpha)[f]_{C^{0,\alpha}(B_\varepsilon(x))} \varepsilon^\alpha < \infty \end{aligned}$$

   And the bound is uniform for any positive distance away from $\partial\Omega$. Notice the small exponent α is crucial, since $|D^2\Gamma(r)| \sim r^{-n}$ itself is not integrable at the origin.

In fact this already sheds light on how we use Hölder Regularity in the proof.

Using Lemma 2.1.1 we know $w \in C^1(\mathbb{R}^n)$, then for η as in (2.3) we define

$$(\partial_i w)_\varepsilon(x) := \int_\Omega \partial_{x_i} \Gamma(x-y)\eta(\frac{|x-y|}{\varepsilon}) f(y)dy$$

Here $(\partial_i w)_\varepsilon \in C^1(\mathbb{R}^n)$ for similar well-definedness reasons as above. In particular, for any $j = 1, \cdots, n$ we compute

$$\begin{aligned} \partial_j(\partial_i w)_\varepsilon(x) &= \int_\Omega \left(\partial_{x_j x_i} \Gamma(x-y)\eta(\frac{|x-y|}{\varepsilon}) + \partial_{x_i}\Gamma(x-y)\eta'(\frac{|x-y|}{\varepsilon})\frac{1}{\varepsilon}\frac{x_j - y_j}{|x-y|} \right) f(y)dy \\ &= \int_\Omega \partial_{x_j} \left(\partial_{x_i}\Gamma(x-y)\eta(\frac{|x-y|}{\varepsilon}) \right) f(y)dy \end{aligned}$$

But in view of (2.4) to incorporate $f(y) - f(x)$ one need to rewrite

$$\begin{aligned} \partial_j(\partial_i w)_\varepsilon(x) &= \int_{\Omega_0} \partial_{x_j} \left(\partial_{x_i}\Gamma(x-y)\eta(\frac{|x-y|}{\varepsilon}) \right) f(y)dy \qquad \text{using } f \equiv 0 \text{ outside } \Omega \\ &= \int_{\Omega_0} \partial_{x_j} \left(\partial_{x_i}\Gamma(x-y)\eta(\frac{|x-y|}{\varepsilon}) \right) (f(y) - f(x))dy + f(x) \int_{\Omega_0} \partial_{x_j} \left(\partial_{x_i}\Gamma(x-y)\eta(\frac{|x-y|}{\varepsilon}) \right) dy \\ &\overset{(1.5)}{=} \int_{\Omega_0} \partial_{x_j} \left(\partial_{x_i}\Gamma(x-y)\eta(\frac{|x-y|}{\varepsilon}) \right) (f(y) - f(x))dy + f(x) \int_{\partial\Omega_0} \partial_{x_i}\Gamma(x-y)\eta(\frac{|x-y|}{\varepsilon})\nu_j(y)dS(y) \end{aligned}$$

Now we define $u(x) :=$ RHS of (2.4). One can estimate

$$\begin{aligned} |u(x) - \partial_j(\partial_i w)_\varepsilon(x)| &\le \int_{\Omega_0} |\partial_{x_j}(\partial_{x_i}\Gamma(x-y)(1 - \eta(\frac{|x-y|}{\varepsilon})))||f(y) - f(x)|dy + |f(x)| \int_{\partial\Omega_0} |\partial_{x_i}\Gamma(x-y)(1-\eta(\frac{|x-y|}{\varepsilon}))|dS(y) \\ &\le \int_{|x-y|\le 2\varepsilon} \left(|\partial_{x_j x_i}\Gamma(x-y)| + \frac{2}{\varepsilon}|\partial_{x_i}\Gamma(x-y)| \right) |f(y) - f(x)|dy \\ &\le C(n)[f]_{C^{0,\alpha}(B_{2\varepsilon}(x))} \int_0^\varepsilon (r^{-n} + \frac{2}{\varepsilon} r^{1-n}) r^\alpha r^{n-1} dr \qquad \text{using Hölder Continuity of } f \\ &\le C(n)[f]_{C^{0,\alpha}(B_{2\varepsilon}(x))} \varepsilon^\alpha \qquad \forall\ \mathrm{dist}(x, \partial\Omega) > 2\varepsilon \end{aligned}$$

In other words, for any $\Omega' \Subset \Omega$

$$|u(x) - \partial_j(\partial_i w)_\varepsilon(x)| \le C(n)[f]_{C^{0,\alpha}(\Omega')} \varepsilon^\alpha \qquad \forall\ \mathrm{dist}(x, \partial\Omega') > 2\varepsilon$$

Hence $\partial_j(\partial_i w)_\varepsilon$ converges uniformly to u in $\overline{\Omega'_{2\varepsilon}}$. Now $\partial_j(\partial_i w)_\varepsilon$ converges locally uniformly to u in Ω as $\varepsilon \to 0$. Using $(\partial_i w)_\varepsilon$ converge locally uniformly to $\partial_i w$ in Lemma 2.1.1 one obtain

$$\partial_{ji} w(x) = \lim_{\varepsilon \to 0} \partial_j(\partial_i w)_\varepsilon(x) = u(x) \qquad \forall\, x \in \Omega$$

Now to see (2.5) take $\Omega_0 = B_R(x)$ for R large so that

$$\Delta w(x) = \int_{B_R(x)} \Delta\Gamma(x-y)(f(y)-f(x))dy + f(x)\int_{\partial B_R(x)} \nabla_x \Gamma(x-y)\cdot \nu(y) dS(y)$$

Notice the first term vanishes because for any $r > 0$

$$\begin{aligned} \Big|\int_{B_R(x)} \Delta\Gamma(x-y)(f(y)-f(x))dy\Big| &= \Big|\int_{B_r(x)} \Delta\Gamma(x-y)(f(y)-f(x))dy\Big| \\ &\le 2[f]_{C^\alpha(B_r(x))} C(n)\int_0^r s^{-n}s^{n-1}s^\alpha ds \le C(n)[f]_{C^\alpha(B_r(x))} r^\alpha \qquad \forall\, r > 0 \end{aligned}$$

LHS is independent of r, hence send $r \to 0$ on RHS to conclude

$$\int_{B_R(x)} \Delta\Gamma(x-y)(f(y)-f(x))dy = 0$$

The second term gives

$$f(x)\int_{\partial B_R(x)} \nabla_x\Gamma(x-y)\cdot\nu(y)dS(y) = f(x)\int_{\partial B_R(x)} \nabla\Gamma(y-x)\cdot\frac{y-x}{|y-x|}dS(y) = f(x) \qquad \text{due to choice of (1.21)}$$

□

2.1.2.1 Dirichlet BVP problem for Poisson's Equation over bounded domain

As an immediate corollary, the Dirichlet BVP problem for Poisson's Equation over bounded domain Ω (with regular boundary) is uniquely solvable if the bounded forcing is of Hölder regularity.

Theorem 2.1.1 ([GT01] Theorem 4.3)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open bounded connected domain. Let $\partial\Omega$ be regular. Assume $f \in L^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$. Then for any continuous boundary data $\varphi \in C(\partial\Omega)$, the Dirichlet BVP*

$$\begin{cases} \Delta u = f & \Omega \\ u = \varphi & \partial\Omega \end{cases}$$

is uniquely solvable, i.e., $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ solution exists.

Proof. Define w as the Newtonian potential with force f (2.1). First note from Lemma 2.1.1 that $w \in C^1(\mathbb{R}^n)$, in particular, continuous up to $\partial\Omega$. Using Theorem 1.5.1, since $\partial\Omega$ is regular, consider $v \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ solution to Laplace BVP

$$\begin{cases} \Delta v = 0 & \Omega \\ v = \varphi - w & \partial\Omega \end{cases}$$

Using Lemma 2.1.2, $f \in L^\infty \cap C^{0,\alpha}(\Omega)$ yields $w \in C^2(\Omega)$ with

$$\Delta w = f \qquad \Omega$$

Hence we define

$$u := v + w$$

Using bounded domain, u is the unique solution to the desired Dirichlet BVP problem. □

2.1.3 Interior $C^{2,\alpha}$ Estimates via Potential Theory

In the following we plan for local Hölder estimates for solution to Poisson's Equation. To begin, we prove the fundamental local Hölder estimates for the Newtonian Potential.

2.1.3.1 Newtonian Potential local $C^{2,\alpha}$ Regularity

In this estimate we work with our best friend, balls. Denote $B_r = B_r(0)$ for $r > 0$ arbitrary. We shall see that $f \in C^\alpha(\overline{B_{2r}})$ for $0 < \alpha < 1$ itself gives local Hölder regularity, which is much of an improvement compared with Lemma 2.1.2 locally!

Lemma 2.1.3 ([GT01] Lemma 4.4)**.** *Let $f \in C^\alpha(\overline{B_{2r}})$ for $\alpha \in (0,1)$. Then Newtonian Potential $w \in C^{2,\alpha}(\overline{B_r})$ and there exists $C = C(n,\alpha) > 0$ s.t. the non-dimensional Hölder norms satisfy*

$$\left\|D^2 w\right\|'_{C^{0,\alpha}(\overline{B_r})} \leq C \left\|f\right\|'_{C^{0,\alpha}(\overline{B_{2r}})} \tag{2.6}$$

i.e.

$$\|\partial_{ij} w\|_{C^0(\overline{B_r})} + r^\alpha [\partial_{ij} w]_{C^{0,\alpha}(\overline{B_r})} \leq C \left(\|f\|_{C^0(\overline{B_{2r}})} + r^\alpha [f]_{C^{0,\alpha}(\overline{B_{2r}})} \right) \qquad \forall\ i,j = 1,\cdots,n$$

The reason why we work with B_r instead of unit balls is because the formula (2.4) is difficult to scale with.

Proof. Recall we already have the formula (2.4). Hence we directly estimate $\|\partial_{ij} w\|_{C^0(\overline{B_r})}$

$$\begin{aligned}
\partial_{ij} w(x) &= \int_{B_{2r}} \partial_{ij}\Gamma(x-y)(f(y) - f(x))dy + f(x) \int_{\partial B_{2r}} \partial_i \Gamma(x-y)\nu_j(y)dS(y) \qquad \forall\ x \in B_r \\
|\partial_{ij} w(x)| &\leq C(n)[f]_{C^{0,\alpha}(\overline{B_{2r}})} \int_{B_{2r}} |x-y|^{-n}|x-y|^\alpha dy \\
&\quad + C(n) \left\|f\right\|_{C^0(\overline{B_r})} r^{1-n} |\partial B_{2r}| \qquad \text{using closest distance between } y \in \partial B_{2r} \text{ and } x \in B_r \text{ is } r \qquad (2.7) \\
&\leq C(n)[f]_{C^{0,\alpha}(\overline{B_{2r}})} \int_0^{2r} s^{-n+\alpha+n-1} ds + C(n) \left\|f\right\|_{C^0(\overline{B_r})} \\
&\leq C(n,\alpha) \left(\|f\|_{C^0(\overline{B_{2r}})} + r^\alpha [f]_{C^{0,\alpha}(\overline{B_{2r}})} \right) \qquad \forall\ x \in B_r
\end{aligned}$$

Now to estimate the Hölder seminorm, we take any two points $x, \overline{x} \in B_r$ s.t. $0 < \delta := |x - \overline{x}| < 2r$, and consider their middle point

$$\xi := \frac{1}{2}(x + \overline{x})$$

Our issue mainly happens when x and $\overline{x}$ are close. Hence to study the behavior there, we essentially study the ball $B_\delta(\xi)$ with radius $\delta = |x - \overline{x}|$ and centered at the middle points, which surely contains the two points x, $\overline{x}$ and sits inside B_{2r}.

Let's compute

$$\begin{aligned}
&\partial_{ij} w(\overline{x}) - \partial_{ij} w(x) \\
&= \int_{B_{2r}} \partial_{ij}\Gamma(\overline{x}-y)(f(y) - f(\overline{x}))dy - \int_{B_{2r}} \partial_{ij}\Gamma(x-y)(f(y)-f(x))dy + \int_{\partial B_{2r}} \left(f(\overline{x})\partial_i\Gamma(\overline{x}-y) - f(x)\partial_i\Gamma(x-y)\right)\nu_j(y)dS(y)
\end{aligned}$$

Always keep in mind that, one aim for the form

$$|\partial_{ij} w(\overline{x}) - \partial_{ij} w(x)| \lesssim C(n,\alpha) \left([f]_{C^{0,\alpha}} |\overline{x} - x|^\alpha + \|f\|_{C^0} \frac{|\overline{x} - x|^\alpha}{r^\alpha} \right)$$

due to our scale in balls of size r.

1. For the boundary portion ∂B_{2r}, we split into two parts which respectively use the Hölder semi-norm of f and the classical derivative of $\partial_i \Gamma$.

$$\begin{aligned}
&\left| \int_{\partial B_{2r}} \left(f(\overline{x})\partial_i\Gamma(\overline{x}-y) - f(x)\partial_i\Gamma(x-y)\right)\nu_j(y)dS(y) \right| \\
&= \left| \int_{\partial B_{2r}} \left((f(\overline{x}) - f(x))\partial_i\Gamma(\overline{x}-y) + f(x)(\partial_i\Gamma(\overline{x}-y) - \partial_i\Gamma(x-y))\right)\nu_j(x)dS(y) \right| \\
&\leq |f(\overline{x}) - f(x)| \left| \int_{\partial B_{2r}} \partial_i\Gamma(\overline{x}-y)\nu_j(y)dS(y) \right| + |f(x)| \int_{\partial B_{2r}} |\partial_i\Gamma(\overline{x}-y) - \partial_i\Gamma(x-y)| dS(y) \\
&\leq C(n)[f]_{C^{0,\alpha}(\overline{B_r})} |\overline{x} - x|^\alpha \qquad \text{the closest distance between } y \in \partial B_r \text{ and } \overline{x} \text{ is } r, \text{ and } r^{1-n}|\partial B_{2r}| = O(1) \qquad (2.8) \\
&\quad + \|f\|_{C^0(\overline{B_r})} |\overline{x} - x| \int_{\partial B_{2r}} |\nabla \partial_i \Gamma(\hat{x} - y)| dS(y) \qquad \text{for } \hat{x} \text{ in between the line segment of } x, \overline{x} \\
&\leq C(n)[f]_{C^{0,\alpha}(\overline{B_r})} |\overline{x} - x|^\alpha + C(n) \|f\|_{C^0(\overline{B_r})} \frac{1}{r} |\overline{x} - x| \qquad \text{closest } y \in \partial B_2 \text{ and } \hat{x} \text{ is } r, \text{ and } r^{-n}|\partial B_2| \sim r^{-1} \qquad (2.9) \\
&\leq C(n)[f]_{C^{0,\alpha}(\overline{B_r})} |\overline{x} - x|^\alpha + C(n,\alpha) \|f\|_{C^0(\overline{B_r})} \frac{|\overline{x} - x|^\alpha}{r^\alpha} \qquad \text{use } \frac{|\overline{x} - x|}{2r} < 1 \text{ and put the } 2^{-\alpha} \text{ into } C(n,\alpha)
\end{aligned}$$

2. For the interior portion B_{2r}, we mainly split into two portions

$$B_\delta(\xi), \qquad B_{2r} \setminus B_\delta(\xi)$$

On $B_\delta(\xi)$ near the singularity, one need to rely on $f \in C^{0,\alpha}$ to obtain the extra power δ^α. On $B_{2r} \setminus B_\delta(\xi)$ away from the singularity, one is allowed to rely on derivatives of Γ which are classical.

$$\begin{aligned}
&|\int_{B_{2r}} \partial_{ij}\Gamma(\overline{x}-y)(f(y)-f(\overline{x}))dy - \int_{B_{2r}} \partial_{ij}\Gamma(x-y)(f(y)-f(x))dy| \\
&\leq |\int_{B_{2r}\setminus B_\delta(\xi)} (\partial_{ij}\Gamma(\overline{x}-y)(f(y)-f(\overline{x})) - \partial_{ij}\Gamma(x-y)(f(y)-f(x)))\, dy| \\
&+ \int_{B_\delta(\xi)} (|\partial_{ij}\Gamma(\overline{x}-y)(f(y)-f(\overline{x}))| + |\partial_{ij}\Gamma(x-y)(f(y)-f(x))|)\, dy
\end{aligned}$$

(a) Let's begin by dealing with the $B_\delta(\xi)$ portion. Consider for example the part at point x

$$\begin{aligned}
\int_{B_\delta(\xi)} |\partial_{ij}\Gamma(x-y)(f(y)-f(x))|dy &\leq C(n)[f]_{C^{0,\alpha}(\overline{B_{2r}})} \int_{B_\delta(\xi)} |x-y|^{-n}|x-y|^\alpha dy \\
&\leq C(n)[f]_{C^{0,\alpha}(\overline{B_{2r}})} \int_{B_{\frac{3}{2}\delta}(x)} |x-y|^{-n}|x-y|^\alpha dy \qquad \text{recenter } B_\delta(\xi) \subseteq B_{\frac{3}{2}\delta}(x) \\
&\leq C(n)[f]_{C^{0,\alpha}(\overline{B_{2r}})} \int_0^{\frac{3}{2}\delta} r^{-n+\alpha+n-1} dr \leq C(n,\alpha)[f]_{C^{0,\alpha}(\overline{B_{2r}})}|\overline{x}-x|^\alpha
\end{aligned}$$

The exact same argument applies to

$$\int_{B_\delta(\xi)} |\partial_{ij}\Gamma(\overline{x}-y)(f(y)-f(\overline{x}))|dy \leq C(n,\alpha)[f]_{C^{0,\alpha}(\overline{B_{2r}})}|\overline{x}-x|^\alpha$$

(b) On the far-away portion, we rearrange

$$\begin{aligned}
&|\int_{B_{2r}\setminus B_\delta(\xi)} (\partial_{ij}\Gamma(\overline{x}-y)(f(y)-f(\overline{x})) - \partial_{ij}\Gamma(x-y)(f(y)-f(x)))\, dy| \\
&= |\int_{B_{2r}\setminus B_\delta(\xi)} ((\partial_{ij}\Gamma(\overline{x}-y) - \partial_{ij}\Gamma(x-y))(f(y)-f(\overline{x})) + \partial_{ij}\Gamma(x-y)(f(x)-f(\overline{x})))\, dy| \\
&\leq \int_{B_{2r}\setminus B_\delta(\xi)} |(\partial_{ij}\Gamma(\overline{x}-y) - \partial_{ij}\Gamma(x-y))(f(y)-f(\overline{x}))|dy + |f(x)-f(\overline{x})|| \int_{B_{2r}\setminus B_\delta(\xi)} \partial_{ij}\Gamma(x-y)dy|
\end{aligned}$$

For the second part we apply Green's Formula to compute

$$\begin{aligned}
|\int_{B_{2r}\setminus B_\delta(\xi)} \partial_{ij}\Gamma(x-y)dy| &= |\int_{\partial(B_{2r}\setminus B_\delta(\xi))} \partial_i\Gamma(x-y)\nu_j(y)dS(y)| \\
&= |\int_{\partial B_{2r}} \partial_i\Gamma(x-y)\nu_j(y)dS(y) - \int_{\partial B_\delta(\xi)} \partial_i\Gamma(x-y)\nu_j(y)dS(y)| \\
&\leq \int_{\partial B_{2r}} |x-y|^{1-n}dS(y) + \int_{\partial B_\delta(\xi)} |x-y|^{1-n}dS(y) \\
&\leq C(n)r^{1-n}|\partial B_{2r}| + C(n)(\frac{\delta}{2})^{1-n}|\partial B_\delta| \\
&\leq C(n) \qquad \text{the shortest distance on } \partial B_{2r} \text{ to } x \text{ is } r\text{, and on } \partial B_\delta(\xi) \text{ to } x \text{ is } \frac{\delta}{2}
\end{aligned}$$

Hence for the second part one obtain

$$|f(x)-f(\overline{x})|| \int_{B_{2r}\setminus B_\delta(\xi)} \partial_{ij}\Gamma(x-y)dy| \leq C(n)[f]_{C^{0,\alpha}(\overline{B_r})}|\overline{x}-x|^\alpha$$

For the first part we make use of differentiability of $\partial_{ij}\Gamma$

$$\begin{aligned}
&\int_{B_{2r}\setminus B_\delta(\xi)} |(\partial_{ij}\Gamma(\overline{x}-y) - \partial_{ij}\Gamma(x-y))(f(y)-f(\overline{x}))|dy \\
&\leq |\overline{x}-x| \int_{B_{2r}\setminus B_\delta(\xi)} |\nabla\partial_{ij}\Gamma(\hat{x}-y)||f(y)-f(\overline{x})|dy \qquad \text{for } \hat{x} \text{ in between the line segment of } x, \overline{x} \\
&\leq C(n)|\overline{x}-x| \int_{|y-\xi|\geq\delta} \frac{|f(y)-f(\overline{x})|}{|\hat{x}-y|^{n+1}}dy \qquad \text{using third derivative estimate of } \Gamma \\
&\leq C(n)|\overline{x}-x|[f]_{C^{0,\alpha}(\overline{B_{2r}})} \int_{|y-\xi|\geq\delta} \frac{|y-\overline{x}|^\alpha}{|\hat{x}-y|^{n+1}}dy
\end{aligned}$$

Let's pause the think about the relations between $y \notin B_\delta(\xi)$ and the distances $|y-\xi|$, $|y-\hat{x}|$, $|y-\overline{x}|$. Since $|y-\xi|$ is centered around ξ, to apply change of variables, one want to establish estimates of the kind

$$|y-\overline{x}| \le C|y-\xi|$$
$$|y-\hat{x}| \ge C|y-\xi|$$

To determine the constants, we investigate the case when y, x, $\hat{x}$, $\overline{x}$, ξ lie on the same straight line, i.e., extremes happen. In the worst case $|y-\overline{x}| = \frac{3}{2}\delta$ yet $|y-\xi| = \delta$ hence

$$|y-\overline{x}| \le \frac{3}{2}|y-\xi|$$

In another worst case $|y-\hat{x}| \to \frac{1}{2}\delta$ and $|y-\xi| = \delta$ hence

$$\frac{1}{|y-\hat{x}|} \le \frac{2}{|y-\xi|}$$

Thus

$$\begin{aligned}
&\int_{B_{2r}\setminus B_\delta(\xi)} |(\partial_{ij}\Gamma(\overline{x}-y) - \partial_{ij}\Gamma(x-y))(f(y)-f(\overline{x}))|dy \\
&\le C(n,\alpha)|\overline{x}-x|[f]_{C^{0,\alpha}(\overline{B_{2r}})} \int_{|y-\xi|\ge\delta} |y-\xi|^{\alpha-n-1}dy \\
&\le C(n,\alpha)|\overline{x}-x|[f]_{C^{0,\alpha}(\overline{B_{2r}})} \int_\delta^\infty r^{\alpha-n-1+n-1}dr \\
&= C(n,\alpha)|\overline{x}-x|[f]_{C^{0,\alpha}(\overline{B_{2r}})} \frac{1}{1-\alpha}\delta^{\alpha-1} \qquad \text{This is where we need strict } \alpha<1! \\
&\le C(n,\alpha)[f]_{C^{0,\alpha}(\overline{B_{2r}})}|\overline{x}-x|^\alpha
\end{aligned}$$

Now one may put everything together and observe

$$\begin{aligned}
|\partial_{ij}w(\overline{x}) - \partial_{ij}w(x)| &\le C(n,\alpha)\left([f]_{C^{0,\alpha}(\overline{B_{2r}})}|\overline{x}-x|^\alpha + \|f\|_{C^0(\overline{B}_r)}\frac{|\overline{x}-x|^\alpha}{r^\alpha}\right) \qquad \forall\, x, \overline{x} \in B_r \\
[\partial_{ij}w]_{C^{0,\alpha}(\overline{B_r})} &\le C(n,\alpha)\left([f]_{C^{0,\alpha}(\overline{B_{2r}})} + \frac{1}{r^\alpha}\|f\|_{C^0(\overline{B}_r)}\right)
\end{aligned}$$

And (2.6) follows. □

In the following, one has immediate corollaries in Hölder Estimates for solutions to Possion's Equation

$$\Delta u = f$$

The following are a priori estimates. One need to assume the existence of solutions in different cases, then with Hölder forcing f, one obtain Hölder estimates.

2.1.3.2 $C^{2,\alpha}$ Estimates for global solution with compact support

In this simplest case we assume for some $0<\alpha<1$ with both compact support solution and compact support forcing ([GT01] Theorem 4.5)

$$u \in C_0^2(\mathbb{R}^n), \qquad f \in C_0^\alpha(\mathbb{R}^n), \qquad \Delta u = f \qquad \mathbb{R}^n \tag{2.10}$$

1. First, we remark that given condition that both u and f have compact support, u must be the Newtonian Potential (2.1) that uniquely solves (2.10)

   $$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y)f(y)dy \qquad \forall\, x \in \mathbb{R}^n$$

   To see this, let supp(f), supp$(u) \subseteq B_R(0)$ for $R>0$ sufficiently large. Then $B_R(0)$ is open bounded domain with smooth boundary. Using our assumption $u \in C_0^2(\mathbb{R}^n)$, indeed $u \in C^2(B_R)\cap C^1(\overline{B_R})$, hence the Green's Representation formula (1.26) applies

   $$\begin{aligned}
   u(x) &= \int_{\partial B_R} (u(y)\frac{\partial\Gamma}{\partial\nu_y}(x-y) - \frac{\partial u}{\partial\nu_y}(y)\Gamma(x-y))dS(y) + \int_{B_R}\Gamma(x-y)\Delta u(y)dy \\
   &= \int_{B_R}\Gamma(x-y)f(y)dy = \int_{\mathbb{R}^n}\Gamma(x-y)f(y)dy \qquad \forall\, x \in B_R
   \end{aligned}$$

   $R>0$ large is arbitrary, hence u as Newtonian Potential is compactly supported global solution to (2.10).

2. Now our estimates follow directly. Let $\text{supp}(u), \text{supp}(f) \subseteq B_R(0)$ for $R > 0$ large. Then

   (a) One directly obtain C^0 estimate.

$$
\begin{aligned}
|u(x)| &\leq \|f\|_{C^0(\overline{B_R})} \int_{B_R(0)} |\Gamma(x-y)| dy \qquad \forall\, x \in B_R \\
|u(x)| &\leq \|f\|_{C^0(\overline{B_R})} \int_{B_{2R}(x)} |\Gamma(x-y)| dy \qquad \text{using } B_R(0) \subseteq B_{2R}(x) \text{ for } x \in B_R(0) \\
&\leq \begin{cases} C(n) \|f\|_{C^0(\overline{B_R})} \int_0^{2R} r^{2-n} r^{n-1} dr = C(n) \|f\|_{C^0(\overline{B_R})} R^2 & n \geq 3 \\ C \|f\|_{C^0(\overline{B_R})} \int_0^{2R} |\log(r)| r dr \leq C \|f\|_{C^0(\overline{B_R})} R^2 |\log(R)| & n = 2 \end{cases} \\
\|u\|_{C^0(\overline{B_R})} &\leq C(n) \|f\|_{C^0(\overline{B_R})} R^2 \qquad n \geq 3
\end{aligned}
$$

   (b) Applying Lemma 2.1.1 one obtain C^1 estimate.

$$
\begin{aligned}
\partial_i u(x) &= \int_{B_R(0)} \partial_{x_i} \Gamma(x-y) f(y) dy \qquad \forall\, x \in B_R \\
|\partial_i u(x)| &\leq C(n) \|f\|_{C^0(\overline{B_R})} \int_{B_{2R}(x)} |\partial_{x_i} \Gamma(x-y)| dy \qquad \text{using } B_R(0) \subseteq B_{2R}(x) \text{ for } x \in B_R(0) \\
&\leq C(n) \|f\|_{C^0(\overline{B_R})} \int_0^{2R} r^{1-n} r^{n-1} dr = C(n) \|f\|_{C^0(\overline{B_R})} R \\
\|\nabla u\|_{C^0(\overline{B_R})} &\leq C(n) \|f\|_{C^0(\overline{B_R})} R
\end{aligned}
$$

   (c) Applying Lemma 2.1.3 one obtain C^2 and $C^{2,\alpha}$ estimate. Due to the compact support, B_{2R} in (2.6) reduces to B_R

$$
\begin{aligned}
\left\|D^2 u\right\|_{C^0(\overline{B_R})} &\leq C(n,\alpha) \left(\|f\|_{C^0(\overline{B_R})} + R^\alpha [f]_{C^{0,\alpha}(\overline{B_R})} \right) \\
[D^2 u]_{C^{0,\alpha}(\overline{B_R})} &\leq C(n,\alpha) \left(\frac{1}{R^\alpha} \|f\|_{C^0(\overline{B_R})} + [f]_{C^{0,\alpha}(\overline{B_R})} \right)
\end{aligned}
$$

   Now, putting together all above, for $n \geq 3$, one obtain estimate for the non-dimensional norm

$$
\begin{aligned}
\|u\|'_{C^{2,\alpha}(\overline{B_R})} &:= \|u\|_{C^0} + R \|\nabla u\|_{C^0} + R^2 \left\|D^2 u\right\|_{C^0} + R^{2+\alpha} [D^2 u]_{C^{0,\alpha}} \\
&\leq C(n) \|f\|_{C^0(\overline{B_R})} R^2 + C(n,\alpha) \|f\|_{C^0(\overline{B_R})} R^2 + C(n,\alpha) [f]_{C^{0,\alpha}(\overline{B_R})} R^{2+\alpha} \\
&\leq C(n,\alpha) \|f\|'_{C^{0,\alpha}(\overline{B_R})} R^2
\end{aligned}
$$

   In particular we've improved the regularity

$$
u \in C_0^{2,\alpha}(\mathbb{R}^n)
$$

2.1.3.3 Interior $C^{2,\alpha}$ Estimates for solutions over open domains (in balls)

Of course one can remove the condition that both u and f have compact support over $\mathbb{R}^n$ into an open domain case. Let $\Omega \subseteq \mathbb{R}^n$ be an open domain. One obtain estimate for C^2 solution and $0 < \alpha < 1$ Hölder force both defined in Ω ([GT01] Theorem 4.6)

$$
u \in C^2(\Omega), \qquad f \in C^\alpha(\Omega), \qquad \Delta u = f \qquad \Omega \tag{2.11}
$$

Compared to the previous problem (2.10), (2.11) specifies a common open set for which u and f are defined, and moreover Ω can be taken to be unbounded. One still has interior estimates in balls, which are our prototype. WLOG assume $0 \in \Omega$.

1. For any $R > 0$, one may consider the solution u restricted to the ball $B_{2R}(0)$. First notice that as in Theorem 2.1.1, u takes the form

$$
u(x) = v(x) + w(x) \qquad \forall\, x \in B_{2R}
$$

   where

$$
\begin{aligned}
\Delta v = 0 \qquad & B_{2R} \\
w := \int_{B_{2R}} & \Gamma(x-y) f(y) dy
\end{aligned}
$$

   This is because we a priori assumed for a solution $u \in C^2(\Omega)$. Thus over B_{2R}, the structure of u follows from the unique solvability of Poisson's equation to the natural boundary condition $u|_{\partial B_{2R}}$.

2. Now let's estimate the size of u.

   (a) For w the Newtonian Potential of u in B_{2R}, we estimate w in B_R

$$
\begin{aligned}
|w(x)| &\leq \|f\|_{C^0(\overline{B_{2R}})} \int_{B_{2R}} |\Gamma(x-y)|dy \qquad \forall\, x \in B_R \\
&\leq \|f\|_{C^0(\overline{B_{2R}})} \int_{B_{3R}(x)} |\Gamma(x-y)|dy \qquad \text{using } B_{2R}(0) \subseteq B_{3R}(x) \text{ for } x \in B_R(0) \\
&\leq C(n)\,\|f\|_{C^0(\overline{B_{2R}})} \cdot \begin{cases} R^2 & n \geq 3 \\ R^2|\log(R)| & n = 2 \end{cases} \\
|\partial_i w(x)| &\leq C(n)\,\|f\|_{C^0(\overline{B_{2R}})}\, R \qquad \text{as in Lemma 2.1.1} \qquad \forall\, x \in B_R \\
|\partial_{ij} w(x)| &\leq C(n,\alpha)\left(\|f\|_{C^0(\overline{B_{2R}})} + R^\alpha [f]_{C^{0,\alpha}(\overline{B_{2R}})}\right) \qquad \text{as in Lemma 2.1.3} \qquad \forall\, x \in B_R \\
[D^2 w]_{C^{0,\alpha}(\overline{B_R})} &\leq C(n,\alpha)\left(\frac{1}{R^\alpha}\|f\|_{C^0(\overline{B_{2R}})} + [f]_{C^{0,\alpha}(\overline{B_{2R}})}\right) \qquad \text{as in Lemma 2.1.3}
\end{aligned}
$$

   Putting together the estimates for Dw, one obtain estimate for non-dimensional norm

$$
\begin{aligned}
R\,\|Dw\|'_{C^{1,\alpha}(\overline{B_R})} &= R\,\|\nabla w\|_{C^0(\overline{B_R})} + R^2\,\left\|D^2 w\right\|_{C^0(\overline{B_R})} + R^{2+\alpha}[D^2 w]_{C^{0,\alpha}(\overline{B_R})} \\
&\leq C(n,\alpha)R^2\left(\|f\|_{C^0(\overline{B_{2R}})} + R^\alpha [f]_{C^{0,\alpha}(\overline{B_{2R}})}\right) = C(n,\alpha)R^2\,\|f\|'_{C^{0,\alpha}(\overline{B_{2R}})} \qquad (2.12)
\end{aligned}
$$

   (b) For $v = u - w$ the harmonic function in B_{2R}, we estimate v in B_R. First we need to control C^0 norm of v, which needs to be distinguished w.r.t. dimension

      i. For $n \geq 3$, trivially

$$
|v(x)| \leq |u(x)| + |w(x)| \leq \|u\|_{C^0(\overline{B_{2R}})} + C(n)\,\|f\|_{C^0(\overline{B_{2R}})}\, R^2 \qquad (2.13)
$$

      ii. For $n = 2$, $R^2|\log(R)|$ replaces R^2 in the classical decomposition. But as suggested in [GT01] Theorem 4.6, One regard

$$
\overline{u}(x_1, x_2, x_3) := u(x_1, x_2)
$$

      for

$$
u_{x_1x_1} + u_{x_2x_2} = f(x_1, x_2) \qquad B_{2R} \subseteq \mathbb{R}^2
$$

      Then $\overline{u}$ solves

$$
\Delta\overline{u} = \overline{u}_{x_1x_1} + \overline{u}_{x_2x_2} + \overline{u}_{x_3x_3} = u_{x_1x_1} + u_{x_2x_2} = f(x_1, x_2) \qquad \forall\, (x_1, x_2, x_3) \in B_{2R} \subseteq \mathbb{R}^3
$$

      By uniqueness of Dirichlet Problem $\overline{u} = \overline{v} + \overline{w}$ where

$$
\begin{aligned}
\Delta\overline{v} &= \overline{v}_{x_1x_1} + \overline{v}_{x_2x_2} + \overline{v}_{x_3x_3} = 0 \\
\overline{w}(x) = \overline{w}(x_1, x_2, x_3) &:= \int_{B_{2R}} \Gamma(x-y) f(y_1, y_2) dy_1 dy_2 dy_3
\end{aligned}
$$

      Now one has improve C^0 estimate

$$
\begin{aligned}
&|\overline{w}(x)| \leq C\,\|f\|_{C^0(\overline{B_{2R}})}\, R^2 \qquad \forall\, (x_1, x_2, x_3) \in B_{2R} \subseteq \mathbb{R}^3 \\
&|\overline{v}(x)| \leq |\overline{u}(x)| + |\overline{w}(x)| = |u(x_1, x_2)| + |\overline{w}(x_1, x_2, x_3)| \leq \|u\|_{C^0(\overline{B_{2R}})} + C\,\|f\|_{C^0(\overline{B_{2R}})}\, R^2 \qquad (2.14)
\end{aligned}
$$

   Now for C^1 and C^2 estimate with $n \geq 2$, one use gradient estimate (1.15) and then higher order gradient estimate (1.18)

$$
\begin{aligned}
|\nabla v(x)| &\leq C(n)\frac{1}{R}\,\|v\|_{C^0(\overline{B_{2R}})} \qquad \forall\, x \in B_R \qquad (1.15) \\
|\partial_{ij} v(x)| &\leq C(n)\frac{1}{R^2}\,\|v\|_{C^0(\overline{B_{2R}})} \qquad \forall\, x \in B_R \qquad (1.18)
\end{aligned}
$$

   In particular one obtain $C^{2,\alpha}$ by doing bit rescaling and apply (1.58)

$$
[\partial_{ij} v]_{C^\alpha(\overline{B_R})} \overset{(1.58)}{\leq} C(n)\frac{1}{R^\alpha}\,\|\partial_{ij} v\|_{C^0(\overline{B_{\frac{3}{2}R}})} \overset{(1.18)}{\leq} C(n)\frac{1}{R^{2+\alpha}}\,\|v\|_{C^0(\overline{B_{2R}})}
$$

   Putting together the estimates for v one obtain estimate for non-dimensional norm

$$
\begin{aligned}
R\,\|Dv\|'_{C^{1,\alpha}(\overline{B_R})} &= R\,\|\nabla v\|_{C^0(\overline{B_R})} + R^2\,\left\|D^2 v\right\|_{C^0(\overline{B_R})} + R^{2+\alpha}[D^2 v]_{C^{0,\alpha}(\overline{B_R})} \\
&\leq C(n)\,\|v\|_{C^0(\overline{B_{\frac{3}{2}R}})} \overset{(2.13),(2.14)}{\leq} C(n)\left(\|u\|_{C^0(\overline{B_{2R}})} + \|f\|_{C^0(\overline{B_{2R}})}\, R^2\right) \qquad (2.15)
\end{aligned}
$$

(c) To put things up, for $n \geq 3$ one decompose

$$u = w + v$$

in the classical sense, while for $n = 2$ one decompose

$$u(x_1, x_2) = \overline{u}(x_1, x_2, x_3) = \overline{v}(x_1, x_2, x_3) + \overline{w}(x_1, x_2, x_3)$$

Since all estimates mentioned above works the same for either decomposition, one treat them equally and obtain estimate

$$\begin{aligned}
\|u\|'_{C^{2,\alpha}(\overline{B_R})} &= \|u\|_{C^0(\overline{B_R})} + R\|\nabla u\|_{C^0(\overline{B_R})} + R^2\left\|D^2 u\right\|_{C^0(\overline{B_R})} + R^{2+\alpha}[D^2 u]_{C^{0,\alpha}(\overline{B_R})} \\
&\leq \|u\|_{C^0(\overline{B_R})} + R(\|\nabla w\|_{C^0(\overline{B_R})} + \|\nabla v\|_{C^0(\overline{B_R})}) \\
&+ R^2\left(\left\|D^2 w\right\|_{C^0(\overline{B_R})} + \left\|D^2 v\right\|_{C^0(\overline{B_R})}\right) + R^{2+\alpha}\left([D^2 w]_{C^{0,\alpha}(\overline{B_R})} + [D^2 v]_{C^{0,\alpha}(\overline{B_R})}\right) \\
&= \|u\|_{C^0(\overline{B_R})} + R\|Dw\|'_{C^{1,\alpha}(\overline{B_R})} + R\|Dv\|'_{C^{1,\alpha}(\overline{B_R})} \\
&\overset{(2.12),(2.15)}{\leq} C(n)\|u\|_{C^0(\overline{B_{2R}})} + C(n)\|f\|_{C^0(\overline{B_{2R}})}R^2 + C(n,\alpha)\|f\|'_{C^{0,\alpha}(\overline{B_{2R}})}R^2 \\
&\leq C(n,\alpha)\left(\|u\|_{C^0(\overline{B_{2R}})} + R^2\|f\|'_{C^{0,\alpha}(\overline{B_{2R}})}\right) \qquad \forall\ B_{2R} \Subset \Omega
\end{aligned} \tag{2.16}$$

In particular we've improved the regularity

$$u \in C^{2,\alpha}(\Omega)$$

2.1.3.4 Compactness

As an immediate corollary, one obtain compactness result for 'being solution to Poisson's Equation' via equicontinuity on compact subdomains of second order derivatives.

Theorem 2.1.2 ([GT01] Corollary 4.7)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open domain. Let $u_n \in C^2(\Omega)$ be uniformly bounded sequence of solutions to*

$$\Delta u_n = f \qquad \Omega$$

for the same force $f \in C^\alpha(\Omega)$. Then there exists a subsequence u_{n_k} s.t.

$$u_{n_k} \to u \qquad \textit{locally uniformly}$$

to some $u \in C^2(\Omega)$ that solves

$$\Delta u = f \qquad \Omega$$

Proof. Since u_m is uniformly bounded, assume there exists $M > 0$ s.t.

$$\|u_m\|_{C^0(\Omega)} \leq M < \infty \qquad \forall\ m$$

Now for any subdomain $\Omega'' \Subset \Omega' \Subset \Omega$, fix some $i,\ j \in \{1, \cdots, n\}$, one use interior second order derivative estimate (2.16) to obtain

$$\|\partial_{ij} u_m\|_{C^0(\overline{\Omega''})} \leq C(\Omega'', \Omega', n, \alpha)\left(\|u_m\|_{C^0(\overline{\Omega'})} + \|f\|_{C^{0,\alpha}(\overline{\Omega'})}\right) \leq C(M + \|f\|_{C^{0,\alpha}(\overline{\Omega'})}) < \infty \qquad \forall\ m$$

Hence $\{\partial_{ij} u_m\}$ is uniformly bounded over Ω''. On the other hand, using Interior Hölder estimate (2.16)

$$\begin{aligned}
|\partial_{ij} u_m(x) - \partial_{ij} u_m(y)| &\leq C(\Omega'', \Omega', n, \alpha)\left(\|u_m\|_{C^0(\overline{\Omega'})} + \|f\|_{C^{0,\alpha}(\overline{\Omega'})}\right)|x-y|^\alpha \\
&\leq C(M + \|f\|_{C^{0,\alpha}(\overline{\Omega'})})|x-y|^\alpha \qquad \forall\ x,\ y \in \Omega'' \qquad \forall\ m
\end{aligned}$$

Hence $\{\partial_{ij} u_m\}$ is uniformly equicontinuous over Ω''. One may do this on any compact subdomain, hence via diagonalization and Arzela-Ascoli, there exists $v_{ij} \in C(\Omega)$ and a subsequence s.t.

$$\partial_{ij} u_{m_k} \to v_{ij} \qquad \text{locally uniformly} \qquad k \to \infty$$

One may pass to another subsequence and converge for another pair of (i, j). Hence after passing to $n \times n$ many subsequences, there exists $v_{ij} \in C(\Omega)$ for $i,\ j = 1, \cdots, n$ s.t. (up to subsequence)

$$\partial_{ij} u_n \to v_{ij} \qquad \text{locally uniformly} \qquad \forall\ i,\ j = 1, \cdots, n$$

Now we do for first order derivatives. Fix $i \in \{1, \cdots, n\}$ one use interior first order derivative estimate (2.16)

$$\|\partial_i u_m\|_{C^0(\overline{\Omega''})} \leq C(\Omega'', \Omega', n)\left(\|u_m\|_{C^0(\overline{\Omega'})} + \|f\|_{C^0(\overline{\Omega'})}\right) \leq C(M + \|f\|_{C^0(\overline{\Omega'})}) < \infty \qquad \forall\ m$$

$$\begin{aligned}|\partial_i u_m(x) - \partial_i u_m(y)| &\leq C(\Omega'', \Omega', n, \alpha)\left(\|u_m\|_{C^0(\overline{\Omega'})} + \|f\|_{C^{0,\alpha}(\overline{\Omega'})}\right)|x-y| \\ &\leq C(M + \|f\|_{C^{0,\alpha}(\overline{\Omega'})})|x-y| \qquad \forall\ x, y \in \Omega'' \qquad \forall\ m\end{aligned}$$

Apply Arzela-Ascoli there exists $v_i \in C(\Omega)$ s.t. up to subsequence

$$\partial_i u_m \to v_i \qquad \text{locally uniformly}$$

One again pass to another subsequence and converge for another i to obtain $v_i \in C(\Omega)$ for $i = 1, \cdots, n$ s.t. up to subsequence

$$\partial_i u_m \to v_i \qquad \text{locally uniformly} \qquad \forall\ i$$

Finally, doing the trivial C^0 estimate and passing to the last subsequence

$$\|u_m\|_{C^0(\overline{\Omega''})} \leq M < \infty \qquad \forall\ m$$

$$|u_m(x) - u_m(y)| \leq C(\Omega'', \Omega', n)\left(\|u_m\|_{C^0(\overline{\Omega'})} + \|f\|_{C^0(\overline{\Omega'})}\right)|x-y| \leq C(M + \|f\|_{C^0(\overline{\Omega'})})|x-y| < \infty \qquad \forall\ x, y \in \Omega'' \qquad \forall\ m$$

One obtain $v \in C(\Omega)$ s.t.

$$u_m \to v \qquad \text{locally uniformly}$$

Recall we're been constantly extracting subsequences. Recall that $C^2(\overline{\Omega'})$ is Banach space for any Ω' compact subdomain, hence

$$\partial_i v = v_i, \qquad \partial_{ij} v = v_{ij} \qquad \forall\ i, j = 1, \cdots, n$$

Thus $v \in C^2(\Omega)$ and

$$u_m \to v, \qquad \partial_i u_m \to \partial_i v, \qquad \partial_{ij} u_m \to \partial_{ij} v \qquad \text{locally uniformly}$$

In particular

$$|\Delta v - f| = \lim_{m \to \infty} |\Delta u_m - f| = 0 \qquad \forall\ x \in \Omega$$

Thus v solves classically the Poisson's Equation

$$\Delta v = f \qquad \Omega$$

□

2.1.4 Boundary Hölder Estimates via Potential Theory

In the following we discuss estimates up to the boundary. Let $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_n > 0\}$ and denote

$$B_R^+ := B_R(0) \cap \mathbb{R}^n_+$$

2.1.4.1 Newtonian Potential Boundary $C^{2,\alpha}$ Regularity

In this estimate we work with our second best friend, balls intersection $\mathbb{R}^n_+$, possibly centered at $0 \in \partial\mathbb{R}^n_+$.

Lemma 2.1.4 ([GT01] Lemma 4.10)**.** *Let $f \in C^\alpha(\overline{B_{2r}^+})$ for $\alpha \in (0,1)$. Let the Newtonian Potential of f in B_{2r}^+ be as in* (2.1)

$$w(x) := \int_{B_{2r}^+} \Gamma(x-y) f(y) dy \qquad \forall\ x \in B_{2r}^+$$

Then $w \in C^{2,\alpha}(\overline{B_r^+})$ and there exists $C = C(n, \alpha) > 0$ s.t. the non-dimensional Hölder norms satisfy

$$\left\|D^2 w\right\|'_{C^{0,\alpha}(B_r^+)} \leq C\,\|f\|'_{C^{0,\alpha}(\overline{B_{2r}^+})} \tag{2.17}$$

i.e.

$$\|\partial_{ij} w\|_{C^0(\overline{B_r^+})} + r^\alpha [\partial_{ij} w]_{C^{0,\alpha}(\overline{B_r^+})} \leq C\left(\|f\|_{C^0(\overline{B_{2r}^+})} + r^\alpha [f]_{C^{0,\alpha}(\overline{B_{2r}^+})}\right) \qquad \forall\ i, j = 1, \cdots, n$$

Proof. Recall the formula (2.4) applied to $\Omega = B_r^+$ and $\Omega_0 = B_{2r}^+$

$$\partial_{ij} w(x) = \int_{B_{2r}^+} \partial_{x_i x_j} \Gamma(x-y)(f(y) - f(x))dy + f(x) \int_{\partial B_{2r}^+} \partial_{x_i} \Gamma(x-y)\nu_j(y)dS(y) \qquad \forall\, x \in B_r^+$$

What portions do we have on ∂B_{2r}^+? Let's decompose

$$\partial B_{2r}^+ = (B_{2r} \cap \partial \mathbb{R}_+^n) \cup \left(\partial B_{2r} \cap \mathbb{R}_+^n\right)$$

The difficulty happens when we're at the first boundary portion $B_{2r} \cap \partial \mathbb{R}_+^n$ essentially because the ball structure fails. Due to this, the essential estimates (2.7) for C^2 and (2.8), (2.9) for $C^{2,\alpha}$ wouldn't work.

What can we do to remove such bad boundary portion? A first notice is that for any $j \neq n$,

$$\nu_j|_{\partial \mathbb{R}_+^n} = 0$$

Also notice divergence theorem holds for the piecewise smooth B_{2r}^+, hence $w \in C^2(B_{2r}^+)$ following Lemma 2.1.2, and thus $\partial_{ij} w = \partial_{ji} w$. This amounts to say the two bad boundary portions

$$\int_{B_{2r} \cap \partial \mathbb{R}_+^n} \partial_{x_i} \Gamma(x-y)\nu_j(y)dS(y) = \int_{B_{2r} \cap \partial \mathbb{R}_+^n} \partial_{x_j} \Gamma(x-y)\nu_i(y)dS(y)$$

So as long as either $i \neq n$ or $j \neq n$, the bad boundary portion vanishes due to the normal. Hence the estimate for $\partial_{ij} w$ for $i \neq n$ or $j \neq n$ follows as in Lemma 2.1.3, and (2.17) follows for i, j not all n.

Now for $\partial_{nn} w$, let's make use of our equation (2.5)

$$\Delta w = f \qquad \forall\, x \in B_r^+$$

One obtain an expression

$$\partial_{nn} w = f - \sum_{i=1}^{n-1} \partial_{ii} w$$

and thus using norm structure

$$\begin{aligned} \|\partial_{nn} w\|'_{C^{0,\alpha}(B_r^+)} &\leq \|f\|'_{C^{0,\alpha}(\overline{B_r^+})} + \sum_{i=1}^{n-1} \|\partial_{ii} w\|'_{C^{0,\alpha}(B_r^+)} \\ &\leq C(n,\alpha)\, \|f\|'_{C^{0,\alpha}(\overline{B_{2r}^+})} \end{aligned}$$

and (2.17) follows. ☐

2.1.4.2 Boundary $C^{2,\alpha}$ Estimate for solutions over B_r^+ half balls

Now we have our boundary estimate for half ball for Newtonian Potential as in (2.17). The question is: How do we go to estimate for solution to Poisson's Equation?

If one want to write as before

$$u = w + v$$

for w Newtonian Potential

$$w(x) := \int_{B_{2r}^+} \Gamma(x-y) f(y)dy$$

Yes w has estimates via (2.17), but what about boundary estimates in v? How do we get estimates as below?

$$r\, \|Dv\|'_{C^{1,\alpha}(\overline{B_r^+})} \leq C(n) \left(\|u\|_{C^0(\overline{B_{2r}^+})} + \|f\|_{C^0(\overline{B_{2r}^+})} r^2 \right)$$

In particular estimation of $\|v\|_{C^0}$ with RHS is fine, but we lost gradient estimates for harmonic functions up to the boundary, which only work for the interior.

A clever strategy to deal with this is to reflect our harmonic function to B_{2r}^-, and **define a harmonic function in B_{2r}, on which we can apply interior gradient estimates**. The technique we use is *Schwarz Reflection.*

But we need to impose further that

$$v = u - w = 0 \qquad T := B_{2r} \cap \partial \mathbb{R}_+^n$$

so that reflection works. Now we have another issue on the boundary value. Yes we can impose the condition that $u = 0$ on T for now, but

$$w(x_1, \cdots, x_{n-1}, 0) = w(x', 0) = \int_{B_{2r}^+} \Gamma(x' - y', -y_n) f(y) dy \neq 0$$

Hence we need to redefine a potential $\tilde{w}$ that carries the estimates from Newtonian Potential (2.17), but also achieves 0 boundary data

$$\tilde{w} = 0 \qquad T$$

Can we do this? Yes we can, by replacing the Kernel with Green's Function (1.34). In fact if we also reflect f to the other side B_{2r}^-, we keep the estimates as in (2.17).

We deliver our estimate ([GT01] Theorem 4.11). Denote $B_{2r} = B_{2r}(0)$, $B_{2r}^+ := B_{2r} \cap \mathbb{R}_+^n$. Assume for some $0 < \alpha < 1$

$$u \in C^2(B_{2r}^+) \cap C^0(\overline{B_{2r}^+}), \qquad u = 0 \qquad T := B_{2r} \cap \partial\mathbb{R}_+^n, \qquad f \in C^\alpha(\overline{B_{2r}^+}), \qquad \Delta u = f \qquad B_{2r}^+ \tag{2.18}$$

1. Denote
$$B_{2r}^- := B_{2r} \cap \mathbb{R}_-^n = \{x \in B_{2r} \mid x_n < 0\}$$
In the first step we define our new potential $\tilde{w}$, then reflect f to B_{2r}^- and see how our estimates (2.17) translates to this case.

   (a) Recall $\tilde{x} = (x_1, \cdots, x_{n-1}, -x_n)$ denotes the reflection point. Recall Green's Function writes (1.34)
   $$G(x, y) := \Gamma(x - y) + \Phi(x, y) = \Gamma(x - y) - \Gamma(\tilde{x} - y) \qquad \forall\ x \in \overline{B_{2r}^+}, \qquad y \in B_{2r}^+, \qquad y \neq x$$
   and we define
   $$\tilde{w}(x) := \int_{B_{2r}^+} G(x, y) f(y) dy = \int_{B_{2r}^+} (\Gamma(x - y) - \Gamma(\tilde{x} - y)) f(y) dy \tag{2.19}$$
   One immediately notice
   $$\begin{aligned}\Delta\tilde{w}(x) &= \Delta_x \left(\int_{B_{2r}^+} \Gamma(x - y) f(y) dy \right) + \Delta_x \left(\int_{B_{2r}^+} \Gamma(\tilde{x} - y) f(y) dy \right) \\ &= f(x) \qquad \forall\ x \in B_{2r}^+ \qquad \text{using (2.5) and } \Delta_x\Gamma = 0 \text{ far from singularity}\end{aligned}$$
   And our friend: 0 boundary value, as argued in Property 1.3.3.
   $$\tilde{w}(x_1, \cdots x_{n-1}, 0) = 0$$

   (b) We define our **reflected force as the trivial reflection**
   $$\tilde{f}(x) := \begin{cases} f(x) = f(x_1, \cdots, x_{n-1}, x_n) & x_n \geq 0 \\ f(\tilde{x}) = f(x_1, \cdots, x_{n-1}, -x_n) & x_n < 0 \end{cases} \qquad \forall\ x \in B_{2r}$$
   $\tilde{f}$ carries the Hölder Regularity. For any $x \in B_{2r}^+$ and $y \in B_{2r}^-$, consider
   $$z = \text{intersection of line segment connecting } x \text{ and } y \text{, with } \partial\mathbb{R}_+^n$$
   Then
   $$\begin{aligned}\frac{|\tilde{f}(x) - \tilde{f}(y)|}{|x - y|^\alpha} &\leq \frac{|\tilde{f}(x) - \tilde{f}(z)|}{|x - y|^\alpha} + \frac{|\tilde{f}(z) - \tilde{f}(y)|}{|x - y|^\alpha} \\ &\leq [f]_{C^0(\overline{B_{2r}^+})} (\frac{|x - z|}{|x - y|})^\alpha + [f]_{C^0(\overline{B_{2r}^+})} (\frac{|z - y|}{|x - y|})^\alpha \\ &\leq 2[f]_{C^0(\overline{B_{2r}^+})} \\ \left\|\tilde{f}\right\|'_{C^{0,\alpha}(\overline{B_{2r}})} &= \left\|\tilde{f}\right\|_{C^0(\overline{B_{2r}})} + r^\alpha [\tilde{f}]_{C^{0,\alpha}(\overline{B_{2r}})} \\ &\leq \|f\|_{C^0(\overline{B_{2r}^+})} + 2r^\alpha [f]_{C^0(\overline{B_{2r}^+})} \leq 2\|f\|'_{C^{0,\alpha}(\overline{B_{2r}^+})}\end{aligned} \tag{2.20}$$

(c) Now with $\tilde{f}$ one can rewrite $\tilde{w}$.

$$\tilde{w}(x) = \int_{B_{2r}^+} \left(\Gamma(x-y) - \Gamma(\tilde{x}-y)\right) f(y)dy = \int_{B_{2r}^+} \left(\Gamma(x-y) - \Gamma(x-\tilde{y})\right) f(y)dy \qquad \text{using radial symmetry of } \Gamma$$

Notice for the second portion, one can shift $\tilde{y}$ to $\tilde{f}(y)$

$$\begin{aligned}
&\int_{B_{2r}^+} \Gamma(x-\tilde{y}) f(y)dy \\
&= \int_{B_{2r}\cap\{y_n>0\}} \Gamma(x'-y', x_n+y_n) f(y', y_n)dy = \int_{B_{2r}\cap\{y_n>0\}} \Gamma(x'-y', x_n-(-y_n)) f(y', -(-y_n))dy \\
&= \int_{B_{2r}\cap\{y_n<0\}} \Gamma(x'-y', x_n-y_n) f(y', -y_n)dy = \int_{B_{2r}^-} \Gamma(x-y)\tilde{f}(y)dy
\end{aligned}$$

Thus $\tilde{w}$ rewrites

$$\begin{aligned}
\tilde{w}(x) &= \int_{B_{2r}^+} \Gamma(x-y) f(y)dy - \int_{B_{2r}^-} \Gamma(x-y)\tilde{f}(y)dy \\
&= 2\int_{B_{2r}^+} \Gamma(x-y) f(y)dy - \int_{B_{2r}} \Gamma(x-y)\tilde{f}(y)dy \qquad \forall\, x \in B_{2r}^+ \\
&=: 2w_1(x) - w_2(x)
\end{aligned}$$

Now for the first piece, we have good estimates as in (2.17). For the second piece we apply interior estimates (2.6) to B_r^+ and B_{2r} so that

$$\begin{aligned}
\left\|D^2\tilde{w}\right\|'_{C^{0,\alpha}(\overline{B_r^+})} &\leq 2\left\|D^2 w_1\right\|'_{C^{0,\alpha}(\overline{B_r^+})} + \left\|D^2 w_2\right\|'_{C^{0,\alpha}(\overline{B_r^+})} \\
&\leq 2C(n,\alpha)\left\|f\right\|'_{C^{0,\alpha}(\overline{B_{2r}^+})} + C(n,\alpha)\left\|\tilde{f}\right\|'_{C^{0,\alpha}(\overline{B_{2r}})} \qquad \text{(2.17) and (2.6)} \\
&\overset{(2.20)}{\leq} 4C(n,\alpha)\left\|f\right\|'_{C^{0,\alpha}(\overline{B_{2r}^+})}
\end{aligned}$$

On the other hand, the first order derivative removes the singularity, hence we estimate directly $\nabla\tilde{w}$. For any $x \in B_r^+$

$$\begin{aligned}
|\nabla\tilde{w}(x)| &\leq 2|\nabla w_1(x)| + |\nabla w_2(x)| \\
&\leq C(n)\left\|\tilde{f}\right\|_{C^0(\overline{B_{2r}})} \int_0^{4r} s^{1-n}s^{n-1}ds \leq C(n)\left\|\tilde{f}\right\|_{C^0(\overline{B_{2r}})} r \\
&\leq C(n)\left\|f\right\|_{C^0(\overline{B_{2r}^+})} r \\
\left\|\nabla\tilde{w}\right\|_{C^0(\overline{B_r^+})} &\leq 2\left\|\nabla w_1\right\|_{C^0(\overline{B_r^+})} + \left\|\nabla w_2\right\|_{C^0(\overline{B_r^+})} \\
&\leq C(n)\left\|f\right\|_{C^0(\overline{B_{2r}^+})} r
\end{aligned}$$

Thus from the first set of estimates one obtain the estimate for non-dimensional norms

$$\begin{aligned}
r\left\|D\tilde{w}\right\|'_{C^{1,\alpha}(\overline{B_r^+})} &\leq r\left(\left\|\nabla\tilde{w}\right\|_{C^0(\overline{B_r^+})} + r\left\|D^2\tilde{w}\right\|'_{C^{0,\alpha}(\overline{B_r^+})}\right) \\
&\leq C(n)\left\|f\right\|_{C^0(\overline{B_{2r}^+})} r^2 + 4C(n,\alpha)\left\|f\right\|'_{C^{0,\alpha}(\overline{B_{2r}^+})} r^2 \\
&\leq C(n,\alpha)\left\|f\right\|'_{C^{0,\alpha}(\overline{B_{2r}^+})} r^2
\end{aligned} \tag{2.21}$$

2. Now in the second step, we define

$$\tilde{v} := u - \tilde{w} \qquad B_{2r}^+$$

Since both $u = \tilde{w}$ on T, one get $\tilde{v} = 0$ on T. Also $\tilde{v}$ is harmonic

$$\Delta\tilde{v} = \Delta u - \Delta\tilde{w} = f - f = 0 \qquad B_{2r}^+$$

One may define the Schwarz Reflection (1.40) $V \in C^2(B_{2r})$ s.t. $V = \tilde{v}$ in B_{2r}^+, and apply Lemma 1.3.3 so that V is harmonic in B_{2r}. Now since we have interior estimates for harmonic function V, one obtain

$$\begin{aligned}
r\left\|D\tilde{v}\right\|'_{C^{1,\alpha}(\overline{B_r^+})} &\leq r\left\|DV\right\|'_{C^{1,\alpha}(\overline{B_r})} \overset{(2.15)}{\leq} C(n)\left\|V\right\|_{C^0(\overline{B_{\frac{3}{2}r}})} \\
&\leq C(n)\left\|\tilde{v}\right\|_{C^0(\overline{B^+_{\frac{3}{2}r}})} \qquad \text{using definition of Schwarz Reflection (1.40)} \\
&\leq C(n)\left(\left\|u\right\|_{C^0(\overline{B^+_{\frac{3}{2}r}})} + \left\|\tilde{w}\right\|_{C^0(\overline{B^+_{\frac{3}{2}r}})}\right) \\
&= C(n)\left(\left\|u\right\|_{C^0(\overline{B^+_{\frac{3}{2}r}})} + \left\|w_1\right\|_{C^0(\overline{B^+_{\frac{3}{2}r}})} + \left\|w_2\right\|_{C^0(\overline{B^+_{\frac{3}{2}r}})}\right) \\
&\leq C(n)\left(\left\|u\right\|_{C^0(\overline{B^+_{2r}})} + C(n)\left\|\tilde{f}\right\|_{C^0(\overline{B^+_{2r}})} r^2\right) \qquad \text{using (2.13), (2.14)} \\
&\leq C(n)\left(\left\|u\right\|_{C^0(\overline{B^+_{2r}})} + C(n)\left\|f\right\|_{C^0(\overline{B^+_{2r}})} r^2\right) \qquad \text{using (2.20)}
\end{aligned} \tag{2.22}$$

3. To obtain the estimate for u, we mimic steps that lead to (2.16). For $n \geq 3$ we define $\tilde{w}$ as in (2.19). For $n = 2$, we view the solution as defined in $n = 3$, and choose $\Gamma(x-y) = \frac{1}{-3\omega_3}|x-y|^{-1}$ as our kernel. The definition (2.19) follows with force f defined in $n = 2$. In both cases we define $\tilde{v} = u - \tilde{w}$.

$$\begin{aligned}
\left\|u\right\|'_{C^{2,\alpha}(\overline{B_r^+})} &= \left\|u\right\|_{C^0(\overline{B_r^+})} + r\left\|\nabla u\right\|_{C^0(\overline{B_r^+})} + r^2\left\|D^2u\right\|_{C^0(\overline{B_r^+})} + r^{2+\alpha}[D^2u]_{C^{0,\alpha}(\overline{B_r^+})} \\
&\leq \left\|u\right\|_{C^0(\overline{B_r^+})} + r\left\|D\tilde{w}\right\|'_{C^{1,\alpha}(\overline{B_r^+})} + r\left\|D\tilde{v}\right\|'_{C^{1,\alpha}(\overline{B_r^+})} \\
&\overset{(2.21),(2.22)}{\leq} \left\|u\right\|_{C^0(\overline{B_r^+})} + C(n,\alpha)\left\|f\right\|'_{C^{0,\alpha}(\overline{B^+_{2r}})} r^2 + C(n)\left(\left\|u\right\|_{C^0(\overline{B^+_{2r}})} + \left\|f\right\|_{C^0(\overline{B^+_{2r}})} r^2\right) \\
&\leq C(n,\alpha)\left(\left\|u\right\|_{C^0(\overline{B^+_{2r}})} + \left\|f\right\|'_{C^{0,\alpha}(\overline{B^+_{2r}})} r^2\right)
\end{aligned} \tag{2.23}$$

In particular we've improved to $C^{2,\alpha}$ regularity up to the boundary

$$u \in C^{2,\alpha}(\overline{B_r^+})$$

2.1.5 Global $C^{2,\alpha}$ Regularity for solution to Dirichlet BVP over B_R balls

Recall that for $\Omega \subseteq \mathbb{R}^n$ open bounded connected, one has well-posedness of Dirichlet Boundary Value problem Theorem 2.1.1. A natural question to ask is, if given $f \in C^{0,\alpha}(\overline{\Omega})$ Hölder forcing and boundary data with sufficient regularity, does our solution u inherit $C^{2,\alpha}$?

For general domains we leave to later. But for Balls, one has trick known as *Kelvin Transform* (1.49). The good thing about the Kelvin Transform (1.48)

$$x \in \mathbb{R}^n \setminus \{0\} \mapsto x^* = \frac{x}{|x|^2} \in \mathbb{R}^n \setminus \{0\}$$

is that this maps the unit ball centered at $e_n = (0, \cdots, 0, 1)$ to the half space with boundary $\{x_n = \frac{1}{2}\}$. Indeed, for the boundary portion, take any $x \in \partial B_1(e_n)$

$$\begin{aligned}
|x - e_n| &= 1 \\
\sum_{i=1}^{n-1} x_i^2 + (x_n - 1)^2 &= 1 \\
|x|^2 - 2x_n &= 0 \\
\frac{1}{2} &= \frac{x_n}{|x|^2} = x_n^*
\end{aligned}$$

Now for the interior, $|x - e_n| < 1$,

$$|x|^2 - 2x_n < 0 \iff \frac{1}{2} < \frac{x_n}{|x|^2} = x_n^*$$

Thus

$$B_1(e_n) \overset{\text{Kelvin Transform (1.48)}}{\longrightarrow} \{x \in \mathbb{R}^n \mid x_n > \frac{1}{2}\} \tag{2.24}$$

Now for a solution to Poisson's Equation u defined on

$$\Delta u = f \qquad B_1(e_n)$$

Using (1.50), we know its Kelvin transform

$$u^*(x) := |x|^{2-n} u(\frac{x}{|x|^2}) \qquad \forall\, x_n > \frac{1}{2}$$

solves the Poisson's equation

$$\Delta u^*(x) = |x|^{-n-2} f(\frac{x}{|x|^2}) \qquad \forall\, x_n > \frac{1}{2}$$

Now one has flat boundary for u^*, and may apply estimates obtained from (2.23).

We deliver our first set of estimates assuming zero boundary data ([GT01] Theorem 4.13). Let B_1 be unit ball in $\mathbb{R}^n$. Assume for some $0 < \alpha < 1$

$$u \in C^2(B_1) \cap C^0(\overline{B_1}), \qquad u = 0 \qquad \partial B_1, \qquad f \in C^\alpha(\overline{B_1}), \qquad \Delta u = f \qquad B_1 \tag{2.25}$$

1. For any boundary point $x_0 \in \partial B_1$, one may do translation so that assume $x_0 = 0$. Also, one may do rotation s.t. $B_1 = B_1(e_n)$. Now as is calculated in (2.24)

$$B_1(e_n) \overset{\text{Kelvin Transform (1.48)}}{\longrightarrow} \{x \in \mathbb{R}^n \mid x_n > \frac{1}{2}\}$$

   One consider u^* as in (1.49) which solves

$$\Delta u^*(x) = |x|^{-n-2} f(\frac{x}{|x|^2}) \qquad \forall\, x_n > \frac{1}{2}$$

2. Now consider the portion of u^* defined in

$$B_r^+ := B_r(\frac{1}{2}e_n) \cap \{x_n > \frac{1}{2}\}$$

   for $0 < r < \frac{1}{8}$ small. On this piece one has estimates via (2.23)

$$\|u^*\|'_{C^{2,\alpha}(\overline{B_r^+})} \le C(n,\alpha) \left(\|u^*\|_{C^0(\overline{B_{2r}^+})} + \left\| |x|^{-n-2} f(\frac{x}{|x|^2}) \right\|'_{C^{0,\alpha}(\overline{B_{2r}^+})} r^2 \right) \tag{2.26}$$

   Notice we've made use that ∂B_1 is mapped to $\{x_n = \frac{1}{2}\}$, and we assumed

$$u^*|_{x_n=\frac{1}{2}} = u|_{\partial B_1} = 0$$

   (a) Let's control RHS of (2.26) via u and f. The question is: what does B_{2r}^+ map to under Kelvin Transform? For any $x \in B_{2r}^+$

$$\begin{aligned} |x - \frac{1}{2}e_n| &< 2r \\ |x|^2 - x_n &< 4r^2 - \frac{1}{4} < 0 \\ 1 &< \frac{x_n}{|x|^2} = \tilde{x}_n \end{aligned}$$

   Thus, using that Kelvin Transform equals to its inverse transformation, and we know one started with $B_1(e_n)$, necessarily

$$B_{2r}^+ \qquad \text{is mapped into } B_1(e_n) \cap \{x_n > 1\} \text{ under Kelvin Transform}$$

   Hence one has control over RHS of (2.26)

$$\|u^*\|_{C^0(\overline{B_{2r}^+})} = \left\| |x|^{2-n} u(\frac{x}{|x|^2}) \right\|_{C^0(\overline{B_{2r}^+})} \le C(n) \|u\|_{C^0(\overline{B_1(e_n)})}$$

$$\left\| |x|^{-n-2} f(\frac{x}{|x|^2}) \right\|_{C^{0,\alpha}(\overline{B_{2r}^+})} r^2 \le C(n) \|f\|'_{C^{0,\alpha}(\overline{B_1(e_n)})} r^2$$

   Both of which one has good control of via assumption (2.25).

(b) Let's bound LHS of (2.26) from below via u. The important thing to understand is, which portion of u are we controlling using this estimate? First of all, $B_1(e_n)$ gets mapped to $\{x_n > \frac{1}{2}\}$, so we want to study which part of $B_1(e_n)$ gets mapped into B_r^+. To see this, for any point

$$x \in B_r^+ = B_r(\frac{1}{2}e_n) \cap \{x_n > \frac{1}{2}\}$$

x necessarily satisfy

$$|x - \frac{1}{2}e_n| < r$$
$$|x|^2 - x_n + \frac{1}{4} < r^2$$
$$1 + (\frac{1}{4} - r^2)\frac{1}{|x|^2} < \frac{x_n}{|x|^2} = \tilde{x}_n$$

Notice $\frac{1}{4} - r^2 > 0$. Since $\frac{1}{2} < |x| < \frac{1}{2} + r$, in order to maximize LHS, $|x|$ may be taken to be $\frac{1}{2}$. Thus one need to ensure

$$1 + (\frac{1}{4} - r^2)\frac{1}{(\frac{1}{2})^2} = 2 - 4r^2 < \tilde{x}_n$$

Since the inverse Kelvin Transform is itself, this is to say

$$B_{4r^2}(2e_n) \cap B_1(e_n) \qquad \text{is mapped into } B_r^+ = B_r(\frac{1}{2}e_n) \cap \{x_n > \frac{1}{2}\} \text{ under Kelvin Transform}$$

Thus one has estimate

$$\|u\|'_{C^{2,\alpha}(\overline{B_{4r^2}(2e_n) \cap B_1(e_n)})} \leq \left\| u(\frac{x}{|x|^2}) \right\|'_{C^{2,\alpha}(\overline{B_r^+})} \leq C(n,r) \|u^*\|'_{C^{2,\alpha}(\overline{B_r^+})}$$

Now following estimates as above, one translate (2.26) to

$$\|u\|'_{C^{2,\alpha}(\overline{B_{4r^2}(2e_n) \cap B_1(e_n)})} \leq C(n,r,\alpha) \left(\|u\|_{C^0(\overline{B_1(e_n)})} + \|f\|'_{C^{0,\alpha}(\overline{B_1(e_n)})} \right) \tag{2.27}$$

In particular, notice putting x_0 at the origin 0 gives the estimate up to the boundary portion near the point $2e_n$. This is due to the nature of Kelvin Transform.

3. Now using ∂B_1 is compact, there exists finitely many balls with fixed radius $4r^2$ for $0 < r < \frac{1}{8}$, and with centers on the boundary s.t.

$$\partial B_1 \subseteq \bigcup_{i=1}^{N} B_{4r^2}(x_i) \qquad \text{for } x_i \in \partial B_1,\ i = 1, \cdots, N$$

Now for each $B_{4r^2}(x_i)$, treat x_i as $2e_n$ and its reflected point w.r.t. the center as 0 the origin, one recover estimate (2.27). Hence

$$\sum_{i=1}^{N} \|u\|'_{C^{2,\alpha}(\overline{B_{4r^2}(x_i) \cap B_1})} \leq C(n,r,\alpha,N) \left(\|u\|_{C^0(\overline{B_1})} + \|f\|'_{C^{0,\alpha}(\overline{B_1})} \right)$$

Now the interior portion is included via

$$B_1 \setminus \left(\bigcup_{i=1}^{N} B_{4r^2}(x_i) \right) \subseteq B_R \Subset B_{R+\varepsilon} \Subset B_1$$

Hence following (2.16)

$$\|u\|'_{C^{2,\alpha}(\overline{B_R})} \leq C(n,R,\varepsilon,\alpha) \left(\|u\|_{C^0(\overline{B_{R+\varepsilon}})} + \|f\|'_{C^{0,\alpha}(\overline{B_{R+\varepsilon}})} \right)$$

Summing the Interior and Boundary estimate up gives the full estimate

$$\begin{aligned} \|u\|'_{C^{2,\alpha}(\overline{B_1})} &\leq \|u\|'_{C^{2,\alpha}(\overline{B_R})} + \sum_{i=1}^{N} \|u\|'_{C^{2,\alpha}(\overline{B_{4r^2}(x_i) \cap B_1})} \\ &\leq C(n,\alpha,r,N,R,\varepsilon) \left(\|u\|_{C^0(\overline{B_1})} + \|f\|'_{C^{0,\alpha}(\overline{B_1})} \right) \end{aligned} \tag{2.28}$$

In particular, we've improved to $C^{2,\alpha}$ regularity over the whole ball

$$u \in C^{2,\alpha}(\overline{B_1})$$

Hence one has $C^{2,\alpha}$ global regularity for Dirichlet Boundary Value Problem.

Theorem 2.1.3 ([GT01] Corollary 4.14; De Silva Analysis II 2025)**.** *Let $B_1 \subseteq \mathbb{R}^n$ denote unit ball. Assume for $0 < \alpha < 1$, $f \in C^{\alpha}(\overline{B_1})$, $\varphi \in C^{2,\alpha}(\overline{B_1})$. Then the Dirichlet Boundary Value Problem*

$$\begin{cases} \Delta u = f & B_1 \\ u = \varphi & \partial B_1 \end{cases}$$

is uniquely solvable for $u \in C^{2,\alpha}(\overline{B_1})$. One has estimate for $C = C(n,\alpha)$

$$\|u\|_{C^{2,\alpha}(\overline{B_1})} \le C \left(\|\varphi\|_{C^{2,\alpha}(\overline{B_1})} + \|f\|_{C^{0,\alpha}(\overline{B_1})} \right) \tag{2.29}$$

Proof. First, using Theorem 2.1.1, we know $u \in C^2(B_1) \cap C^0(\overline{B_1})$ solves the Dirichlet BVP uniquely. Denote $v := u - \varphi$. Then v necessarily solves

$$\begin{cases} \Delta v = f - \Delta\varphi & B_1 \\ v = 0 & \partial B_1 \end{cases}$$

Note the force for v belongs to

$$f - \Delta\varphi \in C^{0,\alpha}(\overline{B_1})$$

Hence v fits into our assumption (2.25) with zero boundary data. Using estimate (2.28) one obtain

$$v \in C^{2,\alpha}(\overline{B_1})$$

Thus

$$u = v + \varphi \in C^{2,\alpha}(\overline{B_1})$$

What estimates do we have?

$$\begin{aligned} \|u\|_{C^{2,\alpha}(\overline{B_1})} &\le \|v\|_{C^{2,\alpha}(\overline{B_1})} + \|\varphi\|_{C^{2,\alpha}(\overline{B_1})} \\ &\overset{(2.28)}{\le} C(n,\alpha) \left(\|v\|_{C^0(\overline{B_1})} + \|f - \Delta\varphi\|_{C^{0,\alpha}(\overline{B_1})} \right) + \|\varphi\|_{C^{2,\alpha}(\overline{B_1})} \\ &\le C(n,\alpha) \left(\|v\|_{C^0(\overline{B_1})} + \|\varphi\|_{C^{2,\alpha}(\overline{B_1})} + \|f\|_{C^{0,\alpha}(\overline{B_1})} \right) \end{aligned}$$

Let's deal with the $\|v\|_{C^0}$ term. Notice v is of 0 boundary data, hence using Green's Representation (1.32), for any $x \in B_1$

$$\begin{aligned} |v(x)| &\le \|f\|_{C^0(\overline{B_1})} \left| \int_{B_2(x)} G(x,y)dy \right| \\ &\le \|f\|_{C^0(\overline{B_1})} \left(\int_{B_2(x)} |\Gamma(x-y)|dy + C(n) \right) \qquad \text{using Property 1.3.2} \\ &\le C(n) \|f\|_{C^0(\overline{B_1})} \qquad \forall\, x \in B_1 \end{aligned}$$

Thus one obtain the estimate (2.29). $\square$

2.2 Singular Integrals

We ask under what conditions is $w := f * \Gamma$ as in (2.1) a classical solution.

Lemma 2.2.1 (Savin Analysis II Spring 2026)**.** *For w the Newtonian Potential (2.1), with $f \in C_0^{0,\alpha}(\mathbb{R}^n)$ Hölder with compact support, and $0 < \alpha < 1$. Then $w \in C^2(\mathbb{R}^n)$ and*

$$\partial_{ij} w = \frac{\delta_{ij}}{n} f + \text{p.v.}\left(f * \partial_{ij}\Gamma\right)(x) \tag{2.30}$$

Compare the formula with (2.4).

Example 2.2.1. *We remark that f continuous fails. For example examine*

$$u(x_1, x_2) = (x_1^2 - x_2^2)(-\log|x|)^{\frac{1}{2}}$$

and consider

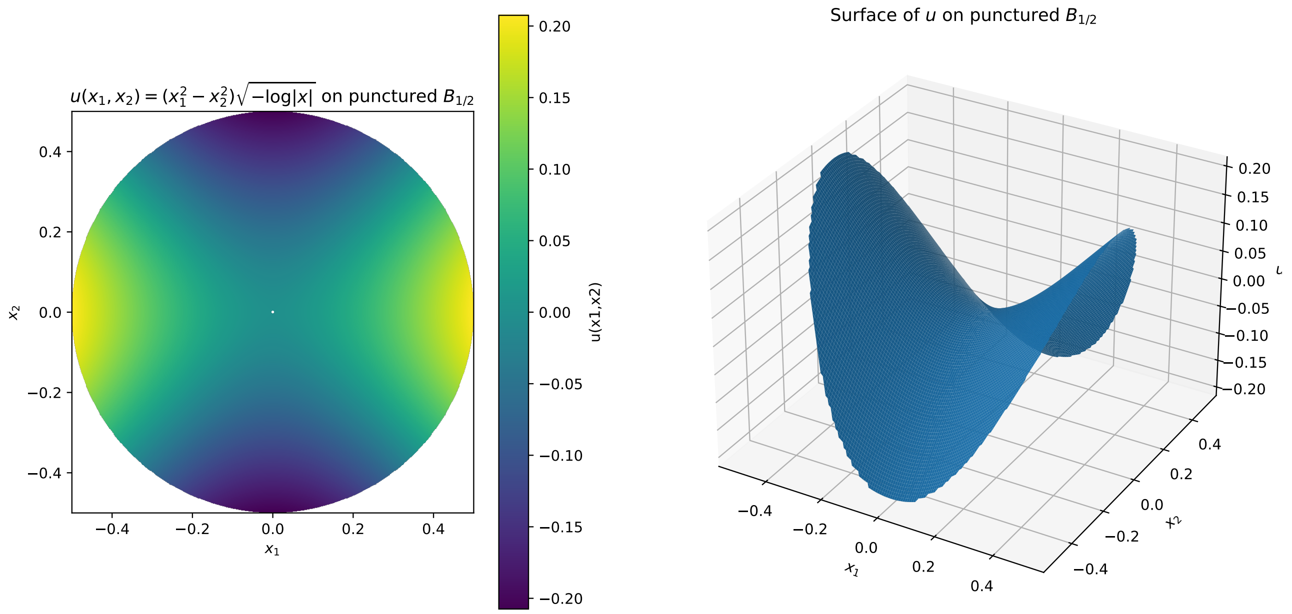
$$f(x) := \begin{cases} \frac{x_2^2 - x_1^2}{2|x|^2}\left(\frac{4}{(-\log|x|)^{\frac{1}{2}}} + \frac{1}{2(-\log|x|)^{\frac{3}{2}}}\right) & B_{1/2}\setminus\{0\} \\ 0 & 0 \end{cases}$$

Such f is continuous but not Hölder of any α.

Now on the region $B_{1/2}\setminus\{0\}$

$$\Delta u = f \qquad B_{1/2}\setminus\{0\}$$

and $u \in C^\infty(B_{1/2}\setminus\{0\}) \cap C^0(B_{1/2})$ but $u \notin C^2(B_{1/2})$. In fact no C^2 solution in $B_{1/2}$ exists for such f. Indeed,



(a) Heatmap of u on $B_{1/2}\setminus\{0\}$. (b) Surface plot of u on $B_{1/2}\setminus\{0\}$.

Figure 2.1: Plots of $u(x_1, x_2) = (x_1^2 - x_2^2)\sqrt{-\log|x|}$ on the punctured disk $B_{1/2}\setminus\{0\}$.

assume there is such $v \in C^2(B_{1/2})$ with $\Delta v = f$ in $B_{1/2}$. Then $w = v - u$ solves

$$\Delta w = 0 \qquad B_{1/2}\setminus\{0\}$$

and w is bounded at 0. Then w can be extended to be a harmonic function in $B_{1/2}$. But this contradicts u is not C^2.

Well-definedness of Singular Integral Operator Tf Let's make sense of the singular integral operator. First of all, the *Cauchy Principal Value* is defined via

$$\text{p.v.}(f * K)(x) := \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B_\varepsilon(0)} K(y) f(x-y)\,dy \tag{2.31}$$

Proposition 2.2.1 (De Silva Analysis II Spring 2025)**.** *Assume that for*

$$K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$$

we know

1. *K is homogeneous of degree $-n$, i.e., there exists $g \in C(\partial B_1)$ s.t.*

$$K(x) = |x|^{-n} g(\frac{x}{|x|}) \tag{2.32}$$

2. *K on the sphere averages out to* 0

$$\int_{\partial B_1} K\, dS = \int_{\partial B_1} g\, dS = 0 \tag{2.33}$$

Consider the singular operator defined via

$$Tf(x) := \text{p.v.}\,(f * K)\,(x) \overset{(2.31)}{=} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} K(y) f(x-y) dy$$

Then Tf is well-defined function, and in fact $Tf \in C(\mathbb{R}^n)$ for any $f \in C_0^{0,\alpha}(\mathbb{R}^n)$ Hölder with compact support, $0 < \alpha < 1$.

Proof. Take $f \in C_0^{0,\alpha}(\mathbb{R}^n)$. Since f is compactly supported, there exists $R > 0$ large s.t.

$$\int_{\mathbb{R}^n \setminus B_\varepsilon(0)} K(y) f(x-y) dy = \int_{B_R \setminus B_\varepsilon(0)} K(y) f(x-y) dy$$

We first compute

$$\int_{B_R \setminus B_\varepsilon(0)} K(y) dy = \int_\varepsilon^R \int_{\partial B_r} K dS\, dr \overset{(2.32)}{=} C_n \int_\varepsilon^R r^{-n} r^{n-1} \int_{\partial B_1} g\, dS dr \overset{(2.33)}{=} 0$$

Now it makes sense to look at the difference

$$\int_{B_R \setminus B_\varepsilon(0)} K(y) f(x-y) dy = \int_{B_R \setminus B_\varepsilon(0)} K(y) \left(f(x-y) - f(x)\right) dy$$

We would like to show this converges uniformly in x as $\varepsilon \to 0$.

But before, this we need to ensure the following integral (which is the candidate for a limit as above) converges

$$|\int_{B_R} K(y)(f(x-y) - f(x)) dy| \leq C(n)[f]_{C^{0,\alpha}} \int_0^R r^{-n} \cdot r^\alpha \cdot r^{n-1} dr = C(n)[f]_{C^{0,\alpha}} R^\alpha < \infty$$

This is a function that is L^∞.

Ok, now we want to show

$$\begin{aligned}
&|\int_{B_R} K(y)(f(x-y) - f(x)) dy - \int_{B_R \setminus B_\varepsilon(0)} K(y)(f(x-y) - f(x)) dy| \\
&= |\int_{B_\varepsilon(0)} K(y)(f(x-y) - f(x)) dy| \\
&\leq \int_{B_\varepsilon(0)} |K(y)||f(x-y) - f(x)| dy \\
&\leq C[f]_{C^{0,\alpha}} \int_0^\varepsilon r^{-n} \cdot r^\alpha \cdot r^{n-1} dr \\
&= C(n,\alpha)[f]_{C^{0,\alpha}} \varepsilon^\alpha
\end{aligned}$$

which converges uniformly in x as $\varepsilon \to 0$. Hence this convergence is uniform on all compact subsets, which means

$$\int_{B_R} K(y)(f(x-y) - f(x)) dy = \lim_{\varepsilon \to 0} \int_{B_R \setminus B_\varepsilon(0)} K(y)(f(x-y) - f(x)) dy = Tf(x) \tag{2.34}$$

Defines $Tf \in C(\mathbb{R}^n)$. □

Representation Formula for $\partial_{ij}w$ In this paragraph we prove Lemma 2.2.1. Notice in our case

$$\begin{aligned}\text{p.v.}(f*\partial_{ij}\Gamma)(x) &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B_\varepsilon(0)}\partial_{ij}\Gamma(y)f(x-y)dy\\ &= \lim_{\varepsilon\to 0} f*(\partial_{ij}\Gamma\chi_{B_\varepsilon^c})(x)\end{aligned}$$

Why does $\partial_{ij}\Gamma$ satisfy assumptions for Proposition 2.2.1? Compute

$$\begin{aligned}\Gamma &= C_n|x|^{2-n}\\ \partial_i\Gamma &= C_n'\frac{x_i}{|x|^n}\\ \partial_{ij}\Gamma &= C_n'\frac{-nx_ix_j+|x|^2\delta_{ij}}{|x|^{n+2}}\end{aligned}$$

Now

$$\int_{\partial B_1}\partial_{ij}\Gamma = C_n'\int_{\partial B_1}\frac{-nx_ix_j+|x|^2\delta_{ij}}{|x|^{n+2}} = 0$$

Thus Proposition 2.2.1 is applicable to $K=\partial_{ij}\Gamma$. Consequently, the second term on RHS of (2.30) is in fact continuous. Since f itself is continuous, once we prove the representation formula, we know that $w\in C^2(\mathbb{R}^n)$.

Proof of Lemma 2.2.1. Recall our best friend to approximate Γ, the $\Gamma_\varepsilon\in C^{1,1}(\mathbb{R}^n)$

$$\Gamma_\varepsilon(x) := \begin{cases}\Gamma & |x|\geq\varepsilon\\ \frac{1}{2n|B_\varepsilon|}(|x|^2-\varepsilon^2)+\Gamma(\varepsilon) & |x|\leq\varepsilon\end{cases}$$

Now define $w^\varepsilon := f*\Gamma_\varepsilon$. We compute

$$\begin{aligned}\partial_{ij}w^\varepsilon(x) &= f*\partial_{ij}(\Gamma_\varepsilon)(x)\\ &= f*(\partial_{ij}(\Gamma_\varepsilon)\chi_{B_\varepsilon})(x)+f*(\partial_{ij}\Gamma\chi_{B_\varepsilon^c})(x)\\ &= f*(\frac{\delta_{ij}}{n|B_\varepsilon|}\chi_{B_\varepsilon})(x)+f*(\partial_{ij}\Gamma\chi_{B_\varepsilon^c})(x)\\ &= \frac{\delta_{ij}}{n}\fint_{B_\varepsilon(x)}f+f*(\partial_{ij}\Gamma\chi_{B_\varepsilon^c})(x)\end{aligned}$$

Now using Lebesgue Differentiation Theorem and that f is continuous, and the definition for (2.31), one obtain

$$\lim_{\varepsilon\to 0}\partial_{ij}w^\varepsilon(x) = \frac{\delta_{ij}}{n}f(x)+\text{p.v.}(f*\partial_{ij}\Gamma)(x)$$

Since w^ε converge uniformly to w and $\partial_{ij}w^\varepsilon(x)$ converge uniformly as well, we know that $\partial_{ij}w$ must coincide with the limit of $\partial_{ij}w^\varepsilon(x)$. □

Properties of $\text{p.v.}(f*\partial_{ij}\Gamma)(x)$ Assume that $u\in C_0^\infty(\mathbb{R}^n)$, then $u=w$. One may apply the Fourier Transform

$$\hat{u}(\xi) := \int_{\mathbb{R}^n}u(x)e^{-ix\cdot\xi}dx$$

so that the inversion writes

$$u(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\hat{u}(\xi)e^{ix\cdot\xi}d\xi$$

Now if we take derivative

$$\begin{aligned}\widehat{\partial_i u}(\xi) &= i\xi_i\hat{u}(\xi)\\ \widehat{\partial_{ii}u}(\xi) &= -\xi_i^2\hat{u}(\xi)\\ \widehat{\Delta u}(\xi) &= -|\xi|^2\hat{u}(\xi)\end{aligned}$$

the equation $\Delta u = f$ under transformation is

$$\widehat{u}(\xi) = -\frac{1}{|\xi|^2}\hat{f}$$

and thus

$$\widehat{\partial_{ij} u}(\xi) = \frac{\xi_i \xi_j}{|\xi|^2} \hat{f}$$

Notice $\frac{\xi_i \xi_j}{|\xi|^2}$ in the front is homogeneous of degree 0. This is multiplying $\hat{f}$ by something that doesn't decay at all. Thus formally

$$\widehat{\partial_{ij}\Gamma}(\xi) = \frac{\xi_i \xi_j}{|\xi|^2} \qquad i \neq j$$
$$\widehat{\partial_{ii}\Gamma}(\xi) = -\frac{1}{n} + \frac{\xi_i^2}{|\xi|^2}$$

$C^{0,\alpha}$ Regularity of Singular Integral Operator Tf

Proposition 2.2.2. *Let $\alpha \in (0,1)$. Take K as in Proposition 2.2.1. If we furthermore assume that K is homogeneous of degree $-n$ with $g \in C^{0,1}(\partial B_1)$ being Lipschitz so that it has decay rate*

$$|\nabla K(x)| \leq \frac{C}{|x|^{n+1}} \tag{2.35}$$

Then for any $f \in C_0^{0,\alpha}(\mathbb{R}^n)$, in fact $Tf \in C^{0,\alpha}(\mathbb{R}^n)$ with

$$[Tf]_{C^{0,\alpha}} \leq C(n,K)[f]_{C^{0,\alpha}}$$

If $\operatorname{supp}(f) \subseteq B_R$, *then*

$$\|Tf\|_{C^{0,\alpha}(\overline{B_R})} \leq C(K,n,R)\,\|f\|_{C^{0,\alpha}(\overline{B_R})}$$

Proof. WLOG assume $[f]_{C^{0,\alpha}} = 1$. Let $\operatorname{supp}(f) \subseteq B_R$. We look at $y, z \in B_R$. Denote $a = |y-z| > 0$.

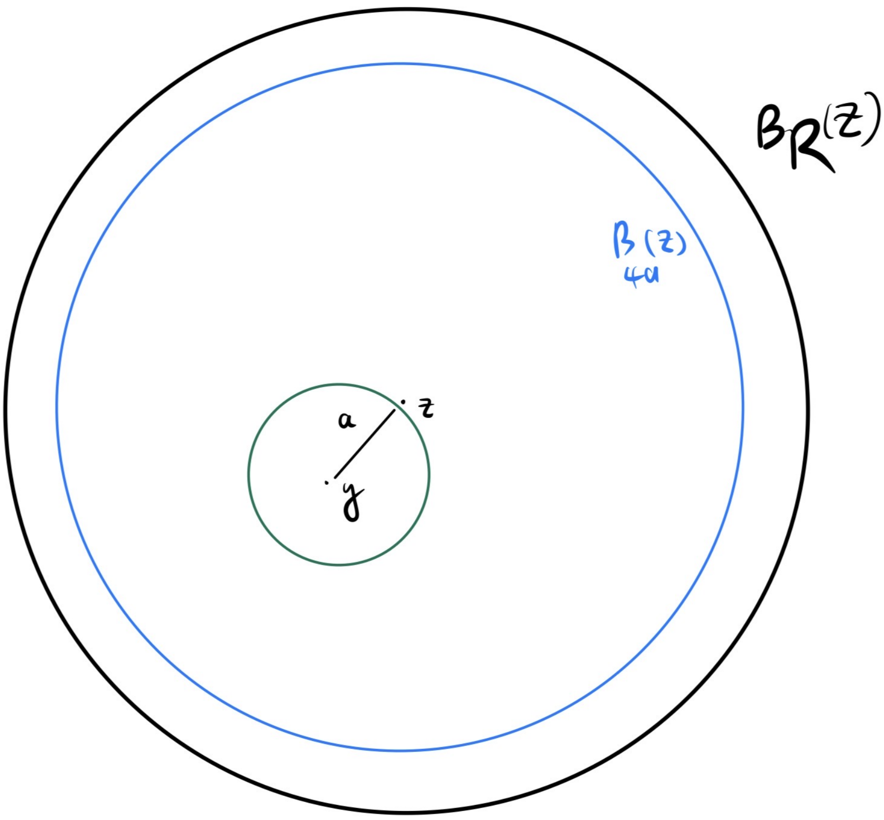


Figure 2.2: Balls for Tf where $a = |y-z|$

Let's write

$$\begin{aligned}
Tf(z) = \text{p.v.}(f*K)(z) &\overset{(2.34)}{=} \int_{B_R} K(x)\,(f(z-x)-f(z))\,dx = \int_{B_R(z)} K(z-x)\,(f(x)-f(z))\,dx \\
&= \int_{B_{4a}(z)} K(z-x)\,(f(x)-f(z))\,dx + \int_{B_{4a}^c(z)} K(z-x)\,(f(x)-f(z))\,dx \\
&= o(a^\alpha) + \int_{B_{4a}^c(z)} K(z-x)\,(f(x)-f(z))\,dx
\end{aligned}$$

In particular, if one consider $a = R$ being large, then the second term vanishes, and one has shown

$$\|Tf\|_{L^\infty(B_R)} \leq CR^\alpha [f]_{C^{0,\alpha}}$$

If we're able to prove the first result, then the second item follows.

On the other hand, we write

$$\begin{aligned}
Tf(y) = \text{p.v.}(f * K)(y) &\overset{(2.34)}{=} \int_{B_R} K(x)\,(f(y-x) - f(y))\,dx = \int_{B_R(y)} K(y-x)\,(f(x) - f(y))\,dx \\
&= \int_{B_a(y)} K(y-x)\,(f(x) - f(y))\,dx + \int_{B_a^c(y)} K(y-x)\,(f(x) - f(z))\,dx \\
&= o(a^\alpha) + \int_{B_{4a}^c(z)} K(y-x)\,(f(x) - f(z))\,dx + \int_{B_{4a}(z)\setminus B_a(y)} K(y-x)\,(f(x) - f(z))\,dx
\end{aligned}$$

Notice the subtlety that once we're on $B_a^c(y)$, the term $\int_{B_a^c(y)} K(y-x)f(y)dx = 0$, thus we can freely replace this $f(y)$ with the $f(z)$ that we want.

Let's analyze the last term

$$\int_{B_{4a}(z)\setminus B_a(y)} K(y-x)\,(f(x) - f(z))\,dx \lesssim a^{-n} \cdot a^\alpha \cdot a^n = o(a^\alpha)$$

Thus we subtract and obtain the difference of the two major terms

$$\begin{aligned}
|Tf(y) - Tf(z)| &= o(a^\alpha) + |\int_{B_{4a}^c(z)} (K(y-x) - K(z-x))\,(f(x) - f(z))\,dx| \\
&\leq o(a^\alpha) + [f]_{C^{0,\alpha}} \int_{B_{4a}^c(z)} \sup_{\xi\in[z,y]} |\nabla K(x-\xi)| \cdot a \cdot |x-z|^\alpha dx \\
&\overset{(2.35)}{\leq} o(a^\alpha) + C \cdot a \int_{4a}^\infty r^{-(1+n)} \cdot r^\alpha \cdot r^{n-1} dr \\
&= o(a^\alpha) + o(a \cdot a^{\alpha-1}) = o(a^\alpha)
\end{aligned}$$

The novelty is that for x is both far away from y and from z, then the distances $|x-y|$ and $|x-z|$ are comparable, and the difference

$$|K(y-x) - K(z-x)| \sim |x|^{-n-1}$$

has faster decay rate compared with the decay rate of the Kernel themselves $K(z-x) \sim |x|^{-n}$, $K(y-x) \sim |x|^{-n}$.

□

2.3 Schauder Estimates via Comparison Principles

In this section we look again at Interior Schauder Estimates in the following form

Theorem 2.3.1 ([FRRO22] Theorem 2.14)**.** *Let $\alpha \in (0,1)$ and $f \in C^{0,\alpha}(\overline{B_1})$. Assume $u \in C^2(B_1)$ solves*

$$\Delta u = f \qquad B_1$$

Then $u \in C^{2,\alpha}(B_1)$ locally, and in particular there exists $C = C(n,\alpha) > 0$

$$\|u\|_{C^{2,\alpha}(\overline{B_{1/2}})} \le C \left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(\overline{B_1})} \right) \tag{2.36}$$

We're essentially redoing the Interior Hölder Estimate in balls that we obtained as in (2.16). But in this section, we assume nothing about Potential Theory. We only use Comparison Principle and the equation itself!

Now before we dive into the proofs, let's see what Theorem 2.3.1 gives us.

Covering Argument First of all, if one has estimate on a ball of small size, via rescaling and covering by finitely many balls one can obtain estimate on a larger interior ball ([FRRO22] Remark 2.15).

In particular, if u solves weakly

$$\Delta u = f \qquad B_1$$

and one already has estimate on B_{r_1} for $r_1 \in (0,1)$ small and $\alpha \in (0,1)$ with universal C

$$\|u\|_{C^{2,\alpha}(\overline{B_{r_1}})} \le C \left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(\overline{B_1})} \right) \tag{2.37}$$

Then we ask: How can one obtain estimate on ball B_{r_2} with $r_2 \in (r_1, 1)$?

$$\|u\|_{C^{2,\alpha}(\overline{B_{r_2}})} \le C(r_1, r_2) \left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(\overline{B_1})} \right) \tag{2.38}$$

We cover via the following.

Covering Argument. Let $r := (1-r_2)r_1$ and consider the open cover $\{B_r(x)\}_{x \in B_{r_2}}$. Since $\overline{B_{r_2}}$ is compact one may cover B_{r_2} with finitely many balls $\{B_r(x_i)\}_{i=1}^N$.

Now, since each $B_{1-r_2}(x_i) \subseteq B_1$, one may apply the estimate (2.37) to each translated and rescaled $B_{1-r_2}(x_i)$ ball. Let's see what our estimate for the rescaled solution

$$\tilde{u}(x) := u(x_i + (1-r_2)x), \qquad \Delta \tilde{u} = (1-r_2)^2 f(x_i + (1-r_2)x) = \tilde{f}(x) \qquad \forall\, x \in B_1$$

satisfies. Plugging in (2.37) one obtain

$$\begin{aligned}
\|\tilde{u}\|_{C^{2,\alpha}(\overline{B_{r_1}})} &= \|u\|_{L^\infty(B_r(x_i))} + (1-r_2)\|\nabla u\|_{L^\infty(B_r(x_i))} + (1-r_2)^2 \left\|D^2 u\right\|_{L^\infty(B_r(x_i))} + (1-r_2)^{2+\alpha}[D^2 u]_{C^{0,\alpha}(\overline{B_r(x_i)})} \\
&\le C\left(\|\tilde{u}\|_{L^\infty(B_1)} + \left\|\tilde{f}\right\|_{C^{0,\alpha}(\overline{B_1})} \right) \\
&= C\left(\|u\|_{L^\infty(B_{1-r_2}(x_i))} + (1-r_2)^2 \|f\|_{C^0(\overline{B_{1-r_2}(x_i)})} + (1-r_2)^{2+\alpha}[f]_{C^{0,\alpha}(\overline{B_{1-r_2}(x_i)})} \right)
\end{aligned}$$

so that

$$\|u\|_{C^{2,\alpha}(\overline{B_r(x_i)})} \le C(r_1,r_2)\left(\|u\|_{L^\infty(B_{1-r_2}(x_i))} + \|f\|_{C^{0,\alpha}(\overline{B_{1-r_2}(x_i)})} \right) \le C(r_1,r_2)\left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(\overline{B_1})} \right)$$

$$\|u\|_{C^{2,\alpha}(\overline{B_{r_2}})} \le \sum_{i=1}^N \|u\|_{C^{2,\alpha}(\overline{B_r(x_i)})} \le NC(r_1,r_2)\left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(\overline{B_1})} \right)$$

□

From *a priori* Estimate to Regularity Improvement Now we demonstrate how a priori estimates (assuming C^2 regularity) actually helps us in upgrading the regularity of a weak solution. In the following we assume for Theorem 2.3.1.

Corollary 2.3.1 ([FRRO22] Corollary 2.16)**.** *Let $\alpha \in (0,1)$ and $f \in C^{0,\alpha}(\overline{B_1})$. Let $u \in L^\infty(B_1) \cap H^1(B_1)$ be weak solution to*

$$\Delta u = f \qquad B_1$$

Then $u \in C^{2,\alpha}(B_1)$ and the same estimate (2.36) holds.

Proof. A standard tool is to convolve the solution with a mollifier, i.e., for $\eta_\varepsilon \in C_0^\infty$ with $\text{supp}(\eta_\varepsilon) \subseteq B_\varepsilon(0)$ and $\int \eta_\varepsilon = 1$, define

$$u_\varepsilon := \eta_\varepsilon * u$$

Since mollification commutes with derivatives, u_ε solves weakly

$$\Delta u_\varepsilon = f_\varepsilon \qquad B_{1-\varepsilon}$$

Mollification $u_\varepsilon \in C^\infty(B_{1-\varepsilon})$, hence to u_ε we can apply our Interior Schaduer Estimate Theorem 2.3.1. In particular for ε small enough, one may consider the solution satisfying in, say, $B_{7/8}$, and obtain via the covering argument the following

$$\|u_\varepsilon\|_{C^{2,\alpha}(\overline{B_{1/2}})} \le C\left(\|u_\varepsilon\|_{L^\infty(B_{1-\varepsilon})} + \|f_\varepsilon\|_{C^{0,\alpha}(\overline{B_{1-\varepsilon}})}\right)$$

A natural and important question to ask is, whether one has uniform bound of the RHS in ε. Indeed

$$\begin{aligned}
|u_\varepsilon(x)| &\le \int_{B_\varepsilon} |u(x-y)|\eta_\varepsilon(y)dy \le \|u\|_{L^\infty(B_1)} && \forall\, x \in B_{1-\varepsilon}\\
|f_\varepsilon(x)| &\le \int_{B_\varepsilon} |f(x-y)|\eta_\varepsilon(y)dy \le \|f\|_{C^0(\overline{B_1})} && \forall\, x \in B_{1-\varepsilon}\\
|f_\varepsilon(x) - f_\varepsilon(y)| &\le \int_{B_\varepsilon} |f(x-z) - f(y-z)|\eta_\varepsilon(z)dz \le [f]_{C^{0,\alpha}(\overline{B_1})}|x-y|^\alpha && \forall\, x, y \in B_{1-\varepsilon}
\end{aligned}$$

Thus one has the desired uniform bound

$$\|u_\varepsilon\|_{C^{2,\alpha}(\overline{B_{1/2}})} \le C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(\overline{B_1})}\right) \qquad \forall\, \varepsilon \ll 1$$

From Oscillation decay implies Hölder Continuity Lemma 1.8.4, we know $u \in C(B_1)$. Thus the mollification u_ε converges locally uniformly on compact subsets to u. Combining the uniform convergence, that C^2 is Banach space and uniform bound in $C^{2,\alpha}$ so that one can apply Arzela-Ascoli, one obtain $u \in C^{2,\alpha}(B_{1/2})$ in the limit and the estimate (2.36). Again via a covering argument one can do for any ball $B_1 \Supset B_\rho \supseteq B_{1/2}$ and thus $u \in C^{2,\alpha}(B_\rho)$ for any $\rho < 1$. □

L^∞ **force** We remark that at least Dini continuity on the force is required to guarantee $u \in C^2$, and $f \in C^{0,\alpha}$ is necessary for $u \in C^{2,\alpha}$. On the other hand if $f \in C$ or merely $f \in L^\infty$, the best one can do is $u \in C^{1,1-\varepsilon}$ for any $\varepsilon \in (0,1)$, not even up to $C^{1,1}$.

Proposition 2.3.1 ([FRRO22] Proposition 2.18). *Let $u \in L^\infty(B_1) \cap H^1(B_1)$ be weak solution to*

$$\Delta u = f \qquad B_1$$

Then $u \in C^{1,1-\varepsilon}(B_1)$ for any $\varepsilon \in (0,1)$, along with the estimate for $C_\varepsilon = C(n,\varepsilon)$

$$\|u\|_{C^{1,1-\varepsilon}(\overline{B_{1/2}})} \le C_\varepsilon\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}\right) \tag{2.39}$$

2.3.1 Campanato's Method

We demonstrate Campanato ([FRRO22] Second Proof of Theorem 2.14; De Silva Analysis II 2025). Let's first make a sequence of simplifications to our target (2.36).

1. If either of RHS of (2.36) is infinity this is trivial. Otherwise upon dividing by big constants

$$\begin{aligned}
\tilde{u} &:= \frac{1}{\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(\overline{B_1})}}\delta u\\
\tilde{f} &:= \frac{1}{\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(\overline{B_1})}}\delta f
\end{aligned}$$

   One may assume

$$\|u\|_{L^\infty(B_1)} \le \delta, \qquad \|f\|_{C^{0,\alpha}(\overline{B_1})} \le \delta$$

   for $\delta = \delta(n,\alpha) > 0$ small to be chosen. This is treating our equation as a small perturbation of Laplace Equation!

2. One wish to bound
$$[D^2u]_{C^{0,\alpha}(\overline{B_{1/2}})} \leq C$$
If so, via Interpolation between
$$\|u\|_{C^2(\overline{B_{1/2}})} \leq C(n)\,\|u\|_{L^\infty(\overline{B_{1/2}})} + [D^2u]_{C^{0,\alpha}(\overline{B_{1/2}})}$$
one recover the result (2.36).

3. Now one need to recall a local polynomial characterisation of $C^{2,\alpha}$ ([FRRO22] A.5).
For universal constants $\rho \in (0, \frac{1}{2})$, $M > 0$, if for any $x \in B_{1/2}$, there exists a sequence of quadratic polynomials $\{P_{x,\rho^k}\}_{k=1}^{\infty}$ defined on B_1 s.t.
$$\left\|u - P_{x,\rho^k}\right\|_{L^\infty(B_{\rho^k}(x))} \leq M\rho^{k(2+\alpha)} \qquad \forall\ k \in \mathbb{N}$$
Then there exists $C = C(n, \alpha, \rho) > 0$ s.t.
$$[D^2u]_{C^{0,\alpha}(\overline{B_{1/2}})} \leq MC$$
Hence our task reduces to constructing the sequence of quadratic polynomials with $\rho^{k(2+\alpha)}$ error in L^∞ to u.

4. In fact it suffices to work with the origin 0 and construct a sequence $P_k := P_{0,\rho^k}$. If so, one can simply do a translation to the polynomials, and one is eligible to do it for any $x \in B_{1/2}$.

Let's see what our task is upon the simplifications.

Lemma 2.3.1 (Campanato)**.** *Let $\alpha \in (0,1)$ and $f \in C^{0,\alpha}(\overline{B_1})$. Assume $u \in C^2(B_1)$ solves*
$$\Delta u = f \qquad B_1$$
with size
$$\|u\|_{L^\infty(B_1)} \leq \delta, \qquad \|f\|_{C^{0,\alpha}(\overline{B_1})} \leq \delta \tag{2.40}$$
for $\delta = \delta(n,\alpha) > 0$ to be chosen. Then there exists some $\rho = \rho(n,\alpha) \in (0, \frac{1}{2})$ and a sequence P_k of quadratic polynomials s.t.
$$\|u - P_k\|_{L^\infty(B_{\rho^k}(0))} \leq \rho^{k(2+\alpha)}, \qquad \Delta P_k = f(0) \qquad \forall\ k \in \mathbb{N} \tag{2.41}$$

And we wish to construct P_k by choosing universal constants δ, ρ.

Proof. One construct via induction.

1. Let's do the base case $k = 0$. We define
$$P_0(x) := \frac{f(0)}{2n}|x|^2$$
Why is this good? Because
$$\begin{aligned} \Delta P_0 &= f(0) \\ |u(x) - \frac{f(0)}{2n}|x|^2| &\leq \|u\|_{L^\infty(B_1)} + \frac{1}{2n}\,\|f\|_{L^\infty(B_1)} \\ &\leq (1 + \frac{1}{2n})\delta \leq 1 \qquad \text{by ensuring } \delta \text{ small enough} \end{aligned}$$
and thus
$$\|u - P_0\|_{L^\infty(B_1(0))} \leq 1$$

2. Now let's do induction! Assume (2.41) holds at step $k \in \mathbb{N}$, i.e., P_k is already constructed. Then one wish to construct P_{k+1} s.t. (2.41) holds at level $k+1$.

   (a) First note what we assumed is at scale B_{ρ^k}. To go into scale $B_{\rho^{k+1}}$, we rescale
$$\tilde{u}(x) := \frac{u(\rho^k x) - P_k(\rho^k x)}{\rho^{k(2+\alpha)}} \qquad \forall\ x \in B_1$$

Now what equation does $\tilde{u}$ satisfy? We look at

$$\begin{aligned}
\Delta\tilde{u}(x) &= \frac{\rho^{2k}}{\rho^{k(2+\alpha)}}\left(\Delta u(\rho^k x) - \Delta P_k(\rho^k x)\right) \\
&= \frac{1}{\rho^{k\alpha}}\left(f(\rho^k x) - f(0)\right) \qquad \text{using equation } \Delta u = f \text{ and inductive assumption } \Delta P_k = f(0) \\
\|\Delta\tilde{u}\|_{L^\infty(B_1)} &\le \frac{1}{\rho^{k\alpha}}\left\|f(\rho^k x) - f(0)\right\|_{L^\infty(B_1)} \le \|f\|_{C^{0,\alpha}(\overline{B_1})} \le \delta \qquad \text{using assumption (2.40)}
\end{aligned}$$

On the other hand

$$\begin{aligned}
\tilde{u}(x) &= \frac{u(\rho^k x) - P_k(\rho^k x)}{\rho^{k(2+\alpha)}} \\
\|\tilde{u}\|_{L^\infty(B_1)} &\le \frac{1}{\rho^{k(2+\alpha)}}\left\|u(\rho^k x) - P_k(\rho^k x)\right\|_{L^\infty(B_1)} = \frac{1}{\rho^{k(2+\alpha)}}\|u - P_k\|_{L^\infty(B_{\rho^k}(0))} \\
&\le 1 \qquad \text{using inductive assumption } \|u - P_k\|_{L^\infty(B_{\rho^k}(0))} \le \rho^{k(2+\alpha)}
\end{aligned}$$

Why do we need such a $\tilde{u}$? Notice this is small perturbation of a harmonic function in B_1. We consider the solution w to

$$\begin{cases} \Delta w = 0 & B_{3/4} \\ w = \tilde{u} & \partial B_{3/4} \end{cases}$$

Then their difference necessarily satisfies

$$\begin{cases} |\Delta(\tilde{u} - w) \le \delta & B_{3/4} \\ \tilde{u} - w = 0 & \partial B_{3/4} \end{cases}$$

Using the Maximum Principle (for weak solutions) (1.87) one obtain the estimate

$$\|\tilde{u} - w\|_{L^\infty(B_{3/4})} \le C(n)\delta \tag{2.42}$$

(b) Using the fact that harmonic functions w are analytic Corollary 1.1.5, there is a quadratic polynomial P_w of degree 2 at the origin, i.e.

$$P_w(x) := w(0) + \nabla w(0)\cdot x + \frac{1}{2}x^T D^2 w(0) x$$

that approximates w in the taylor expansion fashion, i.e.

$$\|w - P_w\|_{L^\infty(B_\rho)} \le C(n)\rho^3 \tag{2.43}$$

Now for any $\rho < \frac{3}{4}$, one naturally wish to ensure

$$\begin{aligned}
\|\tilde{u} - P_w\|_{L^\infty(B_\rho)} &\le \|\tilde{u} - w\|_{L^\infty(B_{3/4})} + \|w - P_w\|_{L^\infty(B_\rho)} \le C(n)(\delta + \rho^3) \qquad (2.42), (2.43) \\
&\le \rho^{2+\alpha} \qquad \text{wish to ensure}
\end{aligned}$$

To do so, we first choose $\rho = \rho(n,\alpha)$ small so that

$$C(n)\rho^3 \le \frac{1}{2}\rho^{2+\alpha}$$

Then we choose $\delta = \delta(n,\alpha) > 0$ small so that

$$C(n)\delta \le \frac{1}{2}\rho^{2+\alpha}$$

Thus

$$\|\tilde{u} - P_w\|_{L^\infty(B_\rho)} \le \rho^{2+\alpha} \tag{2.44}$$

(c) The last question is: what is our P_{k+1} for the next iteration, and why is (2.41) satisfied at the step $k+1$?

We naturally unravel via rescaling

$$\begin{aligned}
\tilde{u}(x) - P_w(x) &= \frac{u(\rho^k x) - P_k(\rho^k x)}{\rho^{k(2+\alpha)}} - P_w(x) \qquad \forall\, x \in B_\rho \\
&= \frac{u(x) - P_k(x)}{\rho^{k(2+\alpha)}} - P_w(\frac{x}{\rho^k}) \qquad \forall\, x \in B_{\rho^{k+1}}
\end{aligned}$$

Using (2.44) one obtain

$$|\frac{u(x)-P_k(x)}{\rho^{k(2+\alpha)}} - P_w(\frac{x}{\rho^k})| \le \rho^{2+\alpha} \qquad \forall\ x \in B_{\rho^{k+1}}$$

$$|u(x) - \left(P_k(x) + \rho^{k(2+\alpha)} P_w(\frac{x}{\rho^k})\right)| \le \rho^{(k+1)(2+\alpha)} \qquad \forall\ x \in B_{\rho^{k+1}}$$

Now if we're to define

$$P_{k+1}(x) := P_k(x) + \rho^{k(2+\alpha)} P_w(\frac{x}{\rho^k})$$

P_{k+1} remains a quadratic polynomial in x and one obtain the targeted estimate

$$\|u - P_{k+1}\|_{L^\infty(B_{\rho^{k+1}})} \le \rho^{(k+1)(2+\alpha)}$$

Now what about the last thing, $\Delta P_{k+1}(0) = f(0)$? One calculate

$$\begin{aligned} \Delta P_{k+1}(x) &= \Delta P_k(x) + \rho^{k(2+\alpha)} \Delta P_w(\frac{x}{\rho^k}) \\ &= \Delta P_k(x) = f(0) \qquad \text{using inductive assumption} \end{aligned}$$

where we computed

$$\begin{aligned} P_w &= w(0) + \sum_i \partial_i w(0) x_i + \frac{1}{2} \sum_i \partial_{ii} w(0) x_i^2 + \sum_{1 \le i < j \le n} \partial_{ij} w(0) x_i x_j \\ \partial_i P_w &= \partial_i w(0) + \partial_{ii} w(0) x_i + \sum_{j \ne i} \partial_{ij} w(0) x_j \\ \partial_{ii} P_w &= \partial_{ii} w(0) \\ \Delta P_w &= \Delta w(0) = 0 \qquad \text{using } w \text{ is harmonic at } 0 \end{aligned}$$

□

2.3.2 Wang's Method

We demonstrate Wang's ([FRRO22] First Proof of Theorem 2.14; [Wan06] Theorem 1).

In the following we assume for f *Dini Continuous* in B_1, i.e., for

$$w(r) := \sup_{\substack{x,\, y \in B_1 \\ |x-y|<r}} |f(x) - f(y)|$$

modulus of continuity of f, one has

$$\int_0^1 \frac{w(r)}{r} dr < \infty \tag{2.45}$$

The proof demonstrates estimate on the growth of D^2u given such Dini continuous forcing data. Notice this is improvement compared to our method provided in Potential Theory (2.4) (which requires $C^{0,\alpha}$ force).

One wish to obtain estimate of the following.

Lemma 2.3.2 (Wang). *Let f be Dini continuous in B_1 with modulus $w(r)$. Assume $u \in C^2(B_1)$ solves*

$$\Delta u = f \qquad B_1$$

Then defining $d := |x-y|$ for $x,\, y \in B_{1/32}$, one has the estimate for $C = C(n) > 0$

$$|D^2u(x) - D^2u(y)| \le C(n) \left(d\, \|u\|_{L^\infty(B_1)} + \int_0^{16d} \frac{w(r)}{r} + d \int_d^1 \frac{w(r)}{r^2} \right) \qquad \forall\ x,\, y \in B_{1/32} \tag{2.46}$$

Notice Dini continuity (2.45) indeed yields finiteness of the RHS

$$\begin{aligned} \int_0^{16d} \frac{w(r)}{r} &\le \int_0^1 \frac{w(r)}{r} < \infty \\ d \int_d^1 \frac{w(r)}{r^2} &\le C \int_d^1 \frac{w(r)}{r} < \infty \end{aligned}$$

Yet if we're additionally given $f \in C^{0,\alpha}$ for $0 < \alpha < 1$, then we recover our desired estimate (2.36) via the following

$$\begin{aligned}
|D^2u(x) - D^2u(y)| &\leq C(n)\left(d\,\|u\|_{L^\infty(B_1)} + [f]_{C^{0,\alpha}(\overline{B_1})}\int_0^{16d} r^{\alpha-1} + d[f]_{C^{0,\alpha}(\overline{B_1})}\int_d^1 r^{\alpha-2}\right)\\
&= C(n,\alpha)\left(d\,\|u\|_{L^\infty(B_1)} + [f]_{C^{0,\alpha}(\overline{B_1})}\frac{d^\alpha}{\alpha} + d[f]_{C^{0,\alpha}(\overline{B_1})}\left(\frac{1}{\alpha-1} + \frac{1}{1-\alpha}d^{\alpha-1}\right)\right)\\
&\leq C(n,\alpha)\left(\|u\|_{L^\infty(B_1)} + \frac{1}{\alpha(1-\alpha)}[f]_{C^{0,\alpha}(\overline{B_1})}\right)d^\alpha\\
[D^2u]_{C^{0,\alpha}(\overline{B_{1/32}})} &\leq C(n,\alpha)\left(\|u\|_{L^\infty(B_1)} + [f]_{C^{0,\alpha}(\overline{B_1})}\right)
\end{aligned}$$

and we conclude via $C^{2,\alpha}$ interpolation.

If we have $f \in C^{0,1}$ then the estimate takes the form

$$\begin{aligned}
|D^2u(x) - D^2u(y)| &\leq C(n)\left(d\,\|u\|_{L^\infty(B_1)} + d[f]_{C^{0,1}(\overline{B_1})} + d[f]_{C^{0,1}(\overline{B_1})}\int_d^1 \frac{1}{r}\right)\\
&= C(n)\left(\|u\|_{L^\infty(B_1)} + [f]_{C^{0,1}(\overline{B_1})} + [f]_{C^{0,1}(\overline{B_1})}|\log(d)|\right)\cdot d\\
[D^2u]_{C^{0,1}(\overline{B_{1/32}})} &\leq C(n)\left(\|u\|_{L^\infty(B_1)} + [f]_{C^{0,1}(\overline{B_1})}(1+|\log(d)|)\right)
\end{aligned}$$

Proof of Lemma 2.3.2. WLOG one may do the estimate at the origin. Let's manipulate the idea of 'zooming in'. Let $\rho := \frac{1}{2}$. We write ρ so as to be clear about the dependence, but we really need $1/2$ to make 'playing with balls' simple.

Consider a sequence of functions u_k for $k \in \mathbb{N}$ that solves

$$\begin{cases} \Delta u_k = f(0) & B_{\rho^k} \\ u_k = u & \partial B_{\rho^k} \end{cases}$$

We claim this sequence of functions is good! And in fact one conclude our estimate using

$$|D^2u(x) - D^2u(0)| \leq |D^2u_k(0) - D^2u(0)| + |D^2u_k(x) - D^2u_k(0)| + |D^2u(x) - D^2u_k(x)| =: \mathbf{I} + \mathbf{II} + \mathbf{III} \quad (2.47)$$

Notice here one has two smallness. For $|x|$ close to 0 which we can control, there should be some k large so that **I**, **II** and **III** are small w.r.t. $|x|$.

1. First notice their difference solves

$$\begin{cases} \Delta(u_k - u) = f(0) - f & B_{\rho^k} \\ u_k - u = 0 & \partial B_{\rho^k} \end{cases}$$

   and we had the estimate on forcing

$$\|f(0) - f\|_{L^\infty(B_{\rho^k})} \leq w(\rho^k)$$

   Then using the rescaled Maximum Principle (1.88) one obtain

$$\|u_k - u\|_{L^\infty(B_{\rho^k})} \overset{(1.88)}{\leq} C\rho^{2k}\,\|f(0) - f\|_{L^\infty(B_{\rho^k})} \leq C\rho^{2k}w(\rho^k) \quad (2.48)$$

   (a) First one notice immediately

$$\|u_k - u_{k+1}\|_{L^\infty(B_{\rho^{k+1}})} \leq \|u_k - u\|_{L^\infty(B_{\rho^{k+1}})} + \|u - u_{k+1}\|_{L^\infty(B_{\rho^{k+1}})} \leq C\rho^{2k}w(\rho^k)$$

   Why do we need consecutive terms in the sequence? Because they're harmonic

$$\Delta(u_k - u_{k+1}) = f(0) - f(0) = 0 \qquad B_{\rho^{k+1}}$$

   Hence one has interior gradient (1.15) and second order derivative estimates (1.18)

$$\begin{aligned}
\|\nabla(u_k - u_{k+1})\|_{L^\infty(B_{\rho^{k+2}})} &\leq C(n)\frac{\|u_k - u_{k+1}\|_{L^\infty(B_{\rho^{k+1}})}}{\rho^{k+1}} \leq C(n,\rho)\rho^k w(\rho^k)\\
\left\|D^2u_k - D^2u_{k+1}\right\|_{L^\infty(B_{\rho^{k+2}})} &\leq C(n,\rho)w(\rho^k)
\end{aligned} \quad (2.49)$$

(b) We further ask: How do we obtain convergence in ∇u_k and $D^2 u_k$ so that looking at the term **I** as in (2.47) makes sense?
Since $u \in C^2$ by our assumption, in particular one has Taylor Expansion estimate

$$|u(x) - P(x)| \le C|x|^3 \qquad \forall\, x \in B_1 \tag{2.50}$$

where $P(x)$ is the quadratic polynomial

$$P(x) := u(0) + \nabla u(0)\cdot x + \frac{1}{2}x^T D^2 u(0)x$$

Notice the polynomial satisfies

$$\Delta P(x) = \Delta\left(u(0) + \nabla u(0)\cdot x + \frac{1}{2}x^T D^2 u(0)x\right) = \Delta u(0) = f(0)$$

Hence the difference is harmonic

$$\Delta\left(u_k - P\right) = f(0) - f(0) = 0 \qquad B_{\rho^k}$$

One therefore has again harmonic function interior estimates

$$\begin{aligned}
\|\nabla u_k - \nabla P\|_{L^\infty(B_{\rho^{k+1}})} &\le C\frac{1}{\rho^k}\|u_k - P\|_{L^\infty(B_{\rho^k})}\\
&\le \frac{C}{\rho^k}\left(\|u_k - u\|_{L^\infty(B_{\rho^k})} + \|u - P\|_{L^\infty(B_{\rho^k})}\right)\\
&\overset{(2.48),(2.50)}{\le} C\left(\rho^k w(\rho^k) + \rho^{2k}\right)
\end{aligned}$$

$$|\nabla u_k(0) - \nabla u(0)| \le \|\nabla u_k - \nabla P\|_{L^\infty(B_{\rho^{k+1}})} + C\rho^{2k}|D^2 u(0)| \to 0 \qquad k\to\infty$$

$$\left\|D^2 u_k - D^2 P\right\|_{L^\infty(B_{\rho^{k+1}})} \le \frac{C}{\rho^{2k}}\left(\|u_k - u\|_{L^\infty(B_{\rho^k})} + \|u - P\|_{L^\infty(B_{\rho^k})}\right) \le C(w(\rho^k) + \rho^k)$$

$$|D^2 u_k(0) - D^2 u(0)| = |D^2 u_k(0) - D^2 P(0)| \le C(w(\rho^k) + \rho^k) \to 0 \qquad k\to\infty$$

Thus one has convergence at the origin

$$\lim_{k\to\infty} \nabla u_k(0) = \nabla u(0) \qquad \lim_{k\to\infty} D^2 u_k(0) = D^2 u(0) \tag{2.51}$$

2. Recall one need to pick k depending on x to make **I**, **II** and **III** small. We make our global choice of

$$\rho^{k+4} \le |x| < \rho^{k+3} \tag{2.52}$$

which we'll need to make sense of later.

3. We estimate the size of **I** via the following.

$$\begin{aligned}
\mathbf{I} = |D^2 u(0) - D^2 u_k(0)| &\overset{(2.51)}{=} \lim_{j\to\infty} |D^2 u_j(0) - D^2 u_k(0)|\\
&\le \sum_{j=k}^{\infty} |D^2 u_j(0) - D^2 u_{j+1}(0)|\\
&\overset{(2.49)}{\le} C(n,\rho)\sum_{j=k}^{\infty} w(\rho^j)
\end{aligned}$$

Now let's pause and observe that for $w(r)$ decreasing in r

$$\begin{aligned}
\sum_{j=k}^{\infty} w(\rho^j) &= \lim_{K\to\infty}\sum_{j=k}^{K} w(\rho^j)\\
&= \lim_{K\to\infty}\sum_{j=k}^{K}\frac{1}{\rho^j - \rho^{j+1}}\int_{\rho^{j+1}}^{\rho^j} w(\rho^j)dr\\
&= \lim_{K\to\infty}\frac{1}{1-\rho}\sum_{j=k}^{K}\int_{\rho^{j+1}}^{\rho^j}\frac{w(\rho^j)}{\rho^j}dr\\
&\le \lim_{K\to\infty}\frac{1}{1-\rho}\sum_{j=k}^{K}\int_{\rho^{j+1}}^{\rho^j}\frac{w(r)}{r}dr \qquad \text{using both } w(r) \text{ and } \frac{1}{r} \text{ are decreasing in } r\\
&= \frac{1}{1-\rho}\int_0^{\rho^k}\frac{w(r)}{r}dr \qquad \text{using } \rho^j \to 0 \text{ as } j\to\infty
\end{aligned}$$

What do we need on k to make RHS small? In our choice (2.52) one has

$$\rho^{k+4} \le |x| \implies \rho^k \le \rho^{-4}|x|$$

Hence the estimate follows

$$\mathbf{I} = |D^2u(0) - D^2u_k(0)| \le C(n,\rho)\int_0^{\rho^{-4}|x|} \frac{w(r)}{r}dr \qquad \forall\ \rho^{k+4} \le |x| < \rho^{k+3} \tag{2.53}$$

4. For **III**, we adopt a similar trick, but the analysis is much more delicate. Fix our x satisfying (2.52). Here we look at another sequence of functions v_ℓ that solves

$$\begin{cases} \Delta v_\ell = f(x) & B_{\rho^\ell}(x) \\ v_\ell = u & \partial B_{\rho^\ell}(x) \end{cases}$$

One wish to estimate **III** via D^2v_k in the triangle inequality

$$\mathbf{III} = |D^2u(x) - D^2u_k(x)| \le |D^2u(x) - D^2v_k(x)| + |D^2v_k(x) - D^2u_k(x)|$$

(a) We claim the first portion follows exactly as in the procedure for **I**. In particular, the difference solves

$$\begin{cases} \Delta(v_\ell - u) = f(x) - f & B_{\rho^\ell}(x) \\ v_\ell - u = 0 & \partial B_{\rho^\ell}(x) \end{cases}$$

Hence applying rescaled Maximum Principle (1.88) again yields

$$\|v_\ell - u\|_{L^\infty(B_{\rho^\ell}(x))} \le C\rho^{2\ell}\,\|f(x) - f\|_{L^\infty(B_{\rho^\ell}(x))} \le C\rho^{2\ell}w(\rho^\ell) \tag{2.54}$$

As in (2.49) one obtain

$$\left\|D^2v_\ell - D^2v_{\ell+1}\right\|_{L^\infty(B_{\rho^{\ell+2}}(x))} \le C(n,\rho)w(\rho^\ell) \tag{2.55}$$

Running the Taylor series expansion around x again gives convergence as in (2.51)

$$\lim_{j\to\infty} D^2v_\ell(x) = D^2u(x)$$

Thus the first portion follows as our previous step for **I**.

(b) For the second portion, however, notice the difference solves

$$\Delta(v_k - u_k) = f(x) - f(0) \qquad B_{\rho^k}(x) \cap B_{\rho^k}(0)$$

But what is the intersection $B_{\rho^k}(x) \cap B_{\rho^k}(0)$? Note as we've chosen (2.52), in particular

$$|x| < \rho^{k+3} < \rho^{k+1} \tag{2.56}$$

Then necessarily

$$|x-y| < \rho^{k+1} \implies |y - 0| \le |x-y| + |x| \overset{(2.56)}{<} \rho^{k+1} + \rho^{k+1} = \rho^k \implies B_{\rho^{k+1}}(x) \subseteq B_{\rho^k}(x) \cap B_{\rho^k}(0)$$

where picking $\rho = \frac{1}{2}$ universal indeed satisfies

$$\rho + \rho = \frac{1}{2} + \frac{1}{2} = 1$$

Thus one obtained interior equation

$$\Delta(v_k - u_k) = f(x) - f(0) \qquad B_{\rho^{k+1}}(x) \tag{2.57}$$

How can one estimate this with constant forcing on the RHS?

i. Notice in general, for constant A

$$\Delta u = A \qquad B_r$$

one subtract $u - \frac{A}{2n}|x|^2$ so that

$$\Delta(u - \frac{A}{2n}|x|^2) = 0 \qquad B_r$$

Applying harmonic estimates (1.18) to this one obtain

$$\left\|D^2(u - \frac{A}{2n}|x|^2)\right\|_{L^\infty(B_{r/2})} \le \frac{C}{r^2}\left\|u - \frac{A}{2n}|x|^2\right\|_{L^\infty(B_r)} \le \frac{C}{r^2}\,\|u\|_{L^\infty(B_r)} + C(n)A$$

$$\left\|D^2u\right\|_{L^\infty(B_{r/2})} \le \left\|D^2(u - \frac{A}{2n}|x|^2)\right\|_{L^\infty(B_{r/2})} + C(n)A \le \frac{C}{r^2}\,\|u\|_{L^\infty(B_r)} + C(n)A$$

ii. Hence the interior second order gradient estimate for (2.57) writes

$$\left\|D^2 v_k - D^2 u_k\right\|_{L^\infty(B_{\rho^{k+2}}(x))} \le C(n,\rho)\left(\rho^{-2k}\left\|v_k - u_k\right\|_{L^\infty(B_{\rho^{k+1}}(x))} + |f(x) - f(0)|\right) \tag{2.58}$$

Now what about the term $\|v_k - u_k\|_{L^\infty(B_{\rho^{k+1}}(x))}$ that appears on the RHS? One use u in the triangle inequality

$$\begin{aligned}\|v_k - u_k\|_{L^\infty(B_{\rho^{k+1}}(x))} &\le \|v_k - u\|_{L^\infty(B_{\rho^{k+1}}(x))} + \|u - u_k\|_{L^\infty(B_{\rho^{k+1}}(x))} \\ &\le \|v_k - u\|_{L^\infty(B_{\rho^{k+1}}(x))} + \|u - u_k\|_{L^\infty(B_{\rho^k})} \qquad \text{using } B_{\rho^{k+1}}(x) \subseteq B_{\rho^k} \\ &\overset{(2.54),(2.48)}{\le} C\rho^{2k} w(\rho^k)\end{aligned}$$

Thus we're happy to conclude for the second portion

$$\begin{aligned}|D^2 v_k(x) - D^2 u_k(x)| &\le \left\|D^2 v_k - D^2 u_k\right\|_{L^\infty(B_{\rho^{k+2}}(x))} \overset{(2.58)}{\le} C(n,\rho)\left(\rho^{-2k}\|v_k - u_k\|_{L^\infty(B_{\rho^{k+1}}(x))} + |f(x) - f(0)|\right) \\ &\le C(n,\rho)\left(w(\rho^k) + w(|x|)\right) \qquad \text{using the above and definition of } w \\ &\le C(n,\rho) w(\rho^k) \qquad \text{using (2.56)}\end{aligned} \tag{2.59}$$

Thus as a conclusion for the two portions, for $\rho^{k+4} \le |x| < \rho^{k+3}$ satisfying (2.52), one obtain

$$\begin{aligned}\mathbf{III} = |D^2 u(x) - D^2 u_k(x)| &\le |D^2 u(x) - D^2 v_k(x)| + |D^2 v_k(x) - D^2 u_k(x)| \\ &\overset{(2.55),(2.59)}{\le} C(n,\rho)\sum_{j=k}^{\infty} w(\rho^k) + C(n,\rho) w(\rho^k) \\ &\overset{(2.53)}{\le} C(n,\rho)\int_0^{\rho^{-4}|x|} \frac{w(r)}{r}\,dr \qquad \forall\, \rho^{k+4} \le |x| < \rho^{k+3}\end{aligned} \tag{2.60}$$

5. For **II**, recall we're keeping k fixed. Define

$$h_j := u_j - u_{j-1} \qquad \forall\, j = 1, \cdots, k$$

Why so? Because h_j are harmonic in B_{ρ^j}, one has the interior higher order gradient estimates (1.18). In particular, one has for any $|x| < \rho^{k+3}$ (as in (2.52))

$$\begin{aligned}|D^2 h_j(x) - D^2 h_j(0)| &\le C|x|\left\|D^3 h_j\right\|_{L^\infty(B_{\rho^{k+3}})} \qquad \forall\, |x| \le \rho^{k+3} \qquad \text{definition of differentiation} \\ &\le C|x|\frac{1}{\rho^{k+2}}\left\|D^2 h_j\right\|_{L^\infty(B_{\rho^{k+2}})} \qquad \text{harmonic function gradient estimate (1.15)} \\ &\le C(n,\rho)|x|\frac{1}{\rho^j}\left\|D^2 h_j\right\|_{L^\infty(B_{\rho^{j+1}})} \qquad \text{using } j \le k \\ &\le C(n,\rho)|x|\frac{w(\rho^j)}{\rho^j} \qquad \text{using second order estimate (2.49)}\end{aligned} \tag{2.61}$$

Then we rewrite

$$\begin{aligned}\mathbf{II} &= |D^2 u_k(x) - D^2 u_k(0)| \\ &\le \sum_{j=1}^{k} |D^2 h_j(x) - D^2 h_j(0)| + |D^2 u_0(x) - D^2 u_0(0)| \\ &\overset{(2.61)}{\le} C(n,\rho)\sum_{j=1}^{k} |x|\frac{w(\rho^j)}{\rho^j} + C|x|\left\|D^3 u_0\right\|_{L^\infty(B_{1/8})} \qquad \text{using definition} \\ &\le C(n,\rho)|x|\left(\sum_{j=1}^{k}\frac{w(\rho^j)}{\rho^j} + \|u_0\|_{L^\infty(B_1)}\right) \qquad \text{using gradient estimate for harmonic functions} \\ &\le C(n,\rho)|x|\left(\sum_{j=1}^{k}\frac{w(\rho^j)}{\rho^j} + \|u_0 - u\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right) \\ &\overset{(2.48)}{\le} C(n,\rho)|x|\left(\sum_{j=1}^{k}\frac{w(\rho^j)}{\rho^j} + w(1) + \|u\|_{L^\infty(B_1)}\right) = C(n,\rho)|x|\left(\sum_{j=0}^{k}\frac{w(\rho^j)}{\rho^j} + \|u\|_{L^\infty(B_1)}\right)\end{aligned}$$

We notice again

$$\begin{aligned}\sum_{j=0}^{k}\frac{w(\rho^j)}{\rho^j} &= \sum_{j=0}^{k}\frac{1}{\rho^j-\rho^{j+1}}\int_{\rho^{j+1}}^{\rho^j}\frac{w(\rho^j)}{\rho^j}dr = \frac{1}{1-\rho}\sum_{j=0}^{k}\int_{\rho^{j+1}}^{\rho^j}\frac{w(\rho^j)}{\rho^{2j}}dr \\ &\leq \frac{1}{1-\rho}\sum_{j=0}^{k}\int_{\rho^{j+1}}^{\rho^j}\frac{w(r)}{r^2}dr \qquad \text{using both } w(r) \text{ and } \frac{1}{r^2} \text{ are decreasing in } r \\ &= \frac{1}{1-\rho}\int_{\rho^{k+1}}^{1}\frac{w(r)}{r^2}dr\end{aligned}$$

We know from condition (2.52) that

$$|x| < \rho^{k+3} < \rho^{k+1}$$

hence the estimate follows

$$\mathbf{II} = |D^2u_k(x) - D^2u_k(0)| \leq C(n,\rho)|x|\left(\|u\|_{L^\infty(B_1)} + \int_{|x|}^{1}\frac{w(r)}{r^2}dr\right) \qquad \forall\ \rho^{k+4} \leq |x| < \rho^{k+3} \tag{2.62}$$

6. In this step we sum up our estimates for **I**, **II** and **III**.

$$\begin{aligned}|D^2u(x) - D^2u(0)| &\leq \mathbf{I} + \mathbf{II} + \mathbf{III} \\ &\overset{(2.53),(2.62),(2.60)}{\leq} C(n,\rho)\left(\int_0^{16|x|}\frac{w(r)}{r}dr + |x|\,\|u\|_{L^\infty(B_1)} + |x|\int_{|x|}^{1}\frac{w(r)}{r^2}dr\right) \qquad \forall\ x \in B_{1/16}\end{aligned}$$

where we need $|x| \leq \frac{1}{16}$ to make sense of $\rho^{-4}|x| \leq 1$. Now upon translation, the result (2.46) holds for x, y in $B_{1/32}$.

□

2.4 Calderón-Zygmund Estimates

The philosophy for Calderón-Zygmund Estimates are that

$$f \in L^p \text{ for } 1 < p < \infty \implies u \in W^{2,p}$$

Estimate for p = 2 Let's estimate for a simple case $p = 2$.

Theorem 2.4.1 ([FRRO22] Remark 2.13)**.** *For any $u \in H^1(B_1)$ and $f \in L^2(B_1)$ that solves weakly*

$$-\Delta u = f \qquad B_1$$

One has $u \in W^{2,2}_{loc}(B_1)$ and the estimate for some $C = C(n) > 0$

$$\|u\|_{W^{2,2}(B_{1/2})} \le C\left(\|u\|_{L^2(B_1)} + \|f\|_{L^2(B_1)}\right) \tag{2.63}$$

Proof. Leveraging density one may do for $u \in C^\infty(B_1)$ so that in particular $D^2u \in L^2$ is well-defined. Then take a cutoff

$$\eta \in C_0^\infty(B_1) \text{ with } \eta \ge 0, \text{ s.t. } \eta = 1 \text{ in } B_{1/2} \text{ and } \eta = 0 \text{ in } B_1 \setminus B_{3/4}$$

One compute

$$\begin{aligned}
\left\|D^2u\right\|^2_{L^2(B_{1/2})} &= \sum_{i,j}\int_{B_{1/2}} |\partial_{ij}u|^2 \le \sum_{i,j}\int_{B_1} |\partial_{ij}(\eta u)|^2 \\
&= -\sum_{i,j}\int_{B_1} \partial_{iij}(\eta u)\partial_j(\eta u) \qquad \text{using } \eta u \text{ of compact support and IBP} \\
&= \sum_{i,j}\int_{B_1} \partial_{ii}(\eta u)\partial_{jj}(\eta u) = \int_{B_1} \Delta(\eta u)^2 \\
&= \int_{B_1} (\partial_i(u\partial_i\eta + \eta\partial_i u))^2 = \int_{B_1} (u\Delta\eta + 2\nabla u\cdot\nabla\eta + \eta\Delta u)^2 \\
&\le C(n)\int_{B_1} u^2|\Delta\eta|^2 + |\nabla u|^2|\nabla\eta|^2 + |\eta|^2|\Delta u|^2 \\
&\le C(n,\eta,\Delta\eta)\int_{B_1} u^2 + (\Delta u)^2 + |\nabla u|^2|\nabla\eta|^2
\end{aligned}$$

Now let's estimate the middle part

$$\begin{aligned}
\int_{B_1} |\nabla u|^2|\nabla\eta|^2 &= -\int_{B_1} u\nabla\cdot\left(|\nabla\eta|^2\nabla u\right) = -\int_{B_1} u\partial_i(|\nabla\eta|^2\partial_i u) \\
&= -\int_{B_1} u\Delta u|\nabla\eta|^2 - \int_{B_1} 2u\partial_i\eta\partial_{ii}\eta\partial_i u \\
&= -\int_{B_1} u\Delta u|\nabla\eta|^2 - \cancel{\int_{B_1} u\partial_i\eta\partial_{ii}\eta\partial_i u} + \int_{B_1} u^2\partial_i(\partial_i\eta\partial_{ii}\eta) + \cancel{\int_{B_1} u\partial_i\eta\partial_{ii}\eta\partial_i u} \\
&\le C(|\nabla\eta|^2, |\Delta\eta|^2, |\nabla\eta||D^3\eta|)\int_{B_1} u^2 + |\Delta u|^2
\end{aligned}$$

∎

Chapter 3

Classical Maximum Principle

In this chapter we study the maximum principle for more general elliptic operators.

Given open domain $\Omega \subseteq \mathbb{R}^n$. Consider an operator of the form

$$\mathcal{L}u = a_{ij}(x)\partial_{ij}u(x) + b_i(x)\partial_i u(x) + c(x)u(x) \qquad \forall\, x \in \Omega \tag{3.1}$$

We assume $a_{ij} = a_{ji}$ is symmetric. We introduce certain terminologies ([GT01])

1. $\mathcal{L}$ is *elliptic at* $x \in \Omega$ if the coefficient matrix $(a_{ij}(x))$ is positive, i.e.

$$0 < \lambda(x)|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda(x)|\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^n \setminus \{0\}$$

   for $\lambda(x)$, $\Lambda(x)$ as the smallest and largest eigenvalue of $(a_{ij}(x))$.

2. $\mathcal{L}$ is elliptic in Ω if $\lambda(x) > 0$ for any $x \in \Omega$.

3. $\mathcal{L}$ is strictly elliptic in Ω if

$$\lambda(x) \ge \lambda_0 > 0 \qquad \forall\, x \in \Omega$$

4. $\mathcal{L}$ is uniformly elliptic in Ω if the ratios are bounded

$$\frac{\Lambda(x)}{\lambda(x)} \le C \qquad \forall\, x \in \Omega \tag{3.2}$$

One also wish to limit the relative importance of the lower order terms $b_i\partial_i u$, cu w.r.t. the principal term $a_{ij}\partial_{ij}u$. One make the assumption that

$$\frac{|b_i(x)|}{\lambda(x)} \le C \qquad \forall\, x \in \Omega, \qquad i = 1, \cdots, n \tag{3.3}$$

The above definitions, are of course, quite general. In most books one discuss in simpler settings.

Remark 3.0.1. *One may do a normalization to simplify the discussion.*

1. *When $\mathcal{L}$ is elliptic in Ω, one may normalize the operator*

$$\tilde{\mathcal{L}} := \frac{1}{\lambda}\mathcal{L}$$

   by dividing by the smallest eigenvalue of $a_{ij}(x)$ at each point. Then

$$0 < |\xi|^2 \le \tilde{a}_{ij}(x)\xi_i\xi_j \le \frac{\Lambda(x)}{\lambda(x)}|\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^n \setminus \{0\}$$

   So immediately $\tilde{\mathcal{L}}$ is strictly elliptic.

2. *Upon normalizing, the assumption (3.3) translates to boundedness of b*

$$|\tilde{b}_i(x)| \le C \qquad \forall\, x \in \Omega, \qquad i = 1, \cdots, n$$

3. *Upon normalizing, if $\mathcal{L}$ is uniformly elliptic in Ω, (3.2) translates to boundedness of a*

$$1 \le |\tilde{a}_{ij}(x)| \le C \qquad \forall\, x \in \Omega, \qquad \forall\, i,\, j = 1, \cdots, n$$

Remark 3.0.2. *Let $\mathcal{L}$ be elliptic in Ω. When a_{ij}, $b_i \in C(\Omega)$, then automatically on any subdomain $\Omega' \Subset \Omega$, a_{ij}, $b_i \in L^\infty(\Omega')$ and a_{ij} stays a positive distance away from 0. In particular, the assumption (3.3) holds and $\mathcal{L}$ is uniformly elliptic on Ω'.*

One shall observe that, for a_{ij} symmetric positive-definite matrix, one may apply Spectral Decomposition that

$$(a_{ij}) = A = Q^T \mathbf{\Lambda} Q$$

where $\mathbf{\Lambda}$ are diagonal eigenvalues and $Q = (q_{ij})$ are corresponding orthonormal eigenvectors. Now

$$a_{ii} = e_i^T A e_i = \sum_{k=1}^n \lambda_k q_{ik}^2$$

Since $\sum_{k=1}^n q_{ik}^2 = 1$, the RHS gives a convex combination of eigenvalues. In particular, any a_{ii} diagonal entry is trapped in between the smallest and the largest eigenvalue $[\lambda, \Lambda]$.

3.1 Maximum Principle

3.1.1 Weak Maximum Principle

It is worth noting that the weak maximum principle requires only the ellipticity condition (not uniform ellipticity). In the following we assume $\mathcal{L}$ **is elliptic in** Ω (Ellipticity suffices!), a bounded domain. Assume (3.3) holds. Assume one has a solution

$$u \in C^2(\Omega) \cap C^0(\overline{\Omega})$$

1. Let's see for a simple prototype ([HL11] Lemma 2.1). Under the assumption

   $$\mathcal{L}u > 0, \qquad c = 0 \qquad \Omega$$

   One can easily conclude that

   $$\sup_{\Omega} u = \sup_{\partial\Omega} u \tag{3.4}$$

   Proof. Assume there exists $x_0 \in \Omega$ interior s.t.

   $$u(x_0) = \sup_{\Omega} u$$

   Then due to interior maximum and $u \in C^2$ around x_0, necessarily

   $$\partial_{ij} u(x_0) \le 0, \qquad \partial_i u(x_0) = 0$$

   Hence

   $$\mathcal{L}u(x_0) = a_{ij}(x_0)\partial_{ij}u(x_0) + b_i(x_0)u(x_0) \le 0$$

   which contradicts $\mathcal{L}u > 0$. □

   Note only the semi positive-definiteness of a_{ij} is used here.

2. Now the important thing is, the result (3.4) remains true even if we assume $\mathcal{L}u \ge 0$ ([GT01] Theorem 3.1; [Eva10] Theorem 6.4.1).

   $$\sup_{\Omega} u = \sup_{\partial\Omega} u \tag{3.5}$$

   Proof. The idea is to introduce a perturbation of the solution. Consider the function $e^{\gamma x_1}$ for γ large to be chosen. Compute

   $$\begin{aligned} \mathcal{L}e^{\gamma x_1} &= a_{11}\gamma^2 e^{\gamma x_1} + b_1 \gamma e^{\gamma x_1} \\ &\ge (\lambda\gamma^2 + b_1\gamma)e^{\gamma x_1} > 0 \end{aligned}$$

   for γ sufficiently large (using assumption (3.3)). Notice here we're just using Ellipticity at the point $(x_1, 0, \cdots, 0)$, and a_{11} is assumed to be positive at this point.

   Now define

   $$u_\varepsilon := u + \varepsilon e^{\gamma x_1}$$

One obtain

$$\mathcal{L}u_\varepsilon = \mathcal{L}u + \varepsilon\mathcal{L}e^{\gamma x_1} > 0 \qquad \forall\, \varepsilon > 0$$

Apply previous result, one know

$$\sup_{\Omega}(u + \varepsilon e^{\gamma x_1}) = \sup_{\partial\Omega}(u + \varepsilon e^{\gamma x_1})$$

Using Ω is bounded, sending $\varepsilon \to 0$ yields (3.4). □

3. Now let us relax the condition to $c \leq 0$, which alternatively require non-negative maximum.

Theorem 3.1.1 ([GT01] Corollary 3.2; [HL11] Theorem 2.3; [Eva10] Theorem 6.4.2)**.** *Let $\mathcal{L}$ be elliptic in bounded open bounded domain Ω. Assume (3.3) holds, and that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. If*

$$\mathcal{L}u \geq 0, \qquad c \leq 0 \qquad \Omega$$

Then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ \tag{3.6}$$

where $u^+ := \max\{u, 0\}$ denotes the non-negative part.

Proof. By assumption u is continuous. Assume the open set $\{u > 0\} \neq \varnothing$.

$$\begin{aligned} \mathcal{L}u &= a_{ij}\partial_{ij}u + b_i\partial_i u + cu \geq 0 \\ \mathcal{L}_0 u &:= a_{ij}\partial_{ij}u + b_i\partial_i u \geq -cu \geq 0 \qquad \text{on } \{u > 0\} \end{aligned}$$

Hence applying the previous result to $\mathcal{L}_0$ on each connected component of $\{u > 0\}$ one obtain

$$\sup_{\Omega} u = \sup_{\Omega\cap\{u>0\}} u = \sup_{\partial(\Omega\cap\{u>0\})} u = \sup_{\partial\Omega} u^+$$

If on the other hand $\Omega \subseteq \{u \leq 0\}$, the inequality (3.6) possibly holds. □

One needs to remark that negativity of c is essential.

Remark 3.1.1 ([HL11] Remark 2.4)**.** *Consider $k \geq 0$ and the problem for $\Omega \subseteq \mathbb{R}^n$ bounded open with $\partial\Omega$ sufficiently regular.*

$$\begin{cases} \Delta u + ku = 0 & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

Then $L^2(\Omega)$ has a complete orthogonal sequence of smooth eigenfunctions indexed by $k \in \mathbb{N}$.

Or, for example, when $n = 2$, $k = 2$ and

$$\Omega = [0, \pi]^2$$

The function

$$u(x, y) = \sin(x)\sin(y)$$

is nontrivial solution.

As in the case for harmonic functions, one has two immediate corollaries: uniqueness to Dirichlet Problem, and a comparison principle.

Corollary 3.1.1 ([GT01] Corollary 3.2)**.** *Let $\mathcal{L}$ be elliptic in Ω bounded open domain with $c \leq 0$ in Ω. Let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$.*

1. *If*

$$\begin{cases} \mathcal{L}u = \mathcal{L}v & \Omega \\ u = v & \partial\Omega \end{cases}$$

Then $u = v$ in Ω.

2. *If*

$$\begin{cases} \mathcal{L}u \geq \mathcal{L}v & \Omega \\ u \leq v & \partial\Omega \end{cases}$$

Then $u \leq v$ in Ω.

The Comparison Principle will be our best friend.

3.1.2 Strong Maximum Principle

In the following one need to consider $\mathcal{L}$ uniformly elliptic in Ω.

Hopf's Lemma One begin with the Hopf Boundary Point Lemma.

Theorem 3.1.2 ([GT01] Lemma 3.4; [HL11] Theorem 2.5, Corollary 2.9; [Eva10] 6.4.2)**.** *Let $\mathcal{L}$ be uniformly elliptic in Ω bounded open domain. Assume (3.3) holds. Let that $u \in C^2(\Omega) \cap C^1(\Omega \cup \{x_0\})$ for some $x_0 \in \partial\Omega$ s.t.*

$$\mathcal{L}u \geq 0 \qquad \Omega$$

and

1. *the boundary point x_0 satisfies an interior ball condition*
2. *$u(x_0) > u(x)$ for any x sufficiently close to x (in particular, in the interior tangent ball)*

Assume also, that either

1. *$c = 0$*
2. *or $c \leq 0$ with $\frac{|c|}{\lambda} \leq C$, and $u(x_0) \geq 0$ (thus x_0 is a non-negative local maximum)*
3. *or $u(x_0) = 0$*

Then the outer normal derivative of u satisfies the strict inequality

$$\frac{\partial u}{\partial \nu}(x_0) > 0 \tag{3.7}$$

Proof. Take $y \in \Omega$ and consider the ball $B_r(y) \subseteq \Omega$ s.t. $\overline{B_r(y)} \cap \partial\Omega = \{x_0\}$. This is valid due to interior ball condition.

For fixed y as center, we construct the Hopf barrier

$$v(x) := e^{-\alpha|x-y|^2} - e^{-\alpha r^2} \qquad \forall\, x \in B_r(y)$$

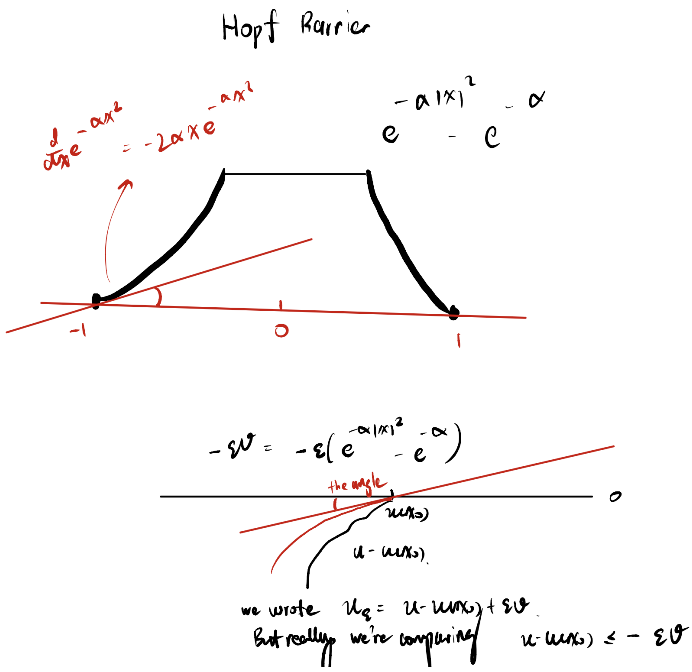


Figure 3.1: Picture for Hopf Barrier 3.1.2

Notice v satisfies (with c included)

$$\partial_i v = -2\alpha(x_i - y_i)e^{-\alpha|x-y|^2}$$

$$\partial_{ij} v = \left(4\alpha^2(x_i - y_i)(x_j - y_j) - 2\alpha\delta_{ij}\right) e^{-\alpha|x-y|^2}$$

$$\mathcal{L}v = \left(\left(4\alpha^2(x_i - y_i)(x_j - y_j) - 2\alpha\delta_{ij}\right) a_{ij} - 2\alpha(x_i - y_i)b_i + c\right) e^{-\alpha|x-y|^2} - ce^{-\alpha r^2}$$

and that

$$v = 0 \qquad \partial B_r(y)$$

Assume for now $c \leq 0$. If one restrict on the annulus

$$B_r(y) \setminus B_{r/2}(y)$$

one may take α sufficiently large to ensure

$$\begin{aligned}
\mathcal{L}v &= \left(\left(4\alpha^2(x_i-y_i)(x_j-y_j) - 2\alpha\delta_{ij}\right)a_{ij} - 2\alpha(x_i-y_i)b_i + c\right)e^{-\alpha|x-y|^2} - ce^{-\alpha r^2} \\
&\geq \left(\left(4\alpha^2(x_i-y_i)(x_j-y_j) - 2\alpha\delta_{ij}\right)a_{ij} - 2\alpha(x_i-y_i)b_i + c\right)e^{-\alpha|x-y|^2} \qquad \text{using } c \leq 0 \\
&\geq \left(\alpha^2 r^2\lambda - 2\alpha(a_{ii} + r|b|) + c\right)e^{-\alpha|x-y|^2} \qquad \text{using we're on } \frac{1}{2}r < |x-y| < r \\
&= \lambda\left(\alpha^2 r^2 - 2\alpha\left(\frac{a_{ii}}{\lambda} + r\frac{|b|}{\lambda}\right) + \frac{c}{\lambda}\right)e^{-\alpha|x-y|^2}
\end{aligned}$$

Now using our assumption on uniform ellipticity, on (3.3), and on $\frac{|c|}{\lambda} \leq C$, one obtain

$$\frac{a_{ii}}{\lambda} \leq \frac{\Lambda}{\lambda} \leq C, \qquad \frac{|b|}{\lambda} \leq C, \qquad \frac{|c|}{\lambda} \leq C$$

Hence one wish to choose α large s.t.

$$\alpha^2 r^2 - 2C(r)\alpha - C \geq 0$$

which is indeed valid for r fixed.

Now we think of sliding $-\varepsilon v$ down to touch the function $u - u(x_0)$ from above at the point x_0. Essentially we use that v creates a positive angle at x_0, which forces u to create a larger angle at x_0, giving us (3.7).

We do our analysis on the annulus $B_r(y) \setminus B_{r/2}(y)$. Since

$$u(x) < u(x_0) \qquad \forall\, x \in B_r(y)$$

Using continuity, one may choose $\varepsilon > 0$ small so that

$$u - u(x_0) < -\varepsilon v(x) \qquad \partial B_{r/2}(y)$$

On the other hand $v = 0$ on $\partial B_r(y)$ indeed gives

$$u - u(x_0) < -\varepsilon v \qquad \partial B_r(y)$$

Notice

$$\mathcal{L}(u - u(x_0) + \varepsilon v) = \mathcal{L}u - cu(x_0) + \varepsilon\mathcal{L}v \geq 0 \qquad B_r(y) \setminus B_{r/2}(y)$$

where we've used either $c = 0$, or $u(x_0) \geq 0$ and $c \leq 0$. Now one obtain via comparison principle (Weak Maximum Principle Corollary 3.1.1) that

$$u - u(x_0) \leq -\varepsilon v \qquad B_r(y) \setminus B_{r/2}(y)$$

Now in particular picking

$$x = x_0 - t\nu \qquad \nu \text{ outer-normal at } x_0$$

one obtain

$$\frac{1}{t}(u(x_0) - u(x_0 - t\nu)) \geq -\varepsilon\frac{1}{t}(v(x_0) - v(x_0 - t\nu))$$

where we've used that $v(x_0) = 0$. Now taking $t \to 0$ yields

$$\liminf_{t\to 0} \frac{1}{t}(u(x_0) - u(x_0 - t\nu)) \geq -\varepsilon\frac{\partial v}{\partial \nu}(x_0)$$

where

$$\frac{\partial v}{\partial \nu}(x_0) = \nabla v(x_0)\cdot\nu = -2\alpha e^{-\alpha r^2}(x_0 - y)\cdot\nu < 0$$

Hence (3.7) follows for the case $c = 0$ or $c \leq 0$ with $\frac{|c|}{\lambda} \leq C$ with $u(x_0) \geq 0$.

Finally, if one only has the condition $u(x_0) = 0$, consider using the operator

$$\mathcal{L} - c^+$$

and the result follows. □

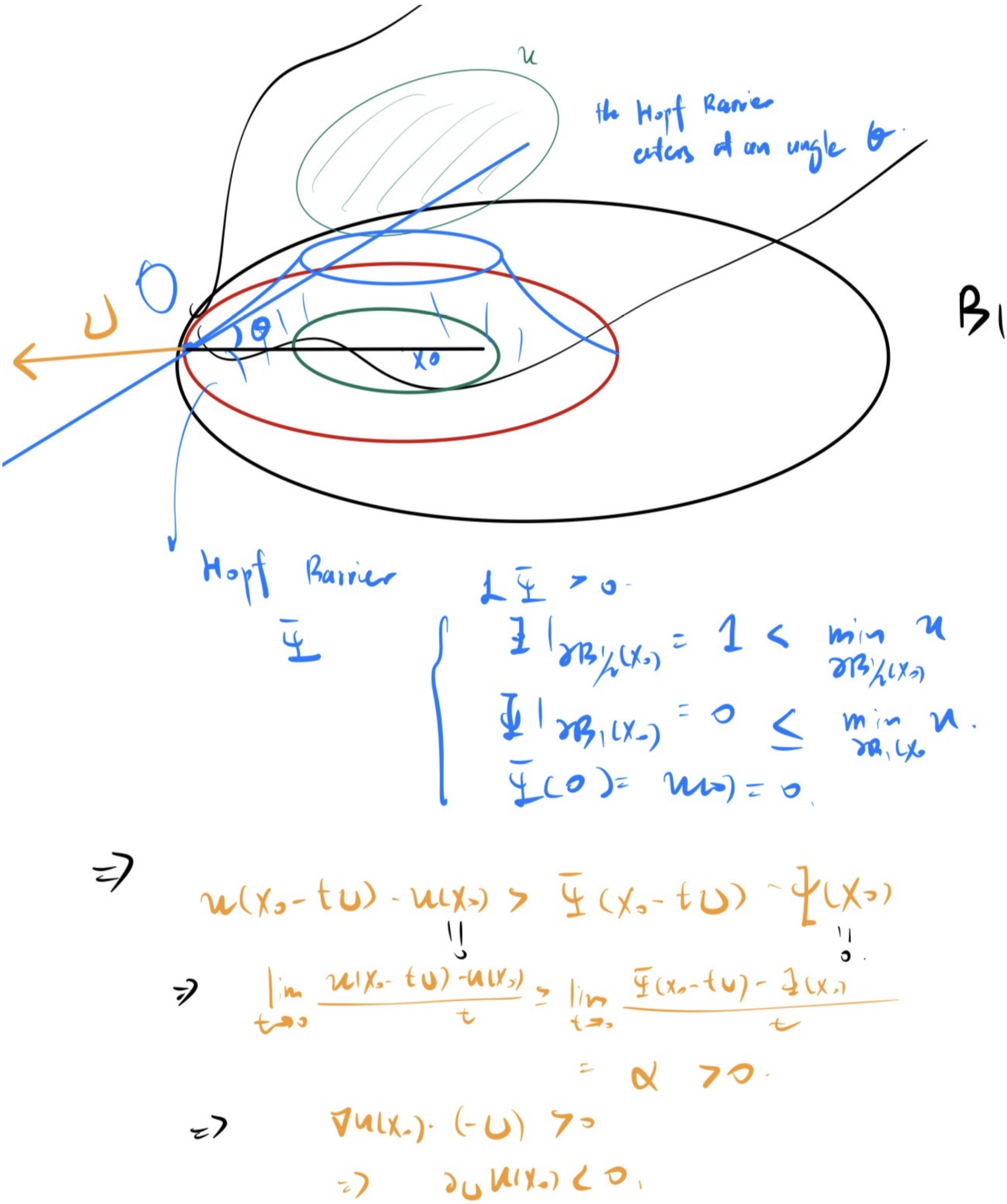


Figure 3.2: Picture for Hopf's Lemma 3.1.2

With Hopf Lemma, one may show for Strong Maximum Principle.

Theorem 3.1.3 ([GT01] Theorem 3.5; [HL11] Theorem 2.7)**.** *Let $\mathcal{L}$ be uniformly elliptic in Ω open domain (not necessarily bounded). Assume $u \in C^2(\Omega)$ solves*

$$\mathcal{L}u \geq 0 \qquad \Omega$$

Then

1. *if $c = 0$, u cannot achieve interior maximum unless u is constant.*
2. *if $c \leq 0$ and $\frac{|c|}{\lambda} \leq C$, u cannot achieve interior non-negative maximum unless u is constant.*

Proof. Assume there exists $x_0 \in \Omega$ interior s.t. $u(x_0)$ achieves a non-negative maximum. Then consider the open set

$$\Omega^- = \{x \in \Omega \mid u(x) < u(x_0)\}$$

Pick any point $y \in \Omega^-$ s.t. $r = \text{dist}(y, \partial\Omega^-) < \frac{1}{2}\text{dist}(y, \partial\Omega)$ (if $\partial\Omega$ exists). Then consider the ball $B_r(y)$, on which

$$u < u(x_0) \qquad x \in B_r(y)$$

Thus one may do Hopf's Lemma Theorem 3.1.2 (with either assumption) at the ball $B_r(y)$ with boundary point x_0 so that

$$\frac{\partial u}{\partial \nu}(x_0) > 0$$

But this is contradiction to an interior maximum. □

One make few observations for the Strong Maximum Principle.

1. Assume one already know u is constant throughout Ω. Then if $c < 0$ for some point, necessarily $u \equiv 0$.
2. Assume u achieves interior maximum at x_0 and $u(x_0) = 0$, then regardless of sign of c, via Hopf Lemma Theorem 3.1.2 one has $u \equiv 0$.

Or one may simply remember the punchline: if $c(x) \leq 0$ and bounded (which is very good assumption), one has for classical solution $\mathcal{L}u \geq 0$ with $u \in C^2(\Omega) \cap C(\overline{\Omega})$

1. If $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ in Ω (by Weak Maximum Principle). In fact, either $u \equiv 0$ or $u < 0$ strictly in Ω (by Strong Maximum Principle). ([HL11] Corollary 2.8)
2. If Ω has interior sphere condition and $u \in C^1(\overline{\Omega})$, then for any $x_0 \in \overline{\Omega}$ that $u(x_0)$ achieves non-negative maximum, necessarily $x_0 \in \partial\Omega$, and
$$\frac{\partial u}{\partial \nu}(x_0) > 0$$
unless u is a constant. ([HL11] Corollary 2.9)

3.1.2.1 Uniqueness of Boundary Value Problems

We see uniqueness of Neumann Boundary Value Problem (up to adding constants).

Corollary 3.1.2 ([GT01] Theorem 3.6)**.** *Let $\mathcal{L}$ be uniformly elliptic in Ω open bounded domain with interior sphere condition. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solve*

$$\begin{cases} \mathcal{L}u = 0 & \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \partial\Omega \end{cases}$$

Assume either

1. *$c = 0$*
2. *or $c \leq 0$ with $\frac{|c|}{\lambda} \leq C$.*

Then u is constant throughout Ω. If $c < 0$ at some point, then $u \equiv 0$ throughout Ω.

Proof. Assume $u \neq$ constant. If u is constant on $\partial\Omega$ then uniqueness of Dirichlet problem forces $u \equiv$ constant in Ω, and we're done. Otherwise one may assume either u or $-u$ achieves non-negative maximum at point $x_0 \in \partial\Omega$. By Strong Maximum Principle, any point in Ω is strictly less than the maximum value, otherwise it's constant and we're done. Now in the worst case, one still has the Hopf Lemma which forces

$$\frac{\partial u}{\partial \nu}(x_0) > 0$$

yielding a contradiction to the Neumann Boundary condition. Hence $u \equiv$ constant in Ω.

On the other hand, if for some $x_1 \in \Omega$, $c(x_1) < 0$, then using u constant throughout Ω

$$\mathcal{L}u(x_1) = c(x_1)u(x_1) = 0 \implies u(x_1) = 0$$

Hence $u \equiv 0$ throughout Ω. □

Or, more generally, one can do for mixed boundary conditions.

Corollary 3.1.3 ([HL11] Section 2.1 Application)**.** *Let $\mathcal{L}$ be uniformly elliptic in Ω open bounded domain with interior sphere condition. Let $u \in C^2(\Omega) \cap C^1\overline{\Omega})$ solve*

$$\begin{cases} \mathcal{L}u = f & \Omega \\ \frac{\partial u}{\partial \nu} + \alpha(x)u = \varphi & \partial\Omega \end{cases}$$

with $f \in C(\overline{\Omega})$ and $\varphi \in C(\partial\Omega)$. Assume

$$c \le 0, \qquad \alpha \ge 0$$

Then

1. *If $c \ne 0$, $\frac{|c|}{\lambda} \le C$, or $\alpha \ne 0$, then $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ has at most one solution.*
2. *If $c \equiv 0$ and $\alpha \equiv 0$, then $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ has at most one solution up to constants.*

Proof. Notice, via linearity, the difference of two possible solutions solve the homogeneous equation. Thus it suffices to consider solution u to the homogeneous equation

$$\begin{cases} \mathcal{L}u = 0 & \Omega \\ \frac{\partial u}{\partial \nu} + \alpha(x)u = 0 & \partial\Omega \end{cases}$$

1. Let $c \ne 0$ or $\alpha \ne 0$. If $u \not\equiv 0$, WLOG assume it has an positive maximum at $x_0 \in \overline{\Omega}$.
   (a) Assume x_0 occurs in the interior. Then by Strong Maximum Principle $u \equiv u(x_0) > 0$ throughout Ω. If $c < 0$ at some point, say $c(x_1) < 0$, then using u is constant

   $$\mathcal{L}u(x_1) = c(x_1)u(x_1) = c(x_1)u(x_0) < 0$$

   yields a contradiction to the equation. If on the other hand $\alpha > 0$ at some point $x_2 \in \partial\Omega$, then at the point x_2

   $$\frac{\partial u}{\partial \nu}(x_2) + \alpha(x_2)u(x_2) = \alpha(x_2)u(x_0) > 0$$

   a contradiction to the boundary condition. Then that x_0 lies in the interior is ruled out.

   (b) Assume $x_0 \in \partial\Omega$, then one directly apply Hopf's Lemma (in view of interior sphere condition) to obtain

   $$\frac{\partial u}{\partial \nu}(x_0) > 0$$

   But then at the point x_0, $u(x_0) > 0$ by assumption, and $\alpha \ge 0$, hence

   $$\frac{\partial u}{\partial \nu}(x_0) + \alpha(x_0)u(x_0) > 0$$

   a contradiction.

   Hence $u \equiv 0$ is the unique solution.
2. Let $c \equiv 0$ and $\alpha \equiv 0$. Then directly using Corollary 3.1.2 with $c = 0$, one obtain u is constant throughout Ω.

□

3.1.2.2 Strong Maximum Principle with relaxed assumption on c

In this section we relax the assumption $c \le 0$ with multiple alternatives.

Serrin: $u \le 0$ A result by Serrin uses the stronger assumption $u \le 0$ in Ω to conclude either $u \equiv 0$ or $u < 0$ in Ω, without mentioning the sign on c.

Theorem 3.1.4 ([HL11] Theorem 2.10)**.** *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be classical subsolution $\mathcal{L}u \ge 0$. Assume $u \le 0$ in Ω, then either $u < 0$ in Ω or $u \equiv 0$.*

Proof. Two methods.

1. If for some interior $x_0 \in \Omega$, $u(x_0) = 0$, then this should be local maximum. Then consider decomposition $c = c^+ - c^-$ so

   $$a_{ij}\partial_{ij}u + b_i\partial_i u - c^- u \ge -c^+ u \ge 0 \qquad \text{using } u \le 0$$

   Since $-c^- \le 0$, one conclude via Strong Maximum Principle that either $u \equiv 0$ or $u < 0$. But $u(x_0) = 0$ yields $u \equiv 0$.

2. Another method considers $v = ue^{-\alpha x_1}$. Assume that a_{ij} is symmetric. Compute

$$\partial_i v = \partial_i u e^{-\alpha x_1} - \alpha u \delta_{1i} e^{-\alpha x_1}$$
$$\partial_{ij} v = \partial_{ij} u e^{-\alpha x_1} - \alpha \partial_i u \delta_{1j} e^{-\alpha x_1} - \alpha \partial_j u \delta_{1i} e^{-\alpha x_1} + \alpha^2 u \delta_{1i}\delta_{1j} e^{-\alpha x_1}$$

so

$$\begin{aligned}\mathcal{L}_0 v &= a_{ij}\partial_{ij} v + b_i \partial_i v \\ &= e^{-\alpha x_1}\mathcal{L}_0 u - 2\alpha a_{1i}\partial_i u e^{-\alpha x_1} - \alpha b_1 u e^{-\alpha x_1} + \alpha^2 a_{11} u e^{-\alpha x_1}\end{aligned}$$

Now add both sides with $2\alpha a_{1i}\partial_i v$ so that

$$\begin{aligned}a_{ij}\partial_{ij} v + (b_i + 2\alpha a_{1i})\partial_i v &= e^{-\alpha x_1}\mathcal{L}_0 u - 2\alpha^2 a_{11} u e^{-\alpha x_1} - \alpha b_1 u e^{-\alpha x_1} + \alpha^2 a_{11} u e^{-\alpha x_1} \\ &= e^{-\alpha x_1}\mathcal{L} u - (\alpha^2 a_{11} + \alpha b_1 + c) v\end{aligned}$$

$$a_{ij}\partial_{ij} v + (b_i + 2\alpha a_{1i})\partial_i v + (\alpha^2 a_{11} + \alpha b_1 + c) v = e^{-\alpha x_1}\mathcal{L} u \geq 0$$

Recall one has a sign on $v \leq 0$ due to $u \leq 0$. Thus for α sufficiently large

$$a_{ij}\partial_{ij} v + (b_i + 2\alpha a_{1i})\partial_i v \geq -(\alpha^2 a_{11} + \alpha b_1 + c) v \geq 0$$

Now conclude via Strong Maximum Principle that either $v \equiv 0$ (thus $u \equiv 0$) or $v < 0$ (thus $u < 0$).

□

Existence of Supersolution Barrier $w > 0$ with $\mathcal{L} w \leq 0$ For the usual Maximum Principle to apply, let Ω be bounded domain.

Proposition 3.1.1 ([HL11] Theorem 2.11)**.** *Assume there exists $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ s.t. $w > 0$ in Ω and $\mathcal{L} w \leq 0$. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be classical subsolution $\mathcal{L} u \geq 0$.*

Then the ratio $\frac{u}{w}$ cannot achieve interior non-negative maximum unless $\frac{u}{w}$ is constant.

Moreover, if $\frac{u}{w}$ achieves non-negative maximum at $x_0 \in \partial\Omega$ and $\frac{u}{w} \not\equiv$ constant, and $\partial\Omega$ admits interior sphere condition at x_0, then

$$\frac{\partial}{\partial \nu}(\frac{u}{w})(x_0) > 0$$

Proof. Consider $\frac{u}{w}$. We compute

$$\partial_i(\frac{u}{w}) = \frac{\partial_i u}{w} - \frac{u\partial_i w}{w^2}$$
$$\partial_{ij}(\frac{u}{w}) = \frac{\partial_{ij} u}{w} - \frac{\partial_i u \partial_j w}{w^2} - \frac{\partial_j u \partial_i w}{w^2} - \frac{u\partial_{ij} w}{w^2} + 2\frac{u\partial_i w \partial_j w}{w^3}$$

Thus using the above

$$\begin{aligned}\mathcal{L}(\frac{u}{w}) &= a_{ij}\partial_{ij}(\frac{u}{w}) + b_i\partial_i(\frac{u}{w}) + c\frac{u}{w} \\ &= \frac{1}{w}\mathcal{L} u - 2\frac{1}{w^2} a_{ij}\partial_i u \partial_j w - \frac{u}{w^2} b_i \partial_i w - \frac{u}{w^2} a_{ij}\partial_{ij} w + 2\frac{u}{w^3} a_{ij}\partial_i w \partial_j w \\ &= \frac{1}{w}\mathcal{L} u - \frac{2}{w} a_{ij}\partial_j w \partial_i(\frac{u}{w}) \cancel{- \frac{2u}{w^3} a_{ij}\partial_i w \partial_j w} - \frac{u}{w}\left(\frac{1}{w} a_{ij}\partial_{ij} w + \frac{1}{w} b_i \partial_i w\right) \cancel{+ 2\frac{u}{w^3} a_{ij}\partial_i w \partial_j w}\end{aligned}$$

One obtain

$$a_{ij}\partial_{ij}(\frac{u}{w}) + (b_i + \frac{2}{w} a_{ij}\partial_j w)\partial_i(\frac{u}{w}) + (\frac{\mathcal{L} w}{w})\frac{u}{w} = \frac{1}{w}\mathcal{L} u \geq 0$$

Now if we want to apply our original Maximum Principle, one need to ensure $w > 0$ and that $\mathcal{L} w \leq 0$. □

Corollary 3.1.4 ([HL11] Remark 2.12)**.** *Let the operator $\mathcal{L}$ in Ω satisfy that there exists $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ s.t. $w > 0$ in Ω and $\mathcal{L} w \leq 0$.*

Then the comparison principle applies to $\mathcal{L}$. In particular, the Dirichlet Boundary value problem

$$\begin{cases}\mathcal{L} u = f & \Omega \\ u = \varphi & \partial\Omega\end{cases}$$

has at most one solution.

Proof. Assume u_1 and u_2 are two solutions to the Dirichlet Boundary Value Problem, then $\mathcal{L}(u_1 - u_2) = 0$, then the ratio $\frac{1}{w}(u_1 - u_2)$ is either constant (thus $u_1 = u_2$ due to boundary condition), or $\frac{u_1 - u_2}{w}$ cannot achieve interior non-negative maximum, which also forces $u_1 = u_2$. □

In the following we discuss when such w exists for certain $\mathcal{L}$. This is Maximum Principle for a Narrow Domain.

Proposition 3.1.2 ([HL11] Proposition 2.13)**.** *Let $\mathcal{L}$ be uniformly elliptic operator with constant $\lambda > 0$. Then there exists $d_0 = d_0(\lambda, \|b\|_{C^0}, \|c^+\|_{C^0}) > 0$ s.t. for any $d < d_0$, if for some e unit direction, the domain Ω is narrow in the sense*

$$|(y - x) \cdot e| < d \qquad \forall\ x, y \in \Omega$$

One has the existence of $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ s.t. $w > 0$ in Ω and $\mathcal{L}w \leq 0$.

Proof. Assume WLOG that $\Omega \subseteq \{0 < x_1 < d\}$. Construct

$$w(x) := e^{\alpha d} - e^{\alpha x_1} > 0 \qquad \forall\ x \in \Omega$$

Now compute

$$\begin{aligned}
\partial_1 w &= -\alpha e^{\alpha x_1} \\
\partial_{11} w &= -\alpha^2 e^{\alpha x_1} \\
\mathcal{L}w &= a_{11}\partial_{11} w + b_1 \partial_1 w + cw = -(\alpha^2 a_{11} + \alpha b_1)e^{\alpha x_1} + c(e^{\alpha d} - e^{\alpha x_1}) \\
&\leq -(\lambda\alpha^2 - \|b\|_{C^0}\,\alpha)e^{\alpha x_1} + \left\|c^+\right\|_{C^0} e^{\alpha d}
\end{aligned}$$

Choose $\alpha = \alpha(\lambda, \|b\|_{C^0}, \|c^+\|_{C^0})$ large so

$$\lambda\alpha^2 - \|b\|_{C^0}\,\alpha \geq 2\left\|c^+\right\|_{C^0}$$

Then choosing $d < d_0 = d_0(\lambda, \|b\|_{C^0}, \|c^+\|_{C^0})$ sufficiently small so that

$$e^{\alpha d} - 2 \leq 0$$

yields the result. □

There's an counter-example for unbounded narrow domain where maximum Principle fails.

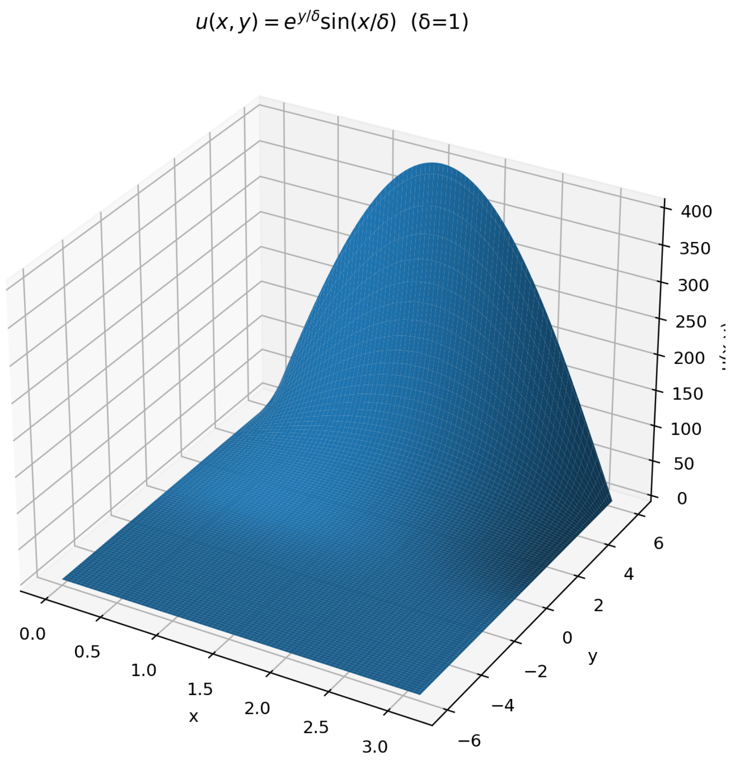


Figure 3.3: Picture for $u(x, y) = e^y \sin(x)$

Let

$$\Omega = (0, \delta\pi) \times \mathbb{R}$$

and

$$u(x, y) = e^{\frac{y}{\delta}} \sin(\frac{1}{\delta}x)$$

so that

$$\partial_x u = \frac{1}{\delta} e^y \cos(\frac{1}{\delta}x)$$
$$\partial_y u = \frac{1}{\delta} e^{\frac{y}{\delta}} \sin(\frac{1}{\delta}x)$$
$$\partial_{xx} u = -\frac{1}{\delta^2} e^y \sin(\frac{1}{\delta}x)$$
$$\partial_{yy} u = \frac{1}{\delta^2} e^y \sin(\frac{1}{\delta}x)$$

Thus

$$\Delta u = 0 \qquad (0, \delta\pi) \times \mathbb{R} \qquad \forall\ \delta > 0$$

But

$$u|_{\partial\Omega} = 0$$

3.2 A Priori Estimates

3.2.1 Dirichlet Estimates

Dirichlet Bound $(c \leq 0)$ Let's first establish the A priori bound for the Dirichlet Problem.

Theorem 3.2.1 ([GT01] Theorem 3.7)**.** *Consider* Ω *open, bounded domain. Let* $\mathcal{L}$ *be elliptic in* Ω*, with bounds on coefficient* (3.3)

$$\frac{|b_i(x)|}{\lambda} \leq \beta \qquad \forall\ x \in \Omega, \qquad \forall\ i = 1, \cdots, n$$

Assume $c \leq 0$ *in* Ω*. Let* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *solve*

$$\mathcal{L}u \geq f$$

for $f \in C(\overline{\Omega})$*. Then one has the estimate*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \sup_{\Omega} \frac{|f^-|}{\lambda} \tag{3.8}$$

for $C = C(\operatorname{diam}(\Omega), \beta) > 0$.

Proof. Since Ω is bounded, WLOG one assume there exists $d \geq \operatorname{diam}(\Omega) > 0$ s.t.

$$\Omega \subseteq \{x \in \mathbb{R}^n \mid 0 < x_1 < d\}$$

Consider barrier model

$$e^{\alpha x_1}$$

Hitting our operator $\mathcal{L}_0 = a_{ij}\partial_{ij} + b_i\partial_i$ on it yields

$$\begin{aligned} \mathcal{L}_0 e^{\alpha x_1} &= \left(a_{11}\alpha^2 + b_1\alpha\right) e^{\alpha x_1} \\ &= \lambda \left(\frac{a_{11}}{\lambda}\alpha^2 + \frac{b_1}{\lambda}\alpha\right) e^{\alpha x_1} \\ &\geq \lambda \left(\alpha^2 - \beta\alpha\right) e^{\alpha x_1} \\ &\geq \lambda \qquad \text{upon choosing } \alpha \geq \beta + 1 \text{ sufficiently large} \end{aligned}$$

Using such $e^{\alpha x_1}$, one construct our barrier as

$$v := \sup_{\partial\Omega} u^+ + \left(e^{\alpha d} - e^{\alpha x_1}\right) \sup_{\Omega} \frac{|f^-|}{\lambda}$$

Assuming $\sup_{\Omega} \frac{|f^-|}{\lambda} < \infty$. Now

$$\begin{aligned} \mathcal{L}v &= \mathcal{L}_0 v + cv \\ &= -\sup_{\Omega} \frac{|f^-|}{\lambda} \mathcal{L}_0 e^{\alpha x_1} + cv \\ &\leq -\lambda \sup_{\Omega} \frac{|f^-|}{\lambda} \qquad \text{using } c \leq 0 \text{ and } \sup_{\partial\Omega} u^+ \geq 0 \end{aligned}$$

Hence

$$\mathcal{L}v \leq f \leq \mathcal{L}u \qquad \Omega$$

On the other hand

$$v \geq u \qquad \partial\Omega \qquad \text{using the portion } \sup_{\partial\Omega} u^+$$

Hence by Comparison Principle

$$v \geq u \qquad \Omega$$

In other words

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C(\beta, d) \sup_{\Omega} \frac{|f^-|}{\lambda}$$

$\square$

In particular, when

$$\mathcal{L}u = f \qquad \Omega$$

One may redo the above for $-u$ to obtain

$$\sup_{\Omega}(-u) \le \sup_{\partial\Omega} u^- + C\sup_{\Omega}\frac{|f^+|}{\lambda}$$

Hence as a combination of both, one obtain

$$\|u\|_{C^0(\overline{\Omega})} \le \sup_{\partial\Omega}|u| + C\sup_{\Omega}\frac{|f|}{\lambda} \tag{3.9}$$

Dirichlet Bound (c not too positive) One can generalize a bit for c without a sign, but not too positive.

Corollary 3.2.1 ([GT01] Corollary 3.8)**.** *Consider Ω open bounded domain. Let $\mathcal{L}$ be elliptic in Ω, with bounds on coefficients*

$$\frac{|b_i|}{\lambda} \le \beta, \qquad \frac{|c^+|}{\lambda} \le \gamma < \frac{1}{C(\mathrm{diam}(\Omega), \beta)}$$

for C as in (3.8). Assume $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ solve

$$\mathcal{L}u = f \qquad \Omega$$

for $f \in C(\overline{\Omega})$. Then one has estimate

$$\|u\|_{C^0(\overline{\Omega})} \le \frac{1}{1 - C\gamma}\left(\sup_{\partial\Omega}|u| + C\sup_{\Omega}\left(\frac{|f|}{\lambda}\right)\right) \tag{3.10}$$

For $C = C(\beta, \mathrm{diam}(\Omega))$ as in (3.8).

Proof. We rewrite $c = c^+ - c^-$. Now

$$\mathcal{L}u = \mathcal{L}_0 u + cu$$
$$\mathcal{L}_0 u - c^- u = f - c^+ u$$

Now apply the Dirichlet bound (3.9) to $\mathcal{L}_0 u - c^- u$ one obtain

$$\begin{aligned}
\sup_{\Omega}|u| &\le \sup_{\partial\Omega}|u| + C\sup_{\Omega}\left(\frac{|f - c^+ u|}{\lambda}\right) \\
&\le \sup_{\partial\Omega}|u| + C\sup_{\Omega}\left(\frac{|f|}{\lambda}\right) + C\sup_{\Omega}(\frac{|c^+|}{\lambda}u) \\
(1 - C\gamma)\sup_{\Omega}|u| &\le \sup_{\partial\Omega}|u| + C\sup_{\Omega}\left(\frac{|f|}{\lambda}\right)
\end{aligned}$$

□

3.2.2 Neumann Estimates

Theorem 3.2.2 ([HL11] Proposition 2.16)**.** *Consider Ω open bounded domain. Let $\mathcal{L}$ be strictly elliptic in Ω, with bounds on coefficient (3.3)*

$$\frac{|b_i(x)|}{\lambda} \le \beta \qquad \forall\ x \in \Omega, \qquad \forall\ i = 1, \cdots, n$$

Assume $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solve

$$\begin{cases} \mathcal{L}u = f & \Omega \\ \frac{\partial u}{\partial \nu} + \alpha(x)u = \varphi & \partial\Omega \end{cases}$$

Assume

$$c(x) \le 0 \qquad \Omega, \qquad \alpha(x) \ge \alpha_0 > 0 \qquad \partial\Omega$$

Then

$$\sup_{\Omega}|u| \le C(\beta, \mathrm{diam}(\Omega), \alpha_0)\left(\|f\|_{C^0(\overline{\Omega})} + \|\varphi\|_{C(\partial\Omega)}\right) \tag{3.11}$$

Proof. 1. We first do the special case

$$c(x) \le -c_0 < 0$$

Define the trivial competitor

$$v := \frac{1}{c_0}\|f\|_{C^0(\overline{\Omega})} + \frac{1}{\alpha_0}\|\varphi\|_{C(\partial\Omega)}$$

where we need to compare with both $\pm u$. Notice

$$\begin{aligned}\mathcal{L}v = c\left(\frac{1}{c_0}\|f\|_{C^0(\overline{\Omega})} + \frac{1}{\alpha_0}\|\varphi\|_{C(\partial\Omega)}\right) \le -\|f\|_{C^0(\overline{\Omega})} \le \mathcal{L}(\pm u)\\ \frac{\partial v}{\partial\nu} + \alpha(x)v = \alpha\left(\frac{1}{c_0}\|f\|_{C^0(\overline{\Omega})} + \frac{1}{\alpha_0}\|\varphi\|_{C(\partial\Omega)}\right) \ge \|\varphi\|_{C(\partial\Omega)} \ge \frac{\partial(\pm u)}{\partial\nu} + \alpha(x)(\pm u)\end{aligned}$$

Essentially one want to make use of the idea of Comparison Principle. We want to show necessarily

$$v \ge \pm u \qquad \Omega$$

To do so, assume $v \pm u$ achieves a negative minimum in $\overline{\Omega}$. Since

$$\mathcal{L}(v \pm u) \le 0$$

Using Weak Minimum Principle the negative minimum is possibly achieved at $x_0 \in \partial\Omega$. Now at such point x_0

$$\frac{\partial v \pm u}{\partial\nu}(x_0) + \alpha(x_0)(v \pm u)(x_0) \le \alpha(x_0)(v \pm u)(x_0) < 0 \qquad \text{using } \alpha \ge \alpha_0 > 0$$

Contradicting our boundary condition

$$\frac{\partial v}{\partial\nu} + \alpha(x)v \ge \frac{\partial(\pm u)}{\partial\nu} + \alpha(x)(\pm u)$$

Hence

$$v \pm u \ge 0 \iff \sup_{\Omega}|u| \le \frac{1}{c_0}\|f\|_{C^0(\overline{\Omega})} + \frac{1}{\alpha_0}\|\varphi\|_{C(\partial\Omega)}$$

2. In the general case assume

$$c(x) \le 0$$

Consider splitting

$$u(x) = z(x)w(x)$$

for some auxiliary function z positive function in $\overline{\Omega}$. Compute

$$\begin{aligned} f = \mathcal{L}u &= a_{ij}\partial_{ij}(zw) + b_i\partial_i(zw) + czw\\ &= a_{ij}(\partial_i(z\partial_j w + w\partial_j z)) + b_i(z\partial_i w + w\partial_i z) + czw\\ &= a_{ij}(\partial_i z\partial_j w + z\partial_{ij}w + \partial_i w\partial_j z + w\partial_{ij}z) + b_i(z\partial_i w + w\partial_i z) + czw\\ &= z\left(a_{ij}\partial_{ij}w + b_i\partial_i w\right) + z(a_{ji}\frac{\partial_j z}{z} + a_{ij}\frac{\partial_j z}{z})\partial_i w + (cz + a_{ij}\partial_{ij}z + b_i\partial_i z)\, w\end{aligned}$$

Hence

$$\begin{aligned}\frac{f}{z} &= a_{ij}\partial_{ij}w + B_i\partial_i w + \left(c + \frac{1}{z}a_{ij}\partial_{ij}z + \frac{1}{z}b_i\partial_i z\right) w\\ \frac{1}{z}\mathcal{L}u &= a_{ij}\partial_{ij}w + B_i\partial_i w + \frac{\mathcal{L}z}{z}w\end{aligned}$$

for

$$B_i := b_i + (a_{ji}\frac{\partial_j z}{z} + a_{ij}\frac{\partial_j z}{z})$$

One obtain a second order linear elliptic equation on w. On the other hand, the boundary condition translates to

$$\begin{aligned}\varphi = \frac{\partial u}{\partial\nu} + \alpha u = \frac{\partial(zw)}{\partial\nu} + \alpha zw = w\frac{\partial z}{\partial\nu} + z\frac{\partial w}{\partial\nu} + \alpha zw\\ = z\left(\frac{\partial w}{\partial\nu} + \alpha w\right) + w\frac{\partial z}{\partial\nu}\end{aligned}$$

Hence

$$\frac{\varphi}{z} = \frac{\partial w}{\partial \nu} + \left(\alpha + \frac{1}{z}\frac{\partial z}{\partial \nu}\right) w \qquad \partial\Omega$$

We claim that one may choose z appropriately so that one return to the first case for the equations w. More precisely, one wish to choose z so that

$$\frac{1}{z}a_{ij}\partial_{ij}z + \frac{1}{z}b_i\partial_i z \leq -c_0 < 0 \qquad \text{for some } c_0 \text{ of our choice}$$
$$|\frac{1}{z}\frac{\partial z}{\partial \nu}| < \frac{1}{2}\alpha_0$$

In this case, one pick

$$z(x) := A + e^{\gamma d} - e^{\gamma x_1}$$

where we assumed WLOG that $\text{diam}(\Omega) < d$ and

$$\Omega \subseteq \{0 < x_1 < d\}$$

Let's see how we choose our A and γ to satisfy what we need.

$$\begin{aligned}
\frac{1}{z}a_{ij}\partial_{ij}z + \frac{1}{z}b_i\partial_i z &= \frac{1}{A + e^{\gamma d} - e^{\gamma x_1}}\left(-a_{11}\gamma^2 e^{\gamma x_1} - b_1\gamma e^{\gamma x_1}\right) \\
&\leq \frac{-(a_{11}\gamma^2 + b_1\gamma)}{A + e^{\gamma d}} \\
&= -\frac{\lambda(\frac{a_{11}}{\lambda}\gamma^2 + \frac{b_1}{\lambda}\gamma)}{A + e^{\gamma d}} \leq -\frac{\lambda(\gamma^2 - \beta\gamma)}{A + e^{\gamma d}} \\
&\leq -\frac{\lambda}{A + e^{\gamma d}} \qquad \text{choosing } \gamma \geq \beta + 1 \text{ sufficiently large} \\
&\leq -\frac{\lambda_0}{A + e^{\gamma d}} := -c_0 < 0 \qquad \text{using } \mathcal{L} \text{ is strictly elliptic}
\end{aligned}$$

On the other hand

$$|\frac{1}{z}\frac{\partial z}{\partial \nu}| \leq |\frac{\gamma e^{\gamma x_1}}{A + e^{\gamma d} - e^{\gamma x_1}}| \leq \underbrace{\frac{\gamma}{A}|e^{\gamma d}| \leq \frac{1}{2}\alpha_0}_{\text{using strict-positivity of } \alpha_0}$$

where we picked A to be sufficiently large depending on α_0, γ, d.

Hence w solves for the first case, and one has the estimate

$$\sup_{\Omega}|w| \leq C(\beta, d, \alpha_0)\left(\left\|\frac{f}{z}\right\|_{C^0(\overline{\Omega})} + \left\|\frac{\varphi}{z}\right\|_{C(\partial\Omega)}\right)$$
$$\sup_{\Omega}|u| \leq C(\beta, d, \alpha_0)\left(\|f\|_{C^0(\overline{\Omega})} + \|\varphi\|_{C(\partial\Omega)}\right)$$

□

3.3 Gradient Estimates

Consider $\Omega \subseteq \mathbb{R}^n$ bounded open. Consider the equation

$$a_{ij}\partial_{ij}u + b_i\partial_i u = f(x,u) \qquad \Omega$$

for $f \in C(\Omega, \mathbb{R})$.

3.3.1 Bernstein's Technique

Theorem 3.3.1 ([HL11] Proposition 2.18)**.** *Let Ω be bounded open. Let $u \in C^3(\Omega) \cap C^1(\overline{\Omega})$ be solution to*

$$a_{ij}\partial_{ij}u + b_i\partial_i u = f(x,u) \qquad \Omega$$

for $a_{ij}, b_i \in C^1(\overline{\Omega})$, $f \in C^1(\overline{\Omega} \times \mathbb{R})$, and a_{ij} uniformly elliptic with constants $\lambda, \Lambda > 0$.

Then there exists $C = C(n, \lambda, \operatorname{diam}(\Omega), \|a_{ij}\|_{C^1}, \|b_i\|_{C^1}, M = \sup_\Omega u, \|f\|_{C^1(\overline{\Omega}\times[-M,M])}) > 0$ s.t.

$$\sup_\Omega |\nabla u| \leq \sup_{\partial\Omega} |\nabla u| + \sup_{\partial\Omega} u + C$$

Proof. Denote $\mathcal{L}u := a_{ij}\partial_{ij}u + b_i\partial_i u$. Now we compute

$$\partial_i(|\nabla u|^2) = \partial_i\Big(\sum_k (\partial_k u)^2\Big) = \sum_k 2\partial_k u \partial_{ik} u$$
$$\partial_{ij}(|\nabla u|^2) = \sum_k 2\left(\partial_{jk}u\partial_{ik}u + \partial_k u \partial_{ijk}u\right)$$

Now how do we know what equation $|\nabla u|^2$ solves? To do so, we differentiate the original equation in x_k

$$\partial_k a_{ij}\partial_{ij}u + \partial_k b_i \partial_i u + a_{ij}\partial_{ijk}u + b_i\partial_{ik}u = \partial_k f + \partial_{n+1}f\partial_k u$$
$$a_{ij}\partial_{ijk}u + b_i\partial_{ik}u = \partial_k f + \partial_{n+1}f\partial_k u - (\partial_k a_{ij}\partial_{ij}u + \partial_k b_i\partial_i u)$$

Now making use of our previous computations, we multiply both sides with $\partial_k u$ and sum in k

$$a_{ij}\partial_{ijk}u\partial_k u + b_i\partial_{ik}u\partial_k u = (\partial_k f + \partial_{n+1}f\partial_k u - (\partial_k a_{ij}\partial_{ij}u + \partial_k b_i\partial_i u))\,\partial_k u \qquad \text{multiply both sides by } \partial_k u$$
$$a_{ij}\partial_{ij}(|\nabla u|^2) + b_i\partial_i(|\nabla u|^2) = 2a_{ij}\nabla\partial_j u \cdot \nabla\partial_i u + \sum_k (\partial_k f + \partial_{n+1}f\partial_k u - (\partial_k a_{ij}\partial_{ij}u + \partial_k b_i\partial_i u))\,\partial_k u$$

Now via uniform ellipticity, then Young's Inequality one obtain

$$\mathcal{L}(|\nabla u|^2) \geq \frac{1}{2}\lambda|D^2u|^2 - C_1|\nabla u|^2 - C_2 \tag{3.12}$$

Then agenda is as follows: One want to add terms to the LHS so that $\mathcal{L}(|\nabla u|^2 + \text{terms}) \geq 0$, and one may apply the Weak Maximum Principle. The two bad terms one needs to fight against are $-C_1|\nabla u|^2$ and $-C_2$.

We first add αu^2

$$\partial_i(u^2) = 2u\partial_i u$$
$$\partial_{ij}(u^2) = 2\partial_i u\partial_j u + 2u\partial_{ij}u$$

so that (renaming the constants) for α large

$$\mathcal{L}(|\nabla u|^2 + \alpha u^2) \geq \frac{1}{4}\lambda|D^2u|^2 + C_1|\nabla u|^2 - C_2u^2 - C_3$$

Now one has the bad term $-C_2u^2$ to fight against. We add $e^{\beta x_1}$

$$\partial_1 e^{\beta x_1} = \beta e^{\beta x_1}$$

so that (renaming the constants) for β large depending on $\sup_\Omega u$

$$\mathcal{L}(|\nabla u|^2 + \alpha u^2 + e^{\beta x_1}) \geq \frac{1}{4}\lambda|D^2u|^2 + C_1|\nabla u|^2 - C_2u^2 - C_3 + \beta^2 a_{11}e^{\beta x_1} + \beta b_1 e^{\beta x_1} \geq 0$$

where we put $\Omega \subseteq \{x_1 > 0\}$. $\square$

3.3.2 Berstein's Technique for Interior Gradient Estimate

Theorem 3.3.2 ([HL11] Proposition 2.19)**.** *Let Ω be bounded open. Let $u \in C^3(\Omega)$ be solution to*

$$a_{ij}\partial_{ij}u + b_i\partial_i u = f(x,u) \qquad \Omega$$

for $a_{ij}, b_i \in C^1(\overline{\Omega})$, $f \in C^1(\overline{\Omega}\times\mathbb{R})$, and a_{ij} uniformly elliptic with constants $\lambda, \Lambda > 0$.

Then for any compact subset $\Omega' \Subset \Omega$, there exists $C = C(n, \lambda, \mathrm{diam}(\Omega), \mathrm{dist}(\Omega', \partial\Omega), \|a_{ij}\|_{C^1}, \|b_i\|_{C^1}, M = \sup_\Omega u, \|f\|_{C^1(\overline{\Omega}\times[-M,M])}) > 0$ s.t.

$$\sup_{\Omega'}|\nabla u| \le \sup_{\partial\Omega} u + C$$

Proof. Instead of considering $|\nabla u|^2$, one would like to multiply with a cutoff $\eta \in C_0^\infty(\Omega)$ with $\eta \ge 0$. Now

$$\partial_i(\eta|\nabla u|^2) = \partial_i\eta|\nabla u|^2 + 2\eta\sum_k \partial_k u\partial_{ik}u$$
$$\partial_{ij}(\eta|\nabla u|^2) = \partial_{ij}\eta|\nabla u|^2 + 2\partial_i\eta\sum_k \partial_k u\partial_{jk}u + 2\partial_j\eta\sum_k \partial_k u\partial_{ik}u + 2\eta\sum_k(\partial_{jk}u\partial_{ik}u + \partial_k u\partial_{ijk}u)$$

Hence in view of our previous computations (3.12), denoting $\mathcal{L} = a_{ij}\partial_{ij} + b_i\partial_i$

$$\begin{aligned}\mathcal{L}(\eta|\nabla u|^2) &= a_{ij}\partial_{ij}(\eta|\nabla u|^2) + b_i\partial_i(\eta|\nabla u|^2)\\ &= 2\eta\mathcal{L}(|\nabla u|^2) + \partial_i\eta(4a_{ij}\sum_k \partial_k u\partial_{jk}u + b_i|\nabla u|^2) + a_{ij}\partial_{ij}\eta|\nabla u|^2 \qquad \text{using } a_{ij} = a_{ji} \text{ symmetric}\\ &\ge \eta(\lambda|D^2u|^2 - C_1|\nabla u|^2 - C_2) - C_3\left(C(\varepsilon)|\nabla u|^2 + \varepsilon|\nabla\eta|^2|D^2u|^2 + |\nabla\eta||\nabla u|^2\right) - C_4|D^2\eta||\nabla u|^2\end{aligned}$$

Now choosing $\varepsilon > 0$ small and picking η s.t.

$$|\nabla\eta|^2 \le C\eta$$

yields non-negativity of the leading order term, hence

$$\mathcal{L}(\eta|\nabla u|^2) \ge \eta\frac{1}{4}\lambda|D^2u|^2 - C_1(|D^2\eta| + |\nabla\eta| + \eta + 1)|\nabla u|^2 - C_2\eta$$

Now for fixed choice of η, add the terms αu^2 and $e^{\beta x_1}$ for α, β large depending on η so that

$$\mathcal{L}(\eta|\nabla u|^2 + \alpha u^2 + e^{\beta x_1}) \ge 0$$

Conclude via Weak Maximum Principle. □

3.3.3 Boundary Gradient Estimate

In the boundary version we forget about Berstein, but still use Classical Maximum Principle. The barrier function one need encodes in a ball to the exterior of Ω, that depends on the distance from a point to the center of the ball.

Theorem 3.3.3 ([HL11] Proposition 2.20)**.** *Let Ω be bounded open, with exterior ball condition. Let $u \in C^2(\Omega) \cap C(\overline{\Omega}) \cap C^1(\Omega \cup \{x_0\})$ be solution to*

$$a_{ij}\partial_{ij}u + b_i\partial_i u = f(x,u) \qquad \Omega$$

for $a_{ij}, b_i \in C(\overline{\Omega})$, $f \in C(\overline{\Omega}\times\mathbb{R})$, and a_{ij} uniformly elliptic with constants $\lambda, \Lambda > 0$.

Then there exists $C = C(n, \lambda, \Lambda, \mathrm{diam}(\Omega), M = \sup_\Omega|u|, \|f\|_{C^0(\Omega\times[-M,M])}, \|\varphi\|_{C^2(\overline{\Omega})}, \|a\|_{C^0}, \|b\|_{C^0}) > 0$ s.t.

$$\left|\frac{\partial u}{\partial\nu}(x_0)\right| \le C$$

Proof. Assume for simplicity that $u = 0$ on $\partial\Omega$. This is done upon subtracting u with $\varphi \in C^2(\overline{\Omega})$ s.t. $u = \varphi$ on $\partial\Omega$.

Denote $F := \sup_\Omega|f(\cdot, u)|$. Denote $\mathcal{L} = a_{ij}\partial_{ij} + b_i\partial_i$ so that

$$\mathcal{L}(\pm u) = \pm f \ge -F \qquad \Omega$$

Now for any $x_0 \in \partial\Omega$, one would like to build a function w s.t.

$$\mathcal{L}w \le -F \qquad \Omega, \qquad w(x_0) = 0, \qquad w \ge 0 \qquad \partial\Omega \tag{3.13}$$

If so, via Weak Maximum Principle

$$\begin{cases} \mathcal{L}(\pm u - w) & \geq 0 \qquad \Omega \\ \pm u - w & \leq 0 \qquad \partial\Omega \end{cases} \implies -w \leq u \leq w \qquad \Omega$$

Now letting $x = x_0 - t\nu$ where ν denotes the outward unit normal of $\partial\Omega$ one obtain

$$\frac{-w(x) + w(x_0)}{t} \leq \frac{u(x_0 - t\nu) - u(x_0)}{t} \leq \frac{w(x) - w(x_0)}{t}$$

Now sending $t \to 0$ yields

$$|\frac{\partial u}{\partial \nu}(x_0)| \leq \frac{\partial w}{\partial \nu}(x_0)$$

In addition to (3.13) one also need a uniform in $x_0 \in \partial\Omega$ bound on $\frac{\partial w}{\partial \nu}$.

We need the *exterior ball condition.* Let $B_R(y)$ with $\overline{B_R(y)} \cap \overline{\Omega} = \{x_0\}$. Now let d denote the distance from x to the center y

$$d(x) := |x - y| - R \qquad \forall\, x \in \overline{\Omega}$$

so that

$$0 \leq d \leq \operatorname{diam}(\Omega) < \infty$$

Let's define

$$w(x) := \psi(d(x))$$

for $\psi \in C^2[0, \infty)$ to be chosen. Compute

$$\begin{aligned} \partial_i w &= \psi'(d(x))\partial_i d \\ \partial_{ij} w &= \psi''(d(x))\partial_i d \partial_j d + \psi'(d(x))\partial_{ij} d \\ \partial_i d(x) &= \frac{x_i - y_i}{|x - y|} \\ \partial_{ij} d(x) &= \frac{\delta_{ij}}{|x - y|} - \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^3} \end{aligned}$$

The good choice about d is that now

$$\begin{aligned} a_{ij}\partial_{ij} d(x) &= a_{ii}\frac{1}{|x - y|} - a_{ij}\frac{(x_i - y_i)(x_j - y_j)}{|x - y|^3} \\ &= a_{ii}\frac{1}{|x - y|} - a_{ij}\frac{1}{|x - y|}\partial_i d(x)\partial_j d(x) \\ &\leq \frac{\Lambda}{|x - y|} - \frac{\lambda}{|x - y|} \qquad \text{using diagonal entries trapped in between smallest and largest eigenvalues} \\ &\leq \frac{\Lambda - \lambda}{R} \end{aligned}$$

so that

$$\begin{aligned} \mathcal{L}w &= a_{ij}\partial_{ij} w + b_i \partial_i w \\ &= a_{ij}\psi''\partial_i d \partial_j d + a_{ij}\psi'\partial_{ij} d + b_i \psi' \partial_i d \\ &\leq a_{ij}\psi''\partial_i d \partial_j d + |\psi'|\frac{\Lambda - \lambda}{R} + \|b\|_{C^0} |\psi'| \end{aligned}$$

We want to make the RHS smaller then $-F$, which is negative. The goal is thus to require

$$\psi'' < 0, \qquad \psi' > 0$$

so that one may bound

$$\mathcal{L}w \leq \lambda\psi'' + \psi'\left(\frac{\Lambda - \lambda}{R} + \|b\|_{C^0}\right) \overset{\text{require}}{\leq} -F$$

It suffices to find ψ a solution to

$$\begin{aligned} \psi'' + A\psi' + B = 0 \\ A &= \frac{1}{\lambda}\left(\frac{\Lambda - \lambda}{R} + \|b\|_{C^0}\right) \\ B &= \frac{1}{\lambda}F \end{aligned}$$

which has solution of the general form

$$\psi(d) = -\frac{B}{A}d(x) + \frac{C_1}{A} - \frac{C_2}{A}e^{-Ad(x)}$$

We need to determine C_1 and C_2.

In view of our requirement (3.13) we need

$$\begin{aligned} w(x_0) = \psi(d(x_0)) = \psi(0) = 0 \\ w = \psi \geq 0 \qquad \forall\ x \in \partial\Omega \\ |\nabla w(x_0)| \leq |\psi'(0)| \leq C \end{aligned}$$

So let's impose conditions

$$\begin{aligned} \psi(0) &= 0 \\ \psi(d) &\geq 0 \qquad \forall\ d \in [0,\infty) \\ \psi'(d) &> 0 \qquad \forall\ d \in [0,\infty) \\ \psi''(d) &< 0 \qquad \forall\ d \in [0,\infty) \end{aligned}$$

This translates to

$$\begin{aligned} C_1 &= C_2 = C \\ \psi(d) &= -\frac{B}{A}d(x) + C\left(\frac{1}{A} - \frac{1}{A}e^{-Ad(x)}\right) \geq 0 \\ \psi'(d) &= -\frac{B}{A} + Ce^{-Ad(x)} = e^{-Ad}\left(C - \frac{B}{A}e^{Ad}\right) > 0 \\ \psi''(d) &= -ACe^{-Ad} < 0 \end{aligned}$$

To ensure the last inequalities one need to take C large so that

$$\begin{aligned} C &> \frac{B}{A}e^{Ad} \qquad \forall\ d \in [0, \mathrm{diam}(\Omega)] \\ C &\geq 2\frac{B}{A}e^{A\mathrm{diam}(\Omega)} \end{aligned}$$

Now since $\psi'(d) > 0$ and $\psi(0) = 0$, automatically $\psi(d) > 0$ is satisfied. □

3.4 Classical ABP Estimate

Consider the upper contact sets for a function $u \in C^2(\Omega)$. This is the part that u is concave

$$\Gamma^+ := \{y \in \Omega \mid u(x) \le u(y) + Du(y) \cdot (x-y) \qquad \forall\, x \in \Omega\}$$

One has

$$D^2 u(y) \le 0 \qquad \forall\, y \in \Gamma^+$$

If u is merely $C(\Omega)$, one define

$$\Gamma^+ := \{y \in \Omega \mid u(x) \le u(y) + p \cdot (x-y) \qquad \forall\, x \in \Omega, \qquad \text{for some } p = p(y) \in \mathbb{R}^n\}$$

Now u is concave in Ω iff $\Gamma^+ = \Omega$.

Let $\Omega \subseteq \mathbb{R}^n$ be bounded domain.

Lemma 3.4.1 (Geometric Lemma; [HL11] 2.24)**.** *For any $g \in L^1_{loc}(\Omega)$, $g \ge 0$. Then for any $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$*

$$\int_{B_M(0)} g \le \int_{\Gamma^+} g(Du) |\det(D^2 u)| \tag{3.14}$$

for

$$M := \frac{\sup\limits_{\Omega} u - \sup\limits_{\partial\Omega} u^+}{d}, \qquad d = \operatorname{diam}(\Omega)$$

Proof. WLOG assume $u \le 0$ on $\partial\Omega$. We first claim

$$\int_{Du(\Gamma^+ \cap \{u>0\})} g \le \int_{\Gamma^+ \cap \{u>0\}} g(Du) |\det(D^2 u)| \tag{3.15}$$

Indeed, for any $\varepsilon > 0$, the function

$$\chi_\varepsilon = Du - \varepsilon Id : \Omega \to \mathbb{R}^n$$

is invertible on Γ^+ since

$$D\chi_\varepsilon = D^2 u - \varepsilon I < 0 \qquad \text{is strictly negative on } \Gamma^+$$

Thus by change of variables

$$\int_{\chi_\varepsilon(\Gamma^+ \cap \{u>0\})} g = \int_{\Gamma^+ \cap \{u>0\}} g(\chi_\varepsilon) |\det(D\chi_\varepsilon)|$$

Passing $\varepsilon \to 0$ yields (3.15).

Second we claim

$$B_M(0) \subseteq Du(\Gamma^+ \cap \{u > 0\}) \tag{3.16}$$

i.e., for any $|a| < M$, there exists $\tilde{x} \in \Omega$ s.t. $u(\tilde{x}) > 0$, $\tilde{x} \in \Gamma^+$ and $Du(\tilde{x}) = a$.

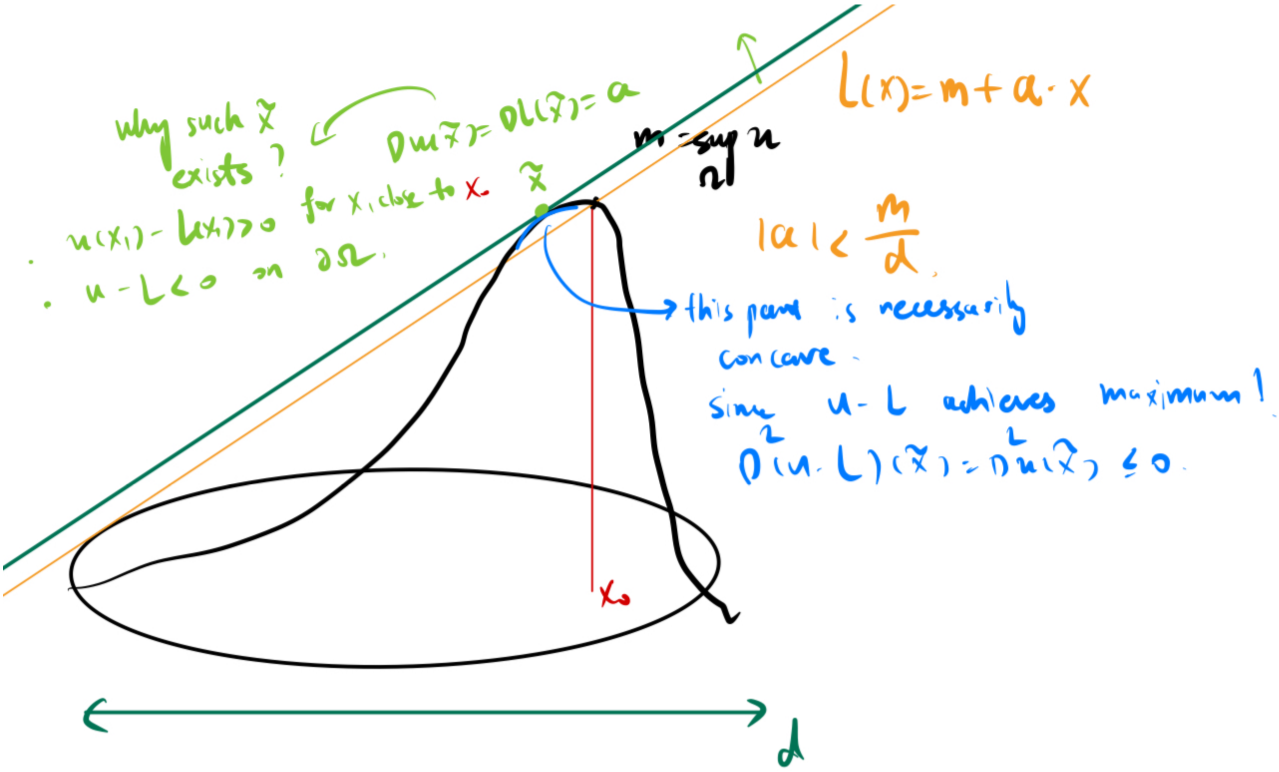


Figure 3.4: Claim (3.16)

To do so, consider any $|a| < M$, and WLOG assume $0 \in \{u > 0\}$ s.t.

$$u(0) = \sup_{\Omega} u = m$$

Then define

$$L(x) = m + a \cdot x$$

What's good about such L? At 0, $L(0) = m$, while

$$L(x) \geq m - |a||x| > 0 \qquad \forall\, x \in \overline{\Omega}, \qquad \text{in particular at } \partial\Omega$$

Now we claim there necessarily exists $\tilde{x} \in \Omega$ s.t. $\nabla u(\tilde{x}) = \nabla L(\tilde{x}) = a$. If $a = 0$ this is done by picking $\tilde{x} = 0$. If $a > 0$, there exists x_1 close to $\tilde{x}$ s.t. $u(x_1) > L(x_1)$. Now

$$\begin{cases} (u - L)(x_1) > 0 & x_1 \in \Omega \\ u - L < 0 & \partial\Omega \end{cases}$$

Then there necessarily exists $\tilde{x} \in \Omega$ where $u - L$ achieves non-negative maximum, so that

$$u(\tilde{x}) - L(\tilde{x}) > 0, \qquad \nabla(u - L)(\tilde{x}) = 0, \qquad D^2(u - L)(\tilde{x}) \leq 0$$

Thus (3.16) is achieved. □

Let's see how we make use of the geometric lemma (3.14). First note for symmetric $A = (a_{ij})$ that is elliptic in Ω (hence positive-definite), denote

$$D^* := (\det(A))^{\frac{1}{n}} = (\prod_{i=1}^{n} \lambda_i(A))^{\frac{1}{n}} \qquad \text{geometric mean of the eigenvalues of } A$$

Notice by the GM-AM inequality, for any B matrix that is nonnegative-definite

$$(\prod_{i=1}^{n} \lambda_i(B))^{\frac{1}{n}} \leq \frac{1}{n} \sum_{i=1}^{n} \lambda_i(B) = \frac{1}{n} \text{Tr}(B)$$

Then for any $x \in \Gamma^+$ so $D^2u(x)$ is non-positive definite ([HL11] Remark 2.25)

$$
\begin{aligned}
|\det(-D^2u(x))| &\leq \frac{1}{\det(a_{ij}(x))}|\det(-AD^2u(x))| = \frac{1}{(D^*)^n}|\det(-AD^2u(x))| \\
&\leq \frac{1}{(D^*)^n}(\frac{1}{n}\mathrm{Tr}(-AD^2u(x)))^n \leq (\frac{-a_{ij}\partial_{ij}u(x)}{nD^*})^n
\end{aligned}
$$

Thus combining with Lemma (3.14) one obtain for $g \geq 0$

$$
\int_{B_M(0)} g \leq \int_{\Gamma^+} g(Du)(\frac{-a_{ij}\partial_{ij}u(x)}{nD^*})^n \tag{3.17}
$$

On the other hand, taking $g = 1$ and using (3.14)

$$
\begin{aligned}
\int_{B_M(0)} 1 = \omega_n|M|^n = \omega_n(\frac{\sup_{\Omega} u - \sup_{\partial\Omega} u^+}{d})^n &\overset{(3.14)}{\leq} \int_{\Gamma^+}|\det(-D^2u)| \\
&\overset{(3.17)}{\leq} \int_{\Gamma^+}(\frac{-a_{ij}\partial_{ij}u(x)}{nD^*})^n
\end{aligned}
$$

$$
\sup_{\Omega} u - \sup_{\partial\Omega} u^+ \leq C(n, \mathrm{diam}(\Omega))\left(\int_{\Gamma^+}(\frac{-a_{ij}\partial_{ij}u(x)}{D^*})^n\right)^{\frac{1}{n}}
$$

Thus if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is solution to

$$
a_{ij}\partial_{ij}u = f \qquad \Omega
$$

where a_{ij} is symmetric, elliptic, one has the ABP estimate

$$
\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + \underbrace{C(n)\left(\int_{\Gamma^+}(\frac{f}{D^*})^n\right)^{\frac{1}{n}}}_{\text{replace with } f \text{ is only place to use equation}} \tag{3.18}
$$

Let's do the general form.

Theorem 3.4.1 ([HL11] Theorem 2.21)**.** *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ solve the elliptic inequality*

$$
\mathcal{L}u = a_{ij}\partial_{ij}u + b_i\partial_i u + cu \geq f, \qquad a_{ij} > 0, \qquad c \leq 0
$$

Then for $C = C(n, \mathrm{diam}(\Omega), \frac{|b|}{D^*}{}_{L^n(\Gamma^+)}) > 0$

$$
\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C\left\|\frac{f^-}{D^*}\right\|_{L^n(\Gamma^+)} \tag{3.19}
$$

Proof. We need to choose properly the g to apply Lemma (3.14).

For $\mu > 0$

$$
\begin{aligned}
-a_{ij}\partial_{ij}u &\leq b_i\partial_i u + cu - f \\
&\leq b_i\partial_i u - f \qquad \text{over } \{u > 0\} \\
&\leq |\nabla u||b| + f^- = |b| \cdot |\nabla u| + \frac{f^-}{\mu} \cdot \mu \\
&\leq \left(|b|^n + (\frac{f^-}{\mu})^n\right)^{\frac{1}{n}} (|\nabla u|^n + \mu^n)^{\frac{1}{n}} \cdot 2^{\frac{n-2}{n}} \qquad \text{Cauchy and Hölder}
\end{aligned}
$$

$$
(-a_{ij}\partial_{ij}u)^n \leq 2^{n-2}\left(|b|^n + (\frac{f^-}{\mu})^n\right)(|\nabla u|^n + \mu^n) \tag{3.20}
$$

Now we choose (this is where we use $\mu > 0$)

$$
g(p) = \frac{1}{|p|^n + \mu^n}
$$

to both ensure $g \in L^1_{\text{loc}}$ integrability, and to balance out the gradient term. Thus

$$
\begin{aligned}
\int_{B_M(0)} \frac{1}{|p|^n + \mu^n} &\overset{(3.14)}{\leq} \int_{\Gamma^+ \cap \{u>0\}} \frac{1}{|Du|^n + \mu^n}|\det(-D^2u)| \\
&\overset{(3.17)}{\leq} \int_{\Gamma^+ \cap \{u>0\}} \frac{1}{|Du|^n + \mu^n}(\frac{-a_{ij}\partial_{ij}u(x)}{nD^*})^n \\
&\overset{(3.20)}{\leq} C(n)\int_{\Gamma^+ \cap \{u>0\}} \frac{1}{(D^*)^n}\left(|b|^n + (\frac{f^-}{\mu})^n\right)
\end{aligned}
$$

Let's also deal the the LHS

$$\int_{B_M(0)} \frac{1}{|p|^n + \mu^n} = C(n) \int_0^M \frac{1}{r^n + \mu^n} r^{n-1} dr = C(n) \log(\frac{M^n + \mu^n}{\mu^n})$$

Thus

$$\log(\frac{M^n + \mu^n}{\mu^n}) \leq C(n) \int_{\Gamma^+ \cap \{u>0\}} \frac{1}{(D^*)^n} \left(|b|^n + (\frac{f^-}{\mu})^n \right)$$
$$M^n \leq \mu^n \left(\exp\left(C(n) \int_{\Gamma^+ \cap \{u>0\}} \left((\frac{|b|}{D^*})^n + (\frac{f^-}{\mu D^*})^n \right) \right) - 1 \right)$$

If $f \neq 0$, choose

$$\mu = \left\| \frac{f^-}{D^*} \right\|_{L^n(\Gamma^+ \cap \{u>0\})}$$

If $f = 0$ let $\mu \to 0$. □

ABP for Prescribed Mean Curvature Equation Consider the Prescribed Mean Curvature Equation

$$(1 + |\nabla u|^2)\Delta u - \partial_i u \partial_j u \partial_{ij} u = nH(x)(1 + |\nabla u|^2)^{\frac{3}{2}} \tag{3.21}$$

for $H \in C(\Omega)$. This is Quasi-linear Equation of the form

$$Qu := a_{ij}(x, u, \nabla u)\partial_{ij} u + b(x, u, \nabla u) = 0$$

where

$$a_{ij}(x, z, p) = (1 + |p|^2)\delta_{ij} - p_i p_j$$
$$b(x, z, p) = -nH(x)(1 + |p|^2)^{\frac{3}{2}}$$

In particular, since $A = a_{ij}$ is symmetric, one may compute its spectrum via the following: for $p \neq 0$, let $v = p$ so

$$Ap = (1 + |p|^2)p - |p|^2 p = p$$

so p is an eigenvector with eigenvalue 1. Now for any $v \perp p$, compute

$$Av = (1 + |p|^2)v - p(p^T v) = (1 + |p|^2)v$$

so $1 + |p|^2$ are eigenvalues of multiplicity $n - 1$. Thus

$$\det(a_{ij}) = (1 + |p|^2)^{n-1}$$
$$D^* = \det(a_{ij})^{\frac{1}{n}} = (1 + |p|^2)^{\frac{n-1}{n}}$$

The computations include the case for $p = 0$ as well.

Now, what can we say about $\frac{b}{D^*}$?

$$\frac{|b(x, z, p)|}{nD^*} \leq \frac{|H(x)|(1 + |p|^2)^{\frac{3}{2}}}{(1 + |p|^2)^{\frac{n-1}{n}}} = |H(x)|(1 + |p|^2)^{\frac{n+2}{2n}}$$

In view of (3.17), the RHS is essentially what we're trying to bound.

Proposition 3.4.1 ([HL11] Proposition 2.27, Corollary 2.28). *Let $u \in C(\Omega) \cap C^2(\Omega)$ be solution to* (3.21). *Then if*

$$H_0 \equiv \int_\Omega |H(x)|^n dx < \omega_n \tag{3.22}$$

one has for $C = C(n, H_0) > 0$

$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u| + C\mathrm{diam}(\Omega)$$

Proof. First consider $Qu \geq 0$ so

$$-a_{ij}(x,u,\nabla u)\partial_{ij}u \leq b(x,u,\nabla u)$$

Plug into (3.17) with $x \in \Gamma^+$ implies $-a_{ij}\partial_{ij}u$ is non-negative definite, thus one can do for any n (no worries on the sign!)

$$\begin{aligned}\int_{B_M(0)} g &\leq \int_{\Gamma^+} g(\nabla u)\left(\frac{-a_{ij}\partial_{ij}u}{nD^*}\right)^n \\ &\leq \int_{\Gamma^+} g(\nabla u)\left(\frac{b(x,u,\nabla u)}{nD^*}\right)^n \\ &\leq \int_{\Gamma^+} g(\nabla u)|H(x)|^n(1+|\nabla u|^2)^{\frac{n+2}{2}}\end{aligned}$$

One has not made the choice of g yet! Let's just choose

$$g(p) = \frac{1}{(1+|p|^2)^{\frac{n+2}{2}}}$$

so that

$$\begin{aligned}\int_{\mathbb{R}^n} g = \int_{\mathbb{R}^n} \frac{1}{(1+|p|^2)^{\frac{n+2}{2}}} &= n\omega_n \int_0^\infty \frac{r^{n-1}}{(1+r^2)^{\frac{n+2}{2}}}dr \\ &= n\omega_n \int_0^{\frac{\pi}{2}} \frac{\tan(\theta)^{n-1}}{(\sec(\theta))^{n+2}}\sec(\theta)^2 d\theta \qquad \text{change } r=\tan(\theta) \\ &= n\omega_n \int_0^{\frac{\pi}{2}} \sin(\theta)^{n-1}\cos(\theta)d\theta = n\omega_n \int_0^1 t^{n-1}dt = \omega_n\end{aligned}$$

Thus continuing from above yields

$$\int_{B_M(0)} g \leq \int_{\Gamma^+} |H(x)|^n \leq H_0 \overset{(3.22)}{<} \omega_n = \int_{\mathbb{R}^n} g$$

Because the inequality is strict, M is bounded by constant that depends on g (which is dimensional). Thus

$$\begin{aligned} M &\leq C(n,H_0) \\ \sup_\Omega u &\leq \sup_{\partial\Omega} u^+ + C(n)\operatorname{diam}(\Omega)\end{aligned}$$

Now use the other part of the equation and consider $-u$ that solves (3.21) with an opposite sign on the RHS. □

ABP for Monge-Ampère Equation Now we consider the equation with solution $u \in C(\overline{\Omega}) \cap C^2(\Omega)$

$$\det(D^2u) = f(x,u,\nabla u)$$

First for $f = f(x)$, choosing $g = 1$ in Lemma (3.14) gives

$$\int_{B_M(0)} 1 \leq \int_{\Gamma^+} |\det(D^2u)| \leq \int_{\Gamma^+} |f|$$

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + \operatorname{diam}(\Omega)\left(\frac{1}{\omega_n}\int_\Omega |f|\right)^{\frac{1}{n}}$$

Doing for $-u$ and the other side of the inequality gives ([HL11] Corollary 2.30)

$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u| + \operatorname{diam}(\Omega)\left(\frac{1}{\omega_n}\int_\Omega |f|\right)^{\frac{1}{n}}$$

One may also consider the **Prescribed Gaussian Curvature Equation**

$$\det(D^2u) = K(x)(1+|\nabla u|^2)^{\frac{n+2}{2}} \tag{3.23}$$

Proposition 3.4.2 ([HL11] Corollary 2.29, Corollary 2.31)**.** *Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ solve (3.23). Assume*

$$K_0 \equiv \int_\Omega |K(x)| < \omega_n \tag{3.24}$$

One has for $C = C(n, K_0) > 0$

$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u| + C\text{diam}(\Omega)$$

Proof. With g to be chosen, compute

$$\int_{B_M(0)} g \leq \int_{\Gamma^+} g(\nabla u)|\det(D^2 u)| \leq \int_{\Gamma^+} g(\nabla u)|K(x)(1+|\nabla u|^2)^{\frac{n+2}{2}}|$$

Let's pick g to cancel out the last term so

$$g(p) := \frac{1}{(1+|p|^2)^{\frac{n+2}{2}}}$$

Due to previous computations

$$\int_{\mathbb{R}^n} g = \omega_n$$

Now

$$\int_{B_M(0)} g \leq \int_{\Gamma^+} |K(x)| \overset{(3.24)}{<} \omega_n = \int_{\mathbb{R}^n} g$$

Thus there exists (due to strict inequality) $C = C(n, K_0) > 0$ s.t.

$$M \leq C(n, H_0)$$

unraveling and doing for the other side yields the result. □

Maximum Principle for Domain with small volume

Theorem 3.4.2 ([HL11] Theorem 2.32)**.** *Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ solve*

$$\mathcal{L}u := a_{ij}\partial_{ij}u + b_i\partial_i u + cu \geq 0 \qquad \Omega$$

with a_{ij} elliptic, $\sup_\Omega |b| + |c| \leq \Lambda$, and $u \leq 0$ on $\partial\Omega$.

Then for $|\Omega|$ small depending on $n, \Lambda, D^*, \text{diam}(\Omega)$ one obtain

$$u \leq 0 \qquad \Omega$$

Proof. If $\sup_\Omega u < 0$ there is nothing to prove. Otherwise note

$$a_{ij}(x)\partial_{ij}u(x) + b_i(x)\partial_i u(x) - c^-(x)u(x) \geq -c^+(x)u(x) \qquad \text{the RHS we treat as force}$$

Now $-c^+ \leq 0$ and one may apply the ABP estimate (3.19) so

$$\begin{aligned}
\sup_\Omega u &\leq \sup_{\partial\Omega} u^+ + C(n, \Lambda, D^*, \text{diam}(\Omega)) \left\| \frac{c^+ u}{D^*} \right\|_{L^n(\Gamma^+)} \\
&\leq C \|c\|_{L^\infty(\Omega)} \sup_\Omega u \cdot |\Omega|^{\frac{1}{n}} \qquad \text{using } u \leq 0 \text{ on } \partial\Omega \\
&\leq \frac{1}{2} \sup_\Omega u \qquad \text{by choosing } |\Omega| \text{ sufficiently small}
\end{aligned}$$

which implies $\sup_\Omega u \leq 0$. □

Chapter 4

Schauder's Approach

4.1 Hölder Spaces

The notations in [GT01] are terrible. We include them all here.

4.1.1 Notations for Different Hölder Norms

([GT01] Section 4.1, Section 4.3, Section 6.1)

Hölder Seminorms Let $x_0 \in \mathbb{R}^n$ and f defined on D bounded set containing x_0. Let $\alpha \in (0,1]$.

$$[f]_{\alpha;x_0} := \sup_D \frac{|f(x)-f(x_0)|}{|x-x_0|^\alpha} \qquad \alpha\text{th Hölder coefficient of } f \text{ at } x_0 \text{ w.r.t. } D$$

$$[f]_{\alpha;D} := \sup_{x,\,y\in D,\ x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} = [f]_{C^{0,\alpha}(\overline{D})} \qquad \alpha\text{th Hölder seminorm in } D$$

f is called *α-Hölder continuous at x_0* if $[f]_{\alpha;x_0} < \infty$. When $\alpha = 1$, f is called *Lipschitz continuous at x_0*.

f is called *uniformly α-Hölder continuous in D* if $[f]_{\alpha;D} < \infty$.

f is called *locally α-Hölder continuous in D* if f is uniformly α-Hölder continuous in any compact subset K of D.

Hölder Spaces. Let $k \in \mathbb{N}$. Let $\Omega \subseteq \mathbb{R}^n$ be open subset. The classical function spaces are

$$C^k(\overline{\Omega}) := \{u : \Omega \to \mathbb{R} \mid D^\beta u \text{ exists and uniformly continuous in } \Omega \text{ for any } |\beta| \leq k\}$$

$$C^k(\Omega) := \{u : \Omega \to \mathbb{R} \mid \text{for any } K \subseteq \Omega \text{ compact subset, } u \in C^k(K)\}$$

So functions in $C^k(\overline{\Omega})$ have continuous kth derivatives all the way up to the boundary. Now for $\alpha \in (0,1]$

$$\begin{aligned}
C^{k,\alpha}(\overline{\Omega}) &:= \{u \in C^k(\overline{\Omega}) \mid [D^\beta u]_{\alpha,\Omega} < \infty,\ \forall\, |\beta| = k\} \\
&= \text{subspace of } C^k(\overline{\Omega}) \text{, whose } k\text{th derivatives are uniformly } \alpha\text{-Hölder continuous in } \Omega \\
C^{k,\alpha}(\Omega) &:= \{u \in C^k(\Omega) \mid [D^\beta u]_{\alpha,K} < \infty,\ \forall\, |\beta| = k,\ \forall\, K \subseteq \Omega \text{ compact}\} \\
&= \text{subspace of } C^k(\Omega) \text{, whose } k\text{th derivatives are locally } \alpha\text{-Hölder continuous in } \Omega
\end{aligned}$$

Also denote

$$C_0^{k,\alpha}(\Omega) := \{u \in C^{k,\alpha}(\Omega) \mid u \text{ has compact support in } \Omega\}$$

Seminorms Let $k \in \mathbb{N}$ and $\alpha \in (0,1]$

$$[u]_{k,0;\Omega} = |D^k u|_{0;\Omega} := \sup_{|\beta|=k} \sup_\Omega |D^\beta u| \qquad \text{supremum norm for all } k\text{th derivative}$$

$$[u]_{k,\alpha;\Omega} = [D^k u]_{\alpha;\Omega} := \sup_{|\beta|=k} [D^\beta u]_{\alpha;\Omega} \qquad \alpha\text{-Hölder seminorm for } k\text{th derivative}$$

Norms Let $k \in \mathbb{N}$ and $\alpha \in (0,1]$

$$\|u\|_{C^k(\overline{\Omega})} = |u|_{k;\Omega} = |u|_{k,0;\Omega} := \sum_{j=0}^k [u]_{j,0;\Omega} = \sum_{j=0}^k |D^j u|_{0;\Omega} = \sum_{j=0}^k \sup_{|\beta|=j} \sup_\Omega |D^\beta u| \qquad \text{norm for } C^k(\overline{\Omega})$$

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} = |u|_{k,\alpha;\Omega} := |u|_{k;\Omega} + [u]_{k,\alpha;\Omega} = |u|_{k;\Omega} + [D^k u]_{\alpha;\Omega} = \sum_{j=0}^k \sup_{|\beta|=j} \sup_\Omega |D^\beta u| + \sup_{|\beta|=k} [D^\beta u]_{\alpha;\Omega} \qquad \text{norm for } C^{k,\alpha}(\overline{\Omega})$$

Nondimensional Norms (Global) Let $k \in \mathbb{N}$ and $\alpha \in (0,1]$. Let Ω be bounded with $d := \operatorname{diam}(\Omega)$.

$$\|u\|'_{C^k(\overline{\Omega})} = |u|'_{k;\Omega} := \sum_{j=0}^k d^j \cdot [u]_{j,0;\Omega} = \sum_{j=0}^k d^j \cdot |D^j u|_{0;\Omega} = \sum_{j=0}^k \operatorname{diam}(\Omega)^j \cdot \sup_{|\beta|=j} \sup_{\Omega} |D^\beta u| \qquad \text{non-dimensional norm for } C^k(\overline{\Omega})$$

$$\|u\|'_{C^{k,\alpha}(\overline{\Omega})} = |u|'_{k,\alpha;\Omega} := |u|'_{k;\Omega} + d^{k+\alpha} \cdot [u]_{k,\alpha;\Omega} = |u|'_{k;\Omega} + d^{k+\alpha} \cdot [D^k u]_{\alpha;\Omega}$$

$$= \sum_{j=0}^k \operatorname{diam}(\Omega)^j \cdot \sup_{|\beta|=j} \sup_{\Omega} |D^\beta u| + \operatorname{diam}(\Omega)^{k+\alpha} \cdot \sup_{|\beta|=k} [D^\beta u]_{\alpha;\Omega} \qquad \text{non-dimensional norm for } C^{k,\alpha}(\overline{\Omega})$$

For $u \in C^{0,\alpha}(\overline{\Omega})$ and $v \in C^{0,\beta}(\overline{\Omega})$, the product $uv \in C^{0,\gamma}(\overline{\Omega})$ where $\gamma = \min(\alpha, \beta)$, and

$$\|uv\|_{C^{0,\gamma}(\overline{\Omega})} \le \max(1 + d^{\alpha+\beta-2\gamma}) \, \|u\|_{C^{0,\alpha}(\overline{\Omega})} \cdot \|v\|_{C^{0,\beta}(\overline{\Omega})}$$
$$\|uv\|'_{C^{0,\gamma}(\overline{\Omega})} \le \|u\|'_{C^{0,\alpha}(\overline{\Omega})} \cdot \|v\|'_{C^{0,\beta}(\overline{\Omega})}$$

Interior Nondimensional Seminorms and Norms. Let $\Omega \subseteq \mathbb{R}^n$ be any proper open subset. Let

$$d_x = \operatorname{dist}(x, \partial\Omega), \qquad d_y = \operatorname{dist}(y, \partial\Omega), \qquad d_{x,y} := \min\{d_x, d_y\} \qquad \forall\, x,\, y \in \Omega \tag{4.1}$$

For $k \in \mathbb{N}$, $\alpha \in (0,1]$, define seminorms

$$[u]^*_{k;\Omega} = [u]^*_{k,0;\Omega} := \sup_{|\beta|=k} \sup_{x\in\Omega} d_x^k \cdot |D^\beta u(x)| \qquad \text{interior non-dimensional supremum norm for } k\text{th derivative}$$

$$[u]^*_{k,\alpha;\Omega} := \sup_{|\beta|=k} \sup_{\substack{x\neq y\\ x,\, y\in\Omega}} d_{x,y}^{k+\alpha} \cdot \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha} \qquad \text{interior non-dimensional } \alpha\text{th Hölder seminorm for } k\text{th derivative}$$

Define norms on subspaces (for which the following are bounded)

$$|u|^*_{k;\Omega} = |u|^*_{k,0;\Omega} := \sum_{j=0}^k [u]^*_{j;\Omega}$$
$$= \sum_{j=0}^k \sup_{|\beta|=j} \sup_{x\in\Omega} d_x^j \cdot |D^\beta u(x)| \qquad \text{interior non-dimensional norm for } C^k(\Omega)$$

$$|u|^*_{k,\alpha;\Omega} := |u|^*_{k;\Omega} + [u]^*_{k,\alpha;\Omega} = \sum_{j=0}^k [u]^*_{j;\Omega} + [u]^*_{k,\alpha;\Omega}$$
$$= \sum_{j=0}^k \sup_{|\beta|=j} \sup_{x\in\Omega} d_x^j \cdot |D^\beta u(x)| + \sup_{|\beta|=k} \sup_{\substack{x\neq y\\ x,\, y\in\Omega}} d_{x,y}^{k+\alpha} \cdot \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha} \qquad \text{interior non-dimensional norm for } C^{k,\alpha}(\Omega)$$

If Ω is bounded and $d = \operatorname{diam}(\Omega)$, then the interior and global norms are related by

$$|u|^*_{k,\alpha;\Omega} \le \max(1, d^{k+\alpha}) |u|_{k,\alpha;\Omega}$$

If $\Omega' \Subset \Omega$ and $\sigma = \operatorname{dist}(\Omega', \partial\Omega)$ then

$$\min(1, \sigma^{k+\alpha}) |u|_{k,\alpha;\Omega} \le |u|^*_{k,\alpha;\Omega}$$

σ-dimensional Interior Seminorms and Norms. Let $\Omega \subseteq \mathbb{R}^n$ be any proper open subset. Let d_x, d_y and $d_{x,y}$ be defined as in (4.1). For $k \in \mathbb{N}$, $\alpha \in (0,1]$, and *a real number* $\sigma \in \mathbb{R}$, define seminorms

$$[f]^{(\sigma)}_{k;\Omega} = [f]^{(\sigma)}_{k,0;\Omega} := \sup_{|\beta|=k} \sup_{x\in\Omega} d_x^{k+\sigma} \cdot |D^\beta f(x)| \qquad \text{interior supremum norm for } k\text{th derivative with dimension } \sigma$$

$$[f]^{(\sigma)}_{k,\alpha;\Omega} := \sup_{|\beta|=k} \sup_{\substack{x\neq y\\ x,\, y\in\Omega}} d_{x,y}^{k+\alpha+\sigma} \cdot \frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\alpha} \qquad \text{interior } \alpha\text{th Hölder seminorm for } k\text{th derivative with dimension } \sigma$$

Define norms

$$|f|^{(\sigma)}_{k;\Omega} := \sum_{j=0}^k [f]^{(\sigma)}_{j;\Omega} = \sum_{j=0}^k \sup_{|\beta|=j} \sup_{x\in\Omega} d_x^{j+\sigma} \cdot |D^\beta f(x)| \qquad \text{interior } \sigma\text{-dimensional norm for } C^k(\Omega)$$

$$|f|^{(\sigma)}_{k,\alpha;\Omega} := |f|^{(\sigma)}_{k;\Omega} + [f]^{(\sigma)}_{k,\alpha;\Omega}$$
$$= \sum_{j=0}^k \sup_{|\beta|=j} \sup_{x\in\Omega} d_x^{j+\sigma} \cdot |D^\beta f(x)| + \sup_{|\beta|=k} \sup_{\substack{x\neq y\\ x,\, y\in\Omega}} d_{x,y}^{k+\alpha+\sigma} \cdot \frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\alpha} \qquad \text{interior } \sigma\text{-dimensional norm for } C^{k,\alpha}(\Omega)$$

One has

$$|fg|^{(\sigma+\tau)}_{0,\alpha;\Omega} \le |f|^{(\sigma)}_{0,\alpha;\Omega} |g|^{(\tau)}_{0,\alpha;\Omega} \qquad \forall\, \sigma + \tau \ge 0$$

4.1.2 Interpolation Inequalities

Lemma 4.1.1 ([GT01] Lemma 6.32)**.** *Let $j, k \in \mathbb{N}$, $\alpha, \beta \in [0, 1]$ s.t.*

$$j + \beta < k + \alpha$$

Let $\Omega \subseteq \mathbb{R}^n$ and $u \in C^{k,\alpha}(\Omega)$. Then for any $\varepsilon > 0$, there exists $C = C(\varepsilon, k, j)$ s.t.

$$[u]^*_{j,\beta;\Omega} \leq C|u|_{0;\Omega} + \varepsilon[u]^*_{k,\alpha;\Omega}$$

In fact

$$|u|^*_{j,\beta;\Omega} \leq C|u|_{0;\Omega} + \varepsilon[u]^*_{k,\alpha;\Omega} \tag{4.2}$$

Let's in fact demonstrate a simpler lemma that's been used.

Lemma 4.1.2 (Interpolation Inequality; De Silva Analysis II Spring 2025)**.** *For $0 < \alpha < 1$, for any $\varepsilon > 0$, there exists $C = C(n, \varepsilon) > 0$ s.t.*

$$\|u\|_{C^2(\overline{\Omega})} \leq C \|u\|_{L^\infty(\Omega)} + \varepsilon[D^2u]_{C^{0,\alpha}(\overline{\Omega})} \tag{4.3}$$

Proof. **Assume Not**. Then there exists $\varepsilon_0 > 0$ and a sequence u_n s.t.

$$\|\partial_{ij}u_n\|_{L^\infty(\Omega)} \geq n \|u_n\|_{L^\infty(\Omega)} + \varepsilon_0[\partial_{ij}u_n]_{C^{0,\alpha}(\overline{\Omega})}$$

Prepare to apply Ascoli-Arzela. Upon normalizing

$$\tilde{u}_n := \frac{1}{\|\partial_{ij}u_n\|_{L^\infty(\Omega)}} u_n$$

one obtain a uniform bound leveraging linearity

$$1 \geq n \|\tilde{u}_n\|_{L^\infty(\Omega)} + \varepsilon_0[\partial_{ij}\tilde{u}_n]_{C^{0,\alpha}(\overline{\Omega})}$$

In particular

$$[\partial_{ij}\tilde{u}_n]_{C^{0,\alpha}(\overline{\Omega})} \leq \frac{1}{\varepsilon_0} \qquad \forall\ n$$

yields uniformly equi-continuous. Together with the uniform bound

$$\|\partial_{ij}\tilde{u}_n\|_{L^\infty(\Omega)} = 1$$

one may apply Arzela-Ascoli so that there exists a subsequence

$$\partial_{ij}\tilde{u}_n \to w_{ij} \qquad \text{uniformly on } \Omega$$

Reach contradiction. On the other hand

$$\|u_n\|_{L^\infty(\Omega)} \leq \frac{1}{n} \to 0$$

implies

$$\tilde{u}_n \to 0 \qquad \text{uniformly on } \Omega$$

Since differentiation commutes with uniform convergence, necessarily

$$w_{ij} = \lim_{n\to\infty} \partial_{ij}\tilde{u}_n = \partial_{ij}(\lim_{n\to\infty} \tilde{u}_n) = 0$$

But this contradicts with our assumption

$$\|\partial_{ij}\tilde{u}_n\|_{L^\infty(\Omega)} = 1$$

Thus

$$\left\|D^2u\right\|_{L^\infty(\Omega)} \leq C \|u\|_{L^\infty(\Omega)} + \varepsilon[D^2u]_{C^{0,\alpha}(\overline{\Omega})}$$

Now following a same procedure as above, one may prove

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C \|u\|_{L^\infty(\Omega)} + \varepsilon \left\|D^2u\right\|_{L^\infty(\Omega)}$$

Whence (4.3) follows. □

The following Compactness result is used in Method of Continuity.

Lemma 4.1.3 ([GT01] Lemma 6.33)**.** *Let $j, k \in \mathbb{N}$, $\alpha, \beta \in [0,1]$ s.t.*

$$j + \beta < k + \alpha$$

Let $\Omega \subseteq \mathbb{R}^n$ and let S be a bounded subset of the Banach Space

$$C_*^{k,\alpha} := \{u \in C^{k,\alpha}(\Omega) \mid |u|^*_{k,\alpha;\Omega} < \infty\}$$

Suppose the functions $f \in S$ are uniformly equi-continuous on $\overline{\Omega}$.

Then S is precompact in $C_*^{j,\beta}$, i.e., for sequence $\{f_n\} \subseteq S$ (which is uniformly bounded), there exists a subsequence f_{n_m} that converges to some $f \in S$ in the norm $|\cdot|^*_{j,\beta}$.

Proof. By Ascoli-Arzela, for any $\{f_n\} \subseteq S$ (in particular uniformly bounded and equi-continuous by assumption) there exists subsequence f_{n_m} and $f \in C_*^{k,\alpha}$ s.t. $f_{n_m} \to f$ uniformly on $\overline{\Omega}$.

We want to **upgrade the convergence from uniform to** $|\cdot|^*_{j,\beta}$. Since S bounded, there exists universal $M > 0$ s.t.

$$|f_{n_m}|^*_{k,\alpha;\Omega} \leq M \qquad \forall\, j$$

Using Interpolation Inequality (4.2), for any $\varepsilon > 0$, there exists $C = C(\varepsilon)$ s.t.

$$\begin{aligned} |f_{n_m} - f|^*_{j,\beta;\Omega} &\leq C|f_{n_m} - f|_{0;\Omega} + \varepsilon |f_{n_m} - f|^*_{k,\alpha;\Omega} \\ &\leq C\varepsilon + \varepsilon(1 + 2M) \qquad \forall\, m \geq N(\varepsilon) \in \mathbb{N} \end{aligned}$$

where we used uniform convergence for the first item, and uniform boundedness of f_{n_m} for the second. □

4.2 A priori Schauder Estimates

4.2.1 Theorems

We demonstrate the big Theory. Let $0 < \alpha < 1$ (< 1 is important as this uses $C^{2,\alpha}$ estimate for Poisson's Equation).

Theorem 4.2.1 (De Silva Analysis II 2025)**.** *Assume that $u \in C^2(B_1)$ solves*

$$\mathcal{L}u := a_{ij}\partial_{ij}u + b_i\partial_i u + cu = f \qquad B_1$$

where $A = (a_{ij}) \geq \lambda I$ is uniformly elliptic, and

$$a_{ij},\ b_i,\ c,\ f \in C^{0,\alpha}(\overline{B_1})$$

Then $u \in C^{2,\alpha}(\overline{B_{1/2}})$, and there exists $C = C(n, \alpha, \lambda, a_{ij}, b_i, c) > 0$ s.t.

$$\|u\|_{C^{2,\alpha}(\overline{B_{1/2}})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(\overline{B_1})}\right) \tag{4.4}$$

Let us also write a boundary version for the the above. Denote $B_1' \subseteq \mathbb{R}^{n-1}$ as the ball in one-dimension lower.

Theorem 4.2.2 (De Silva Analysis II 2025)**.** *Assume $\Omega = \{x_n > g(x')\}$ with $g \in C^{2,\alpha}(\overline{B_1'})$ and $g(0) = 0$. Assume that $u \in C^2(\Omega \cap B_1) \cap C^0(\overline{\Omega} \cap B_1)$ continuous up to the boundary, solves*

$$\mathcal{L}u = a_{ij}\partial_{ij}u + b_i\partial_i u + cu = f \qquad B_1 \cap \Omega$$

where $A = (a_{ij}) \geq \lambda I$ uniformly elliptic, and

$$a_{ij},\ b_i,\ c,\ f \in C^{0,\alpha}(\overline{B_1 \cap \Omega})$$

Also assume $u = \varphi \in C^{2,\alpha}(\partial\Omega \cap B_1)$ boundary data.
Then $u \in C^{2,\alpha}(\overline{\Omega} \cap B_1)$, and there exists $C = C(n, \alpha, \lambda, a_{ij}, b_i, c) > 0$ s.t.

$$\|u\|_{C^{2,\alpha}(\overline{B_{1/2}\cap\Omega})} \leq C\left(\|u\|_{L^\infty(B_1\cap\Omega)} + \|f\|_{C^{0,\alpha}(\overline{B_1\cap\Omega})} + \|\varphi\|_{C^{2,\alpha}(\overline{\partial\Omega\cap B_1})}\right) \tag{4.5}$$

4.2.2 Tools

Rescaling Assume

$$\mathcal{L}u = f \qquad B_r(0)$$

Then what does rescaling give us? Set

$$\tilde{u}(x) := u(rx) \qquad \forall\ x \in B_1$$

We ask what equation $\tilde{u}$ solves. To see this, scale back

$$\tilde{u}(\frac{x}{r}) = u(x) \qquad \forall\ x \in B_r$$

and plug in the original equation

$$\begin{aligned}
\frac{1}{r^2}a_{ij}(x)\partial_{ij}\tilde{u}(\frac{x}{r}) + \frac{1}{r}b_i(x)\partial_i\tilde{u}(\frac{x}{r}) + c(x)\tilde{u}(\frac{x}{r}) = f(x) \qquad \forall\ x \in B_r \\
\frac{1}{r^2}a_{ij}(ry)\partial_{ij}\tilde{u}(y) + \frac{1}{r}b_i(ry)\partial_i\tilde{u}(y) + c(ry)\tilde{u}(y) = f(ry) \qquad \forall\ y \in B_1 \\
a_{ij}(ry)\partial_{ij}\tilde{u}(y) + rb_i(ry)\partial_i\tilde{u}(y) + r^2c(ry)\tilde{u}(y) = r^2f(ry) \qquad \forall\ y \in B_1
\end{aligned}$$

Now $\tilde{u}$ solves the equation

$$\tilde{\mathcal{L}}\tilde{u}(y) := \tilde{a}_{ij}(y)\partial_{ij}\tilde{u}(y) + \tilde{b}_i(y)\partial_i\tilde{u}(y) + \tilde{c}(y)\tilde{u}(y) = \tilde{f}(y) \qquad \forall\ y \in B_1 \tag{4.6}$$

with

$$\begin{aligned}
\tilde{a}_{ij}(y) &:= a_{ij}(ry) \\
\tilde{b}_i(y) &:= rb_i(ry) \\
\tilde{c}(y) &:= r^2c(ry) \\
\tilde{f}(y) &:= r^2f(ry)
\end{aligned}$$

How are the norms related?

$$\begin{aligned}
\|\tilde{a}_{ij}\|_{C^{0,\alpha}(\overline{B_1})} &= \|\tilde{a}_{ij}\|_{L^\infty(B_1)} + [\tilde{a}_{ij}]_{C^{0,\alpha}(\overline{B_1})} = \|a_{ij}\|_{L^\infty(B_r)} + r^\alpha [a_{ij}]_{C^{0,\alpha}(\overline{B_r})} \\
\left\|\tilde{b}_i\right\|_{C^{0,\alpha}(\overline{B_1})} &= \left\|\tilde{b}_i\right\|_{L^\infty(B_1)} + [\tilde{b}_i]_{C^{0,\alpha}(\overline{B_1})} = r\,\|b_i\|_{L^\infty(B_r)} + r^{1+\alpha}[b_i]_{C^{0,\alpha}(\overline{B_r})} \\
\|\tilde{c}\|_{C^{0,\alpha}(\overline{B_1})} &= \|\tilde{c}\|_{L^\infty(B_1)} + [\tilde{c}]_{C^{0,\alpha}(\overline{B_1})} = r^2\,\|c\|_{L^\infty(B_r)} + r^{2+\alpha}[c]_{C^{0,\alpha}(\overline{B_r})} \\
\left\|\tilde{f}\right\|_{C^{0,\alpha}(\overline{B_1})} &= \left\|\tilde{f}\right\|_{L^\infty(B_1)} + [\tilde{f}]_{C^{0,\alpha}(\overline{B_1})} = r^2\,\|f\|_{L^\infty(B_r)} + r^{2+\alpha}[f]_{C^{0,\alpha}(\overline{B_r})}
\end{aligned}$$

Affine Transformation Assume that u solves constant coefficient equation

$$\mathcal{L}_0 u := a_{ij}(0)\partial_{ij} u = f_0 \qquad \Omega, \qquad a_{ij}(0) = a_{ji}(0) \tag{4.7}$$

which is uniformly elliptic

$$\lambda|\xi|^2 \le a_{ij}(0)\xi_i\xi_j \le \Lambda|\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^n$$

Let P be a constant matrix that denotes a nonsingular linear transformation

$$\begin{aligned}
P : \Omega \subseteq \mathbb{R}^n &\to \tilde{\Omega} \subseteq \mathbb{R}^n \\
x &\mapsto y = Px = (\sum_j P_{ij}x_j)_i
\end{aligned}$$

and define the transformed function $\overline{u}$ as

$$u(x) = \overline{u}(y) = \overline{u} \circ P(x)$$

Now what does $\overline{u}$ solves?

$$\begin{aligned}
\partial_i u(x) &= \partial_i(\overline{u} \circ P(x)) = \partial_k \overline{u}(y)\partial_i(P(x))^k = \partial_k \overline{u}(y)\partial_i(\sum_\ell P_{k\ell}x_\ell) = \partial_k\overline{u}(y)P_{ki} \\
\partial_{ij} u(x) &= \partial_j((\partial_k \overline{u}) \circ P(x))P_{ki} = (\partial_k\partial_\ell\overline{u})(y)P_{\ell j}P_{ki} \\
&= (P^T)_{ik}(\partial_{k\ell}\overline{u})(y)P_{\ell j} \\
D^2u(x) &= P^T D^2\overline{u}(y)P
\end{aligned}$$

Applying $A_0 = (a_{ij}(0))$ on both sides gives

$$a_{ij}(0)\partial_{ij}u(x) = P_{ki}a_{ij}(0)P^T_{j\ell}(\partial_{k\ell}\overline{u})(y)$$

so $\overline{u}$ solves

$$\overline{a}_{k\ell}\overline{u}_{k\ell}(y) = \overline{f}(y) \qquad \forall\; y \in \tilde{\Omega}$$

where

$$\begin{aligned}
\overline{a}_{k\ell} &:= P_{ki}a_{ij}(0)P^T_{j\ell} \\
\overline{A} &= PA_0P^T \\
\overline{f}(y) &= f_0(P^{-1}y)
\end{aligned}$$

Ok, now the agenda is, we want to transform so that it reduces to the Laplace Operator, i.e.

$$\begin{aligned}
\delta_{k\ell} &= P_{ki}a_{ij}(0)P^T_{j\ell} \\
I &= PA_0P^T
\end{aligned}$$

Such affine transformation P exists because A_0 is symmetric and positive-definite. In particular, using spectral decomposition (since symmetric)

$$A_0 = Q^T\Lambda_0 Q \qquad \Lambda_0 = \mathrm{diag}(\lambda_1, \cdots, \lambda_n) \text{ eigenvalues of } A_0,\ Q \text{ orthogonal matrix so } Q^TQ = I$$

Now due to positive definiteness, all eigenvalues for A_0 are positive, thus one define

$$\Lambda_0 = \Lambda_0^{\frac{1}{2}} I \Lambda_0^{\frac{1}{2}} \qquad \Lambda_0^{\frac{1}{2}} := \mathrm{diag}(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n})$$

so that

$$I = PA_0P^T = PQ^T\Lambda_0QP^T = (\Lambda_0^{\frac{1}{2}}QP^T)^T I(\Lambda_0^{\frac{1}{2}}QP^T)$$
$$P := \Lambda_0^{-\frac{1}{2}}Q^T$$

In particular, since the orthogonal matrix Q preserves length

$$\Lambda^{-\frac{1}{2}}|x| \le |y| = |Px| = |\Lambda_0^{-\frac{1}{2}}Q^Tx| \le \lambda^{-\frac{1}{2}}|x|$$

How do the norms change? For $C = C(n, \lambda, \Lambda, \Omega, \tilde{\Omega}, \alpha) > 0$

$$\frac{1}{C}\|u\|_{C^{2,\alpha}(\Omega)} \le \|\overline{u}\|_{C^{2,\alpha}(\tilde{\Omega})} \le C\|u\|_{C^{2,\alpha}(\Omega)}$$
$$\frac{1}{C}\|f_0\|_{C^{0,\alpha}(\Omega)} \le \|\overline{f}\|_{C^{0,\alpha}(\tilde{\Omega})} \le C\|f_0\|_{C^{0,\alpha}(\Omega)}$$

In particular, as along as $A_0 \ge \lambda I$ is satisfied, the Schauder estimates for Poisson's Equation transforms to the equation (4.7) ([GT01] Lemma 6.1).

$$\|u\|_{C^{2,\alpha}(\overline{B_{1/2}})} \le C(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(\overline{B_1})}) \tag{4.8}$$

4.2.3 A priori Interior Estimate

The following is the key a prior estimate for Interior Schauder Theory. Here we assume u to already lie within the space $C^{2,\alpha}$, and derive the bounds on $\|u\|_{C^{2,\alpha}}$.

Lemma 4.2.1 (De Silva Analysis II 2025)**.** *Assume $u \in C^{2,\alpha}(B_1)$ solves*

$$\mathcal{L}u := a_{ij}\partial_{ij}u + b_i\partial_iu + cu = f \qquad B_1$$

where $A = (a_{ij}) \ge \lambda I$ is uniformly elliptic, and

$$a_{ij},\ b_i,\ c,\ f \in C^{0,\alpha}(\overline{B_1})$$

Then there is a smallness universal $\delta = \delta(n, \alpha) > 0$ and $C = C(n, \lambda, \delta) > 0$ universal constant s.t. for any data small enough

$$\|a_{ij} - \delta_{ij}\|_{C^{0,\alpha}(\overline{B_1})},\ \|b_i\|_{C^{0,\alpha}(\overline{B_1})},\ \|c\|_{C^{0,\alpha}(\overline{B_1})} \le \delta \tag{4.9}$$

One has

$$\|u\|_{C^{2,\alpha}(\overline{B_{1/2}})} \le C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(\overline{B_1})}\right) \tag{4.10}$$

Proof. **Method of Freezing Coefficients**. The idea is using $a_{ij}(0) = \delta_{ij}$ Identity matrix. So we rearrange and treat the rest as force

$$a_{ij}(0)\partial_{ij}u = (a_{ij}(0) - a_{ij})\partial_{ij}u - b_i\partial_iu - cu + f := g$$

Applying (4.8) $C^{2,\alpha}$ estimate for constant coefficient operator gives

$$\|u\|_{C^{2,\alpha}(\overline{B_{1/2}})} \le C\left(\|u\|_{L^\infty(B_{3/4})} + \|g\|_{C^{0,\alpha}(\overline{B_{3/4}})}\right)$$

Let's unravel g using smallness assumption and interpolation inequality

$$\begin{aligned}\|g\|_{C^{0,\alpha}(\overline{B_{3/4}})} &\le \|(a_{ij}(0) - a_{ij})\partial_{ij}u\|_{C^{0,\alpha}(\overline{B_{3/4}})} + \|b_i\partial_iu\|_{C^{0,\alpha}(\overline{B_{3/4}})} + \|cu\|_{C^{0,\alpha}(\overline{B_{3/4}})} + \|f\|_{C^{0,\alpha}(\overline{B_{3/4}})} \\ &\overset{(4.9)}{\le} \delta\left(\|D^2u\|_{C^{0,\alpha}(\overline{B_{3/4}})} + \|\nabla u\|_{C^{0,\alpha}(\overline{B_{3/4}})} + \|u\|_{C^{0,\alpha}(\overline{B_{3/4}})}\right) + \|f\|_{C^{0,\alpha}(\overline{B_{3/4}})} \\ &\overset{(4.3)}{\le} \delta\left(C\|u\|_{L^\infty(B_{3/4})} + [D^2u]_{C^{0,\alpha}(\overline{B_{3/4}})}\right) + \|f\|_{C^{0,\alpha}(\overline{B_{3/4}})}\end{aligned}$$

We're tempted to squeeze $[D^2u]_{C^{0,\alpha}}$ to the LHS by choosing the coefficient in front to be small. But we cannot, because the domain on which we evalutae $[D^2u]_{C^{0,\alpha}}$ increased from $B_{1/2}$ to $B_{3/4}$.

How to remedy for this? One try to rewrite

$$\|u\|_{C^{2,\alpha}(\overline{B_{1/2}})} \le C\left((1+\delta)\|u\|_{L^\infty(B_{3/4})} + \|f\|_{C^{0,\alpha}(\overline{B_{3/4}})}\right) + C\delta \sup_{B_{1/8}(x)\subseteq B_{3/4}} [D^2u]_{C^{0,\alpha}(\overline{B_{1/8}(x)})}$$

$$[D^2u]_{C^{0,\alpha}(\overline{B_{1/2}})} \le C(\|u\|_{L^\infty(B_{3/4})} + \|f\|_{C^{0,\alpha}(\overline{B_{3/4}})}) + \mu \sup_{B_{1/8}(x)\subseteq B_{3/4}} [D^2u]_{C^{0,\alpha}(\overline{B_{1/8}(x)})} \tag{4.11}$$

where we've chosen $\delta \leq 1$, and then denotes $\mu := C\delta$ to be chosen small, universal.

Rescaling. Now we use that u solves the equation on $B_r \subseteq B_1$. Let's conduct rescaling back to unit ball

$$\tilde{u}(x) := u(rx) \qquad \forall\ x \in B_1$$

so that $\tilde{u}$ solves (4.6)

$$\tilde{\mathcal{L}}\tilde{u}(y) := \tilde{a}_{ij}(y)\partial_{ij}\tilde{u}(y) + \tilde{b}_i(y)\partial_i\tilde{u}(y) + \tilde{c}(y)\tilde{u}(y) = \tilde{f}(y) \qquad \forall\ y \in B_1$$

with

$$\begin{aligned}\tilde{a}_{ij}(y) &:= a_{ij}(ry)\\ \tilde{b}_i(y) &:= rb_i(ry)\\ \tilde{c}(y) &:= r^2c(ry)\\ \tilde{f}(y) &:= r^2f(ry)\end{aligned}$$

Can we ensure $\tilde{u}$ still satisfies the assumptions (4.9)? Indeed

$$\begin{aligned}\|\tilde{a}_{ij} - \delta_{ij}\|_{C^{0,\alpha}(\overline{B_1})} &\leq \|a_{ij} - \delta_{ij}\|_{L^\infty(B_r)} + r^\alpha\|a_{ij} - \delta_{ij}\|_{C^{0,\alpha}(\overline{B_r})} \leq \delta\\ \left\|\tilde{b}_i\right\|_{C^{0,\alpha}(\overline{B_1})} &\leq r\|b_i\|_{L^\infty(B_r)} + r^{1+\alpha}[b_i]_{C^{0,\alpha}(\overline{B_r})} \leq \delta\\ \|\tilde{c}\|_{C^{0,\alpha}(\overline{B_1})} &= r^2\|c\|_{L^\infty(B_r)} + r^{2+\alpha}[c]_{C^{0,\alpha}(\overline{B_r})} \leq \delta\end{aligned}$$

Defining the Iteration.

Now running (4.6) again yields

$$\begin{aligned}[D^2\tilde{u}]_{C^{0,\alpha}(\overline{B_{1/2}})} &\leq C(\|\tilde{u}\|_{L^\infty(B_{3/4})} + \left\|\tilde{f}\right\|_{C^{0,\alpha}(\overline{B_{3/4}})}) + \mu \sup_{B_{1/8}(x)\subseteq B_{3/4}}[D^2\tilde{u}]_{C^{0,\alpha}(\overline{B_{1/8}(x)})}\\ r^{2+\alpha}[D^2u]_{C^{0,\alpha}(\overline{B_{r/2}})} &\leq C\left(\|u\|_{L^\infty(B_{3r/4})} + r^{2+\alpha}\|f\|_{C^{0,\alpha}(\overline{B_{3r/4}})}\right) + \mu r^{2+\alpha}\sup_{B_{r/8}(x)\subseteq B_{3r/4}}[D^2u]_{C^{0,\alpha}(\overline{B_{r/8}(x)})}\\ [D^2u]_{C^{0,\alpha}(\overline{B_{r/2}})} &\leq Cr^{-3}\left(\|u\|_{L^\infty(B_{3r/4})} + \|f\|_{C^{0,\alpha}(\overline{B_{3r/4}})}\right) + \mu\sup_{B_{r/8}(x)\subseteq B_{3r/4}}[D^2u]_{C^{0,\alpha}(\overline{B_{r/8}(x)})} \qquad \text{be generous...}\end{aligned}$$

But this doesn't have to be done at the origin. In fact, picking any $B_r(x_0) \Subset B_1$ would work. Also, leveraging the linearity of the equation, one may divide by a huge constant and ensure

$$\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(\overline{B_1})} = 1$$

Therefore one obtain

$$[D^2u]_{C^{0,\alpha}(\overline{B_{r/2}(x_0)})} \leq Cr^{-3} + \mu\sup_{B_{r/8}(x)\subseteq B_{3r/4}(x_0)}[D^2u]_{C^{0,\alpha}(\overline{B_{r/8}(x)})}$$

Notice for the supremum over the larger ball, it is essentially achieved somewhere, say on the ball $B_{r/8}(x_1)$. Thus the above gives

$$[D^2u]_{C^{0,\alpha}(\overline{B_{r/2}(x_0)})} \leq Cr^{-3} + \mu[D^2u]_{C^{0,\alpha}(\overline{B_{r/8}(x_1)})}$$

This takes the form of iteration! In the following, take $r = 1$ and $x_0 = 0$ to start with. There exists a sequence of points $\{x_n\} \subseteq B_1$ and radius s.t.

$$r_k := 2^{-2k-1} \qquad \forall\ k \geq 0$$

and

$$x_{k+1} \in B_{3r_{k+1}/2}(x_k) \qquad \forall\ k \geq 0$$

along with the iteration

$$[D^2u]_{C^{0,\alpha}(\overline{B_{r_k}(x_k)})} \leq Cr_k^{-3} + \mu[D^2u]_{C^{0,\alpha}(\overline{B_{r_{k+1}}(x_{k+1})})} \qquad \forall\ k \geq 0$$

Denote

$$a_k := [D^2u]_{C^{0,\alpha}(\overline{B_{r_k}(x_k)})}$$

and the iteration writes

$$a_k \leq M2^{6k} + \mu a_{k+1} \tag{4.12}$$

where $M = 8C$ is universal constant.

Choice of parameter. We claim that, one may choose μ sufficiently small, universal, s.t.

$$[D^2u]_{C^{0,\alpha}(\overline{B_{1/2}})} = a_0 \le C \qquad \text{universally bounded in } u$$

Assume not! So no matter what μ small we pick, no matter what C large we take, there is always a function u s.t. $a_0 > C$. We want to argue this cannot happen, by construct a sequence a_k that blows up as $k \to \infty$. Why is this a contradiction? Because u are assumed to be $C^{2,\alpha}$.

Now μ and C we're free to pick. **We claim** that there is a large enough C and $\sigma > 0$ to be determined, s.t. as along as

$$a_0 \ge C$$

one has

$$a_k \ge C\sigma^k \qquad \forall\ k \ge 0$$

To show the existence of C and σ, one need to do induction. Assume for k, then for $k+1$ to hold, one need to ensure

$$\begin{aligned} a_{k+1} &\overset{(4.12)}{\ge} \frac{1}{\mu}\left(a_k - M2^{6k}\right) \\ &\ge \frac{1}{\mu}\left(C\sigma^k - M2^{6k}\right) \qquad \text{using inductive hypothesis} \end{aligned}$$

Now the only thing one want to ensure is the following

$$\begin{aligned} \frac{1}{\mu}\left(C\sigma^k - M2^{6k}\right) &\ge C\sigma^{k+1} \\ C\sigma^k(1-\mu\sigma) &\ge M2^{6k} \end{aligned}$$

Let's say one pick

$$\mu = 2^{-7}, \qquad 2^6 \le \sigma < \mu^{-1}$$

Then it suffices to pick C so large s.t.

$$C > \frac{M}{1-\mu\sigma} = \frac{2^7M}{2^7-\sigma}$$

Therefore $a_k \to \infty$ and we reach a contradiction.

□

4.2.4 A priori Boundary Estimates

Lemma 4.2.2 (De Silva Analysis II 2025; [GT01] Lemma 6.4)**.** *Assume $u \in C^{2,\alpha}(\overline{B_1^+})$ solves*

$$\mathcal{L}u := a_{ij}\partial_{ij}u + b_i\partial_i u + cu = f \qquad B_1^+$$

where $A = (a_{ij}) \ge \lambda I$ is uniformly elliptic, and

$$a_{ij},\ b_i,\ c,\ f \in C^{0,\alpha}(\overline{B_1^+})$$

Also assume $u = \varphi$ on the boundary $\partial(B_1^+)$

Then there is a smallness universal $\delta = \delta(n,\alpha) > 0$ and $C = C(n,\lambda,\delta) > 0$ universal constant s.t. for any data small enough

$$\|a_{ij}-\delta_{ij}\|_{C^{0,\alpha}(\overline{B_1^+})},\ \|b_i\|_{C^{0,\alpha}(\overline{B_1^+})},\ \|c\|_{C^{0,\alpha}(\overline{B_1^+})} \le \delta$$

One has

$$\|u\|_{C^{2,\alpha}(\overline{B_{1/2}^+})} \le C\left(\|u\|_{L^\infty(B_1^+)} + \|f\|_{C^{0,\alpha}(\overline{B_1^+})} + \|\varphi\|_{C^{2,\alpha}(\overline{B_1^+})}\right) \tag{4.13}$$

$C^{k,\alpha}$ Domains We define domains of class $C^{k,\alpha}$.

Definition 4.2.1 ($C^{k,\alpha}$ domains; [GT01] Section 6.2)**.** *A $\Omega \subseteq \mathbb{R}^n$ bounded domain is of class $C^{k,\alpha}$ for $k \in \mathbb{N}$, $\alpha \in [0,1]$, if*

for any $x_0 \in \partial\Omega$, there exists a ball $B_r(x_0)$ around x_0 and a bijection Ψ from $B_r(x_0)$ onto $D \subseteq \mathbb{R}^n$

$$\begin{aligned} \Psi : B_r(x_0) &\to D \subseteq \mathbb{R}^n \\ x &\mapsto y = \Psi(x) \end{aligned} \tag{4.14}$$

s.t.

1. $\Psi(B_r(x_0) \cap \Omega) \subseteq \mathbb{R}^n_+$.

2. $\Psi(B_r(x_0) \cap \partial\Omega) \subseteq \partial\mathbb{R}^n_+$. *And in particular one may take* $\Psi(x_0) = 0 \in \mathbb{R}^n$.

3. $\Psi \in C^{k,\alpha}(B_r(x_0); D)$ *and* $\Psi^{-1} \in C^{2,\alpha}(D; B_r(x_0))$.

We say Ψ *straightens the boundary near* x_0.

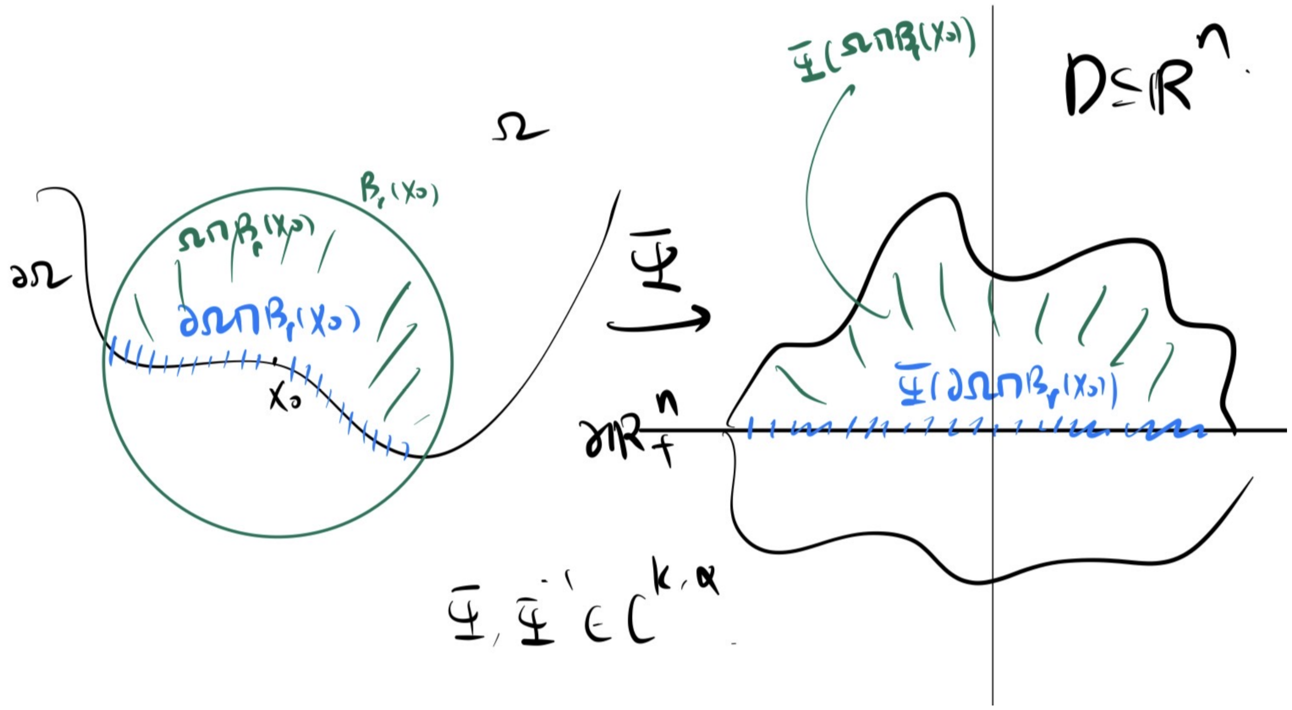


Figure 4.1: $C^{2,\alpha}$ Flattening the boundary

In particular, using both Ψ and Ψ^{-1} are in particular Lipschitz, there exists universal $K = K(\Psi, \Omega) > 0$ s.t. for any $x, y \in \Omega$

$$\frac{1}{K}|x - y| \leq |\Psi(x) - \Psi(y)| \leq K|x - y|$$

Domain Transformation Assume we're given $\partial\Omega$ a $C^{2,\alpha}$ domain, by which we mean for any $x_0 \in \partial\Omega$, there exists a neighborhood N around x_0 and a $C^{2,\alpha}$ diffeomorphism

$$\Psi(x) = (\psi^1(x), \cdots, \psi^N(x))$$

defined on $B_\rho(x_0) \Subset N$, that straightens the boundary in N as in (4.14).

Assume for $x_0 \in \partial\Omega$ and $\rho > 0$ small, the equation is satisfied in

$$\mathcal{L}u = f \qquad B_\rho(x_0) \cap \Omega$$

We ask under the transformation

$$u(x) = \tilde{u}(y) = \tilde{u} \circ \Psi(x) \qquad \forall\, x \in \Omega \cap B_\rho(x_0)$$

what equation $\tilde{u}$ in $\Psi(B_\rho(x_0) \cap \Omega) \subseteq D \cap \mathbb{R}^n_+$ solves, and how the norms of the coefficients change.

We compute

$$\begin{aligned}
\partial_i(\tilde{u} \circ \Psi)(x) &= \partial_k \tilde{u}(y) \partial_i \psi^k(x) \\
\partial_{ij}(\tilde{u} \circ \Psi)(x) &= \partial_{k\ell}\tilde{u}(y)\partial_i\psi^k(x)\partial_j\psi^\ell(x) + \partial_k\tilde{u}(y)\partial_{ij}\psi^k(x)
\end{aligned}$$

so that

$$\begin{aligned}
a_{ij}\partial_{ij}u(x) &= a_{ij}(x)\partial_{ij}(\tilde{u} \circ \Psi)(x) \\
&= a_{ij}(x)\partial_i\psi^k(x)\partial_j\psi^\ell(x)\partial_{k\ell}\tilde{u}(y) + a_{ij}(x)\partial_{ij}\psi^k(x)\partial_k\tilde{u}(y) \\
b_i(x)\partial_i u(x) &= b_i(x)\partial_i\psi^k(x)\partial_k\tilde{u}(y)
\end{aligned}$$

The equation therefore writes

$$\begin{aligned}\mathcal{L}u &= a_{ij}\partial_{ij}u(x) + b_i\partial_i u(x) + cu = a_{ij}(x)\partial_{ij}(\tilde{u}\circ\Psi)(x) + b_i(x)\partial_i(\tilde{u}\circ\Psi)(x) + c(\tilde{u}\circ\Psi)(x)\\ &= a_{ij}(x)\partial_i\psi^k(x)\partial_j\psi^\ell(x)\partial_{k\ell}\tilde{u}(y) + \left(a_{ij}(x)\partial_{ij}\psi^k(x) + b_i(x)\partial_i\psi^k(x)\right)\partial_k\tilde{u}(y) + c(x)\tilde{u}(y)\end{aligned}$$

Thus $\tilde{u}$ solves

$$\tilde{\mathcal{L}}\tilde{u}(y) := \tilde{a}_{k\ell}(y)\partial_{k\ell}u(y) + \tilde{b}_k(y)\partial_k\tilde{u}(y) + \tilde{c}(y)\tilde{u}(y) = \tilde{f}(y) \qquad \forall\, y \in \Psi(\Omega\cap B_\rho(x_0))$$

where

$$\begin{aligned}\tilde{a}_{k\ell}(y) &:= a_{ij}(\Psi^{-1}(y))\partial_i\psi^k(\Psi^{-1}(y))\partial_j\psi^\ell(\Psi^{-1}(y))\\ \tilde{b}_k(y) &:= a_{ij}(\Psi^{-1}(y))\partial_{ij}\psi^k(\Psi^{-1}(y)) + b_i(\Psi^{-1}(y))\partial_i\psi^k(\Psi^{-1}(y))\\ \tilde{c}(y) &:= c(\Psi^{-1}(y))\\ \tilde{f}(y) &:= f(\Psi^{-1}(y))\end{aligned}$$

To ensure the coefficients in $\tilde{\mathcal{L}}$ remain Hölder $C^{0,\alpha}$, one really need to ensure $\Psi \in C^{2,\alpha}$ due to the term $\partial_{ij}\psi^k$ built in the definition of $\tilde{b}_k$.

Assume our original $A = (a_{ij})$ is uniformly elliptic with ellipticity coefficients

$$\lambda|\xi|^2 \le a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^n$$

Does uniform ellipticity of $\tilde{A}$ adapt from that of A? Note

$$\begin{aligned}\tilde{a}_{k\ell}(y) &:= \partial_i\psi^k a_{ij}\partial_j\psi^\ell\\ (\tilde{A}(y))_{k\ell} &= (D\Psi(x)A(x)D\Psi^T(x))_{k\ell}\end{aligned}$$

So for any $\eta \in \mathbb{R}^n$

$$\begin{aligned}\eta^T\tilde{A}\eta &= \eta^T D\Psi(x)A(x)D\Psi^T(x)\eta = (D\Psi^T\eta)^T A(D\Psi^T\eta)\\ \lambda|D\Psi^T\eta|^2 &\le \eta^T\tilde{A}\eta \le \Lambda|D\Psi^T\eta|^2\end{aligned}$$

Now how does the norm $|D\Psi^T\eta|$ compare with $|\eta|$? Using Ψ is C^1 diffeomorphism, denote the universal bound as K

$$\sup_{x\in B_r(x_0)}|D\Psi(x)|,\ \sup_{y\in D}|D\Psi^{-1}(y)| \le K$$

Then

$$\begin{aligned}|D\Psi^T\eta|^2 &\le K^2|\eta|^2\\ |\eta|^2 &= |(D\Psi^T)^{-1}D\Psi^T\eta|^2 \le K^2|D\Psi^T\eta|^2\end{aligned}$$

Thus the uniform elliptic constants for $\tilde{A}$ writes

$$\frac{1}{K^2}\lambda|\eta|^2 \le \eta^T\tilde{A}\eta \le \Lambda K^2|\eta|^2 \qquad \forall\, \eta \in \mathbb{R}^n$$

How does Hölder norms of the Coefficients change? A first remark, under change of variables

$$\begin{aligned}\left\|g\circ\Psi^{-1}\right\|_{C^{0,\alpha}(\overline{D})} &= \left\|g\circ\Psi^{-1}\right\|_{L^\infty(D)} + [g\circ\Psi^{-1}]_{C^{0,\alpha}(\overline{D})}\\ &\le \|g\|_{L^\infty(B_r(x_0))} + [g]_{C^{0,\alpha}(\overline{B_r(x_0)})}\left\|D\Psi^{-1}\right\|^\alpha_{L^\infty(D)}\\ &\le (1+K^\alpha)\|g\|_{C^{0,\alpha}(\overline{B_r(x_0)})}\end{aligned}$$

Then

$$\begin{aligned}\|\tilde{a}_{k\ell}\|_{C^{0,\alpha}(\overline{D})} &\le \left\|a_{ij}\circ\Psi^{-1}\right\|_{C^{0,\alpha}(\overline{D})}\left\|\partial_i\psi^k\circ\Psi^{-1}\right\|_{C^{0,\alpha}(\overline{D})}\left\|\partial_j\psi^\ell\circ\Psi^{-1}\right\|_{C^{0,\alpha}(\overline{D})}\\ &\le (1+K^\alpha)^3\|D\Psi\|^2_{C^{0,\alpha}(\overline{B_r(x_0)})}\|a_{ij}\|_{C^{0,\alpha}(\overline{B_r(x_0)})}\\ &\le C(\alpha, \|D\Psi\|_{C^{0,\alpha}}, \left\|D\Psi^{-1}\right\|_{L^\infty})\|a_{ij}\|_{C^{0,\alpha}(\overline{B_r(x_0)})}\\ \left\|\tilde{b}_k\right\|_{C^{0,\alpha}(\overline{D})} &\le (1+K^\alpha)^2\left(\left\|D^2\Psi\right\|_{C^{0,\alpha}(\overline{B_r(x_0)})}\|a_{ij}\|_{C^{0,\alpha}(\overline{B_r(x_0)})} + \|D\Psi\|_{C^{0,\alpha}}\|b_i\|_{C^{0,\alpha}(\overline{B_r(x_0)})}\right)\\ &\le C(\alpha, \|\Psi\|_{C^{2,\alpha}}, \left\|D\Psi^{-1}\right\|_{L^\infty})\left(\|a_{ij}\|_{C^{0,\alpha}(\overline{B_r(x_0)})} + \|b_i\|_{C^{0,\alpha}(\overline{B_r(x_0)})}\right)\\ \|\tilde{c}\|_{C^{0,\alpha}(\overline{D})} &\le (1+K^\alpha)\|c\|_{C^{0,\alpha}(\overline{B_r(x_0)})}\\ \left\|\tilde{f}\right\|_{C^{0,\alpha}(\overline{D})} &\le (1+K^\alpha)\|f\|_{C^{0,\alpha}(\overline{B_r(x_0)})}\end{aligned}$$

4.2.5 A prior Global Estimates

Theorem 4.2.3 ([GT01] Theorem 6.6)**.** *Let Ω be $C^{2,\alpha}$ domain. Let $u \in C^{2,\alpha}(\overline{\Omega})$ be solution of*

$$\mathcal{L}u = a_{ij}\partial_{ij}u + b_i\partial_i u + cu = f \qquad \Omega$$

where $A = (a_{ij}) \geq \lambda I$ is uniformly elliptic, and

$$a_{ij},\ b_i,\ c,\ f \in C^{0,\alpha}(\overline{\Omega})$$

Also let $\varphi \in C^{2,\alpha}(\overline{\Omega})$ and assume $u = \varphi$ on $\partial\Omega$.

Then for $C = C(n, \alpha, \lambda, \Omega, \|a_{ij}\|_{C^{0,\alpha}}, \|b_i\|_{C^{0,\alpha}}, \|c\|_{C^{0,\alpha}})$

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\left(\|u\|_{L^\infty(\Omega)} + \|\varphi\|_{C^{2,\alpha}(\overline{\Omega})} + \|f\|_{C^{0,\alpha}(\overline{\Omega})}\right) \tag{4.15}$$

We claim it suffices to prove for the case $u = 0$ on $\partial\Omega$ and so $\varphi = 0$. Indeed, if we let $v = u - \varphi$, then

$$\mathcal{L}v = f - \mathcal{L}\varphi \equiv f' \in C^{0,\alpha}(\overline{\Omega})$$

Then using conclusion (4.15)

$$\|v\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\left(\|v\|_{L^\infty(\Omega)} + \|f'\|_{C^{0,\alpha}(\overline{\Omega})}\right)$$

and thus

$$\begin{aligned}\|u\|_{C^{2,\alpha}(\overline{\Omega})} &\leq \|v\|_{C^{2,\alpha}(\overline{\Omega})} + \|\varphi\|_{C^{2,\alpha}(\overline{\Omega})}\\ &\leq C\left(\|v\|_{L^\infty(\Omega)} + \|\varphi\|_{C^{2,\alpha}(\overline{\Omega})} + \|f'\|_{C^{0,\alpha}(\overline{\Omega})}\right)\\ &\leq C\left(\|u\|_{L^\infty(\Omega)} + \|\varphi\|_{C^{2,\alpha}(\overline{\Omega})} + \|f\|_{C^{0,\alpha}(\overline{\Omega})}\right)\end{aligned}$$

4.3 Existence

We demonstrate our ultimate goal: the Existence of Solution to Dirichlet Boundary Value Problem.

Theorem 4.3.1. *Assume Ω is $C^{2,\alpha}$, $f \in C^{0,\alpha}(\overline{\Omega})$, $\varphi \in C^{2,\alpha}(\partial\Omega)$.*

Assume coefficients $A = (a_{ij}) \geq \lambda I$, either $c \leq 0$ or $c \leq c_0(\lambda, \Lambda, \Omega)$ (s.t. the maximum principle holds) and

$$a_{ij},\ b_i,\ c \in C^{0,\alpha}(\overline{\Omega})$$

Then there exists a unique $u \in C^{2,\alpha}(\overline{\Omega})$ that solves

$$\begin{cases} \mathcal{L}u = f & \Omega \\ u = \varphi & \partial\Omega \end{cases}$$

and

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\left(\|f\|_{C^{0,\alpha}(\overline{\Omega})} + \|\varphi\|_{C^{2,\alpha}(\partial\Omega)}\right) \tag{4.16}$$

By a solution to the Dirichlet Boundary Value Problem we mean a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. But the statement says u is in fact $C^{2,\alpha}$ all the way up to the boundary if $\varphi \in C^{2,\alpha}(\partial\Omega)$.

If a $C^{2,\alpha}$ solution exists, then it will satisfy the estimate (4.16) in view of the a prior $C^{2,\alpha}$ interior and boundary estimates, in particular (4.15).

Remark 4.3.1 (De Silva Analysis II Spring 2025)**.** *Notice that in a priori estimates, $\|u\|_{L^\infty(\Omega)}$ **on RHS can be bounded by f and φ***

$$\|u\|_{L^\infty(\Omega)} \leq C\left(\|\varphi\|_{C^{2,\alpha}(\overline{\Omega})} + \|f\|_{C^{0,\alpha}(\overline{\Omega})}\right)$$

This is quite a subtlety, by it is achieved by the maximum principle (3.8), or essentially (3.9).

$$\|u\|_{C^0(\overline{\Omega})} \leq \sup_{\partial\Omega}|u| + C(\lambda)\sup_{\Omega}|f|$$

Let's achieve this again by hand. As long as one has a barrier ψ s.t.

$$\begin{cases} \mathcal{L}\psi < -\delta_0 & \Omega \\ \psi > \delta_0 & \partial\Omega \end{cases}$$

then one may choose t_0 big s.t.

$$t_0\delta_0 = \|f\|_{L^\infty} + \|\varphi\|_{L^\infty}$$

Now as one lower $t \to t_0$, the first touching point of $t\psi$ and u must occur in the interior (as $|u| < t\delta_0$ on $\partial\Omega$), and at such a touching point

$$\mathcal{L}(t\psi) \leq -t\delta_0 \leq -t_0\delta_0 \leq -\|f\|_{L^\infty} \leq \mathcal{L}u$$

contradicting that $t\psi - u$ has a minimum at such touching point. Thus the strong minimum principle says

$$t\psi > u \qquad \forall\ t \geq t_0 \qquad \Omega$$

which is to say

$$\|u\|_{L^\infty} \leq C\left(\|f\|_{L^\infty} + \|\varphi\|_{L^\infty}\right)$$

We remark that the assumption on c is important.

Example 4.3.1. *Consider*

$$\begin{cases} u'' + u = 0 & (0, \pi) \\ u(0) = 0 \\ u(\pi) = 1 \end{cases}$$

Notice there is no general solution, since the formula takes form

$$u(t) = A\sin(t) + B\cos(t)$$

But

$$0 = u(0) = B$$
$$1 = u(\pi) = -B$$

yields contradiction.

Now after subtracting a $C^{2,\alpha}$ extension of the boundary data φ from u, one may assume that $\varphi = 0$.

4.3.1 Method of Continuity

Contraction Mapping Principle Let X be normed vector space. $T : X \to X$ is a contraction mapping if there exists $\theta \in \mathbb{R}$ s.t.

$$\|Tx - Ty\| \leq \theta \|x - y\| \qquad \forall\, x, y \in X$$

Theorem 4.3.2 ([GT01] Theorem 5.1)**.** *A contraction mapping T in a Banach Space X has a unique fixed point, i.e., there exists unique $x \in X$ s.t.*

$$Tx = x$$

Proof. Take $x_0 \in X$ and define $x_n := T^n x_0$. For any $n \geq m$

$$\begin{aligned}\|x_n - x_m\| &\leq \sum_{j=m+1}^{n} \|x_j - x_{j-1}\| = \sum_{j=m+1}^{n} \left\|T^{j-1}x_1 - T^{j-1}x_0\right\| \\ &\leq \sum_{j=m+1}^{n} \theta^{j-1} \|x_1 - x_0\| \leq \frac{\theta^m}{1-\theta} \|x_1 - x_0\| \to 0 \qquad m \to \infty\end{aligned}$$

So x_n is Cauchy sequence and there exists $x \in X$ s.t. (because X is complete), due to T continuous

$$Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x$$

□

Method of Continuity In order to prove existence, one need to use the following functional analysis tool.

Theorem 4.3.3 ([GT01] Theorem 5.2)**.** *Let X be a Banach Space, Y a normed vector space, and $L_0, L_1 \in \mathcal{L}(X, Y)$ bounded linear operators.*

For any $t \in [0, 1]$ denote

$$L_t := (1 - t)L_0 + tL_1 : X \to Y$$

and suppose there exists constant $C > 0$ s.t.

$$\|x\|_X \leq C \|L_t x\|_Y \qquad \forall\, x \in X,\ \forall\, t \in [0, 1] \tag{4.17}$$

Then L_1 maps X onto Y iff L_0 maps X onto Y.

In particular, since the relation (4.17) implies all L_t are injective, as long as L_0 is invertible, we know L_1 is invertible.

The proof relies on the contraction mapping principle.

Proof. Suppose L_s is onto for some $s \in [0, 1]$. Then since the estimate (4.17) holds at s, we know L_s is in fact invertible, so linear operator as following exists

$$L_s^{-1} : Y \to X$$

Now for any $t \in [0, 1]$ and $y \in Y$, solving the equation

$$L_t x = y$$

is equivalent to solving

$$\begin{aligned}L_s x &= y + (L_s - L_t)x \\ &= y + (t - s)L_0 x - (t - s)L_1 x \\ x &= L_s^{-1} y + (t - s)L_s^{-1}(L_0 - L_1)x\end{aligned}$$

Consider the linear operator

$$Tx := L_s^{-1} y + (t - s)L_s^{-1}(L_0 - L_1)x$$

Now this is a contraction mapping if

$$|s - t| < \delta = \frac{1}{C(\|L_0\| + \|L_1\|)}$$

because for any x_1, x_2

$$\begin{aligned}\|Tx_1 - Tx_2\|_X = \left\|(t-s)L_s^{-1}(L_0 - L_1)(x_1 - x_2)\right\|_X &< \frac{1}{C(\|L_0\| + \|L_1\|)}\left\|L_s^{-1}(L_0 - L_1)(x_1 - x_2)\right\|_X \\ &\overset{(4.17)}{\leq} \frac{1}{\|L_0\| + \|L_1\|}\|(L_0 - L_1)(x_1 - x_2)\|_Y \qquad \text{using assumption at } s \\ &\leq \frac{\|L_0 - L_1\|}{\|L_0\| + \|L_1\|}\|x_1 - x_2\|_X\end{aligned}$$

Therefore by Theorem 4.3.2, one know there exist a unique $x \in X$ s.t.

$$x = Tx$$

which is equivalent to say (as our deduction)

$$L_t x = y$$

provided $|s - t| < \delta$. But this δ is universal, so one may divide $[0, 1]$ into sub-intervals of length less than δ, and thus the mapping L_t is onto for all $t \in [0, 1]$ iff it is onto for any fixed $t \in [0, 1]$, in particular, at $t = 0$ or $t = 1$. □

4.3.2 Proof of Existence

In the proof we consider the Banach space

$$X = C_0^{2,\alpha} := \{u \in C^{2,\alpha}(\overline{\Omega}) \mid u = 0 \qquad \partial\Omega\}$$

and

$$Y = C^{0,\alpha}(\overline{\Omega})$$

We consider the family of operators

$$L_t := (1 - t)\Delta + t\mathcal{L}$$

Once we ensure two things

1. $L_0 = \Delta$ is invertible.
2. The estimates (4.17) hold
$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\,\|(1 - t)\Delta + t\mathcal{L}(u)\|_{C^{0,\alpha}}$$
In particular, this is ensured by our a prior estimate for L_1 (4.15) (assuming we already have a solution $u \in C_0^{2,\alpha}$).

Then we may apply the Method of Continuity Theorem 4.3.3 so that L_1 is invertible, i.e., for any force in $C^{0,\alpha}(\overline{\Omega})$ there is a unique $u \in C_0^{2,\alpha}$.

Recall we've assumed for zero boundary data because we can always add back.

Proof of Existence Theorem in balls B_1 We let $\Omega = B_1$.

Proof of Theorem 4.3.1 in Balls. **Invertibility of L_0.** Note $L_0 = \Delta$. We know the operator Δ is invertible in B_1 due to $C^{2,\alpha}$ estimates up to the boundary for Laplacian (2.29). This is essentially done by the Kelvin Transform.

The estimates for all $t \in [0, 1]$. But we also have the a prior estimates! For both interior and boundary versions, which gives us (note we've assumed zero boundary data)

$$\begin{aligned}\|u\|_{C^{2,\alpha}(\overline{B_1})} &\overset{(4.16)}{\leq} C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(\overline{B_1})}\right) \\ &\overset{(3.9)}{\leq} C\,\|f\|_{C^{0,\alpha}(\overline{B_1})}\end{aligned}$$

Thus the method of continuity guarantees existence in $\overline{B_1}$.

□

Proof of Existence with continuous boundary data $\varphi \in C^0(\partial B_1)$

Corollary 4.3.1 ([GT01] Lemma 6.10). *For $\varphi \in C^0(\partial B_1)$, there is still classical solution $u \in C^{2,\alpha}(B_1) \cap C^0(\overline{B_1})$ to Theorem 4.3.1.*

We approximate φ with a sequence $\varphi_k \in C^{2,\alpha}$ that converges uniformly to φ.

The upshot is: **We only constructed convergence in boundary data. What can we say about the convergence in solutions u_k?**

Then we solve the problem $u_k \in C^{2,\alpha}(\overline{B_1})$

$$\begin{cases} \mathcal{L}u_k = f & B_1 \\ u_k = \varphi_k & \partial B_1 \end{cases}$$

and since we've shown existence on balls with $C^{2,\alpha}$ boundary data, we know $u_k \in C^{2,\alpha}(\overline{B_1})$ and

$$\begin{cases} \mathcal{L}(u_k - u_j) = 0 & B_1 \\ u_k - u_j = \varphi_k - \varphi_j & \partial B_1 \end{cases}$$

Thus by the Maximum Principle

$$\|u_k - u_j\|_{L^\infty(B_1)} \le \|\varphi_k - \varphi_j\|_{C(\partial B_1)}$$

where the latter being Cauchy implies $\{u_k\}$ is a Cauchy sequence and hence it converges uniformly to some u which must be continuous up to the boundary, and coincides with φ there.

By **interior a priori estimates**

$$\|u_k\|_{C^{2,\alpha}(\overline{B_{r/2}(x_0)})} \le C\left(\|u_k\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(\overline{B_r(x_0)})}\right)$$

where $\|u_k\|_{L^\infty(B_1)}$ is uniformly bounded. Thus by Ascoli-Arzela D^2u_k converges uniformly up to subsequence, so $u \in C^{2,\alpha}(B_1) \cap C^0(\overline{B_1})$ and solves

$$\mathcal{L}u = f \qquad B_1$$

C^2 solution with no sign/smallness assumption on c are $C^{2,\alpha}$

Corollary 4.3.2 ([GT01] Lemma 6.16). *If $u \in C^2(\Omega)$ solves*

$$\mathcal{L}u = f \qquad \Omega$$

with Hölder coefficients, but with no sign/smallness assumption on c, then we conclude $u \in C^{2,\alpha}(\Omega)$.

Indeed, for any $B_r(x_0) \subseteq \Omega$, one may rescale it back to the unit ball. For r sufficiently small,

$$\tilde{c}(x) := r^2 c(x_0 + rx) \qquad \forall\ x \in B_1$$

will essentially satisfy the smallness assumption. Then one can solve the rescaled equation with continuous data $\tilde{u}(x) := u(x_0 + rx)$ in ∂B_1, i.e., search for $w \in C^{2,\alpha}(B_1) \cap C^0(\overline{B_1})$ s.t.

$$\begin{cases} \mathcal{L}w = 0 & B_1 \\ w = \tilde{u} & \partial B_1 \end{cases}$$

Is the problem solvable? Yes, by our previous Corollary 4.3.1. And by uniqueness, $w = \tilde{u}$ in B_1. Since w are $C^{2,\alpha}$, one do this locally in any ball $B_r(x_0) \subseteq \Omega$ and conclude $u \in C^{2,\alpha}(\Omega)$.

Proof of Existence in domain Diffeomorphic to a Ball For $\Psi = (\psi^1, \cdots, \psi^n)$ a $C^{2,\alpha}$ diffeomorphism between Ω and a ball B_1, and assume u solves the original equation

$$\mathcal{L}u = f \qquad \Omega$$

Then the deformed solution $\tilde{u}$ defined via

$$u(x) = \tilde{u} \circ \Psi(x)$$

solves in B_1 the equation

$$\tilde{\mathcal{L}}\tilde{u} = a_{ij}\partial_i\psi^k\partial_j\psi^\ell\partial_{k\ell}\tilde{u} + (a_{ij}\partial_{ij}\psi^k + b)i\partial_i\psi^k)\partial_k\tilde{u} + c\tilde{u} = \tilde{f} \qquad B_1$$

Since this is $C^{2,\alpha}$ diffeomorphism, the coefficients remain $C^{0,\alpha}$, and the Existence Theorem on B_1 applies. Thus there exists unique $\tilde{u} \in C^{2,\alpha}(\overline{B_1})$. u is unique and satisfies the same estimates.

Proof of Existence in general $C^{2,\alpha}$ Domain In order to apply Method of Continuity 4.3.3, one need to ensure Δ is invertible in a regular domain.

From Theorem 2.1.1, one know that over Ω bounded regular (all points on boundary one can build a barrier)

$$\begin{cases} \Delta u = f & \Omega \\ u = \varphi & \partial\Omega \end{cases}$$

for $f \in C^{0,\alpha}(\overline{\Omega})$ and $\varphi \in C(\partial\Omega)$, there exists a unique solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

Is this enough to ensure $L_0 = \Delta$ is invertible? NO! Because in the domain of our operator, we need $u \in C^{2,\alpha}(\overline{\Omega})$ globally.

Now with the help of Corollary 4.3.1, we know for our classical solution u, it is in fact $C^{2,\alpha}$ in the interior.

It suffices to show such u is $C^{2,\alpha}$ all the way up to the boundary. How do we do the boundary estimate? ([GT01] Lemma 6.18)

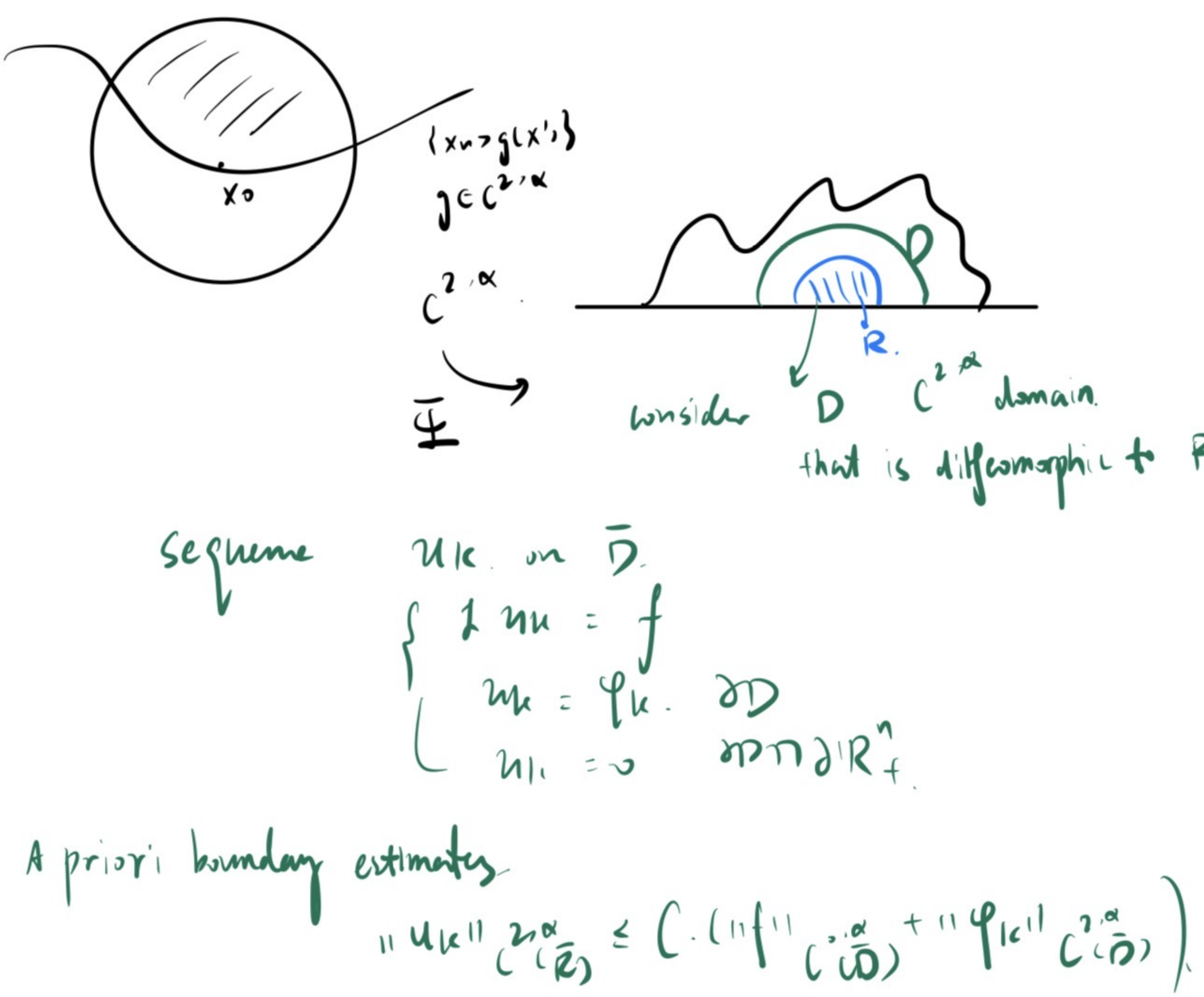


Figure 4.2: $C^{2,\alpha}$ all the way to boundary

Up to covering, let $x_0 \in \partial\Omega$ and take ball $B_r(x_0)$ around. Using $\partial\Omega$ is $C^{2,\alpha}$ domain, flatten out the boundary via Ψ that is $C^{2,\alpha}$. Now for any open region $R \Subset \Psi(B_r(x_0) \cap \overline{\Omega})$ with a piece of boundary portion lying on $\partial\mathbb{R}^n_+$, take any D a $C^{2,\alpha}$ domain that contains R.

Now take φ_k a sequence of $C^{2,\alpha}$ boundary data converging uniformly to u (transformed to define in $\Psi(B_r(x_0) \cap \overline{\Omega})$) on ∂D. Let $u_k \in C^{2,\alpha}(\overline{D})$ be solutions to

$$\begin{cases} \mathcal{L} u_k = f & D \\ u_k = \varphi_k & \partial D \end{cases}$$

which exist uniquely due to 'Existence Theorem for domains diffeomorphic to balls'. By the Maximum Principle

$$\|u_k - u_j\|_{L^\infty(D)} \le \|\varphi_k - \varphi_k\|_{L^\infty(\partial D)}$$

to u_k is Cauchy hence it converges uniformly to some u that is continuous up to the boundary ∂D and coincides with u there.

Recall one has the a priori Boundary Estimates (4.13)

$$\|u_k\|_{C^{2,\alpha}(\overline{R})} \leq C\left(\|u_k\|_{L^\infty(D)} + \|f\|_{C^{0,\alpha}(\overline{D})} + \|\varphi_k\|_{C^{2,\alpha}(\overline{D})}\right)$$

Thus D^2u_k converges uniformly up to subsequences via Ascoli-Arzela, and the limit $u \in C^{2,\alpha}(\overline{R})$, giving the $C^{2,\alpha}$ estimate on the boundary.

4.4 Higher Regularity

Interior Higher Regularity We state the interior Regularity theorem.

Theorem 4.4.1 ([GT01] Theorem 6.17). *Let $u \in C^2(\Omega)$ be solution to*

$$\mathcal{L}u = f \qquad \Omega$$

with $f, a_{ij}, b_i, c \in C^{k,\alpha}(\Omega)$. Then $u \in C^{k+2,\alpha}(\Omega)$.

Assume for simplicity take $b_i = c = 0$.

Base Step: for $k = 1$ we cannot simply differentiate the equation. Therefore we work with difference quotient

$$D_\ell^h v(x) := \frac{v(x + he_\ell) - v(x)}{h}$$

What does $D_\ell^h v$ solve?

First of all we add and subtract the two equations to formulate in the discrete difference setting

$$\begin{aligned}
a_{ij}(x + he_\ell)\partial_{ij}u(x + he_\ell) &= f(x + he_\ell) \\
a_{ij}(x)\partial_{ij}u(x) &= f(x) \\
a_{ij}(x + he_\ell)\partial_{ij}u(x + he_\ell) - a_{ij}(x)\partial_{ij}u(x) &= f(x + he_\ell) - f(x) \\
a_{ij}(x + he_\ell)\left(\partial_{ij}u(x + he_\ell) - \partial_{ij}u(x)\right) + \left(a_{ij}(x + he_\ell) - a_{ij}(x)\right)\partial_{ij}u(x) &= f(x + he_\ell) - f(x) \\
a_{ij}(x + he_\ell)\partial_{ij}\left(D_\ell^h u(x)\right) + D_\ell^h a_{ij}(x)\partial_{ij}u(x) &= D_\ell^h f(x)
\end{aligned}$$

Hence we treat RHS as forcing

$$a_{ij}(x + he_\ell)\partial_{ij}\left(D_\ell^h u(x)\right) = D_\ell^h f(x) - D_\ell^h a_{ij}(x)\partial_{ij}u(x) \tag{4.18}$$

Now applying Interior Schauder Estimate

$$\left\|D_\ell^h u\right\|_{C^{2,\alpha}(\overline{B_{1/2}})} \leq C\left(\underbrace{\left\|D_\ell^h f\right\|_{C^{0,\alpha}}}_{\text{need } C^{1,\alpha} \text{ of foce}} + \underbrace{\left\|D_\ell^h a_{ij}\right\|_{C^{0,\alpha}}}_{\text{need } C^{1,\alpha} \text{ Regularity of } a_{ij}} \|u\|_{C^{2,\alpha}}\right)$$

Since all RHS are bounded in the Hölder norms, this is a uniform bound on $C^{2,\alpha}$ norm on the sequence $D_\ell^h u$. Due to the Hölder exponent there is equi-continuity, thus using Ascoli-Arzela up to subsequence we know that $\{D_\ell^h u\}$ converges locally uniformly to $\partial_\ell u$, and that $\partial_\ell u \in C^{2,\alpha}$. Using $C^{k,\alpha}$ are Banach spaces and $e_\ell \in \mathbb{S}^{n-1}$ arbitrary one get $u \in C^{3,\alpha}$.

Inductive Step: Now we're allowed to differentiate the equation. Assume the data are $C^{2,\alpha}$, and we already know that $u \in C^{3,\alpha}$. Then

$$\begin{aligned}
\partial_\ell(a_{ij}\partial_{ij}u) &= \partial_\ell f \\
a_{ij}\partial_{ij}(\partial_\ell u) &= \partial_\ell f - \partial_\ell a_{ij}\partial_{ij}u
\end{aligned}$$

Since we're assuming all coefficients belong to $C^{2,\alpha}$, here

$$\partial_\ell f \in C^{1,\alpha}, \qquad \partial_\ell a_{ij}\partial_{ij}u \in C^{1,\alpha}$$

Hence $\partial_\ell u$ solves the equation again with $C^{1,\alpha}$ coefficients, meaning $\partial_\ell u \in C^{3,\alpha}$ by our inductive hypothesis. Thus $u \in C^{4,\alpha}$. Now one can keep differentiating the equation.

Global Higher Regularity

Theorem 4.4.2 ([GT01] Theorem 6.19). *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ solve*

$$\begin{cases} \mathcal{L}u = f & \Omega \\ u = \varphi & \partial\Omega \end{cases}$$

where $\varphi \in C^{k+2,\alpha}(\overline{\Omega})$, and $f, a_{ij}, b_i, c \in C^{k,\alpha}(\overline{\Omega})$. Then $u \in C^{k+2,\alpha}(\overline{\Omega})$.

First flatten out the boundary and call the domain G with a boundary portion $T \subseteq \{x_n = 0\}$. Now the equation for differential quotients (4.18) makes sense for any $\ell = 1, \cdots, n-1$ on the portion (for $|h| < h_0$)

$$G' := \{x \in G \mid \text{dist}(x, \partial G \setminus T) > h_0\}$$

which has a hyperplane portion $T' \subseteq G'$.

Boundary a priori estimates apply there, and then one repeat the argument for interior higher regularity base case (assuming all data are $C^{1,\alpha}$) to gain $\partial_\ell u \in C^{2,\alpha}(G' \cup T')$ for any $1 \le \ell \le n-1$.

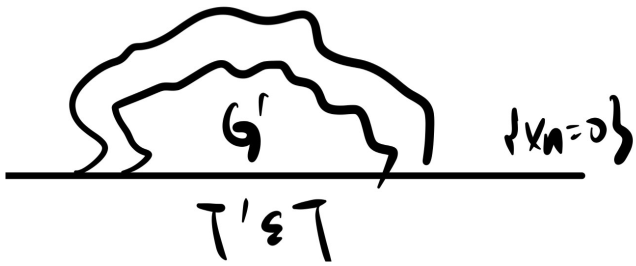


Figure 4.3: Boundary Portion

What about $\partial_n u$? **There one cannot do difference quotient because $x + he_n$ for $h < 0$ is not in the domain!** We use the equation satisfied on the boundary

$$a_{nn}\partial_{nn}u = f - (\mathcal{L}u - a_{nn}\partial_{nn}u))$$

Since we assume $C^{1,\alpha}$ data, and we've shown that all mixed second order derivatives are $C^{1,\alpha}$ up to the boundary potion, all things on RHS are in $C^{1,\alpha}$. Thus $\partial_{nn}u \in C^{1,\alpha}$ meaning $\partial_n u \in C^{2,\alpha}(G' \cup T')$. Thus $u \in C^{3,\alpha}(G' \cup T')$.

To upgrade to higher regularity, each time we differentiate the equation in tangential direction and gain regularity, then use the equation to bound the normal derivative.

4.5 Fredholm Alternative

We ask what happens when one does not have the sign assumption.

Let

$$\mathcal{L}u = a_{ij}\partial_{ij}u + b_i\partial_i u + cu$$

where $a_{ij},\ b_i,\ c \in C^{0,\alpha}(\overline{\Omega})$.

If $c \le 0$, then we know

$$\begin{aligned} \mathcal{L} : C_0^{2,\alpha}(\overline{\Omega}) &\to C^{0,\alpha}(\overline{\Omega}) \\ u &\mapsto \mathcal{L}u \end{aligned}$$

is invertible for Ω $C^{2,\alpha}$ domain, using Existence Theorem 4.3.1. Also, the map

$$T : C^{0,\alpha}(\overline{\Omega}) \overset{\mathcal{L}^{-1}}{\to} C_0^{2,\alpha}(\overline{\Omega}) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$$

is compact, by the usual argument based on Ascoli-Arzela.

Then, if c does not have a sign assumption, we can write

$$\mathcal{L}u = a_{ij}\partial_{ij}u + b_i\partial_i u + (c - \lambda)u + \lambda u$$

where $\lambda = \sup_{\overline{\Omega}} c$. Now if we let

$$\tilde{\mathcal{L}}u := a_{ij}\partial_{ij}u + b_i\partial_i u + (c - \lambda)u$$

such coefficient in front of u has the sign convention. The problem transforms to ask if

$$\mathcal{L}u = \tilde{\mathcal{L}}u + \lambda u = f$$

where $\tilde{\mathcal{L}}$ now has a sign convention, is solvable.

Denote $\tilde{T}$ as the operator corresponding to $\mathcal{L}$. Then we ask whether the following is solvable

$$(I + \lambda \tilde{T})u = \tilde{T}f$$

The main point is that when an operator $\tilde{T}$ is compact, its range is almost finite dimensional in the sense that $\tilde{T}(B_1)$ is included in an ε-neighborhood of a finite dimensional space. It follows that the analysis for the range and kernel of $(\lambda^{-1}\tilde{T} + I$ is similar to the finite dimensional case: it is invertible iff the homogeneous equation has a unique solution $u = 0$.

Using Fredholm Theory Theorem 6.1.2, one can in fact show that the problem admits a unique solution except for a sequence $\lambda_k \to 0$ where λ_k are eigenvalues of $\tilde{\mathcal{L}}$ s.t.

$$\begin{cases} \tilde{\mathcal{L}}u_k + \lambda_k u_k = 0 & \Omega \\ u_k = 0 & \partial\Omega \end{cases}$$

Theorem 4.5.1 ([GT01] Theorem 6.15)**.** *Let*

$$\mathcal{L} = a_{ij}\partial_{ij} + b_i\partial_i + c$$

be strictly elliptic operator with coefficients in $C^{0,\alpha}(\overline{\Omega})$, over $C^{2,\alpha}$ domain.

Then

1. *either the homogeneous problem*
$$\begin{cases} \mathcal{L}u = 0 & \Omega \\ u = 0 & \partial\Omega \end{cases}$$
has only the trivial solution, in which case the inhomogeneous problem
$$\begin{cases} \mathcal{L}u = f & \Omega \\ u = \varphi & \partial\Omega \end{cases}$$
has a unique $C^{2,\alpha}(\overline{\Omega})$ solution for all $f \in C^{0,\alpha}(\overline{\Omega})$, $\varphi \in C^{2,\alpha}(\overline{\Omega})$

2. *or the homogeneous problem*
$$\begin{cases} \mathcal{L}u = 0 & \Omega \\ u = 0 & \partial\Omega \end{cases}$$
has nontrivial solution, which forms a finite dimensional subspace of $C^{2,\alpha}(\overline{\Omega})$.

Proof. First note the homogeneous problem is equivalent to

$$\begin{cases} \mathcal{L}v = f - \mathcal{L}\varphi & \Omega \\ v = 0 & \partial\Omega \end{cases}$$

Thus it suffices to restrict $\mathcal{L}$ to

$$C_0^{2,\alpha}(\overline{\Omega}) := \{u \in C^{2,\alpha}(\overline{\Omega}) \mid u = 0 \ \partial\Omega\}$$

Let $\sigma \geq \sup_{\Omega} c$ be any number. If $\sigma = 0$ then we're done. Otherwise

$$\mathcal{L}_\sigma := \mathcal{L} - \sigma : C_0^{2,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega})$$

is also invertible. Furthermore the inverse mapping $\mathcal{L}_\sigma^{-1}$, by a priori global estimates (4.15), and bounding $\|u\|_{L^\infty}$ by boundary data (3.8), we know $\mathcal{L}_\sigma^{-1}$ is a compact operator from $C^{0,\alpha}(\overline{\Omega})$ into $C^{0,\alpha}(\overline{\Omega})$.

Consider then the equation

$$u + \sigma\mathcal{L}_\sigma^{-1}u = \mathcal{L}_\sigma^{-1}f \qquad f \in C^{0,\alpha}(\overline{\Omega})$$

Using Fredholm Alternative Theorem 6.1.2 for the compact operator $\mathcal{L}_\sigma^{-1}$, we know either

$$u + \sigma\mathcal{L}_\sigma^{-1}u = 0$$

only has trivial solution 0, or $\ker(I + \sigma\mathcal{L}_\sigma^{-1})$ is a finit dimensional subspace of $C^{0,\alpha}(\overline{\Omega})$ (Theorem 6.1.3). ☐

The importance of the alternative for the Dirichlet Problem is that it shows **uniqueness** to be a sufficient condition for existence.

Remark that $C^2(\overline{\Omega})$ solutions of $\mathcal{L}u = f$ are also in $C^{2,\alpha}(\overline{\Omega})$, and hence the nullspace of $\mathcal{L}$ in $C^2(\overline{\Omega})$ is also finite dimensional.

Remark also the set Σ of real values σ for which the homogeneous problem

$$\begin{cases} \mathcal{L}u - \sigma u = 0 & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

has nontrivial solution is at most countable and discrete (Theorem 6.1.3).

Furthermore, if $\sigma \notin \Sigma$, any solution of the Dirichlet Problem

$$\begin{cases} \mathcal{L}_\sigma u = f & \Omega \\ u = \varphi & \partial\Omega \end{cases}$$

satisfies the estimate

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \le C\left(\|\varphi\|_{C^{2,\alpha}(\overline{\Omega})} + \|f\|_{C^{0,\alpha}(\overline{\Omega})}\right)$$

In general we can guarantee existence to an equation $\mathcal{L}u = f$ as long as we can bound $\|u\|_{L^\infty}$.

Chapter 5

Function Spaces

5.1 Sobolev Spaces

5.1.1 Basics for Sobolev Space

Weak Derivative Let $u, v \in L^1_{\text{loc}}(\Omega)$. We say v is a weak derivative of u if

$$\int_\Omega u\partial_i\varphi = -\int_\Omega \varphi v \qquad \forall\ \varphi \in C_0^\infty(\Omega)$$

We denote $v = \partial_i u$.

Lemma 5.1.1. *One has uniqueness of weak derivative.*

Proof. If both $v_1, v_2 \in L^1_{\text{loc}}(U)$ satisfies

$$\int_U u\, D^\alpha \phi\, dx = (-1)^{|\alpha|} \int_U v_1\, \phi\, dx = (-1)^{|\alpha|} \int_U v_2\, \phi\, dx \qquad \forall\ \phi \in C_0^\infty(U)$$

Then $\int_U (v_1 - v_2)\, \phi\, dx = 0$ for any test function ϕ. Hence $v_1 = v_2$ a.e. □

Lemma 5.1.2. *Let $\partial_i u$ be weak derivative of u. Let η_ε be mollifier, then*

$$\partial_i(u * \eta_\varepsilon) = (\partial_i u) * \eta_\varepsilon$$

Proof. Note

$$\partial_i(u * \eta_\varepsilon) = u * (\partial_i \eta_\varepsilon)$$

For any $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega u\partial_i\phi = (-1)\int_\Omega \partial_i u(y)\phi(y)$$

Apply $\phi(y) = \eta_\varepsilon(x-y)$ for fixed $x \in \Omega_\varepsilon$ so

$$\begin{aligned}(\partial_i u) * \eta_\varepsilon(x) &= \int_\Omega \partial_i u(y)\eta_\varepsilon(x-y) \\ &= -\int_\Omega u\partial_{y_i}(\eta_\varepsilon(x-y)) = \int_\Omega u(y)\partial_i\eta_\varepsilon(x-y) \\ &= (u * \partial_i\eta_\varepsilon)(x)\end{aligned}$$

□

Lemma 5.1.3 (Stability under weak convergence)**.** *Let u_k be sequence s.t.*

$$u_k \rightharpoonup u, \qquad \partial_i u_k \rightharpoonup v$$

Then $\partial_i u = v$.

Proof. For any $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega u_k\partial_i\varphi = -\int_\Omega \partial_i u_k \varphi$$

Take limit on both sides, then use uniqueness of weak derivatives. □

Sobolev Space Let $1 \leq p \leq \infty$. Define

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) \mid D^\alpha u \text{ weak derivatives exist up to order } |\alpha| \leq k \text{ and are } L^p\}$$

Define

$$\|u\|_{W^{k,p}(\Omega)} := \|u\|_{L^p} + \sum_{i=1}^n \|\partial_i u\|_{L^p} + \cdots + \sum_{|\alpha|=k} \|D^\alpha u\|_{L^p}$$

$W^{k,p}(\Omega)$ are Banach space under norm convergence.

$W_0^{k,p}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ under the norm $W^{k,p}(\Omega)$.

If $p = 2$, denote $H^k = W^{k,2}$. $H^1(\Omega)$ are Hilbert space

$$(u, v) := \int_\Omega uv + \int_\Omega \nabla u \cdot \nabla v$$

Smooth Approximations Let $1 \leq p < \infty$. Take $u \in W^{k,p}(\Omega)$.

1. Mollification $u * \eta_\varepsilon$ converges in $W^{k,p}_{\text{loc}}$ to u.
2. For $\Omega \subseteq \mathbb{R}^n$ bounded open, there exists $u_m \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$ s.t. $u_m \to u$ in $W^{k,p}(\Omega)$ ([Eva10] Theorem 5.3.2).

   Proof. Let $\Omega = \bigcup_{i=1}^\infty \Omega_i$ for $\Omega_i = \{\text{dist}(x, \partial\Omega) > \frac{1}{i}\}$. Choose smooth partition of unity w.r.t. open sets $V_i = \Omega_{i+3} \setminus \Omega_{i+1}$, $V_0 = \Omega_3$

   $$0 \leq \xi_i \leq 1, \qquad \xi_i \in C_0^\infty(V_i), \qquad \sum_{i=0}^\infty \xi_i = 1 \qquad \Omega$$

   so that

   $$u = \sum_{i=0}^\infty \xi_i u$$

   For $u \in W^{k,p}$, $\xi_i u$ is supported on V_i. For $\delta > 0$, mollify for ε_i small so

   $$\|\eta_{\varepsilon_i} * (\xi_i u) - \xi_i u\|_{W^{k,p}(\Omega)} \leq \frac{1}{2^{i+1}} \delta \qquad \forall\, i \geq 0$$
   $$\text{supp}(\eta_{\varepsilon_i} * (\xi_i u)) \subseteq \Omega_{i+4} \setminus \overline{\Omega}_i \qquad \forall\, i \geq 1$$

   Now for any $\Omega' \Subset \Omega$

   $$\left\| \sum_{i=0}^\infty \eta_{\varepsilon_i} * (\xi_i u) - u \right\|_{W^{k,p}(\Omega')} \leq \sum_{i=0}^\infty \|\eta_{\varepsilon_i} * (\xi_i u) - \xi_i u\|_{W^{k,p}(\Omega)} \leq \delta$$

   Take sup in Ω'. □

3. For $\Omega \subseteq \mathbb{R}^n$ bounded with Lipschitz boundary, there exists $u_m \in C^\infty(\overline{\Omega})$ s.t. $u_m \to u$ in $W^{k,p}(\Omega)$.

Extension Let $\Omega \subseteq \mathbb{R}^n$ be bounded. If $\partial\Omega$ is nice, we can extend functions in $W^{1,p}(\Omega)$ to functions in $W^{1,p}(\mathbb{R}^n)$ that preserve the weak derivatives across $\partial\Omega$. Let $1 \leq p \leq \infty$.

Theorem 5.1.1 ([Eva10] Theorem 5.4.1)**.** *If $\partial\Omega$ is Lipschitz, for any $\tilde{\Omega}$ s.t. $\Omega \Subset \tilde{\Omega}$, one may define Extension Operator as a bounded linear operator*

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$$
$$u \mapsto Eu$$

s.t. $Eu = u$ a.e. in Ω, $\text{supp}(Eu) \subseteq \tilde{\Omega}$, and there exists $C = C(p, \Omega, \tilde{\Omega}) > 0$

$$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\Omega)} \tag{5.1}$$

Proof. After bi-Lipschitz transformation consider $\Omega \subseteq B_2^+$, $\partial\Omega = B_1 \cap \{x_n = 0\}$ flat. We reflect u evenly in the x_n variable via

$$\overline{u}(x) := \begin{cases} u(x) & B_1^+ \\ -3u(x', -x_n) + 4u(x', -\frac{1}{2}x_n) & B_1^- \end{cases}$$

so that in B_1^-

$$\frac{\partial}{\partial x_n}\overline{u} = 3\partial_{x_n} u(x', -x_n) - 2\partial_{x_n} u(x', -\frac{1}{2}x_n), \qquad \frac{\partial}{\partial x_n}\overline{u}\bigg|_{\{x_n=0\}} = \partial_{x_n} u(x', 0)$$

and all tangential derivatives match. Go back to original domain, cover boundary with finitely many balls, then patch up using partition of unity. □

Trace One may assign boundary values to $u \in W^{1,p}(\Omega)$. Let $1 \le p < \infty$.

Theorem 5.1.2 ([Eva10] Theorem 5.5.1). *If $\partial\Omega$ is Lipschitz, one may define Trace Operator as bounded linear operator*

$$\begin{aligned} T : W^{1,p}(\Omega) &\to L^p(\partial\Omega) \\ u &\mapsto Tu \end{aligned}$$

s.t. $Tu = u|_{\partial\Omega}$ for $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, and there exists $C = C(p, \Omega) > 0$ s.t.

$$\|Tu\|_{L^p(\partial\Omega)} \le C \|u\|_{W^{1,p}(\Omega)} \tag{5.2}$$

Proof. After bi-Lipschitz deformation assume that locally Ω is $B_1' \times [0,1]$. First take $u \in C^1(\overline{\Omega})$. We want to have the estimate. In this case define $Tu := u|_{\partial\Omega}$. Look at ball of boundary with radius $r > 0$. For any $\xi \in C_0^\infty(B_r)$ s.t. $\xi \ge 0$ and $\xi = 1$ on $B_{r/2}$ look at

$$\begin{aligned} \int_{\{x_n=0\}\cap B_{r/2}} |u|^p dx' &\le \int_{\{x_n=0\}\cap B_r} \xi |u|^p \, dx' \qquad \text{let's do integration by parts the other way around} \\ &= -\int_{B_r^+} (\xi |u|^p)_{x_n} \, dx \qquad \text{here } \xi \text{ is useful due to compactly supported} \\ &= -\int_{B_r^+} (|u|^p \xi_{x_n} + p|u|^{p-1}(\text{sign} u) u_{x_n} \xi) \, dx \\ &\le C \int_{B_r^+} (|u|^p + |\nabla u|^p) \, dx \qquad \text{Young's Inequality} \end{aligned}$$

Now by density and flattening out the boundary one can conclude. □

Theorem 5.1.3 ([Eva10] Theorem 5.5.2). *If $\partial\Omega$ Lipschitz, $u \in W_0^{1,p}(\Omega)$ iff $Tu = 0$ on $\partial\Omega$.*

5.1.2 Gagliardo-Nirenberg-Sobolev

We demonstrate the Sobolev Inequality.

Theorem 5.1.4 ([Eva10] Theorem 5.6.1). *Let $1 \le p < n$. Let $u \in W_0^{1,p}(\mathbb{R}^n)$. Then there exists $C = C(n,p)$ s.t.*

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\right)^{\frac{1}{p^*}} \le C \left(\int_{\mathbb{R}^n} |\nabla u|^p\right)^{\frac{1}{p}}, \qquad p^* = \frac{np}{n-p} \tag{5.3}$$

5.1.2.1 Direct Proof

Scaling A simple scaling argument demonstrates the choice of p^*. Take $u \in C_0^\infty(\mathbb{R}^n)$. We want to inequality to be dilation-invariant, in particular, for

$$u_\lambda(x) = u(\lambda x)$$

we want the same form to take place. Assume for some q.

$$\begin{aligned} \int_{\mathbb{R}^n} |u_\lambda(x)|^q dx &= \int_{\mathbb{R}^n} |u(\lambda x)|^q dx = \lambda^{-n} \int_{\mathbb{R}^n} |u(y)|^q dy \\ \|u_\lambda\|_{L^q(\mathbb{R}^n)} &\le \lambda^{-\frac{n}{q}} \|u\|_{L^q(\mathbb{R}^n)} \\ &\underbrace{\le C\lambda^{-\frac{n}{q}} \|\nabla u\|_{L^p}}_{\text{we want to ensure this}} \\ &= C\lambda^{-\frac{n}{q}} \left(\int_{\mathbb{R}^n} |\nabla u|^p\right)^{\frac{1}{p}} \end{aligned}$$

But for the gradient, we scale as

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla(u_\lambda(x))|^p dx &= \int_{\mathbb{R}^n} |\nabla(u(\lambda x))|^p dx = \lambda^p \lambda^{-n} \int_{\mathbb{R}^n} |\nabla u(y)|^p dy \\ \left(\int_{\mathbb{R}^n} |\nabla u(y)|^p dy\right)^{\frac{1}{p}} &= \lambda^{\frac{n}{p}-1} \left(\int_{\mathbb{R}^n} |\nabla u_\lambda(x)|^p dx\right)^{\frac{1}{p}} \end{aligned}$$

Therefore one need

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \le C\lambda^{\frac{n}{p}-\frac{n}{q}-1} \|\nabla u_\lambda\|_{L^p}$$

Since we would like to have the inequality that is invariant under dilation, the natural choice is

$$\frac{n}{p} - \frac{n}{q} - 1 = 0$$

We solve for q

$$\begin{aligned}\frac{n}{q} &= \frac{n}{p} - 1 = \frac{n-p}{p}\\ \frac{1}{q} &= \frac{n-p}{np}\\ q &= \frac{np}{n-p}\end{aligned}$$

We thus define the Sobolev Conjugate as

$$p^* = \frac{np}{n-p} \qquad \forall\ 1 \leq p < n$$

Direct Proof of Sobolev Inequality We first do the case for $p = 1$.

Proof of (5.3) *for* $p = 1$. Up to smooth approximation assume $u \in C_0^\infty(\mathbb{R}^n)$. Since we're working with compact support, for any $x \in \mathbb{R}^n$ fixed, for any $i = 1, \cdots, n$

$$\begin{aligned}u(x) &\leq \int_{-\infty}^{x_i} \partial_i u(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n) dy_i\\ |u(x)| &\leq \underbrace{\int_{-\infty}^{\infty} |\nabla u(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)|\, dy_i}_{\text{note we only integrated out } x_i\text{, this is still a function in other variables}}\\ |u(x)|^{\frac{1}{n-1}} &\leq \left(\int_{-\infty}^{\infty} |\nabla u(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)|\, dy_i\right)^{\frac{1}{n-1}} \qquad \forall\ i = 1, \cdots, n\end{aligned}$$

Now we 'raise to the power n' by multiplying together all of them, but each integrated in a difference y_i.

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |\nabla u(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)|\, dy_i\right)^{\frac{1}{n-1}}$$

Notice both sides are functions in the variable x. For each coordinate variable x_i, the RHS as product of n functions has only $n-1$ of them including x_i as a variable.

Hence if we integrate both sides in x_i variable, the one function on the RHS whose y_i get integrated out can be viewed as 'constant' in x_i, then pulls out the product directly. In particular **let's first integrate in** x_1

$$\begin{aligned}&\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1\\ &\leq \left(\int_{-\infty}^{\infty} |\nabla u(y_1, x_2, \cdots, x_n)| dy_1\right)^{\frac{1}{n-1}} \cdot \underbrace{\int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |\nabla u(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)|\, dy_i\right)^{\frac{1}{n-1}} dx_1}_{\text{But how do we deal with this bunch?}}\end{aligned}$$

Again, since the RHS is integration in x_1 only, all other variables are treated as constants. Thus the RHS is a product of $n-1$ functions in x_1. How do one usually bound the L^1 integral of a product of $n-1$ functions? One may use the generalized Hölder inequality with exponents

$$\underbrace{\frac{1}{n-1} + \cdots + \frac{1}{n-1}}_{n-1 \text{ many of them}} = 1$$

so that we raise to the power $n-1$ for each of them

$$\begin{aligned}&\int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |\nabla u(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)|\, dy_i\right)^{\frac{1}{n-1}} dx_1\\ &\leq \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla u(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)| dy_i dx_1\right)^{\frac{1}{n-1}}\end{aligned}$$

In particular, we swapped the order of product and integration in x_1. Now we collect all of them

$$\begin{aligned}&\int_{-\infty}^{\infty}|u(x)|^{\frac{n}{n-1}}dx_1\\&\le\left(\int_{-\infty}^{\infty}|\nabla u(y_1,x_2,\cdots,x_n)|dy_1\right)^{\frac{1}{n-1}}\cdot\prod_{i=2}^{n}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\nabla u(x_1,\cdots,x_{i-1},y_i,x_{i+1},\cdots,x_n)|dy_idx_1\right)^{\frac{1}{n-1}}\end{aligned}$$

Now we integrate in x_2, and follow the same logic of pulling out the one on RHS whose y_2 got integrated out

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u(x)|^{\frac{n}{n-1}}dx_1dx_2\\&\le\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\nabla u(x_1,y_2,\cdots,x_n)|dy_2dx_1\right)^{\frac{1}{n-1}}\\&\cdot\underbrace{\int_{-\infty}^{\infty}\underbrace{\left(\int_{-\infty}^{\infty}|\nabla u(y_1,x_2,\cdots,x_n)|dy_1\right)^{\frac{1}{n-1}}}_{\text{note this still has variable in }x_2\text{ so we put it under }dx_2}\cdot\prod_{i=3}^{n}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\nabla u(x_1,\cdots,x_{i-1},y_i,x_{i+1},\cdots,x_n)|dy_idx_1\right)^{\frac{1}{n-1}}dx_2}_{\text{But for this huge bunch we can again apply generalized Hölder as }L^1\text{ integral of product of }n-1\text{ functions}}\\&\le\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\nabla u(x_1,y_2,\cdots,x_n)|dy_2dx_1\right)^{\frac{1}{n-1}}\\&\cdot\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\nabla u(y_1,x_2,\cdots,x_n)|dy_1dx_2\right)^{\frac{1}{n-1}}\cdot\prod_{i=3}^{n}\left(\iiint_{-\infty}^{\infty}|\nabla u(x_1,x_2,\cdots,y_i,\cdots,x_n)|dx_1dx_2dy_i\right)^{\frac{1}{n-1}}\\&=\underbrace{\left(\iint_{-\infty}^{\infty}|\nabla u(y_1,y_2,x_3,\cdots,x_n)|dy_1dy_2\right)^{\frac{2}{n-1}}}_{\text{we combine the first two integrals}}\cdot\prod_{i=3}^{n}\left(\iiint_{-\infty}^{\infty}|\nabla u(x_1,x_2,\cdots,y_i,\cdots,x_n)|dx_1dx_2dy_i\right)^{\frac{1}{n-1}}\end{aligned}$$

We integrate inductively up until x_{n-1} to obtain (we denote $x'\in\mathbb{R}^{n-1}$)

$$\int_{\mathbb{R}^{n-1}}|u(x)|^{\frac{n}{n-1}}dx'\le\int_{\mathbb{R}^{n-1}}|\nabla u(y',x_n)|dy'\cdot\left(\int_{\mathbb{R}^n}|\nabla u(x)|dx\right)^{\frac{1}{n-1}}$$

Finally we integrate in x_n to kill the game

$$\begin{aligned}\int_{\mathbb{R}^n}|u|^{\frac{n}{n-1}}dx&\le\left(\int_{\mathbb{R}^n}|\nabla u(x)|dx\right)^{\frac{n}{n-1}}\\\left(\int_{\mathbb{R}^n}|u|^{\frac{n}{n-1}}dx\right)^{\frac{n-1}{n}}&\le\int_{\mathbb{R}^n}|\nabla u(x)|dx\end{aligned}\tag{5.4}$$

Notice in particular, for $p=1$, the constant in front is 1. □

Next we do the case for general $1\le p<n$.

Proof of (5.3) *for* $1<p<n$. We work with $|u|^\gamma$ for $\gamma>1$ to be chosen. Apply our previous result to (5.4) to get

$$\begin{aligned}\left(\int_{\mathbb{R}^n}|u|^{\frac{\gamma n}{n-1}}\right)^{\frac{n-1}{n}}&\le\int_{\mathbb{R}^n}|\nabla(|u(x)|^\gamma)|dx\\&=\gamma\int_{\mathbb{R}^n}|u|^{\gamma-1}|\nabla u|dx\end{aligned}$$

Recall we want the RHS with ∇u in L^p. To do so we use Hölder

$$\int_{\mathbb{R}^n}|u|^{\gamma-1}|\nabla u|dx\le\left(\int_{\mathbb{R}^n}|u|^{(\gamma-1)p'}\right)^{\frac{1}{p'}}\cdot\left(\int_{\mathbb{R}^n}|\nabla u|^p\right)^{\frac{1}{p}}$$

where $p'=\frac{p}{p-1}$. Ok, regardless of the power for the integral, at least we want the power of the integrands to match. To do so we require

$$\frac{\gamma n}{n-1}=(\gamma-1)p'=\frac{(\gamma-1)p}{p-1}$$

and we want to solve for γ

$$\begin{aligned} 1-\frac{1}{\gamma} &= \frac{n}{n-1}\frac{p-1}{p} = \frac{np-n}{np-p} \\ \frac{1}{\gamma} &= 1-\frac{np-n}{np-p} = \frac{n-p}{np-p} \\ \gamma &= \frac{n-1}{n-p}p \end{aligned}$$

And thus we find out the exponent to be precisely the Sobolve conjugate

$$\frac{\gamma n}{n-1} = \frac{n-1}{n-p}p \cdot \frac{n}{n-1} = \frac{np}{n-p} = p^*$$

It suffices to verify the powers of the integral

$$\begin{aligned} \frac{n-1}{n} - \frac{1}{p'} &= \frac{n-1}{n} - \frac{p-1}{p} \\ &= \frac{np-p-np+n}{np} = \frac{n-p}{np} = \frac{1}{p^*} \end{aligned}$$

Thus we have estimate

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}}\right)^{\frac{n-p}{np}} \le \frac{n-1}{n-p}p \cdot \left(\int_{\mathbb{R}^n} |\nabla u|^p\right)^{\frac{1}{p}} \tag{5.5}$$

Notice in particular, the constant in front blows up as $n \to p$. □

5.1.2.2 Isoperimetric Inequality Interpretation

An interesting inequality and useful inequality.

Lemma 5.1.4. *For $a(t)$ non-negative decreasing, one has for any $n > 1$*

$$\int_0^\infty a(t)^{\frac{n-1}{n}} dt \ge \left(\int_0^\infty a(t)t^{\frac{1}{n-1}}\right)^{\frac{n-1}{n}} \tag{5.6}$$

Proof. Since $a(t)$ is non-negative decreasing, for any $t \in (0, \infty)$

$$\begin{aligned} \int_0^\infty a(s)^{\frac{n-1}{n}} ds &\ge \int_0^t a(s)^{\frac{n-1}{n}} ds \qquad \text{use non-negative} \\ &\ge t \cdot a(t)^{\frac{n-1}{n}} \qquad \text{use decreasing} \end{aligned}$$

Therefore

$$\begin{aligned} \left(\int_0^\infty a(s)^{\frac{n-1}{n}} ds\right)^{\frac{1}{n-1}} &\ge t^{\frac{1}{n-1}} a(t)^{\frac{1}{n}} \qquad \text{raising to } \frac{1}{n-1} \text{ power} \\ a(t)^{\frac{n-1}{n}} \left(\int_0^\infty a(s)^{\frac{n-1}{n}} ds\right)^{\frac{1}{n-1}} &\ge t^{\frac{1}{n-1}} a(t) \qquad \text{multiplying both sides by } a(t)^{\frac{n-1}{n}} \\ \left(\int_0^\infty a(s)^{\frac{n-1}{n}} ds\right)^{\frac{n}{n-1}} &\ge \int_0^\infty t^{\frac{1}{n-1}} a(t) \qquad \text{integrate in } t \end{aligned}$$

□

Isoperimetric Inequality We demonstrate that for $p = 1$, the Sobolev Inequality is in fact a corollary of the isoperimetric inequality, that works for BV functions (in particular $W^{1,1} \subsetneq \text{BV}$).

Theorem 5.1.5 (Isoperimetric Inequality)**.** *There exists $C(n) > 0$ s.t. for any $E \subseteq \mathbb{R}^n$ bounded Caccioppoli*

$$|E|^{\frac{n-1}{n}} \le C \int_{\mathbb{R}^n} |\nabla \chi_E| \tag{5.7}$$

Loosely speaking, if one view $u = \chi_E$, then this recovers the Sobolev inequality for $p = 1$

$$|E|^{\frac{n-1}{n}} = \left(\int_{\mathbb{R}^n} \chi_E\right)^{\frac{n-1}{n}} \le C \int_{\mathbb{R}^n} |\nabla \chi_E|$$

Ok but this is not rigorous at all. Let's in fact assume for (5.7) and show the Sobolev Inequality for $p = 1$.

Proof of (5.3) *for* $p = 1$ *using Isoperimetric Inequality (Savin Analysis II 2026).* Upon smooth approximation, let $u \in C_0^1(\mathbb{R}^n)$. Then using Sard's Lemma, we know for a.e. $t \in \mathbb{R}$, the level set

$$\{u > t\}$$

has C^1 boundary. In particular the isoperimetric inequality (5.7) indeed holds for these level sets. Also, for E with C^1 boundary, from the theory of sets of finite perimeter, we know that

$$\mathcal{H}^{n-1}(\partial E) = \int_{\mathbb{R}^n} |\nabla \chi_E| = P(E, \mathbb{R}^n)$$

WLOG assume $u \geq 0$. We use the Coarea formula

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla u| &= \int_0^\infty \mathcal{H}^{n-1}(\partial\{u > t\}) dt \\
&\overset{(5.7)}{\geq} c \int_0^\infty |\{u > t\}|^{\frac{n-1}{n}} dt \\
&\overset{(5.6)}{\geq} c \left(\int_0^\infty |\{u > t\}| t^{\frac{1}{n-1}} \right)^{\frac{n-1}{n}}
\end{aligned}$$

where we used the fact that $a(t) = |\{u > t\}|$ is non-negative and decreasing. Now notice in particular using theory of distribution function

$$\frac{n}{n-1} \int_0^\infty |\{u > t\}| t^{\frac{1}{n-1}} dt = \int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx$$

Thus we recover (5.3) for $p = 1$. □

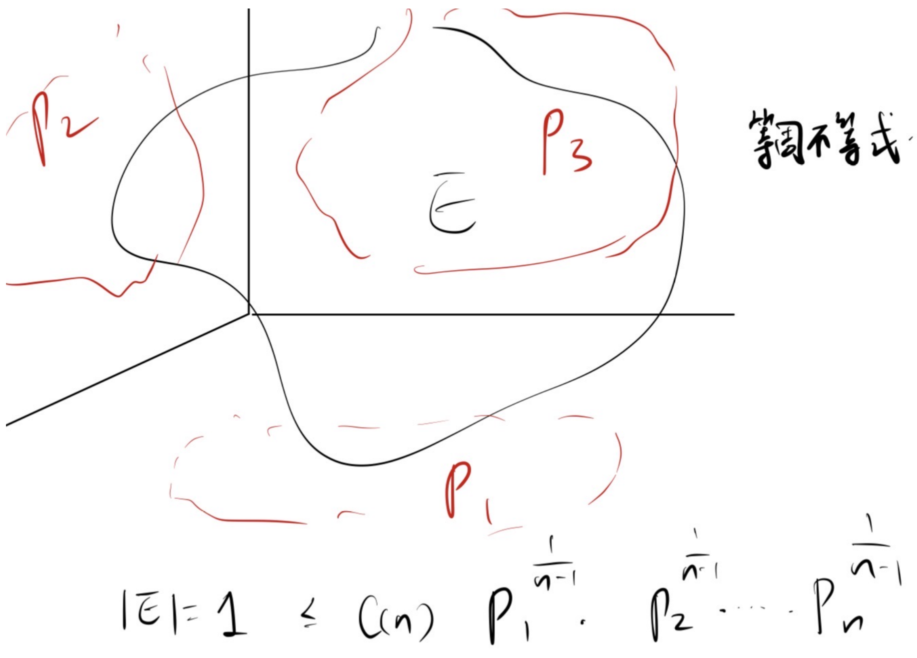


Figure 5.1: Isoperimetric Inequality

We give a quick sketch of a possible sloppy proof for the isoperimetric inequality.

Idea of (5.7). We claim that for f_i, $i = 1, \cdots, n$ functions with each f_i independent of the x_i variable, then

$$\int_{\mathbb{R}^n} \prod_{i=1}^n f_i dx \leq \prod_{i=1}^n \|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})}$$

Notice this is essentially the key step in the direct proof.

Now apply this to $f_i = P_i$ as projections of χ_E along the x_i variable to obtain

$$|E| = \int_{\mathbb{R}^n} \chi_{\mathcal{E}} = \int_{\mathbb{R}^n} \prod_{i=1}^n P_i \leq \prod_{i=1}^n \left(\int_{\mathbb{R}^{n-1}} P_i \right)^{\frac{1}{n-1}}$$
$$\leq \prod_{i=1}^n (\mathcal{H}^{n-1}(\partial E))^{\frac{1}{n-1}} = \mathcal{H}^{n-1}(\partial E)^{\frac{n}{n-1}}$$

□

5.1.3 Morrey's Inequality

We compute for $\gamma > 0$

$$u(x) = |x|^\gamma$$

so

$$|\nabla u| \sim |x|^{\gamma-1}$$
$$|\nabla u|^p \sim |x|^{(\gamma-1)p}$$

If we want this to be integrable, one need

$$\int_{B_1} |x|^{(\gamma-1)p} dx = C \int_0^1 r^{(\gamma-1)p} r^{n-1} dr < \infty$$

Therefore

$$(\gamma - 1)p + n - 1 > -1$$
$$\gamma > \frac{p-n}{p} = 1 - \frac{n}{p}$$

Hence for $u \in W^{1,p}$, necessarily one needs $\gamma > 1 - \frac{n}{p}$.

Oscillation Control in the Integral Sense

Lemma 5.1.5. *Let $1 \leq p < \infty$. There exists $C = C(n) > 0$ s.t. for any $u \in W^{1,p}(\mathbb{R}^n)$*

$$⨍_{B_r(x)} |u(y) - u(x)| dy \leq C \int_{B_r(x)} \frac{|\nabla u(y)|}{|x-y|^{n-1}} dy \qquad \forall\ B_r(x) \subseteq \mathbb{R}^n \tag{5.8}$$

Proof. Upon smooth approximation take $u \in C^1(\mathbb{R}^n)$. Take arbitrary $\omega \in \mathbb{S}^{n-1}$, and any $0 < s < r$. We compute

$$u(x + s\omega) - u(x) = \int_0^s \frac{d}{dt}(u(x + t\omega)) dt = \int_0^s \nabla u(x + t\omega) \cdot \omega dt$$
$$|u(x + s\omega) - u(x)| \leq \int_0^s |\nabla u(x + t\omega)| dt$$

The question is how one would like to integrate. **First we integrate in** $\omega \in \mathbb{S}^{n-1}$

$$\int_{\mathbb{S}^{n-1}} |u(x + s\omega) - u(x)|\, d\mathcal{H}^{n-1} \leq \int_0^s \underbrace{\int_{\mathbb{S}^{n-1}} |\nabla u(x + t\omega)|\, d\mathcal{H}^{n-1}}_{\text{we change of variables back}} dt$$
$$= \int_0^s \int_{\partial B_t(x)} |\nabla u(y)| d\mathcal{H}^{n-1} \cdot t^{-(n-1)} dt = \int_0^s \int_{\partial B_t(x)} \frac{|\nabla u(y)|}{|x-y|^{n-1}} d\mathcal{H}^{n-1} dt$$
$$= \int_{B_s(x)} \frac{|\nabla u(y)|}{|x-y|^{n-1}} dy \qquad \text{using Coarea Formula}$$
$$\leq \underbrace{\int_{B_r(x)} \frac{|\nabla u(y)|}{|x-y|^{n-1}} dy}_{\text{now this is independent of } s}$$

On the other hand, we can change of variables on the LHS.

$$\int_{\mathbb{S}^{n-1}} |u(x + s\omega) - u(x)|\, d\mathcal{H}^{n-1} = \frac{1}{s^{n-1}} \int_{\partial B_s(x)} |u(y) - u(x)|\, d\mathcal{H}^{n-1}(y)$$

The fun step is to now plug in above.

$$\frac{1}{s^{n-1}}\int_{\partial B_s(x)}|u(y)-u(x)|\,d\mathcal{H}^{n-1}(y)\le\int_{B_r(x)}\frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy$$
$$\int_{\partial B_s(x)}|u(y)-u(x)|\,d\mathcal{H}^{n-1}\le s^{n-1}\int_{B_r(x)}\frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy$$

Now we integrate both sides in s from 0 to r.

$$\begin{aligned}\int_{B_r(x)}|u(y)-u(x)|dy&=\int_0^r\int_{\partial B_s(x)}|u(y)-u(x)|\,d\mathcal{H}^{n-1}ds\\&\le\int_0^r s^{n-1}ds\underbrace{\int_{B_r(x)}\frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy}_{\text{this is independent of } s}\\&=\frac{r^n}{n}\int_{B_r(x)}\frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy\\ \fint_{B_r(x)}|u(y)-u(x)|dy&\le\frac{1}{n\omega_n}\int_{B_r(x)}\frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy\end{aligned}$$

□

Morrey's Inequality

Theorem 5.1.6 ([Eva10] Theorem 5.6.4)**.** *Let $n<p<\infty$. Let $u\in W^{1,p}(\mathbb{R}^n)$. Then there exists $C=C(n,p)>0$ s.t.*

$$\|u\|_{C^{0,\alpha}(\mathbb{R}^n)}\le C\,\|u\|_{W^{1,p}(\mathbb{R}^n)}\qquad\alpha=1-\frac{n}{p}>0\tag{5.9}$$

Proof. Upon smooth approximation assume $u\in C^1(\mathbb{R}^n)$.

We first bound the sup norm. For any $x\in\mathbb{R}^n$

$$\begin{aligned}|u(x)|&\le\fint_{B_1(x)}|u(x)-u(y)|dy+\fint_{B_1(x)}|u(y)|dy\\&\overset{(5.8)}{\le}C\fint_{B_1(x)}\frac{|\nabla u(y)|}{|x-y|^{n-1}}dy+\|u\|_{L^p}\\&\le C\,\|\nabla u\|_{L^p}\cdot\left(\int_{B_1(x)}\frac{1}{|x-y|^{(n-1)\frac{p}{p-1}}}dy\right)^{\frac{p-1}{p}}+\|u\|_{L^p}\end{aligned}$$

We need to use our assumption $p>n$ to conclude that

$$\begin{aligned}\int_{B_1(x)}\frac{1}{|x-y|^{(n-1)\frac{p}{p-1}}}dy&=C(n)\int_0^1 r^{n-1}r^{-\frac{p}{p-1}(n-1)}dr\\&=C(n)\int_0^1 r^{-\frac{n-1}{p-1}}dr\le C(n,p)\qquad\text{using the sharp } p>n\end{aligned}$$

Next we bound the Hölder seminorm. For any $x,\,y\in\mathbb{R}^n$, take $r=|x-y|$. We work in the intersection

$$W=B_r(x)\cap B_r(y)$$

Now

$$\begin{aligned}|u(x)-u(y)|&\le\fint_W|u(x)-u(z)|dz+\fint_W|u(y)-u(z)|dz\\&\le C\left(\fint_{B_r(x)}|u(x)-u|dz+\fint_{B_r(y)}|u(y)-u|dz\right)\\&\overset{(5.8)}{\le}C\left(\int_{B_r(x)}\frac{|\nabla u(z)|}{|x-z|^{n-1}}dz+\int_{B_r(y)}\frac{|\nabla u(z)|}{|y-z|^{n-1}}dz\right)\end{aligned}$$

It suffices to look at one of the RHS. In fact the reason we require $p > n$ is exactly the same as above. It's just now in this case we need to compute explicitly the dependence on r

$$\begin{aligned}\int_{B_r(x)} \frac{|\nabla u(z)|}{|x-z|^{n-1}} dz &\leq C \|\nabla u\|_{L^p} \left(\int_0^r s^{n-1} \cdot s^{-(n-1)\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\ &\leq C \|\nabla u\|_{L^p} \cdot \left(\int_0^r s^{-\frac{n-1}{p-1}} \right)^{\frac{p-1}{p}}\end{aligned}$$

We just need to compute

$$\left(\int_0^r s^{-\frac{n-1}{p-1}} \right)^{\frac{p-1}{p}} = C(n,p) r^{\frac{p-n}{p-1} \cdot \frac{p-1}{p}} = C(n,p) \cdot r^{1-\frac{n}{p}}$$

Now recall $r = |x - y|$. This concludes the proof. □

5.1.4 Poincaré Inequality

5.1.4.1 Poincaré

Poincaré for $p = 1$ We first do a direct proof of Poincaré for $p = 1$.

Theorem 5.1.7 (Savin Analysis II 2026)**.** *Let $u \in W^{1,1}(B_1)$. Then for $C = C(n) > 0$*

$$\int_{B_1 \times B_1} |u(x) - u(y)| dx dy \leq C \int_{B_1} |\nabla u| dx \tag{5.10}$$

In particular

$$\int_{B_1} |u - ⨍_{B_1} u| dx \leq C \int_{B_1} |\nabla u| dx \tag{5.11}$$

Proof. In the first step we essentially repeat (5.8). Take any $x \in B_1$ and write $y = x + s\omega$ with $\omega \in \mathbb{S}^{n-1}$. For each direction $\omega \in \mathbb{S}^{n-1}$, we consider the radius

$$r_\omega = \sup\{s > 0 \mid x + s\omega \in B_1\}$$

which is the largest radius along the direction ω s.t. $x + s\omega$ hits ∂B_1.

Our first target is to bound

$$\int_{B_1} |u(y) - u(x)| dy$$

To do so, look at

$$|u(x + s\omega) - u(x)| \leq \int_0^s |\nabla u(x + t\omega)| dt \tag{5.12}$$

We multiply both sizes by s^{n-1} and integrate in $\omega \in \mathbb{S}^{n-1}$ to obtain

$$\begin{aligned}\int_0^{r_\omega} s^{n-1} |u(x+s\omega) - u(x)| ds &\leq \int_0^{r_\omega} s^{n-1} \int_0^s |\nabla u(x+t\omega)| dt\, ds \\ \int_{\mathbb{S}^{n-1}} \int_0^{r_\omega} s^{n-1} |u(x+s\omega) - u(x)| ds d\mathcal{H}^{n-1} &\leq \int_{\mathbb{S}^{n-1}} \int_0^{r_\omega} s^{n-1} \int_0^s |\nabla u(x+t\omega)| dt\, ds\, d\mathcal{H}^{n-1} \\ \underbrace{\int_{B_1} |u(y) - u(x)| dy}_{\text{because the ball } B_1 \text{ is star-shaped}} &\leq \int_{\mathbb{S}^{n-1}} \int_0^{r_\omega} s^{n-1} \int_0^s |\nabla u(x+t\omega)| dt\, ds\, d\mathcal{H}^{n-1}\end{aligned}$$

On the RHS, however, the trick is to use Fubini.

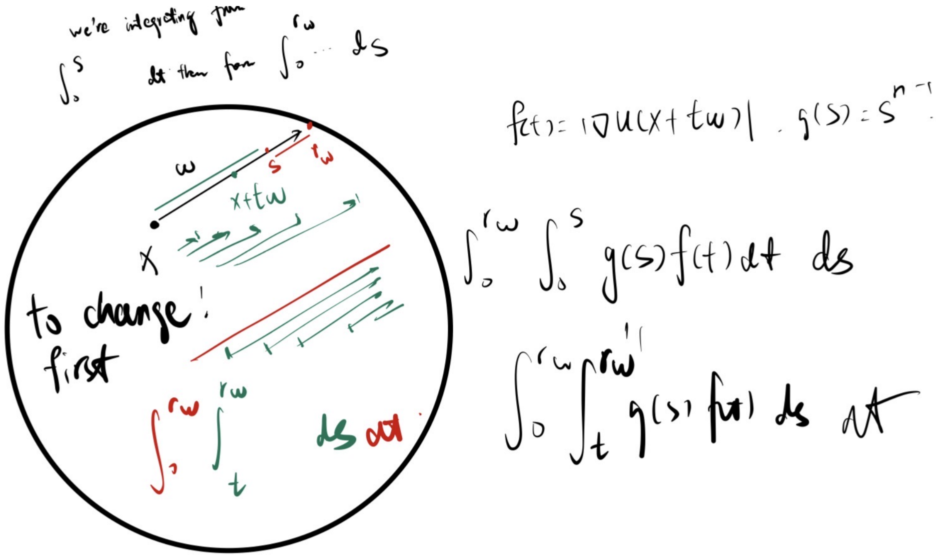


Figure 5.2: Fubini

For fixed ω, apply **Fubini**

$$\int_0^{r_\omega} s^{n-1} \int_0^s |\nabla u(x+t\omega)| dt\, ds = \int_0^{r_\omega} |\nabla u(x+t\omega)| \int_t^{r_\omega} s^{n-1} ds\, dt$$
$$= \frac{1}{n} \int_0^{r_\omega} |\nabla u(x+t\omega)| (r_\omega^n - t^n) dt$$

Thus

$$\int_{B_1} |u(y)-u(x)| dy \le \frac{1}{n} \int_{\mathbb{S}^{n-1}} \underbrace{\int_0^{r_\omega} |\nabla u(x+t\omega)| (r_\omega^n - t^n)}_{\text{just use rays } r_\omega \text{ cannot be longer than 2}} dt\, d\mathcal{H}^{n-1}$$
$$\le \frac{2^n}{n} \int_{\mathbb{S}^{n-1}} \int_0^{r_\omega} |\nabla u(x+t\omega)| dt d\mathcal{H}^{n-1}$$
$$= \frac{2^n}{n} \underbrace{\int_{B_1} \frac{|\nabla u(y)|}{|x-y|^{n-1}} dy}_{\text{rewrite in Cartesian}}$$

Now integrate in the other variable $x \in B_1$ to conclude

$$\int_{B_1 \times B_1} |u(y)-u(x)| dx dy \le \frac{2^n}{n} \int_{B_1} \underbrace{\left(\int_{B_1} \frac{|\nabla u(y)|}{|x-y|^{n-1}} dy \right)}_{\text{convolution } (|\nabla u(y)| * \frac{1}{|y|^{n-1}})(x)} dx$$
$$\le \frac{2^n}{n} \underbrace{\left(\int_{B_1} \frac{1}{|y|^{n-1}} dy \right)}_{\text{integrable}} \cdot \left(\int_{B_1} |\nabla u(y)| dy \right)$$
$$\le C(n) \int_{B_1} |\nabla u(y)| dy$$

We used the Young's convolution Inequality

$$\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q} \qquad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

This concludes (5.10).

For (5.11), we simply divide by $|B_1|$ and use triangle-inequality

$$\int_{B_1} |u(x) - \fint_{B_1} u(y) dy| dx = \int_{B_1} |\fint_{B_1} (u(x)-u(y)) dy| dx \le \int_{B_1} \fint_{B_1} |u(x)-u(y)| dy dx$$
$$\le C \int_{B_1} |\nabla u|$$

□

Poincaré for $p \geq 1$ Now we collect the Poincaré Inequality for $p \geq 1$.

Corollary 5.1.1 (Savin Analysis II 2026)**.** *Consider any $1 \leq p < \infty$. Let $u \in W^{1,p}(B_1)$. Then for $C = C(n,p) > 0$*

$$\left(\int_{B_1} |u - \fint_{B_1} u|^p dx\right)^{\frac{1}{p}} \leq C \left(\int_{B_1} |\nabla u|^p dx\right)^{\frac{1}{p}} \tag{5.13}$$

Proof. At step (5.12) one may raise both sides to power p so that

$$\begin{aligned}
|u(x+s\omega) - u(x)|^p &\leq \left(\int_0^s |\nabla u(x+t\omega)|dt\right)^p \\
&\leq \underbrace{\left(s^{\frac{p-1}{p}} \cdot \left(\int_0^s |\nabla u(x+t\omega)|^p dt\right)^{\frac{1}{p}}\right)^p}_{\text{Hölder}} \\
&= s^{p-1} \cdot \int_0^s |\nabla u(x+t\omega)|^p dt \\
&\leq \int_0^s |\nabla u(x+t\omega)|^p dt
\end{aligned}$$

Now repeat the exact same argument to conclude

$$\int_{B_1 \times B_1} |u(y) - u(x)|^p dx dy \leq C(n) \int_{B_1} |\nabla u|^p dy$$

Now one need the Jensen's Inequality

$$\begin{aligned}
\int_{B_1} |u - \fint_{B_1} u|^p dx &= \int_{B_1} |\fint_{B_1} u(x) - u(y) dy|^p dx \\
&\leq \int_{B_1} \fint_{B_1} |u(x) - u(y)|^p dx dy \\
&\leq C(n) \int_{B_1} |\nabla u|^p dy
\end{aligned}$$

Take $\frac{1}{p}$ on both sides to conclude. □

Poincaré on balls B_r We collect as well the rescaled Poincaré on balls of radius $r > 0$, of quite much importance.

Corollary 5.1.2 ([Eva10] Theorem 5.8.2)**.** *For $1 \leq p < \infty$. Let $u \in W^{1,p}(\mathbb{R}^n)$. There exists $C = C(n,p) > 0$ s.t.*

$$\left\| u - \fint_{B_r(x)} u \right\|_{L^p(B_r(x))} = \left(\int_{B_r(x)} |u - \fint_{B_r(x)} u|^p dy\right)^{\frac{1}{p}} \leq C(n,p) \cdot r \cdot \left(\int_{B_r(x)} |\nabla u|^p\right)^{\frac{1}{p}} = C \cdot r \cdot \|\nabla u\|_{L^p(B_r(x))} \tag{5.14}$$

Proof. It suffices to show

$$\int_{B_r} |u - \fint_{B_r} u|^p \leq C(n,p) \cdot r^p \cdot \int_{B_r} |\nabla u|^p$$

Define $u_r(x) = u(rx)$ for any $x \in B_1$ and look at

$$\begin{aligned}
\int_{B_r} |u - \fint_{B_r} u|^p &= \int_{B_1} |u_r - \fint_{B_1} u_r|^p \cdot r^n \\
&\overset{(5.13)}{\leq} C(n,p) \cdot r^n \cdot \int_{B_1} |\nabla u_r|^p dx \\
&= C(n,p) \cdot r^p \cdot \int_{B_r} |\nabla u|^p dx
\end{aligned}$$

□

5.1.4.2 Pay in Measure for $W^{1,p}$

We follow [Moo12] Section 5.1 the pay in measure remarks.

An important feature of $W^{1,p}$ for $p > 1$ is that functions must pay in measure to jump from 0 to 1, while $W^{1,1}$ functions can have arbitrarily fast jumps.

Example 5.1.1. *Consider functions u_ε which are 1 on B_1 and 0 outside $B_{1+\varepsilon}$, with*

$$|\nabla u_\varepsilon| \leq \frac{1}{\varepsilon}$$

Then

$$\int_{\mathbb{R}^n} |\nabla u_\varepsilon|^p = \int_{B_{1+\varepsilon}\setminus B_1} |\nabla u_\varepsilon|^p \leq C\varepsilon^{-p} \cdot \varepsilon^n \cdot \varepsilon^{n-1} = C\varepsilon^{1-p}$$

So for any $p > 1$, as $\varepsilon \to 0$ the $W^{1,p}$ norm of u_ε blows up. But u_ε have bounded $W^{1,1}$ norm with arbitrarily quick jumps.

We record the De Giorgi Isoperimetric Inequality for $p = 2$.

Lemma 5.1.6 ([FRRO22] Lemma 3.15)**.** *Let $u \in H^1(B_1)$. Denote*

$$A = \{u < 0\} \cap B_1, \qquad D = \{u \geq \frac{1}{2}\} \cap B_1, \qquad E = \{0 < u < \frac{1}{2}\} \cap B_1$$

Then there exists $c = c(n) > 0$ s.t.

$$c|A|^2|D|^2 \leq |E| \int_{B_1} |\nabla u|^2 \tag{5.15}$$

In other words, $|E|$ and the size of $\|\nabla u\|_{L^2}$ cannot both be small.

Proof. WLOG consider $u = 0$ on A, $u = \frac{1}{2}$ on D, whose L^2 norm of gradient is indeed controlled by the original u. Denote u_{B_1} as the average in B_1. Now for any $x \in D$ and $y \in A$, we observe

$$\begin{aligned}
\frac{1}{2} &= |u(x) - u(y)| \\
|A| &\leq 2\int_A |u(x) - u(y)|dy \qquad \text{integrate in } y \in A \\
|A||D| &\leq 2\int_A \int_D |u(x) - u(y)|dxdy \\
&\leq 2\int_A \int_D |u(x) - u_{B_1}| + |u_{B_1} - u(y)|dxdy \\
&\leq 4|B_1| \int_{B_1} |u - u_{B_1}|dx \leq C\int_{B_1} |\nabla u| \\
&\leq C|E|^{\frac{1}{2}} \left(\int_{B_1} |\nabla u|^2\right)^{\frac{1}{2}}
\end{aligned}$$

where in the last step we used Poincaré. Now using construction for u, $|\nabla u|$ is only supported on E, and we can conclude. □

5.1.5 Rellich-Kondrachov Compactness

Definition 5.1.1. *Let X, Y be Banach Space. We say X compactly embedds in Y if*

$$\|u\|_Y \leq C\,\|u\|_X \qquad \forall\ u \in X$$

and any uniformly bounded sequence in X contains a convergent subsequence in Y.

Theorem 5.1.8 ([Eva10] Theorem 5.7)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be bounded open set with $\partial\Omega$ Lipschitz. Let $1 \leq p < n$. Then $W^{1,p}(\Omega) \Subset L^q(\Omega)$ for any $1 \leq q < p^*$.*

Proof. We first use Extension Theorem so that one may WLOG assume $\{u_m\} \subseteq W^{1,p}(\mathbb{R}^n)$ have compact support in $V \subseteq \mathbb{R}^n$ bounded open. By assumption we have a uniformly bounded sequence in $W^{1,p}$

$$\sup_m \|u_m\|_{W^{1,p}(V)} \leq M$$

Our goal is to extract a convergent subsequence in $L^q(V)$ for any $1 \leq q < p^*$. The main idea is to mollify, and then use Ascoli-Arzela. So for $\varepsilon > 0$, we let

$$u_m^\varepsilon := \eta_\varepsilon * u_m \qquad \operatorname{supp}(u_m) \subseteq V$$

Be careful now we have $\varepsilon \to 0$ and $m \to \infty$ two sequences to play with.

Step 1: We want to show $u_m^\varepsilon \to u_m$ in $L^q(V)$ uniformly in m as $\varepsilon \to 0$. Let's first do $q = 1$. Assume u_m smooth by smooth approximation. Then

$$\begin{aligned} u_m^\varepsilon(x) - u_m(x) &= \frac{1}{\varepsilon^n}\int_{B_\varepsilon(x)} \eta(\frac{x-z}{\varepsilon})(u_m(z) - u_m(x))dz = \underbrace{\int_{B_1(0)} \eta(y)(u_m(x-\varepsilon y) - u_m(x))dy}_{\text{change of variables}} \\ &= \int_{B_1(0)} \eta \int_0^1 \frac{d}{dt}(u_m(x-\varepsilon t y))dtdy = -\varepsilon \int_{B_1(0)} \eta \int_0^1 \nabla u_m(x-\varepsilon t y)\cdot y dtdy \end{aligned}$$

$$\begin{aligned} \int_V |u_m^\varepsilon - u_m| dx &\leq \varepsilon \int_V |\nabla u_m(z)|dz \\ &\leq C(V)\varepsilon \|\nabla u_m\|_{L^p(V)} \qquad \text{Hölder} \\ &\leq CM\cdot\varepsilon \end{aligned}$$

RHS is independent of m thus this convergence is uniform in m. Now for general $1 \leq q < p^*$, we use **Interpolation Inequality for L^p norm** so

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \leq \|u_m^\varepsilon - u_m\|_{L^1(V)}^\theta \|u_m^\varepsilon - u_m\|_{L^{p^*}(V)}^{1-\theta} \qquad \frac{1}{q} = \theta + \frac{1-\theta}{p^*}$$

Notice this is the step that requires $q < p^*$! We only obtained uniform convergence in L^1, so as long as $\theta > 0$ uniform boundedness in L^{p^*} by Sobolev Inequality (5.3) suffices. BUT we need $\theta > 0$, which means we cannot conclude anything for $q = p^*$. We take away

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \leq C(n,q,V)M\varepsilon \tag{5.16}$$

Step 2: We want to show for any $\varepsilon > 0$ fixed, $\{u_m^\varepsilon\}$ is uniformly bounded and equi-continuous.

$$\begin{aligned} |u_m^\varepsilon(x)| &\leq \int_{B_\varepsilon(x)} \eta_\varepsilon(x-y)|u_m(y)|dy \\ \sup_m \|u_m^\varepsilon\|_{L^\infty(V)} &\leq C(n)\frac{1}{\varepsilon^n}\sup_m \|u_m\|_{L^1(V)} \leq C(n,V)\frac{1}{\varepsilon^n}M \\ \sup_m \|\nabla u_m^\varepsilon\|_{L^\infty(V)} &\leq C(n,V)\frac{1}{\varepsilon^{n+1}}M \end{aligned}$$

Moreover for any $h > 0$

$$\begin{aligned} |u_m^\varepsilon(x+h) - u_m^\varepsilon(x)| &\leq |\nabla u^\varepsilon(x)||h| \\ \sup_m \|u_m^\varepsilon(\cdot + h) - u_m^\varepsilon\|_{L^\infty(V)} &\leq C(n,V)\frac{1}{\varepsilon^{n+1}}M\cdot|h| \end{aligned}$$

Step 3: For any $\delta > 0$ fixed, we want to extract a convergent subsequence $\{u_{m_j}\}$ s.t.

$$\limsup_{j,k\to\infty} \left\|u_{m_j} - u_{m_k}\right\|_{L^q(V)} \leq \delta \tag{5.17}$$

First, in view of Step 1 (5.16), we choose $\varepsilon > 0$ small so

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \leq \frac{\delta}{2} \qquad \forall\ m \tag{5.18}$$

Since for any $\varepsilon > 0$, $\{u_m^\varepsilon\}$ is uniformly bounded and equi-continuous, apply Ascoli-Arzela so that $\{u_{m_j}^\varepsilon\}$ converges uniformly on V. In particular for L^q norm

$$\limsup_{j,k\to\infty} \left\|u_{m_j}^\varepsilon - u_{m_k}^\varepsilon\right\|_{L^q(V)} = 0$$

Combining with (5.18) one satisfy (5.17).

Finally, choose a sequence of δ approaching 0, and conclude via diagonalization argument.

□

Let's make some remarks.

Remark 5.1.1. *For any $1 \leq p < \infty$, $W^{1,p} \Subset L^p$. Notice for $p > n$, the compact embedding follows from Morrey's Inequality and Ascoli-Arzela.*

Sharpness Take any $u \in W^{1,p}(B_1)$ non-zero, and define dilations

$$u_\varepsilon(x) = u(\frac{x}{\varepsilon}) \qquad \forall\ x \in B_\varepsilon$$

Then

$$\begin{aligned}
\int_{B_\varepsilon} |u_\varepsilon|^{p^*} &= \varepsilon^n \int_{B_1} |u|^{p^*} \\
\|u_\varepsilon\|_{L^{p^*}(B_\varepsilon)} &= \varepsilon^{\frac{n}{p^*}} \|u\|_{L^{p^*}(B_1)} \\
\int_{B_\varepsilon} |\nabla u_\varepsilon|^p &= \varepsilon^{n-p} \int_{B_1} |\nabla u|^p dx \\
\|\nabla u_\varepsilon\|_{L^p(B_\varepsilon)} &= \varepsilon^{\frac{n}{p}-1} \|\nabla u\|_{L^p(B_1)}
\end{aligned}$$

Notice

$$\begin{aligned}
\frac{n}{p} - 1 &= \frac{n}{p^*} \\
p^* &= \frac{np}{n-p}
\end{aligned}$$

Thus if we consider the family

$$v_\varepsilon(x) := \frac{1}{\varepsilon^{\frac{n}{p^*}}} u(\frac{x}{\varepsilon}) \qquad \forall\ x \in B_\varepsilon$$

We obtain a family uniformly bounded in $W^{1,p}$ norm, but also uniformly bounded in L^{p^*} norm. In fact since L^{p^*} norm is a nonzero constant, there does not exist any subsequence that converges in L^{p^*}. However v_ε indeed converges to 0 a.e. and in any L^p norm with $1 \leq p < p^*$.

5.2 Growth of Local Intergrals

In this section we dicuss tools we use.

5.2.1 Campanato

For any $x \in B_1$ and $r > 0$, denote

$$u_{x,r} := \fint_{B_r(x)} u$$

Theorem 5.2.1 ([HL11] Theorem 3.1)**.** *If $u \in L^2(B_1)$ satisfies for certain $\alpha \in (0,1)$*

$$\fint_{B_r(x)} |u - u_{x,r}|^2 \le r^{2\alpha} \qquad \forall\ x \in B_{1/2},\ r \in (0, \frac{1}{2}) \tag{5.19}$$

Then $u \in C^{0,\alpha}(\overline{B_{1/2}})$. Moreover there exists $C = C(n, \alpha) > 0$ s.t.

$$\|u\|_{C^{0,\alpha}(\overline{B_{1/2}})} \le C \left(1 + \|u\|_{L^2(B_1)}\right) \tag{5.20}$$

Proof. Take any $x \in B_{1/2}$. Consider sequence $r_k = \frac{1}{2^k}$ for $k \ge 1$. Now

$$\begin{aligned}
|u_{x,r_k} - u_{x,r_{k+1}}|^2 &\le 2\left(|u_{x,r_k} - u(x)|^2 + |u(x) - u_{x,r_{k+1}}|^2\right) \\
&\le 2\left(\fint_{B_{r_k}} |u_{x,r_k} - u|^2 + \fint_{B_{r_{k+1}}} |u_{x,r_{k+1}} - u|^2\right) \\
&\overset{(5.19)}{\le} 2\left(r_k^{2\alpha} + r_{k+1}^{2\alpha}\right) \\
&= 2(1 + \frac{1}{2^{2\alpha}}) r_k^{2\alpha} = C(\alpha) r_k^{2\alpha}
\end{aligned} \tag{5.21}$$

At each point $x \in B_{1/2}$, the sequence $\{u_{x,r_k}\}_{k\ge1}$ converges as $r_k \to 0$. We define the pointwise limit as

$$\hat{u}(x) := \lim_{k\to\infty} u_{x,r_k}$$

By Lebesgue Differentiation, we know for a.e. $x \in B_{1/2}$

$$u(x) = \hat{u}(x)$$

We want to show $\hat{u} \in C(\overline{B_{1/2}})$. First note for each k, u_{x,r_k} is a continuous function defined on $\overline{B_{1/2}}$. Also, the RHS of (5.21) is independent of x. Hence this is uniform convergence in x, and thus as the uniform limit of uniformly continuous functions defined over $\overline{B_{1/2}}$, we know $\hat{u} \in C(\overline{B_{1/2}})$.

For the estimate of $\|\hat{u}\|_{C^0(\overline{B_{1/2}})}$, for any k fixed (say 1), consider

$$\begin{aligned}
\hat{u}(x) &\le |\fint_{B_{r_k}(x)} u - \hat{u}| + |\fint_{B_{r_k}(x)} u| \\
&\le C(n)\left(\sum_{j\ge k} |u_{x,r_j} - u_{x,r_{j+1}}| + \frac{1}{r_k^n} \|u\|_{L^2(B_{r_k}(x))} r_k^{\frac{n}{2}}\right) \\
&\le C(n,k,\alpha)\left(\sum_{j\ge k} r_j^\alpha + \|u\|_{L^2(B_1)}\right) \\
&\le C(1 + \|u\|_{L^2(B_1)}) \qquad \forall\ x \in B_{1/2}
\end{aligned}$$

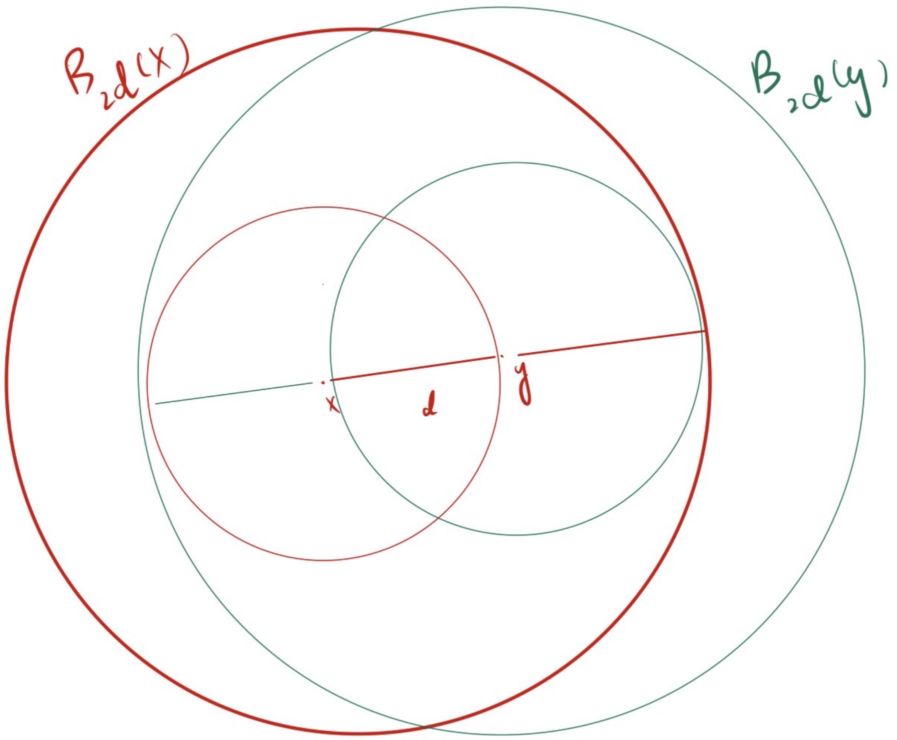


Figure 5.3: Growth of Local Integral Hölder semi-norm balls

For the Hölder semi-norm, consider two cases. Take $x \neq y \in B_{1/2}$, and denote $d = |x-y| > 0$

1. Assume $d < \frac{1}{100}$. Consider

$$\begin{aligned}|u(x)-u(y)| &\leq |u(x)-u_{x,2d}| + |u_{x,2d}-u_{y,2d}| + |u_{y,2d}-u(y)| \\ &\leq C(\alpha,n)d^\alpha + |u_{x,2d}-u_{y,2d}|\end{aligned}$$

How to estimate the last term? Consider any $\xi \in B_{2d}(x) \cap B_{2d}(y)$. Notice that $B_d(x)$, $B_d(y) \subseteq B_{2d}(x) \cap B_{2d}(y)$.

$$\begin{aligned}|u_{x,2d}-u_{y,2d}|^2 &\leq 2\left(|u_{x,2d}-u(\xi)|^2 + |u(\xi)-u_{y,2d}|^2\right) \\ &= 2\fint_{B_{2d}(x)\cap B_{2d}(y)} \left(|u_{x,2d}-u(\xi)|^2 + |u(\xi)-u_{y,2d}|^2\right) \\ &\leq 2\frac{|B_{2d}|}{|B_d|}\left(\fint_{B_{2d}(x)} |u_{x,2d}-u|^2 + \fint_{B_{2d}(y)} |u_{y,2d}-u|^2\right) \qquad \text{this is the Key Step} \\ &\leq C(n)d^{2\alpha}\end{aligned}$$

Thus

$$|u(x)-u(y)| \leq C(n,\alpha)|x-y|^\alpha$$

2. Assume $d \geq \frac{1}{100}$. Then we directly use supremum norm to control. The subtlety is that here one need the L^2 norm of u

$$\begin{aligned}|u(x)-u(y)| &\leq 2\,\|u\|_{C^0(\overline{B_{1/2}})} \leq C(1+\|u\|_{L^2(B_1)}) \\ &\leq C(n,\alpha)(1+\|u\|_{L^2(B_1)})d^\alpha \\ &= C(n,\alpha)(1+\|u\|_{L^2(B_1)})|x-y|^\alpha\end{aligned}$$

□

Corollary 5.2.1 ([HL11] Corollary 3.2). *If $u \in H^1_{loc}(B_1)$ satisfies for $\alpha \in (0,1)$*

$$\fint_{B_r(x)} |\nabla u|^2 \leq r^{2\alpha-2} \qquad \forall\ x \in B_{1/2},\ r \in (0, \frac{1}{2})$$

Then $u \in C^{0,\alpha}(\overline{B_{1/2}})$. Moreover the estimate (5.20) remains true.

Proof. Apply Poincaré

$$
\begin{aligned}
\int_{B_r(x)} |u - u_{x,r}|^2 &\le C(n) r^2 \int_{B_r(x)} |\nabla u|^2 \\
&\le C(n) r^{2+n+2\alpha-2}
\end{aligned}
$$

$$
⨍_{B_r(x)} |u - u_{x,r}|^2 \le C(n) r^{2\alpha}
$$

Then apply Theorem 5.2.1. □

5.2.2 Calderon Zygmund Decomposition

Theorem 5.2.2 ([HL11] Lemma 3.7). *Let $f \ge 0$, $f \in L^1(Q_0)$, and assume for some fixed positive constant*

$$
⨍_{Q_0} f < \alpha
$$

Then there exists a sequence of non-overlapping dyadic cubes $\{Q_j\} \subseteq Q_0$ s.t.

$$
f(x) \le \alpha \qquad a.e.\ Q_0 \setminus \bigcup_j Q_j, \qquad \alpha \le ⨍_{Q_j} f < 2^n \alpha \qquad \forall\ Q_j \tag{5.22}
$$

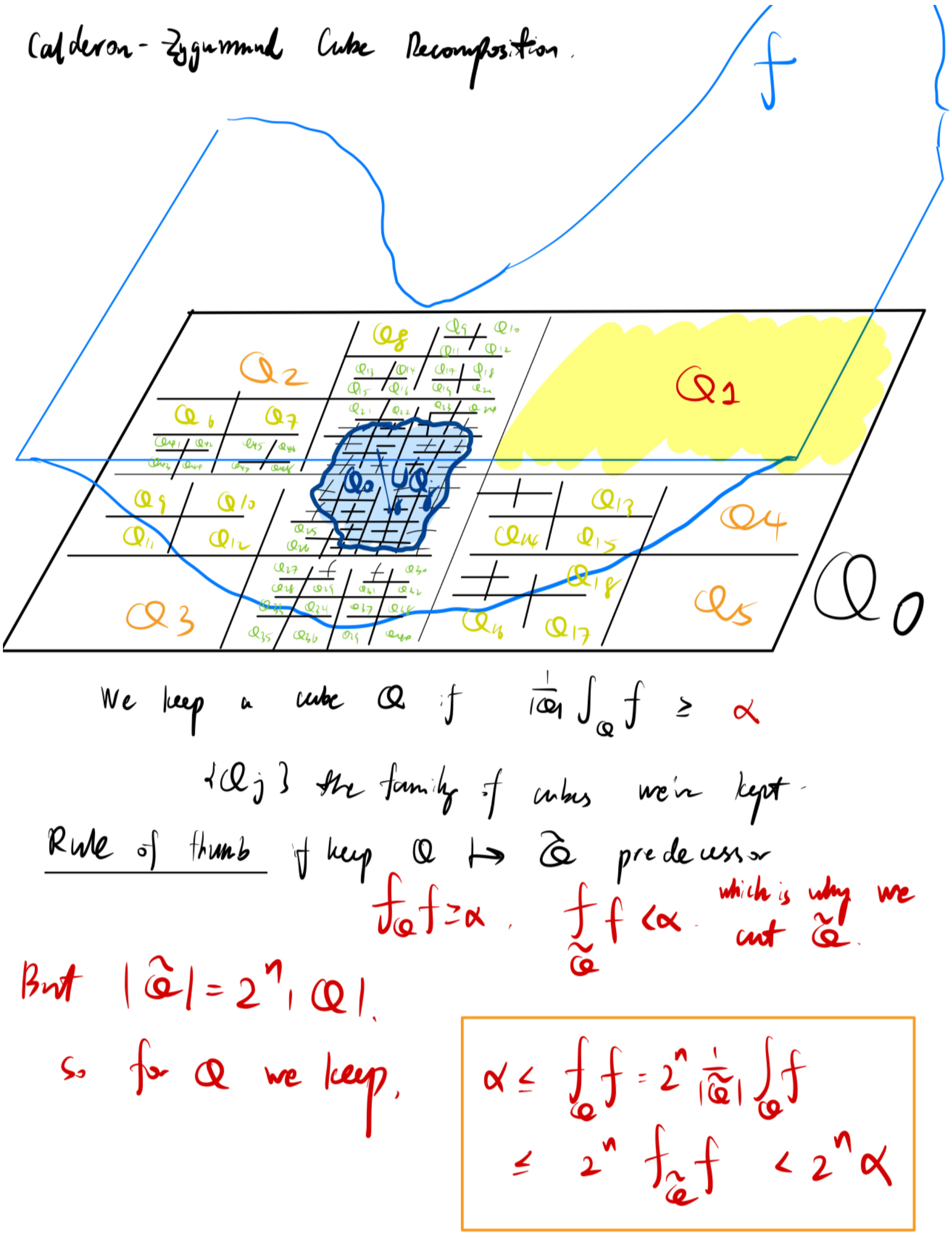


Figure 5.4: Calderon Zygmund

Proof. Let's describe our algorithm.

1. We pick Q if
$$\fint_Q f \geq \alpha$$
2. If not, we cut dyadically the cube Q into 2^n subcubes. Then repeat the algorithm on each of the subcubes.

Consider the collection of cubes we've chosen $\{Q_j\}$. They're non-overlapping by construction, and at most countable.

Notice the algorithm necessarily runs infinitely onwards, since if at some step $\fint_Q f \geq \alpha$ for all cubes, then the assumption $\fint_{Q_0} f < \alpha$ does not hold true.

For each Q, one define its predecessor as
$$Q \mapsto \tilde{Q} \qquad Q \text{ is cut directly from } \tilde{Q}$$
Now for each $Q_j \in \{Q_j\}$ that we pick, observe
$$\fint_{Q_j} f \geq \alpha \qquad \text{as required by the algorithm}$$
$$\fint_{\tilde{Q}_j} f < \alpha \qquad \text{this is why we cut } \tilde{Q}_j \text{ in the first place}$$
Now
$$|\tilde{Q}_j| = 2^n |Q_j|$$
so the second half of (5.22) holds
$$\alpha \leq \fint_{Q_j} f = \frac{2^n}{|\tilde{Q}_j|} \int_{Q_j} f \leq 2^n \fint_{\tilde{Q}_j} f < 2^n \alpha$$
For the first half, for a.e. $x \in Q_0 \setminus \bigcup_j Q_j$, by the algorithm, there exists an infinite sequence of cubes Q^i (which we do not pick) containing x so
$$\fint_{Q^i} f < \alpha \qquad \forall\, i$$
Now by Lebesgue Differentiation Theorem
$$f(x) = \lim_{i \to \infty} \fint_{Q^i} f \leq \alpha \qquad \text{a.e. } x \in Q_0 \setminus \bigcup_j Q_j$$
□

5.2.3 John-Nirenberg and BMO Space

John-Nirenberg

Theorem 5.2.3 ([HL11] Theorem 3.5)**.** *Let $u \in L^1(\Omega)$ satisfy*
$$\fint_{B_r(x)} |u - u_{x,r}| \leq M \qquad \forall\, B_r(x) \subseteq \Omega \tag{5.23}$$
Then there exists positive p_0, $C > 0$ depending only on n s.t.
$$\fint_{B_r(x)} e^{\frac{p_0}{M}|u - u_{x,r}|} \leq C \tag{5.24}$$
Proof. We first simply our proof into cubes. Assume $\Omega = Q_0$ and we change our assumption (5.23) so that
$$\fint_Q |u - u_Q| \leq M \qquad \forall\, Q \subseteq Q_0 \text{ cubes}$$
One also assume WLOG that $M = 1$.

We claim it suffices to prove the following. There exists $c_1(n)$, $c_2(n)$ dimensional constants s.t.
$$|\{x \in Q \mid |u - u_Q| > t\}| \leq c_1 \cdot |Q| \cdot e^{-c_2 t} \qquad \forall\, t > 0 \tag{5.25}$$

Indeed, one may use distribution function to compute

$$\begin{aligned}\int_Q e^{p_0|u-u_Q|} &= p_0\int_0^\infty e^{p_0 t}|\{x\in Q \mid |u(x)-u_Q|>t\}|dt\\ &\overset{(5.25)}{\le} c_1p_0\int_0^\infty e^{p_0t}|Q|e^{-c_2t}dt = c_1p_0|Q|\int_0^\infty e^{-\frac{c_2}{2}t}dt\\ &\le C|Q|\end{aligned}$$

where we've chosen $p_0 = \frac{1}{2}c_2$ and C as the rest.

Now we apply Calderon-Zygmund to prove (5.25). Using (5.23), one may choose $\alpha > 1$ s.t.

$$⨍_{Q_0} |u-u_{Q_0}| \le 1 < \alpha$$

We claim it suffices to show there exists a sequence of 'sequence of cubes' $\{Q_j^{(k)}\}_{j=1}^\infty \subseteq Q_0$ s.t.

$$\sum_j |Q_j^{(k)}| \le \frac{1}{\alpha^k}|Q_0| \qquad \forall\ k\ge 1 \tag{5.26}$$

$$|u(x)-u_{Q_0}| \le 2^n k\cdot\alpha \qquad \text{a.e. } x\in Q_0\setminus\bigcup_j Q_j^{(k)} \tag{5.27}$$

Let's see why. The condition (5.27) implies

$$\{x\in Q_0 \mid |u(x)-u_{Q_0}|>2^nk\cdot\alpha\} \overset{(5.27)}{\subseteq} \bigcup_j Q_j^{(k)}$$

$$|\{x\in Q_0 \mid |u(x)-u_{Q_0}|>2^nk\cdot\alpha\}| \le \sum_j |Q_j^{(k)}| \overset{(5.26)}{\le} \frac{1}{\alpha^k}|Q_0| \tag{5.28}$$

Why is (5.28) useful? For any $t>0$, there exists $k\in\mathbb{N}$ s.t.

$$2^nk\cdot\alpha \le t < 2^n(k+1)\cdot\alpha$$

Now

$$|\{x\in Q_0 \mid |u(x)-u_{Q_0}|>t\}| \le |\{x\in Q_0 \mid |u(x)-u_{Q_0}|>2^nk\cdot\alpha\}| \overset{(5.28)}{\le} \frac{1}{\alpha^k}|Q_0|$$

We need to bound $\frac{1}{\alpha^k}$. Notice

$$t < 2^n(k+1)\cdot\alpha \iff \frac{1}{2^n\alpha}t < k+1$$

$$\alpha^{-k} = \alpha\cdot\alpha^{-(k+1)} = \alpha e^{-(k+1)\log\alpha} < \alpha e^{-\frac{\log\alpha}{2^n\alpha}t}$$

Hence one conclude (5.25) with α universal.

$$|\{x\in Q_0 \mid |u(x)-u_{Q_0}|>t\}| \le \alpha e^{-\frac{\log\alpha}{2^n\alpha}t}\cdot|Q_1|$$

Now we use induction to show existence of $\{Q_j^{(k)}\}_{j=1}^\infty$.
In fact, in addition to satisfying (5.26) and (5.27), one also want to ensure for technical reasons that

$$|u_{Q_j^{(k)}} - u_{Q_0}| \le 2^nk\cdot\alpha \qquad \forall\ j \tag{5.29}$$

1. For the **base case**, apply Calderon-Zygmund Theorem 5.2.2 to the function $|u-u_{Q_0}|$ so that there exists non-overlapping cubes $\{Q_j^{(1)}\}_{j=1}^\infty \subseteq Q_0$ s.t.

$$\alpha \le ⨍_{Q_j^{(1)}} |u-u_{Q_0}| < 2^n\alpha \qquad \forall\ Q_j^{(1)} \tag{5.30}$$

$$|u(x)-u_{Q_0}| \le \alpha \qquad \text{a.e.} \qquad x\in Q_0\setminus\bigcup_j Q_j^{(1)} \tag{5.31}$$

(a) (5.27) is automatically satisfied at $k = 1$ due to (5.31).

(b) To ensure (5.29), notice

$$|u_{Q_j^{(1)}} - u_{Q_0}| = |\fint_{Q_j^{(1)}} u - u_{Q_0}| \le \fint_{Q_j^{(1)}} |u - u_{Q_0}| \overset{(5.30)}{<} 2^n\alpha$$

(c) To see (5.26), notice

$$|Q_j^{(1)}| \overset{(5.30)}{\le} \frac{1}{\alpha}\int_{Q_j^{(1)}} |u - u_{Q_0}|$$

$$\sum_j |Q_j^{(1)}| \le \frac{1}{\alpha}\sum_j \int_{Q_j^{(1)}} |u - u_{Q_0}| \le \frac{1}{\alpha}\int_{Q_0} |u - u_{Q_0}| \le \frac{1}{\alpha}|Q_0|$$

2. **Now assume for** $k-1$, so we've chosen the cubes $\{Q_j^{(k-1)}\}$. By assumption (5.23) for John-Nirenberg, we know

$$\fint_{Q_j^{(k-1)}} |u - u_{Q_j^{(k-1)}}| \le 1 < \alpha \qquad \forall\, j \tag{5.32}$$

Thus for any $Q_j^{(k-1)}$, one may apply Calderon-Zygmund Decomposition. For each j we apply, thus there exists a sequence $\{Q_{j,i}^{(k)}\}_{i=1}^{\infty} \subseteq Q_j^{(k-1)}$ s.t.

$$\alpha \le \fint_{Q_{j,i}^{(k)}} |u - u_{Q_j^{(k-1)}}| < 2^n\alpha \qquad \forall\, Q_{j,i}^{(k)} \tag{5.33}$$

$$|u(x) - u_{Q_j^{(k-1)}}| \le \alpha \qquad \text{a.e.} \qquad x \in Q_j^{(k-1)} \setminus \bigcup_i Q_{j,i}^{(k)} \tag{5.34}$$

We collect all such cubes and denote

$$\{Q_j^{(k)}\} := \bigcup_j \{Q_{j,i}^{(k)}\}_{i=1}^{\infty}$$

(a) Now for any j

$$|Q_{j,i}^{(k)}| \overset{(5.33)}{\le} \frac{1}{\alpha}\int_{Q_{j,i}^{(k)}} |u - u_{Q_j^{(k-1)}}|$$

$$\sum_i |Q_{j,i}^{(k)}| \le \sum_i \frac{1}{\alpha}\int_{Q_{j,i}^{(k)}} |u - u_{Q_j^{(k-1)}}| \le \frac{1}{\alpha}\int_{Q_j^{(k-1)}} |u - u_{Q_j^{(k-1)}}| \overset{(5.32)}{\le} \frac{1}{\alpha}|Q_j^{(k-1)}|$$

$$\sum_j |Q_j^{(k)}| = \sum_j \sum_i |Q_{j,i}^{(k)}| \le \sum_j \frac{1}{\alpha}|Q_j^{(k-1)}| \underbrace{\le \frac{1}{\alpha}\cdot\frac{1}{\alpha^{k-1}}|Q_0|}_{\text{using inductive hypothesis}} = \frac{1}{\alpha^k}|Q_0|$$

Thus we've checked (5.26) at level k.

(b) On the other hand, from inductive hypothesis we know

$$|u(x) - u_{Q_0}| \le 2^n(k-1)\cdot\alpha \qquad \text{a.e. } x \in Q_0 \setminus \bigcup_j Q_j^{(k-1)} \tag{5.35}$$

So on this set $Q_0 \setminus \bigcup_j Q_j^{(k-1)}$

$$|u(x) - u_{Q_0}| \le 2^n k \cdot \alpha$$

is automatically satisfied. It suffices to consider the set

$$\bigcup_j Q_j^{(k-1)} \setminus \bigcup_\ell Q_\ell^{(k)}$$

But

$$\bigcup_\ell Q_\ell^{(k)} = \bigcup_\ell \bigcup_i Q_{\ell,i}^{(k)}$$

so

$$\begin{aligned}\bigcup_j Q_j^{(k-1)} \setminus \bigcup_\ell Q_\ell^{(k)} &= \bigcup_j Q_j^{(k-1)} \setminus \left(\bigcup_\ell \bigcup_i Q_{\ell,i}^{(k)} \right) \\ &\subseteq \bigcup_j Q_j^{(k-1)} \setminus \bigcup_i Q_{j,i}^{(k)}\end{aligned}$$

Thus on this set we can use (5.34).

$$\begin{aligned}|u(x) - u_{Q_0}| \le & \underbrace{|u(x) - u_{Q_j^{(k-1)}}|}_{\text{use Calderon-Zygmund at } k \text{ (5.34)}} + \underbrace{|u_{Q_j^{(k-1)}} - u_{Q_0}|}_{\text{use inductive hypothesis (5.29)}} \\ &\le \alpha + 2^n(k-1)\cdot\alpha \\ &\le 2^n k \cdot \alpha \qquad \text{a.e. } x \in \bigcup_j Q_j^{(k-1)} \setminus \bigcup_\ell Q_\ell^{(k)}\end{aligned}$$

Thus (5.27) is satisfied at k.

(c) It suffices to ensure (5.29). Now for any $Q_j^{(k)}$, WLOG assume of the form $Q_{j,i}^{(k)}$ that comes from $Q_j^{(k-1)}$, one obtain

$$\begin{aligned}|u_{Q_j^{(k)}} - u_{Q_j^{(k-1)}}| = |u_{Q_{j,i}^{(k)}} - u_{Q_j^{(k-1)}}| &= \left| \fint_{Q_{j,i}^{(k)}} u - u_{Q_j^{(k-1)}} \right| \\ &\le \fint_{Q_{j,i}^{(k)}} |u - u_{Q_j^{(k-1)}}| \\ &\overset{(5.33)}{<} 2^n\alpha \\ |u_{Q_j^{(k)}} - u_{Q_0}| \le |u_{Q_j^{(k)}} - u_{Q_j^{(k-1)}}| + |u_{Q_j^{(k-1)}} - u_{Q_0}| &\le 2^n\alpha + 2^n(k-1)\cdot\alpha = 2^n k\cdot\alpha\end{aligned}$$

Thus (5.29) is satisfied at k.

This concludes the proof. □

BMO Space In fact we define

$$\begin{aligned}[u]_{\mathrm{BMO}(\mathbb{R}^n)} &:= \sup_{B_r(x)\subseteq\mathbb{R}^n} \left\{ \fint_{B_r(x)} |u - u_{x,r}| dy \right\} \\ &= \inf\{M > 0 \mid (5.23) \text{ holds in } \mathbb{R}^n\}\end{aligned}$$

We define

$$\mathrm{BMO}(\mathbb{R}^n) := \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \mid [u]_{\mathrm{BMO}(\mathbb{R}^n)} < \infty\}$$

Proposition 5.2.1 ([Eva10] 5.8.1)**.** *If $u \in W^{1,n}(\mathbb{R}^n)$, then $u \in \mathrm{BMO}(\mathbb{R}^n)$. Moreover there exists $C = C(n) > 0$ s.t.*

$$[u]_{\mathrm{BMO}(\mathbb{R}^n)} \le C(n) \|\nabla u\|_{L^n(\mathbb{R}^n)}$$

Proof. We use Poincaré with $p = 1$.

$$\begin{aligned}\fint_{B_r(x)} |u - u_{x,r}| &\le C\cdot r\cdot \fint_{B_r(x)} |\nabla u| \\ &\le C\cdot r\cdot \frac{1}{|B_r(x)|}\left(\int_{B_r(x)} |\nabla u|^n\right)^{\frac{1}{n}} \cdot \left(\int_{B_r(x)} 1\right)^{\frac{n-1}{n}} \qquad \text{Hölder} \\ &\le C\cdot r\cdot \left(\fint_{B_r(x)} |\nabla u|^n\right)^{\frac{1}{n}} \\ &\le C(n)\left(\int_{B_r(x)} |\nabla u|^n\right)^{\frac{1}{n}} \le C(n)\|\nabla u\|_{L^n(\mathbb{R}^n)}\end{aligned}$$

□

Example 5.2.1. $\log(|x|) \in \mathrm{BMO}(\mathbb{R}^n)$. *One need to use that for any* $r > 0$

$$\log(|rx|) = \log(r) + \log(|x|)$$

Hence it suffices to consider B *as ball of radius* 1 *centered at* x_0. *If* $|x_0| \leq 1$, *then*

$$\int_B |\log(|x|)| \leq C$$

by noticing

$$\int_0^1 \log(r)dr = \cancel{r\log(r)|_0^1} - \int_0^1 1 = -1$$

If $|x_0| \geq 1$

$$\int_B |\log(|x|) - \log|x_0|| \leq C$$

Notice $\log(|x|) \notin L^\infty(\mathbb{R}^n)$, but on the other hand

$$[u]_{\mathrm{BMO}(\mathbb{R}^n)} \leq 2\,\|u\|_{L^\infty}$$

Chapter 6

H^1 Theory

6.1 Existence and Uniqueness of Weak Solutions

In this section one consider elliptic operator of the following form

$$\mathcal{L}u := -\partial_j(a_{ij}\partial_i u) + b_i\partial_i u + cu$$

with $a_{ij} = a_{ji}$ symmetric and the operator uniformly elliptic

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \qquad \forall\, x \in \Omega, \qquad \forall\, \xi \in \mathbb{R}^n$$

For simplicity we assume a_{ij}, b_i, $c \in L^\infty(\Omega)$.

Definition 6.1.1 (Weak Solution). *For f_i, $g \in L^2(\Omega)$, we say $u \in H^1(\Omega)$ is a weak (generalized) solution to the inhomogeneous equation*

$$\mathcal{L}u = g - \partial_i f_i \qquad \Omega$$

if

$$\int_\Omega a_{ij}\partial_i u\partial_j v + b_i\partial_i uv + cuv = \int_\Omega gv + f_i\partial_i v \qquad \forall\, v \in H_0^1(\Omega)$$

Moreover, if we impose boundary data $\varphi \in H^1(\Omega)$, we say u is weak (generalized) solution to the generalized Dirichlet Problem

$$\begin{cases} \mathcal{L}u = g - \partial_i f_i & \Omega \\ u = \varphi & \partial\Omega \end{cases} \tag{6.1}$$

if u is weak solution to $\mathcal{L}u = g - \partial_i f_i$ in Ω and $u - \varphi \in H_0^1(\Omega)$.

Remark 6.1.1. *Note for $\varphi \in H^1(\Omega)$, one redefined $\tilde{u} = u - \varphi$ so that for u a solution to (6.1), $\tilde{u}$ solves*

$$\begin{cases} \mathcal{L}\tilde{u} = g - \partial_i f_i - \mathcal{L}\varphi & \Omega \\ \tilde{u} = 0 & \partial\Omega \end{cases}$$

Hence it makes sense to assume for zero boundary data. In other words, $u \in H_0^1(\Omega)$.

Notice that in this case, both sides of the equation make sense in the duality pairing between H^{-1} and H_0^1.

6.1.1 First Existence and Lax-Milgram

6.1.1.1 Lax-Milgram

One view the above as bilinear operator.

$$B(u,v) := \int_\Omega a_{ij}\partial_i u\partial_j v + b_i\partial_i uv + cuv \qquad \forall\, u,\, v \in H_0^1(\Omega) \tag{6.2}$$

Hence we call u a weak solution (with zero boundary data) to $\mathcal{L}u = g - \partial_i f_i$ if

$$B(u,v) = \langle g - \partial_i f_i, v\rangle \qquad \forall\, v \in H_0^1(\Omega)$$

where the RHS denotes duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Let's quote the essential tool.

Lemma 6.1.1 (Lax-Milgram; [Eva10] Theorem 6.1; [GT01] Theorem 5.8)**.** *Let $B : H \times H \to \mathbb{R}$ be bilinear functional over H Hilbert space. Assume*

1. *B is bounded (continuous), i.e., $|B(u, v)| \leq C \|u\| \|v\|$ for any u, $v \in H$*
2. *B is strongly coercive, i.e., there exists $c > 0$*

$$c \|u\|^2 \leq |B(u, u)| \qquad \forall\ u \in H$$

Let $f \in H^*$ be any bounded linear functional over H. Then there exists unique $u \in H$ s.t.

$$B(u, v) = \langle f, v \rangle \qquad \forall\ v \in H$$

Proof. Recall Riesz Representation. Note for $u \in H$ fixed, $B(u, \cdot)$ is bounded linear functional. By Riesz, there exists unique $w \in H$ s.t.

$$B(u, v) = (w, v) \qquad \forall\ v \in H$$

where the RHS denotes the inner product induced by H. One wish to write $w = Au$ for A an invertible (linear isomorphism) mapping. Is it?

1. We check A is bounded and linear. Indeed

$$(A(\alpha u_1 + \beta u_2), v) = B(\alpha u_1 + \beta u_2, v) = \alpha(Au_1, v) + \beta(Au_2, v)$$

and

$$\begin{aligned} \|Au\|^2 &= (Au, Au) = B(u, Au) \leq C \|u\| \|Au\| \\ \|Au\| &\leq C \|u\| \end{aligned}$$

2. Let's use the coercivity condition.

$$\begin{aligned} c \|u\|^2 &\leq B(u, u) = (Au, u) \leq \|Au\| \|u\| \\ c \|u\| &\leq \|Au\| \end{aligned}$$

Hence if $Au = 0$, necessarily $u = 0$, and this gives $\ker(A) = \{0\}$, thus injectivity of A. On the other hand, if Au_k is Cauchy in H norm, then the inequality implies u_k is Cauchy, hence by completeness there exists $u \in H$ s.t. $u_k \to u$. Now by continuity of A

$$Au = A(\lim_{k \to \infty} u_k) = \lim_{k \to \infty} Au_k$$

so the limit exists in H. Thus the range $R(A)$ is complete subset of a complete normed vector space, thus $R(A)$ is closed.

3. We check in fact $R(A) = H$. Assume not, then $R(A)$ as a closed proper subset, and H admits a decomposition $H = R(A) \oplus R(A)^{\perp}$. Take $\tilde{w} \in R(A)^{\perp}$. Then

$$c \|\tilde{w}\|^2 \leq B(\tilde{w}, \tilde{w}) = (A\tilde{w}, \tilde{w}) = 0$$

Which forces $\tilde{w} = 0$.

Thus A is bounded linear bijection between H. Now consider the linear functional $f \in H^*$. By Riesz, there exists unique $w \in H$ s.t.

$$\langle f, v \rangle = (w, v) \qquad \forall\ v \in H$$

Since A is linear bijection, A^{-1} is linear mapping. Since A^{-1} is bounded due to coercivity conditon, A defines a linear isomorphism, hence there exists unique $u \in H$ s.t. $Au = w$. ☐

Remark 6.1.2. *Notice if B is symmetric, then $B(u, u)$ itself defines a new inner product on H, and Riesz direct applies. The power of Lax-Milgram lies in that, it does not require any symmetry of B.*

6.1.1.2 Energy Estimate and first Existence Theorem

The question now leads to, for which kind of coefficients of the elliptic operator are B strongly coercive?

Let B be defined as in (6.2). A trivial energy estimate ([Eva10] Theorem 6.2;; [GT01] Lemma 8.4) gives

$$\begin{aligned}
\lambda \int_\Omega |\nabla u|^2 &\leq \int_\Omega a_{ij}\partial_i u \partial_j u = B(u,u) - \int_\Omega b_i \partial_i u u - cu^2 \\
&\leq B(u,u) + \|b\|_\infty \int_\Omega |\nabla u||u| + \|c\|_\infty \int u^2 \\
&\leq B(u,u) + C\varepsilon \int_\Omega |\nabla u|^2 + \frac{C}{\varepsilon}\int_\Omega u^2 + C\int_\Omega u^2 \\
\frac{\lambda}{2}\int_\Omega |\nabla u|^2 &\leq B(u,u) + C\int_\Omega u^2 \\
\theta \|u\|^2_{H_0^1(\Omega)} &\leq B(u,u) + \gamma \int_\Omega u^2
\end{aligned}$$

For some $\theta = \theta(n, \lambda, \|b\|_\infty, \|c\|_\infty) > 0$ and $\gamma = \gamma(n, \|b\|_\infty, \|c\|_\infty) > 0$. Hence if we instead consider

$$B_\mu(u,v) := B(u,v) + \mu \int_\Omega uv \qquad \forall\ u,\ v \in H_0^1(\Omega)$$

for any $\mu \geq \gamma$, the bilinear form B_μ is strongly coercive.

Theorem 6.1.1 ([Eva10] Theorem 6.3)**.** *There is universal $\gamma > 0$ s.t. for any $\mu \geq \gamma$ the weak generalized Dirichlet problem*

$$\begin{cases} \mathcal{L}u + \mu u = g - \partial_i f_i & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

admits unique weak solution $u \in H_0^1(\Omega)$. In particular, if one denote $\mathcal{L}_\mu := \mathcal{L} + \mu I$, the map

$$\begin{aligned} \mathcal{L}_\mu^{-1} : H^{-1} &\to H_0^1 \\ g - \partial_i f_i &\mapsto u \end{aligned}$$

Defines a bounded linear operator (due to strong coercivity condition).

Note in particular for Dirichlet Energy, $B(u,v) := \int_\Omega \nabla u \cdot \nabla v$, using Poincaré, any $\mu \geq 0$ suffices.

6.1.2 Fredholm Alternative Theory

6.1.2.1 Fredholm Alternative

Definition 6.1.2. *Let X, Y be normed vector spaces. A bounded linear operator $K : X \to Y$ is compact if for any $\{u_k\} \subseteq X$ bounded sequence, the sequence $\{Ku_k\} \subseteq Y$ is precompact in Y, i.e., there exists a subsequence u_{k_j} s.t. $\{Ku_{k_j}\}$ converges in Y.*

In the following we demonstrate the Fredholm Alternative Theory. The Theory concerns compact linear operators from a space into itself as an extension of the theory of linear mappings in finite dimensional spaces.

Theorem 6.1.2 ([GT01] Theorem 5.3)**.** *Let T be compact linear mapping of a normed vector space V into itself.*

Then

1. *either*

$$\ker(I - T) \neq \{0\}$$

i.e., the homogeneous equation $x - Tx = 0$ has a nontrivial solution $x \in V$

2. *or $I - T$ is a linear isomorphism from V to itself, i.e., for each $y \in V$, the equation*

$$x - Tx = y$$

has a uniquely determined solution $x \in V$. In this case, $(I - T)^{-1} : V \to V$ remains a bounded operator.

A technical lemma of Riesz for normed vector spaces.

Lemma 6.1.2 (Riesz Lemma; [GT01] Lemma 5.4)**.** *Let V be normed vector space and $W \subsetneq V$ a proper closed subspace. Then for any $\theta < 1$, there exists $x_\theta \in V$ s.t. $\|x_\theta\| = 1$ and*

$$\operatorname{dist}(x_\theta, W) = \inf_{y \in W} \|y - x_\theta\| \geq \theta \tag{6.3}$$

Proof. Since W is proper closed subspace, for any $x \in V \setminus W$, $d := \operatorname{dist}(x, W) > 0$. Using definition of infimum and the fact that W is linear subspace, for any $\theta < 1$, there exists $y_\theta \in W$ s.t.

$$\|x - y_\theta\| \leq \frac{d}{\theta}$$

We define

$$x_\theta := \frac{x - y_\theta}{\|x - y_\theta\|} \qquad \text{so that} \qquad \|x_\theta\| = 1$$

Now

$$\begin{aligned}\operatorname{dist}(x_\theta, W) = \inf_{y \in W} \|x_\theta - y\| &= \frac{1}{\|x - y_\theta\|} \inf_{y \in W} \|x - y_\theta - \|x - y_\theta\| \, y\| \\ &\geq \frac{1}{\|x - y_\theta\|} \operatorname{dist}(x, W) = \frac{d}{\|x - y_\theta\|} \geq \theta\end{aligned}$$

□

Proof of Theorem 6.1.2. In this proof we denote $S = I - T$.

1. We begin with investigating $\ker(S)$. We claim that there exists constant $C > 0$ s.t.

   $$\operatorname{dist}(x, \ker(S)) \leq C \|Sx\| \qquad \forall\, x \in V \tag{6.4}$$

   Assume for contradiction, then there exists a sequence $\{x_n\} \subseteq V$ s.t. $\|Sx_n\| = 1$ and $d_n := \operatorname{dist}(x_n, \ker(S)) \to \infty$. The clever construction is to take a sequence $\{y_n\} \subseteq \ker(S)$ that is close to the projection of x_n onto $\ker(S)$. In particular, using $\ker(S)$ is linear subspace, one may take $y_n \in \ker(S)$ s.t.

   $$d_n \leq \|x_n - y_n\| \leq 2d_n \qquad \forall\, n$$

   We want to construct a bounded sequence. To do so, simply take

   $$z_n := \frac{x_n - y_n}{\|x_n - y_n\|} \qquad \text{so that} \qquad \|z_n\| = 1, \qquad \|Sz_n\| = \frac{1}{\|x_n - y_n\|} \|Sx_n\| = \frac{1}{d_n} \to 0$$

   Hence $Sz_n \to 0$ in norm. On the other hand, using $\|z_n\|$ is bounded, the trick of compact operator is to extract a strongly convergent subsequence

   $$Tz_{n_j} \to z_0 \in V$$

   Hence

   $$z_{n_j} = Sz_{n_j} + Tz_{n_j} \to z_0 \in V$$

   Using linearity and boundedness of S, necessarily $z_0 \in \ker(S)$. Where do we seek for contradiction? Now consider the distance

   $$\operatorname{dist}(z_n, \ker(S)) = \inf_{y \in \ker(S)} \|z_n - y\| = \frac{1}{\|x_n - y_n\|} \inf_{y \in \ker(S)} \|x_n - y_n - \|x_n - y_n\| \, y\|$$

   Now since $\ker(S)$ is a linear subspace, y_n, $y \in \ker(S)$, indeed

   $$\operatorname{dist}(z_n, \ker(S)) \geq \frac{1}{\|x_n - y_n\|} \operatorname{dist}(x_n, \ker(S)) = \frac{d_n}{\|x_n - y_n\|} \geq \frac{1}{2} \qquad \forall\, n$$

   Hence this is contradiction to $z_n \to z_0 \in \ker(S)$.

2. Next we investigate the Range of S. **We claim $R(S)$ is closed subspace of V.** To see this, take any sequence $\{x_n\} \subseteq V$ s.t. $\{Sx_n\}$ converges to some $y \in V$. We want to show there exists $x \in V$ s.t. $Sx = y$. We take $\{y_n\} \subseteq \ker(S)$ as above s.t.

   $$d_n = \operatorname{dist}(x_n, \ker(S)) \leq \|x_n - y_n\| \leq 2d_n$$

   Using (6.4), since $\{Sx_n\}$ is convergent, hence bounded, we know d_n are bounded in n, thus $w_n := x_n - y_n$ are bounded. Using T is compact operator, there exists a subsequence s.t. $\{Tw_{n_j}\}$ converges to some $w_0 \in V$. Now

   $$w_{n_j} = Sw_{n_j} + Tw_{n_j} = Sx_{n_j} + Tw_{n_j} \to y + w_0 \in V$$

   Using Linearity and boundedness of S we know $Sw_{n_j} \to S(y + w_0)$. On ther other hand

   $$Sw_{n_j} = Sx_{n_j} \to y$$

   Thus $y \in R(S)$.

3. We claim $\ker(S) = \{0\}$ implies that $R(S) = V$. First consider a non-increasing sequence of linear spaces $R^j := R(S^j)$. We claim there exists $\ell \in \mathbb{N}$ s.t. the sequence terminates $R^\ell = R^j$ for any $j \geq \ell$. Assume not, then one constructs a proper nested sequence R^j. Now for each $R^j \supsetneq R^{j+1}$, using (6.3) one may construct $x_j \in R^j$ s.t. $\|x_j\| = 1$ and
$$\operatorname{dist}(x_j, R^{j+1}) \geq \frac{1}{2}$$
Now for any $n < m$
$$\|Tx_n - Tx_m\| = \underbrace{\|x_n - Sx_n - x_m + Sx_m\| = \|x_n - (Sx_n + x_m - Sx_m)\|}_{\text{using the structure of } S = I - T} \geq \operatorname{dist}(x_n, R^{n+1}) \geq \frac{1}{2}$$
But $\{x_j\}$ is bounded sequence, and in this construction Tx_j has no convergent subsequence. Hence a contradiction to compactness of T.

Now we use $\ker(S) = \{0\}$. Take any $y \in V$, now $S^\ell y \in R^\ell = R^{\ell+1}$, hence there exists $x \in V$ s.t.
$$S^\ell y = S^{\ell+1} x \implies S^\ell(y - Sx) = 0$$
Since $\ker(S) = \{0\}$, this forces $y = Sx$.

4. We claim $R(S) = V$ implies $\ker(S) = \{0\}$. Consider a non-decreasing sequence of linear spaces $N^j := \ker(S^j)$. We claim there exists $\ell \in \mathbb{N}$ s.t. $N^\ell = N^j$ for any $j \geq \ell$. Assume not, for each $N^j \subsetneq N^{j+1}$, using (6.3) one construct $x_{j+1} \in N^{j+1}$ s.t. $\|x_{j+1}\| = 1$ and
$$\operatorname{dist}(x_{j+1}, N^j) \geq \frac{1}{2}$$
Now for any $n < m$
$$Tx_n - Tx_m = \|x_m - (x_n - Sx_n + Sx_m)\| \geq \frac{1}{2}$$
A contradiction to T compact.

Now we use $R(S) = V$. Take any $x \in N^\ell$ so $S^\ell x = 0$. Since $x \in R(S) = V = R(S^\ell)$, there exists $y \in V$ s.t. $S^\ell y = x$, hence $S^{2\ell} y = 0$ so $y \in N^{2\ell}$. But $N^{2\ell} = N^\ell$, hence $0 = S^\ell y = x$.

5. Now in the case $\ker(S) = \{0\}$ and $R(S) = V$, $y = (I - T)^{-1} x$ is well-defined linear map. Now in view of (6.4)
$$\left\|(I - T)^{-1} y\right\| \leq C \|y\| \qquad \forall\, y \in V$$
So S^{-1} is bounded linear operator. Thus S is invertible, i.e., a linear isomorphism between V.

□

6.1.2.2 Spectral Theory

Definition 6.1.3. *1. $\lambda \in \mathbb{R}$ is eigenvalue of T if there exists non-zero element $x \in V$ (eigenvector) s.t.*
$$Tx = \lambda x$$

2. Eigenvectors corresponding to different eigenvalues are linearly independent.

Proof. Assume v_1 and v_2 are eigenvectors corresponding to $\lambda_1 \neq \lambda_2$
$$Tv_1 = \lambda_1 v_1, \qquad Tv_2 = \lambda_2 v_2$$
Assume $c_1 v_1 + c_2 v_2 = 0$ with $c_1, c_2 \neq 0$. Then plugging $v_1 = \frac{-c_2}{c_1} v_2$ into the first expression and using linearity of T gives $\lambda_1 = \lambda_2$. For general n do induction. □

3. We define the multiplicity of λ as dimension of the eigenspace
$$\ker(\lambda I - T)$$

4. If $\lambda \in \mathbb{R} \setminus \{0\}$ is not an eigenvalue of T, then by Theorem 6.1.2 the resolvent operator $(\lambda I - T)^{-1}$ is well-defined, bounded linear operator of V.

Spectral Theory for Compact Operators

Theorem 6.1.3 ([GT01] Theorem 5.5)**.** *Let T be compact linear mapping of a normed vector space V into itself.*

Then

1. *For any $\lambda \neq 0$ as eigenvalue of T, $\ker(\lambda I - T)$ is finite dimensional.*

   Proof. Assume not, so $\dim(\ker(\lambda I - T)) = \infty$. In a infinite-dimensional normed vector space, the unit ball is not compact, hence one may find a sequence $\{x_n\} \subseteq \ker(\lambda I - T)$ s.t. $\|x_n\| = 1$ yet there does not exist any convergent subsequence. By then using T is compact, for the bounded sequence x_n, there exists convergent subsequence $Tx_{n_j} \to y$. But then

   $$\lambda x_{n_j} = (\lambda I - T)x_{n_j} + Tx_{n_j} = Tx_{n_j} \to y$$

   one has a contradiction. □

2. *The set of eigenvalues for T is at most countable.*

   Proof. Let K denote the underlying field of V. It suffices to prove for any $n \in \mathbb{N} \setminus \{0\}$,

   $$S_n := \{\lambda \in K \mid \lambda \text{ eigenvalue of } T,\ |\lambda| \geq \frac{1}{n}\}$$

   is finite. Assume not, so for some n_0 fixed, S_{n_0} is infinite, and one may take a sequence of distinct eigenvalues $\lambda_k \in S_{n_0}$. For each k, take some v_k corresponding eigenvector. Consider the increasing sequence of linear subspaces

   $$M_n := \operatorname{Span}\{v_1, \cdots, v_n\}$$

   Since we assumed λ_k to be infinitely many, hence infinitely-many linearly-independent eigenvectors, we know M_n is always proper closed subset of V, and $M_{n-1} \subsetneq M_n$. Thus using (6.3) one construct $x_n \in M_n$ s.t. $\|x_n\| = 1$ and

   $$\operatorname{dist}(x_n, M_{n-1}) \geq \frac{1}{2}$$

   Now for $n > m$ one consider

   $$\|Tx_n - Tx_m\| = \|\lambda_n x_n - (\lambda_n - T)x_n - \lambda_m x_m - (\lambda_m - T)x_m\|$$

   Since $x_n = \sum_{i=1}^n c_i v_i$ for $v_i \in \ker(\lambda_i I - T)$

   $$(\lambda_n - T)x_n = \sum_{i=1}^{n-1} (\lambda_n - \lambda_i)c_i v_i \in M_{n-1}$$

   Where the latter indeed belongs to $M_m \subseteq M_{n-1}$. Thus

   $$\|Tx_n - Tx_m\| \geq |\lambda_n| \operatorname{dist}(x_n, M_{n-1}) \geq \frac{1}{2n_0}$$

   Hence Tx_n does not possess any convergent subsequence, contradicting compactness of T. □

3. *The countable set of eigenvalues have no limit points except possibly at $\lambda = 0$. If not, the set of eigenvalues is finite.*

   Proof. Assume $\lambda_k \to \lambda \neq 0$ finitely many, then $S_{\frac{1}{|\lambda|}}$ contains infinitely many eigenvalues, a contradiction to the previous result. □

6.1.2.3 Fredholm Alternative in Hilbert Spaces

Definition 6.1.4. *Let T be bounded linear operator on Hilbert Space H. The adjoint of T is $T^* : H \to H$ s.t.*

$$(T^* y, x) = (y, Tx) \qquad \forall\ x, y \in H$$

We collect some facts

1. If T is compact, T^* is compact as well ([GT01] Lemma 5.9).

2. For any bounded linear operator $T : H \to H$ ([GT01] Lemma 5.10),

$$\overline{R(T)} = \ker(T^*)^\perp$$

Proof. Assume $y \in R(T)$ so there exists $x \in H$ s.t. $Tx = y$, then for any $v \in \ker(T^*)$

$$(y, v) = (Tx, v) = (x, T^*v) = 0$$

Hence $y \in \ker(T^*)^\perp$. Since the RHS is closed, $\overline{R(T)} \subseteq \ker(T^*)^\perp$. On the other hand, suppose $y \notin \overline{R(T)}$, by projection theorem there exists unique decomposition

$$y = y_1 + y_2$$

where $y_1 \in \overline{R(T)}$ and $y_2 \in \overline{R(T)}^\perp \setminus \{0\}$. Now for any $x \in H$

$$0 = (y_2, Tx) = (T^*y_2, x)$$

Hence $y_2 \in \ker(T^*)$. Thus using $y_1 \in \overline{R(T)} \subseteq \ker(T^*)^\perp$

$$(y_2, y) = (y_2, y_1) + \|y_2\|^2 = \|y_2\|^2 \neq 0$$

Hence $y \notin \ker(T^*)^\perp$. □

In particular, for $I - T$ where T is compact, since Theorem 6.1.2 shows $R(I - T)$ is closed, one obtain

$$R(I - T) = \ker(I - T^*)^\perp$$

Theorem 6.1.4 ([GT01] Theorem 5.11)**.** *Let $T : H \to H$ be compact operator from H Hilbert Space to itself. Then there exists a countable set $\Lambda \subseteq \mathbb{R} \setminus \{0\}$, with no limit point, except possibly at 0, s.t.*

1. *for any $\lambda \in \Lambda$,*

$$0 < \dim(\ker(\lambda I - T)) = \dim(\ker(\lambda I - T^*)) < \infty$$

and

(a) *The equation*

$$(\lambda - T)x = y$$

is solvable iff y is orthogonal to $\ker(\lambda I - T^*)$, i.e.

$$(y, v) = 0 \qquad \forall\ (\lambda - T^*)v = 0$$

(b) *The equation*

$$(\lambda I - T^*)x = y$$

is solvable iff y is orthogonal to $\ker(\lambda I - T)$, i.e.

$$(y, v = 0) \qquad \forall\ (\lambda I - T)v = 0$$

2. *for any $\lambda \notin \Lambda \cup \{0\}$, the equations*

$$(\lambda I - T)x = y, \qquad (\lambda I - T^*)x = y$$

are uniquely solvable for any $y \in H$, and the resolvent operators $(\lambda I - T)^{-1}$, $(\lambda I - T^*)^{-1}$ are bounded.

6.1.3 Second/Third Existence and Fredholm Alternative

6.1.3.1 Second Existence Theory

Let's see how we make use of the Fredholm Alternative. We look at the most general Dirichlet Problem with zero boundary data, for some $F \in H^{-1}(\Omega)$

$$\begin{cases} \mathcal{L}u = F & \Omega \\ u = 0 & \partial\Omega \end{cases} \tag{6.5}$$

From First Existence Theorem 6.1.1, we know there exists certain $\gamma = \gamma(\mathcal{L}) \geq 0$ s.t. the operator

$$\mathcal{L}_\gamma := \mathcal{L} + \gamma I : H_0^1 \to H^{-1}$$

is invertible, i.e.

$$\mathcal{L}_\gamma^{-1} : H^{-1} \to H_0^1$$

is well-defined bounded linear operator. First, notice if $\gamma = 0$ we've happy. Hence in the following we assume $\gamma > 0$. Here $I : H_0^1 \to H^{-1}$ is understood as natural imbedding. In view of our problem (6.5), we seek for $u \in H_0^1(\Omega)$ that solves

$$\mathcal{L}_\gamma u = \gamma I u + F$$

Since $\gamma u + F \in H^{-1}$, we invert so

$$u = \mathcal{L}_\gamma^{-1}(\gamma I u + F)$$

u now solves

$$u - \gamma \mathcal{L}_\gamma^{-1} I u = \mathcal{L}_\gamma^{-1} F$$

Define $K := \gamma \mathcal{L}_\gamma^{-1} I$. One may thus view K as a bounded linear operator

$$K : H_0^1 \to H_0^1$$

In fact it also makes sense to view K as bounded linear operator from L^2 to L^2. We do not specify the inner product that choose, as $(\cdot, \cdot)$ denoting both H^1 and L^2 inner product shall work. I believe in Evans they used L^2.

The key observation is as following.

Lemma 6.1.3 ([GT01] Lemma 8.5)**.** *The operator K is compact.*

Proof. We decompose $I = I_1 I_2$ where

$$\begin{aligned} I_2 : H_0^1 &\to L^2 \\ u &\mapsto u \\ I_1 : L^2 &\to H^{-1} \\ u &\mapsto \ell_u(v) := \int_\Omega uv \qquad \forall\, v \in H_0^1 \end{aligned}$$

By Rellich Compactness Theorem, H^1 compactly embedds into L^2 in any dimension, hence the embedding I_2 is compact. Since I_1 is bounded linear, thus continuous, and continuous composition with compact operator remains compact. L_γ^{-1} is again bounded linear, thus its composition with I remains compact. □

Define $h := \mathcal{L}_\gamma^{-1} F \in H_0^1$. Now one may apply Fredholm Alternative to look at the equation

$$u - Ku = h$$

Using Theorem 6.1.4

1. either $u - Ku = h$ is uniquely solvable for any h, in which case our Dirichlet Problem (6.5) is uniquely solvable for $u \in H_0^1(\Omega)$ given any $F \in H^{-1}(\Omega)$.
2. Or the homogeneous equation

   $$u - Ku = 0$$

   admits nontrivial solution $u \neq 0$.

In the first case, one obtain the nice bound ([GT01] Corollary 8.7)

$$\begin{aligned} \|u\|_{H_0^1} = \left\|(I-K)^{-1} h\right\|_{H_0^1} &\le \left\|(I-K)^{-1}\right\| \|h\|_{H_0^1} \\ &\le \frac{1}{1 - \|K\|} \left\|\mathcal{L}_\gamma^{-1} F\right\|_{H_0^1} \\ &\le \frac{\left\|\mathcal{L}_\gamma^{-1}\right\|}{1 - \|K\|} \|F\|_{H^{-1}} \end{aligned} \tag{6.6}$$

The Alternative In the following we investigate the second case.

Let's first study what the adjoint of K, $K^*: H_0^1 \to H_0^1$ looks like. Recall $\mathcal{L}: H_0^1 \to H^{-1}$, and our bilinear form looks like

$$B(u,v) = \langle \mathcal{L}u, v\rangle = \int_\Omega a_{ij}\partial_i u \partial_j v + b_i \partial_i u v + cuv$$

What if we think of an operator acting on v, and then pairing with u, which gives the same outcome? In this case one define a formal adjoint

$$\mathcal{L}^* v := -\partial_i(a_{ij}\partial_j v) - \partial_i(b_i v) + cv$$

so that $\mathcal{L}^*: H_0^1 \to H^{-1}$ satisfies

$$\langle \mathcal{L}^* v, u\rangle := B(u,v) = \langle \mathcal{L}u, v\rangle \qquad \forall\ u \in H_0^1$$

Now, since $\mathcal{L}^*$ and $\mathcal{L}$ give the same bilinear form, the $\gamma = \gamma(\mathcal{L}) = \gamma(\mathcal{L}^*) > 0$ remains invariant.

Lemma 6.1.4. *$K^* = \gamma(\mathcal{L}_\gamma^*)^{-1} I : H_0^1 \to H_0^1$ is the adjoint of K.*

Proof. For any $u \in H_0^1$, $w = \gamma(\mathcal{L}_\gamma^*)^{-1} I u$ defines the unique weak solution to

$$(\mathcal{L}^* + \gamma)w = \gamma u$$

i.e., w uniquely solves

$$\int_\Omega a_{ij}\partial_i v \partial_j w + b_i \partial_i v w + (c+\gamma)vw = \langle \mathcal{L}_\gamma^* w, v\rangle = \gamma(v,u) \qquad \forall\ v \in H_0^1$$

where $(\cdot,\cdot)$ denotes the inner product induced by the Hilbert space. In particular, if one choose $v = \gamma \mathcal{L}_\gamma^{-1} I z = Kz$ to be unique weak solution to

$$(\mathcal{L} + \gamma)v = \gamma z$$

then for our particular choice of w

$$\int_\Omega a_{ij}\partial_i v \partial_j w + b_i \partial_i v w + (c+\gamma)vw = \langle \mathcal{L}_\gamma v, w\rangle = \gamma(z,w)$$

But notice by definition of $\mathcal{L}_\gamma^*$

$$\langle \mathcal{L}_\gamma^* w, v\rangle = \langle \mathcal{L}_\gamma v, w\rangle$$

Hence

$$(Kz, u) = (v,u) = (z,w) = (z, \gamma(\mathcal{L}_\gamma^*)^{-1} u)$$

And thus $\gamma(\mathcal{L}_\gamma^*)^{-1} = K^*$. □

Now we're ready to say something more in the second case ([Eva10] Theorem 6.2.4).

1. We know $0 < \dim\ker(I-K) = \dim\ker(I-K^*) < \infty$. Now $\ker(I-K)$ are precisely the vector space of weak solutions to the homogeneous equation

$$\begin{cases} \mathcal{L}u = 0 & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

Hence the dimension of such space is finite, and equals the dimension of the space of weak solutions to the homogeneous equation

$$\begin{cases} \mathcal{L}^* v = 0 & \Omega \\ v = 0 & \partial\Omega \end{cases} \tag{6.7}$$

2. We know $R(I-K) = \ker(I-K^*)^\perp$. Hence $u - Ku = h$ has a solution, i.e., a solution to (6.5) exists (no matter unique or not), iff

$$(h,v) = 0 \qquad \forall\ v - K^* v = 0$$

But by extending the definition of adjoint,

$$(h,v) = \langle \frac{1}{\gamma}KF, v\rangle = \frac{1}{\gamma}\langle F, K^* v\rangle = \frac{1}{\gamma}\langle F, v\rangle$$

Thus

$$0 = \langle F, v\rangle \qquad \forall\ v \text{ weak solution to (6.7)}$$

6.1.3.2 Third Existence Theory

Let's fix our operator $\mathcal{L}$, $F \in H^{-1}(\Omega)$ and consider the problem

$$\begin{cases} \mathcal{L}u = \lambda u + F & \Omega \\ u = 0 & \partial\Omega \end{cases} \tag{6.8}$$

We take $\gamma = \gamma(\mathcal{L}) \geq 0$ from First Existence Theorem 6.1.1. WLOG assume $\gamma > 0$. Notice also due to First Existence Theorem, necessarily

$$-\lambda \geq \gamma$$

WLOG we take $0 > \lambda + \gamma$. Then

$$\begin{aligned} \mathcal{L}_\gamma u &= (\gamma + \lambda) I u + F \\ u &= (\gamma + \lambda)\mathcal{L}_\gamma^{-1} I u + \mathcal{L}_\gamma^{-1} F \\ u &= \frac{\gamma + \lambda}{\gamma} K u + \mathcal{L}_\gamma^{-1} F \\ \frac{\gamma}{\gamma + \lambda} u - K u &= \frac{\gamma}{\gamma + \lambda} \mathcal{L}_\gamma^{-1} F \end{aligned}$$

Notice by Fredholm-Alternative, the equation is uniquely solvable iff the homogeneous equation has only the trivial solution, i.e.

$$\frac{\gamma}{\gamma + \lambda} u - K u = 0 \iff u = 0$$

This is to say, iff $\frac{\gamma}{\gamma+\lambda}$ is not eigenvalue to the operator K. Since for K compact, the set of eigenvalues is at most countable, and the only possible limit concentrates at 0, one obtain ([Eva10] Theorem 6.2.5)

1. There exists at most a countable set $\Lambda \subseteq \mathbb{R}$ s.t. (6.8) has a unique weak solution $u \in H_0^1(\Omega)$ for any $F \in H^{-1}(\Omega)$ iff $\lambda \notin \Lambda$.

2. The set Λ is either finite, or $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ where

$$\lambda_k \to \infty$$

6.2 Maximum Principle and Dirichlet Problem

We consider
$$\mathcal{L}u = \partial_i(a_{ij}\partial_j u + b_i u) + c_i\partial_i u + du$$
with measurable coefficients s.t.
$$\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2$$
and other coefficients bounded.

We say u solves
$$\mathcal{L}u = f + \text{div}g \qquad \Omega$$
if
$$\int a_{ij}\partial_j u\partial_i\varphi + b_i u\partial_i\varphi - c_i\partial_i u\varphi - du\varphi + \int f\varphi - \partial_i g\partial_i\varphi = 0 \qquad \forall\ \varphi \in H_0^1$$

We remark that existence, uniqueness and maximum principle holds under the sign condition convention that
$$\text{div}b + d \leq 0$$
i.e.
$$\int d\varphi - b_i\partial_i\varphi \leq 0 \qquad \forall\ \varphi \geq 0 \qquad \varphi \in H_0^1$$

Maximum Principle Let's assume
$$\mathcal{L}u = \partial_i(a_{ij}\partial_j u)$$

Theorem 6.2.1 (Maximum Principle)**.** *If $\mathcal{L}u \geq 0$ and $u^+ \in H_0^1$. Then $u \leq 0$ in Ω.*

Proof. Take u^+ itself as test function s.t. $u^+ \geq 0$
$$\begin{aligned} 0 &\geq \int a_{ij}\partial_i u\partial_j u^+ = \int a_{ij}\partial_i u^+\partial_j u^+ \\ &\geq \lambda\int |\nabla u^+|^2 \end{aligned}$$
□

Dirichlet Problem

Theorem 6.2.2. *For any $\varphi \in H^1(\Omega)$, f, $g \in L^2$, there exists a unique $u \in H^1(\Omega)$ s.t.*
$$\begin{cases} \mathcal{L}u = f + \text{div}g & \Omega \\ u - \varphi \in H_0^1 \end{cases}$$
Moreover
$$\|u\|_{H^1} \leq C\left(\|f\|_{L^2} + \sum_i \|g_i\|_{L^2} + \|\varphi\|_{H^1}\right)$$

If A is symmetric, u comes from minimizing
$$E(u) = \int \frac{1}{2}(\nabla u)^T A\nabla u + fu - g\cdot\nabla u \qquad u - \varphi \in H_0^1$$

6.3 Eigenvalue Problem

We consider divergence form uniformly elliptic operator

$$\mathcal{L}u = -\mathrm{div}(A\nabla u)$$

where $A = a_{ij} \in C^\infty(\overline{\Omega})$. Assume symmetry and uniform ellipticity.

The operator $\mathcal{L}$ is thus symmetric, and the associated bilinear form writes

$$B(u,v) := \int_\Omega a_{ij}\partial_i u \partial_j v \qquad \forall\ u,\ v \in H_0^1(\Omega)$$

Assume Ω is connected.

Theorem 6.3.1 ([Eva10] Theorem 6.5.1)**.** *Let $\mathcal{L}$ be symmetric uniformly elliptic operator as above.*

1. *Each eigenvalue of $\mathcal{L}$ is real.*
2. *There exists a sequence of eigenvalues*
$$\Sigma = \{\lambda_k\}_{k=1}^\infty$$
counting multiplicity s.t.
$$0 < \lambda_1 \le \lambda_2 \le \cdots$$
and $\lambda_k \to \infty$.
3. *Corresponding to Σ, there exists an orthonormal basis $\{w_k\}_{k=1}^\infty$ of $L^2(\Omega)$ where $w_k \in H_0^1(\Omega)$ is an eigenfunction to λ_k*
$$\begin{cases} \mathcal{L}w_k = \lambda_k w_k & \Omega \\ w_k = 0 & \partial\Omega \end{cases}$$
From Differentiability we know $w_k \in C^\infty(\Omega)$.

We define

$$\lambda_1 > 0$$

as the principal eigenvalue of $\mathcal{L}$ in Ω.

Theorem 6.3.2 (Variational Principle for Principal Eigenvalue; [Eva10] Theorem 6.5.2)**.** *Let $\mathcal{L}$ be symmetric uniformly elliptic operator as above.*

1. *λ_1 is computed via the Rayleigh Quotient*
$$\begin{aligned} \lambda_1 &= \min\{B(u,u) \mid u \in H_0^1(\Omega),\ \|u\|_{L^2(\Omega)} = 1\} \\ &= \min_{u \in H_0^1(\Omega),\ u \neq 0} \frac{B(u,u)}{\|u\|_{L^2(\Omega)}^2} \end{aligned}$$
2. *The minimum is attained by w_1 that is strictly positive in Ω, that solves*
$$\begin{cases} \mathcal{L}w_1 = \lambda_1 w_1 & \Omega \\ w_1 = 0 & \partial\Omega \end{cases}$$
3. *The principal eigenvalue λ_1 is simple, in other words, if $u \in H_0^1(\Omega)$ is any weak solution of*
$$\begin{cases} \mathcal{L}u = \lambda_1 u & \Omega \\ u = 0 & \partial\Omega \end{cases}$$
Then u is a multiple of w_1.

6.4 Differentiability

In this section we consider the equation

$$\mathcal{L}u = -\partial_j(a_{ij}\partial_i u) + b_i\partial_i u + cu = f$$

where a_{ij} is symmetric and uniformly elliptic with parameter $\lambda > 0$. Let $f \in L^2(\Omega)$.

H^2 Differentiability

Theorem 6.4.1 ([Eva10] Theorem 6.3.1; [GT01] Theorem 8.8)**.** *Let $u \in H^1(\Omega)$ be weak solution to*

$$\mathcal{L}u = f \qquad \Omega$$

Let $a_{ij} \in C^{0,1}$, b_i, $c \in L^\infty$ and $f \in L^2$. Then $u \in H^2_{loc}(\Omega)$, and in particular for any $\Omega' \Subset \Omega$

$$\|u\|_{H^2(\Omega')} \leq C(\Omega', \Omega, n, \lambda, \|a_{ij}\|_{C^{0,1}}, \|b\|_\infty, \|c\|_\infty)\left(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}\right)$$

Proof. Let $v \in C_0^\infty(\Omega)$ to be chosen. Then using that u is a solution

$$\int_\Omega a_{ij}\partial_i u\partial_j v = \int_\Omega fv - (b_i\partial_i uv + cuv)$$

Take any $\Omega' \Subset \Omega$, and take $2|h| < \mathrm{dist}(\Omega', \Omega)$. If we instead plug in the test function as difference quotient $\nabla_k^{-h}v(x) = \frac{v(x)-v(x-he_k)}{h}$, then using discrete integration by parts

$$\begin{aligned}\int_\Omega a_{ij}\partial_i u\nabla_k^{-h}\partial_j v &= -\int_\Omega \nabla_k^h(a_{ij}\partial_i u)\partial_j v\\ &= \int_\Omega (f - b_i\partial_i u - cu)\,\nabla_k^{-h}v\end{aligned}$$

We look at the term with highest derivatives. Using Product rule (note the second term on RHS is precisely where we need differentiability of a)

$$\nabla_k^h(a_{ij}\partial_i u) = a_{ij}(x + he_k)\nabla_k^h\partial_i u(x) + \partial_i u(x)\nabla_k^h a_{ij}(x)$$

One has

$$\int_\Omega a_{ij}(x + he_k)\nabla_k^h\partial_i u(x)\partial_j v = -\int_\Omega \partial_i u(x)\nabla_k^h a_{ij}(x)\partial_j v(x) - \int_\Omega (f - b_i\partial_i u - cu)\,\nabla_k^{-h}v$$

The point is now, both $\partial_j v$ and $\nabla_k^{-h}v$ one has control via $\|\nabla v\|$, and all derivatives on RHS are of lower orders. One may thus deduce a uniform estimate on the difference quotients for $\partial_i u$, leading to second order derivatives estimates.

To make the idea precise, let $\eta \in C_0^\infty(\Omega)$ with $0 \leq \eta \leq 1$, $\eta = 1$ on Ω', $|\nabla\eta| \leq \frac{2}{\mathrm{dist}(\Omega', \partial\Omega)}$. Now we set

$$v = \eta^2\nabla_k^h u$$

as the difference quotient and compute

$$\begin{aligned}\partial_j(\eta^2\nabla_k^h u) &= 2\eta\partial_j\eta\nabla_k^h u + \eta^2\nabla_k^h\partial_j u\\ \nabla_k^{-h}(\eta^2\nabla_k^h u)(x) &= \eta^2(x - he_k)\nabla_k^{-h}\nabla_k^h u(x) + \nabla_k^{-h}\eta^2(x)\nabla_k^h u(x)\end{aligned}$$

Let $K = K(\Omega, \Omega', n, \|a\|_{C^{0,1}}, \|b\|_\infty, \|c\|_\infty)$ be the constant that bounds all coefficients of the equation, as well as η. So the equation writes via Young's (for ε small universal)

$$\begin{aligned}\int_\Omega a_{ij}(x + he_k)\eta^2\nabla_k^h\partial_i u(x)\nabla_k^h\partial_j u &= -\int_\Omega 2\eta\partial_j\eta a_{ij}(x + he_k)\nabla_k^h\partial_i u(x)\nabla_k^h u\\ &\quad - \int_\Omega \partial_i u(x)\nabla_k^h a_{ij}(x)(2\eta\partial_j\eta\nabla_k^h u + \eta^2\nabla_k^h\partial_j u)\\ &\quad - \int_\Omega (f - b_i\partial_i u - cu)\,(\eta^2(x - he_k)\nabla_k^{-h}\nabla_k^h u(x) + \nabla_k^{-h}\eta^2(x)\nabla_k^h u(x))\\ &\leq K\varepsilon\left(\int_\Omega |\nabla_k^h\partial_i u(x)|^2 + \int_\Omega |\nabla_k^{-h}\nabla_k^h u(x)|^2\right)\\ &\quad + KC(\varepsilon)\left(\int_\Omega |\nabla_k^h u|^2 + \int_\Omega |\nabla u|^2 + \int_\Omega u^2 + \int_\Omega f^2\right)\\ \frac{1}{2}\lambda\int_{\Omega'} |\nabla^h\nabla u|^2 &\leq K(\|u\|^2_{H^1(\Omega)} + \|f\|^2_{L^2(\Omega)})\end{aligned}$$

where the last inequality uses uniform ellipticity, and choosing ε small depending on K, λ. Since this estimate is uniform in h, one has $u \in H^2(\Omega')$ along with the desired estimate. □

In fact one may replace the above $\|u\|_{H^1(\Omega)}$ with $\|u\|_{L^2(\Omega)}$.

Lemma 6.4.1 ([GT01] Exercise 8.2)**.** *Let $u \in H^1(\Omega)$ be weak solution to $\mathcal{L}u = f$ in Ω with λ uniform elliptic constant, $a_{ij}, b_i, c \in L^\infty$, and $f \in L^2$. Then for any $\Omega' \Subset \Omega$ one has*

$$\|u\|_{H^1(\Omega')} \le C\left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right)$$

Proof. Caccioppoli. □

Domain Deformation Divergence Equations preserve naturally when changing variables. Assume u solves

$$\mathcal{L}u = -\text{div}(A(x)u(x)) + b(x)\cdot\nabla u(x) + c(x)u(x) = 0, \qquad \Omega_x \subseteq \mathbb{R}^n$$

i.e.,

$$0 = \int_{\Omega_x} \nabla v(x)^T A(x)\nabla u(x) + b(x)\cdot\nabla u(x)v(x) + cuv \qquad \forall\ v \in H_0^1(\Omega_x)$$

Consider a bi-Lipschitz transformation

$$\begin{aligned}\Psi : \Omega_x &\to \Omega_y\\ x &\mapsto y = \Psi(x) = (\psi^1(x), \cdots, \psi^n(x))\end{aligned}$$

Then we consider

$$u(x) = \tilde{u}(y) = \tilde{u}(\Psi(x))$$

We compute

$$\begin{aligned}\partial_i u(x) &= \partial_k \tilde{u}(\Psi(x))\partial_i\psi^k(x)\\ \nabla u(x) &= D\Psi(x)\nabla\tilde{u}(\Psi(x))\end{aligned}$$

Similar for any $v \in H_0^1(\Omega_x)$ as test function and

$$v(x) = \tilde{v}(y) = \tilde{v}(\Psi(x))$$

it obeys the same rule. Thus

$$\begin{aligned}\nabla v(x)^T A(x)\nabla u(x) &= \nabla\tilde{v}(\Psi(x))^T D\Psi(x)^T A(x) D\Psi(x)\nabla u(x)\\ b(x)\cdot\nabla u(x)v(x) &= b(x)^T D\Psi(x)\nabla\tilde{u}(\Psi(x))\tilde{v}(\Psi(x))\\ c(x)u(x)v(x) &= c(x)\tilde{u}(\Psi(x))\tilde{v}(\Psi(x))\end{aligned}$$

But remember there is Jacobian as a result of change of variables under the integral

$$|\frac{\partial\Psi^{-1}}{\partial y}(y)| = |\det(D\Psi^{-1}(y))| > 0$$

So we define

$$\begin{aligned}&\tilde{A}(y) := D\Psi(\Psi^{-1}(y))^T A(\Psi^{-1}(y)) D\Psi(\Psi^{-1}(y))|\det(D\Psi^{-1}(y))|\\ &\tilde{b}(y) = D\Psi(\Psi^{-1}(y))b(\Psi^{-1}(y))|\det(D\Psi^{-1}(y))|\\ &\tilde{c}(y) = c(\Psi^{-1}(y))|\det(D\Psi^{-1}(y))|\end{aligned}$$

and notice $\tilde{u}$ solves

$$0 = \int_{\Omega_y} \nabla\tilde{v}(y)^T\tilde{A}(y)\nabla\tilde{u}(y) + \tilde{b}(y)\cdot\nabla\tilde{u}(y)\tilde{v}(y) + \tilde{c}(y)\tilde{u}(y)\tilde{v}(y)dy \qquad \forall\ \tilde{v} \in H_0^1(\Omega_y)$$

In other words

$$\tilde{\mathcal{L}}\tilde{u} := -\text{div}(\tilde{A}(y)\nabla\tilde{u}(y)) + \tilde{b}\cdot\nabla\tilde{u}(y) + \tilde{c}\tilde{u} = 0 \qquad \Omega_y$$

Notice $\tilde{\mathcal{L}}$ remains a uniformly-elliptic operator.

6.5 Schauder Theory for Divergence Form Equations

Theorem 6.5.1 (Savin Analysis II 2026)**.** *Assume*

$$\mathcal{L}u = f + \text{div}g \qquad B_1$$

where $\mathcal{L}u := \partial_i(a_{ij}\partial_j u)$ *with* $a_{ij} \in C^{0,\alpha}$, $g \in C^{0,\alpha}$ *for* $0 < \alpha < 1$*. Let* $f \in L^p$ *for* $p > n$ *with*

$$0 < \alpha \leq 1 - \frac{n}{p}$$

Then $u \in C^{1,\alpha}(B_1)$ *and*

$$\|u\|_{C^{1,\alpha}(\overline{B_{1/2}})} \leq C\left(\|u\|_{L^2} + \|f\|_{L^p} + \|g\|_{C^{0,\alpha}}\right) \tag{6.9}$$

Lemma 6.5.1. *Assume* $a_{ij}(0) = \delta_{ij}$, $g(0) = 0$*, and for* $\delta > 0$ *universal to be chosen*

$$\|a_{ij} - \delta_{ij}\|_\infty + \|u\|_{L^2} + \|f\|_{L^p} + \|g\|_{C^{0,\alpha}} \leq \delta$$

We claim there exists $\rho \in (0,1)$ *universal and a sequence of bounded linear function* ℓ_r *s.t.*

$$\fint_{B_r} (u - \ell_r)^2 dx \leq r^{2+2\alpha} \tag{6.10}$$

where $r = \rho^k$ *for any* $k \in \mathbb{N}$.

Proof. For $k = 0$ done. Assume for $r = \rho^k$. Rescale

$$\tilde{u}(x) := \frac{1}{r^{1+\alpha}}(u - \ell_r)(rx) \qquad \forall\, x \in B_1$$

Rescaling.
What does $\tilde{u}$ solve? Rewrite

$$u(x) = r^{1+\alpha}\tilde{u}(\frac{x}{r}) + \ell_r(x) \qquad \forall\, x \in B_r$$

In particular let

$$\ell_r(x) = p_r + q_r \cdot x$$

so that

$$\begin{aligned}
\partial_j u(x) &= r^\alpha \partial_j \tilde{u}(\frac{x}{r}) + \partial_j \ell_r(x) \\
\partial_j u(ry) &= r^\alpha \partial_j \tilde{u}(y) + (q_r)_j \qquad y \in B_1 \\
a_{ij}(ry)\partial_j u(ry) &= r^\alpha a_{ij}(ry)\partial_j \tilde{u}(y) + a_{ij}(ry)(q_r)_j
\end{aligned}$$

Now be careful the next derivative is in ∂_{x_i}.
For rescaling $x = ry$,

$$\partial_{x_i} = \frac{1}{r}\partial_{y_i}$$

Thus

$$\begin{aligned}
\partial_{x_i}\left(a_{ij}(ry)\partial_j u(ry)\right) &= \partial_{x_i}\left(r^\alpha a_{ij}(ry)\partial_j \tilde{u}(y) + a_{ij}(ry)(q_r)_j\right) \\
\frac{1}{r}\partial_{y_i}\left(a_{ij}(ry)\partial_j u(ry)\right) &= \frac{1}{r}\partial_{y_i}\left(r^\alpha a_{ij}(ry)\partial_j \tilde{u}(y) + a_{ij}(ry)(q_r)_j\right)
\end{aligned}$$

But the LHS write

$$\begin{aligned}
\frac{1}{r}\partial_{y_i}\left(a_{ij}(ry)\partial_j u(ry)\right) &= \partial_{x_i}(a_{ij}(x)\partial_j u(x)) = f(x) + \text{div}g(x) \\
&= f(ry) + \frac{1}{r}\text{div}_y(g(ry))
\end{aligned}$$

while RHS writes

$$\frac{1}{r}\partial_{y_i}\left(r^\alpha a_{ij}(ry)\partial_j \tilde{u}(y) + a_{ij}(ry)(q_r)_j\right) = r^{\alpha-1}\partial_{y_i}\left(\tilde{a_{ij}}(y)\partial_j \tilde{u}(y)\right) + r^{-1}\partial_{y_i}\left(\tilde{a_{ij}}(y)\partial_j \ell_r\right)$$

where

$$\tilde{a}_{ij}(y) := a_{ij}(ry) \qquad \forall\, y \in B_1$$

so aligning both yields

$$\begin{aligned}
f(ry) + \frac{1}{r}\mathrm{div}_y(g(ry)) &= r^{\alpha-1}\partial_{y_i}\left(\tilde{a_{ij}}(y)\partial_j\tilde{u}(y)\right) + r^{-1}\partial_{y_i}\left(\tilde{a_{ij}}(y)\partial_j\ell_r\right) \\
\partial_{y_i}\left(\tilde{a_{ij}}(y)\partial_j\tilde{u}(y)\right) &= r^{1-\alpha}f(ry) + r^{-\alpha}\mathrm{div}_y(g(ry)) - r^{-\alpha}\partial_{y_i}\left(\tilde{a_{ij}}(y)\partial_j\ell_r\right) \\
&= r^{1-\alpha}f(ry) + r^{-\alpha}\mathrm{div}_y\left(g(ry) - \tilde{a}_{ij}(y)\nabla\ell_r\right) \\
&= \tilde{f}(y) + \mathrm{div}_y\tilde{g}(y)
\end{aligned}$$

where

$$\tilde{f}(y) = r^{1-\alpha}f(ry), \qquad \tilde{g}(y) = r^{-\alpha}\left(g(ry) - (\tilde{A}(y) - I)\nabla\ell_r\right)$$

Since $\tilde{g}$ is under divergence and $\nabla\ell_r$ are constants, we will squeeze in $\tilde{A}(0) = I$ inside. So the final rescaled equation solves

$$\partial_i(\tilde{a}_{ij}\partial_j\tilde{u}) = \tilde{f} + \mathrm{div}\tilde{g} \qquad B_1 \tag{6.11}$$

f **Forcing Term**.
Look at

$$\int_{B_1}|\tilde{f}|^p = r^{(1-\alpha)p}\int_{B_1}|f(ry)|^p dy = r^{(1-\alpha)p-n}\int_{B_r}|f(x)|^p dx$$

Now assume $f \in L^p$, but when is the factor in the front finite? We need

$$(1-\alpha)p - n > 0 \iff \alpha < 1 - \frac{n}{p}$$

In particular for the integrability of f, $p > n$ can be expected.
Thus

$$\|f\|_{L^p} \leq \delta r^{\frac{(1-\alpha)p-n}{p}} \leq C\delta$$

g **Term**.
Look at

$$\begin{aligned}
\|\tilde{g}\|_{C^{0,\alpha}(B_1)} &\leq C\left(\|g\|_{C^{0,\alpha}(B_r)} + \|a_{ij} - \delta_{ij}\|_{L^\infty(B_r)}|\nabla\ell_r|\right) \\
&\leq C\delta
\end{aligned}$$

Using we have a bounded sequence of linear functions, and our initial smallness assumption.

Construct w **a harmonic replacement in** $B_{1/2}$.
Now let w be harmonic replacement of $\tilde{u}$ in $B_{1/2}$ with same boundary data as $\tilde{u}$ on $\partial B_{1/2}$, so

$$\tilde{u} - w \in H_0^1(B_{1/2})$$

Now we see what $\tilde{u} - w$ solves

$$\begin{aligned}
\partial_i(\tilde{a}_{ij}\partial_j(\tilde{u} - w)) &\overset{(6.11)}{=} \tilde{f} + \mathrm{div}\tilde{g} - \partial_i((\tilde{a}_{ij} - \delta_{ij})\partial_j w + \cancel{\delta_{ij}\partial_j w}) \qquad \text{using } w \text{ harmonic} \\
&= \tilde{f} + \mathrm{div}(\tilde{g} - (A - I)\nabla w)
\end{aligned}$$

Notice

$$\int_{B_{1/2}}|\nabla w|^2 \leq \underbrace{\|\tilde{u}\|^2_{H^1(B_{1/2})} \leq C}_{\text{Caccioppoli}}$$

simply because w is minimizer to Dirichlet energy as harmonic function, and the latter is bounded due to Caccioppoli. Thus with force small

$$\left\|\tilde{f} + \mathrm{div}(\tilde{g} - (A - I)\nabla w)\right\|_{H^{-1}} \leq C\delta$$

One obtain immediately via the a prior bound (6.6) that

$$\|\tilde{u} - w\|_{H_0^1(B_{1/2})} \leq C\delta$$

Choice of next iteration Thus for $\rho < 1/2$

$$⨍_{B_\rho}|\tilde{u} - w|^2 \leq C\delta^2\rho^{-n}$$

Take the linear part of w

$$\ell_w := w(0) + \nabla w(0) \cdot x$$

so that by C^2 interior estimate

$$\|w - \ell_w\|_{L^\infty(B_\rho)} \le \left\|D^2 w\right\|_\infty \rho^2 \le C\rho^2$$

Thus

$$\begin{aligned}
\fint_{B_\rho} (\tilde{u} - \ell_w)^2 &\le \fint_{B_\rho} |\tilde{u} - w|^2 + \fint_{B_\rho} |w - \ell_w|^2 \\
&\le C\delta^2 \rho^{-n} + C\rho^4 \\
&\overset{\text{want to choose}}{\le} \rho^{2+2\alpha}
\end{aligned}$$

By first choosing ρ small then δ small.

Thus

$$\begin{aligned}
\fint_{B_\rho} (\frac{1}{r^{1+\alpha}} u(rx) - \frac{1}{r^{1+\alpha}} \ell_r(rx) - \ell_w(x))^2 &\le \rho^{2+2\alpha} \\
\fint_{B_\rho} |u(rx) - \ell_r(rx) - r^{1+\alpha}\ell_w(x)|^2 &\le (r\rho)^{2+2\alpha} \\
\fint_{B_{r\rho}} |u(x) - \ell_r - r^{1+\alpha}(\frac{x}{r})|^2 &\le (r\rho)^{2+2\alpha}
\end{aligned}$$

But recall $r\rho = \rho^{k+1}$. so we've verified at $k+1$ for (6.10). $\square$

6.6 De Giorgi Theory

We study

$$\mathcal{L}u := \operatorname{div}(A(x)\nabla u(x)) \qquad \forall\, x \in B_1$$

for A uniformly elliptic, i.e., there exists $\lambda, \Lambda > 0$ s.t.

$$\lambda I \leq |A(x)| \leq \Lambda I \qquad \forall\, x \in B_1$$

Most importantly, we assume no regularity on A, i.e., we only need $A \in L^\infty(B_1)$.

6.6.1 From L^2 to L^∞

Theorem 6.6.1 ($L^2 \to L^\infty$; De Silva Analysis II 2025)**.** *Assume $u \in H^1(B_1)$ subsolution*

$$\mathcal{L}u \geq 0 \qquad \text{in } B_1 \text{ in the weak sense}$$

Then for $u^+ := \max\{u, 0\}$

$$\left\|u^+\right\|_{L^\infty(B_{1/2})} \leq C(n, \lambda, \Lambda) \left\|u^+\right\|_{L^2(B_1)} \tag{6.12}$$

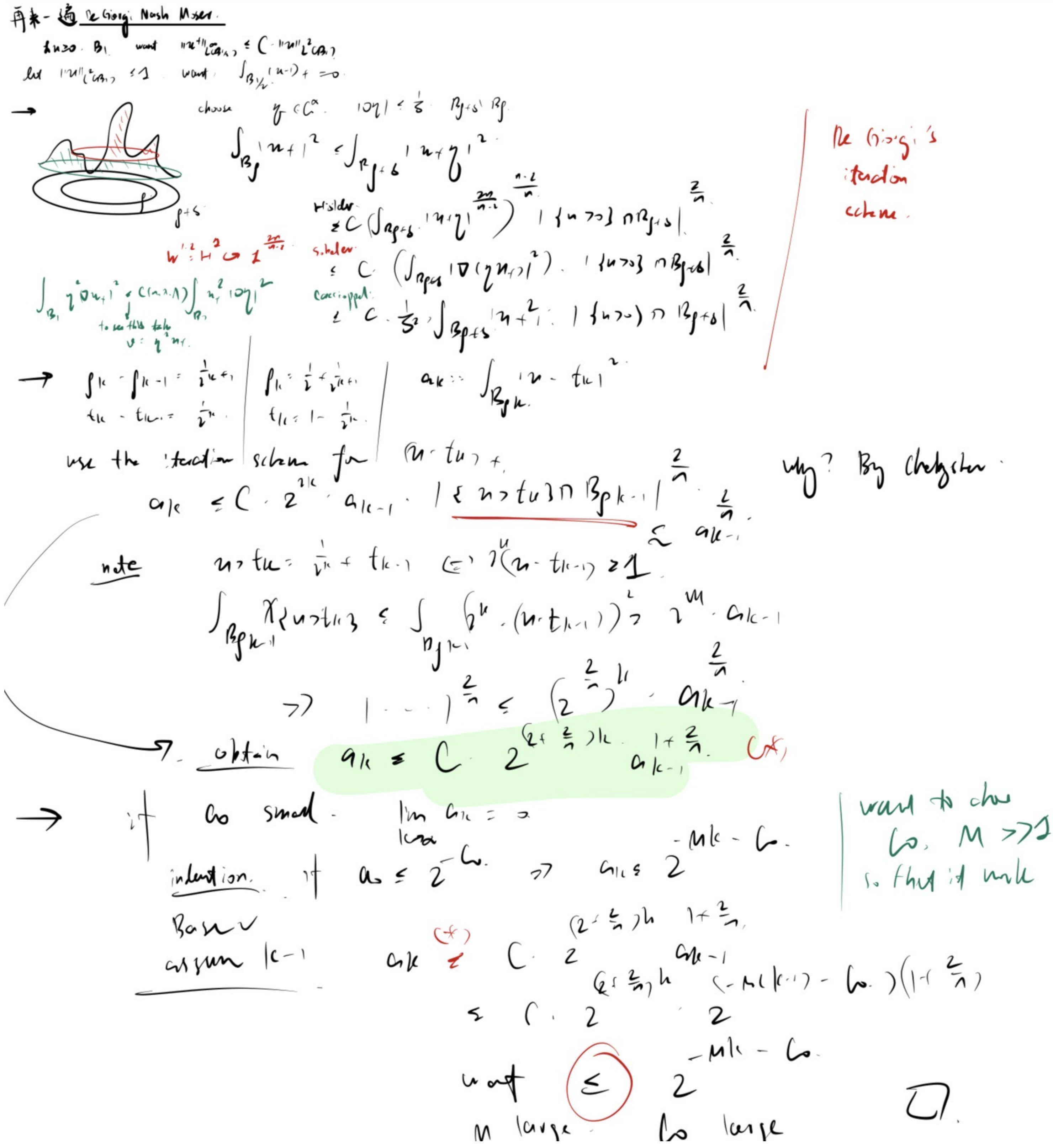


Figure 6.1: De Giorgi Nash Moser

Proof. 1. Leveraging linearity, it suffices to assume

$$\|u^+\|^2_{L^2(B_1)} \leq 2^{-C_0}$$

for $C_0 \gg 1$ to be chosen, and we wish to prove

$$\|u^+\|_{L^\infty(B_{1/2})} \leq 1$$

The first clever thought of De Giorgi is the folllowinng: Instead of pointwise estimate, we alternatively prove

$$\int_{B_{1/2}} ((u-1)^+)^2 = 0 \tag{6.13}$$

The famous De Giorgi iteration exploits the above.

2. In this reduction step, we deduce a 'model' for our iteration scheme. Let us first work with two parameters s.t. $\frac{1}{2} < \rho < \rho + \delta < 1$. In this model, we choose a smart cutoff function $\eta \in C_0^\infty(B_{\rho+\delta})$ s.t.

$$\eta \equiv 1 \qquad B_\rho; \qquad |\nabla \eta| \le \frac{C}{\delta} \qquad B_{\rho+\delta} \setminus B_\rho; \qquad \eta = 0 \qquad \partial B_{\rho+\delta}$$

Let's play with

$$\begin{aligned}
\int_{B_\rho} (u^+)^2 &\le \int_{B_{\rho+\delta}} (\eta u^+)^2 \qquad \text{Choice of cutoff} \\
&\le \left(\int_{B_{\rho+\delta}} (\eta u^+)^{2\cdot\frac{n}{n-2}}\right)^{\frac{n-2}{n}} \left(\int_{B_{\rho+\delta}} \mathbb{1}_{\{u^+>0\}}\right)^{\frac{2}{n}} = \left(\int_{B_{\rho+\delta}} (\eta u^+)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} |\{u>0\} \cap B_{\rho+\delta}|^{\frac{2}{n}} \qquad \text{Hölder} \\
&\le C(n) \left(\int_{B_{\rho+\delta}} |\nabla(\eta u^+)|^2\right) |\{u>0\} \cap B_{\rho+\delta}|^{\frac{2}{n}} \qquad \text{Sobolev } W^{1,2} \hookrightarrow L^{\frac{2n}{n-2}} \text{ and so } \eta u^+ \in H_0^1 \\
&\le C(n,\lambda,\Lambda) \left(\int_{B_{\rho+\delta}} |\nabla \eta|^2 (u^+)^2\right) |\{u>0\} \cap B_{\rho+\delta}|^{\frac{2}{n}} \qquad \text{Caccioppoli (6.15)} \\
&\le \frac{C}{\delta^2} \left(\int_{B_{\rho+\delta}} (u^+)^2\right) |\{u>0\} \cap B_{\rho+\delta}|^{\frac{2}{n}} \qquad \text{Choice of cutoff}
\end{aligned} \tag{6.14}$$

In order to estimate (6.13), we need to choose ρ as a sequence to approach $\frac{1}{2}$, as well as a sequence of functions to approach $u-1$.

3. In this step we prove the Caccioppoli Estimate used above. This is the only place where we've used the equation! Recall we say that u is a subsolution, $\mathcal{L}u \ge 0$ in B_1 if

$$\int_{B_1} (\nabla v) A(x) \nabla u \le 0 \qquad \forall\, v \in H_0^1(B_1), \qquad v \ge 0$$

The clever choice of De Giorgi's test function is the following

$$v := \eta^2 u^+ \qquad \forall\, \eta \in C_0^\infty(B_1)$$

Testing against u yields

$$\begin{aligned}
0 \ge \underbrace{\int_{B_1} \nabla(\eta^2 u^+) A(x) \nabla u^+}_{\text{this is where we use the equation}} &= \int_{B_1} 2\eta u^+ \nabla \eta A(x) \nabla u^+ + \eta^2 \nabla u^+ A(x) \nabla u^+ \\
&\ge -\Lambda \left(\varepsilon \int_{B_1} \eta^2 |\nabla u^+|^2 + \frac{1}{C(\varepsilon)} \int_{B_1} |\nabla \eta|^2 (u^+)^2\right) + \lambda \int_{B_1} \eta^2 |\nabla u^+|^2 \qquad \varepsilon\text{-Young's Inequality}
\end{aligned}$$

Let's put terms involving $\eta^2|\nabla u^+|^2$ on the LHS, and choose $\varepsilon \ll 1$ to absorb the same term to the LHS. We obtain

$$\int_{B_1} \eta^2 |\nabla u^+|^2 \le C(\lambda,\Lambda) \int_{B_1} |\nabla \eta|^2 (u^+)^2 \tag{6.15}$$

This is the Beautiful Caccioppoli's Estimate :). It mainly says u cannot jump too quickly. Notice that in Caccioppoli we require subsolution since u^+ is a subsolution. And in step 2 we're applying to

$$\begin{aligned}
\int_{B_{\rho+\delta}} |\nabla(\eta u^+)|^2 &= \int_{B_{\rho+\delta}} |(\nabla \eta) u^+ + \eta (\nabla u^+)|^2 \le C \left(\int_{B_{\rho+\delta}} |(\nabla \eta) u^+|^2 + \int_{B_{\rho+\delta}} |\eta (\nabla u^+)|^2\right) \\
&\overset{(6.15)}{\le} C(\lambda,\Lambda) \int_{B_{\rho+\delta}} |(\nabla \eta) u^+|^2
\end{aligned}$$

4. In this step we define precisely the De Giorgi's Iteration. We pick domain shrinkage as

$$\rho_k := \frac{1}{2} + \frac{1}{2^{k+1}} \qquad \delta_k := \frac{1}{2^{k+1}} \qquad \forall\ k \geq 0$$

so that

$$\rho_k + \delta_k = \rho_{k-1} \qquad \forall\ k \geq 1$$

On the other hand, we pick truncation of function at the same rate

$$t_k := 1 - \frac{1}{2^k} \qquad \forall\ k \geq 0$$

and define our objects on both sides (6.14) as the following

$$\begin{aligned} a_k &:= \int_{B_{\rho_k}} ((u - t_k)^+)^2 \qquad \forall\ k \geq 0 \\ A_k &:= \{u > t_k\} \cap B_{\rho_{k-1}} \qquad \forall\ k \geq 1 \end{aligned}$$

In particular we record our gaps for domain shrinkage and truncation upwards

$$\rho_k - \rho_{k-1} = \frac{1}{2^{k+1}}, \qquad t_k - t_{k-1} = \frac{1}{2^k}$$

Therefore (6.14) applied to $u - t_k$ rewrites

$$\begin{aligned} a_k &\leq C 2^{2(k+1)} \left(\int_{B_{\rho_{k-1}}} ((u - t_k)^+)^2 \right) |A_k|^{\frac{2}{n}} \\ &\leq C 2^{2k} a_{k-1} |A_k|^{\frac{2}{n}} \qquad \text{using } t_k \text{ is increasing} \end{aligned} \tag{6.16}$$

How lovely would it be if we can get rid of $|A_k|$ using a_{k-1}! Indeed, note

$$t_{k-1} = 1 - \frac{1}{2^{k-1}} = 1 - \frac{2}{2^k} = t_k - \frac{1}{2^k}$$

so

$$u > t_k \iff u > t_{k-1} + \frac{1}{2^k} \iff 2^k (u - t_{k-1}) > 1$$

Chebyshev now gives

$$|A_k| = \int_{B_{\rho_{k-1}}} \mathbb{1}_{\{u > t_k\}} \overset{\text{Chebyshev}}{\leq} \int_{B_{\rho_{k-1}}} \mathbb{1}_{\{u > t_k\}} 2^k (u - t_{k-1})^+ \leq 2^k a_{k-1}$$

Thus (6.16) rewrites

$$\begin{aligned} a_k &\leq C 2^{2k} a_{k-1} (2^k a_{k-1})^{\frac{2}{n}} \\ &\leq C (2^{2 + \frac{2}{n}})^k a_{k-1}^{1 + \frac{2}{n}} \end{aligned} \tag{6.17}$$

5. In this step we conclude via an induction argument. In note of (6.13), observe

$$\int_{B_{1/2}} ((u - 1)^+)^2 \leq a_k \qquad \forall\ k \geq 0$$

It all boils down to show using (6.17) that

$$\lim_{k \to \infty} a_k = 0$$

To do so, we claim that there exists dimensional constants M, $C_0 \gg 1$ s.t.

$$a_k \leq 2^{-Mk - C_0} \qquad \forall\ k \geq 0$$

At base step we assumed

$$a_0 = \int_{B_1} (u^+)^2 \leq 2^{-C_0}$$

for $C_0 \gg 1$ to be chosen. Now assume we have already shown

$$a_{k-1} \le 2^{-M(k-1)-C_0}$$

For the next step we use (6.17)

$$\begin{aligned} a_k &\le C(2^{\frac{2n+2}{n}})^k a_{k-1}^{1+\frac{2}{n}} \\ &\le C(2^{\frac{2n+2}{n}})^k 2^{-\frac{n+2}{n}M(k-1)-\frac{n+2}{n}C_0} \qquad \text{using inductive assumption} \\ &= C2^{(\frac{2n+2}{n}-\frac{n+2}{n}M)k} \cdot 2^{\frac{n+2}{n}M-\frac{n+2}{n}C_0} \end{aligned}$$

It suffices to choose M, $C_0 \gg 1$ so that

$$\begin{aligned} \frac{2n+2}{n} - \frac{n+2}{n}M &\le -M \\ \frac{n+2}{n}M - \frac{n+2}{n}C_0 &\le -C_0 \end{aligned}$$

Rearranging yields

$$\begin{aligned} M &\ge n+1 \\ C_0 &\ge \frac{n+2}{2}M \end{aligned}$$

We're done :))

□

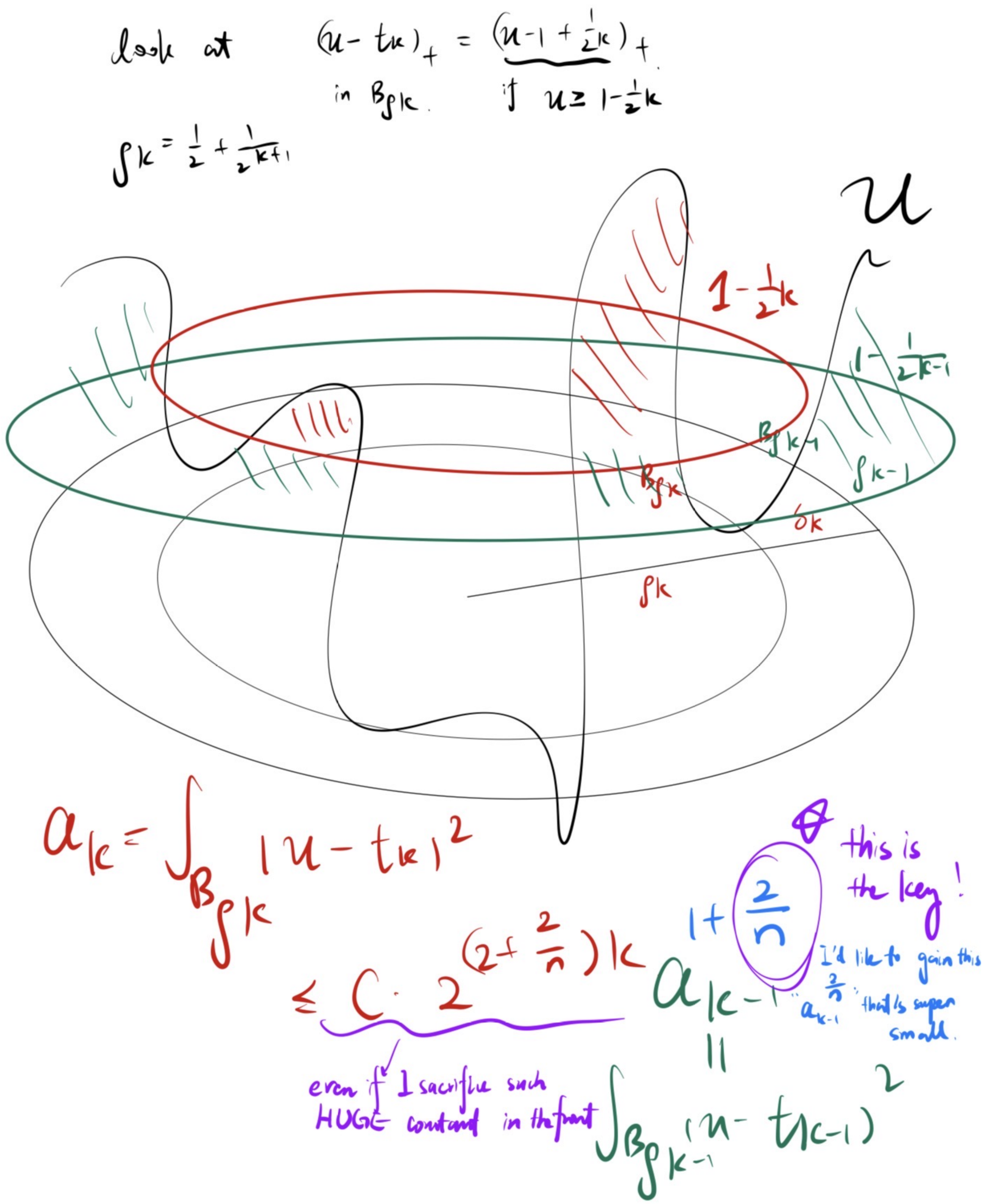


Figure 6.2: Local Boundedness

6.6.2 From L^∞ to C^α

6.6.2.1 Weak Harnack for Subsolution Version 1 (De Giorgi Oscillation)

The following is also known as the De Giorgi Oscillation Lemma. It in fact assumes for two

1. $\|u\|_\infty \leq 1$ (which in our case is given by Local Boundedness (6.12)).
2. Some set that touches zero of positive measure, so solution gets attracted downwards.

Theorem 6.6.2 (Weak Harnack for Subsolution Version 1; De Silva Analysis II 2025, [FRRO22] Lemma 3.14, 3.15)**.** *Assume $u \in H^1(B_1)$ subsolution*

$$\mathcal{L}u \geq 0 \qquad \text{in } B_1 \text{ in the weak sense}$$

Then for any $\delta > 0$ s.t.

$$|\{u \leq 0\} \cap B_{1/2}| \geq \delta |B_{1/2}| \tag{6.18}$$

the positive measure attracts our solution downwards, i.e., there exists $0 < c(n, \lambda, \Lambda, \delta) < 1$ s.t.

$$\left\|u^+\right\|_{L^\infty(B_{1/4})} \leq (1 - c(\delta)) \left\|u^+\right\|_{L^\infty(B_{3/4})} \tag{6.19}$$

Proof. 1. Leveraging linearity and Local Boundedness 6.6.1, it suffices to assume

$$\left\|u^+\right\|_{L^\infty(B_{3/4})} = 1$$

and we wish to show that there exists $c(\delta) > 0$

$$u^+ \leq 1 - c(\delta) \qquad \forall\, x \in B_{1/4}$$

One might naively ask: why is this a problem? As a tryout, let's directly estimate

$$\begin{aligned}
\left\|u^+\right\|_{L^\infty(B_{1/4})} &\leq C \left\|u^+\right\|_{L^2(B_{1/2})} = C \left(\int_{B_{1/2}} (u^+)^2 \right)^{\frac{1}{2}} && \text{use Local Boundedness Theorem 6.6.1} \\
&\leq C \left\|u^+\right\|_{L^\infty(B_{1/2})} |\{u > 0\} \cap B_{1/2}|^{\frac{1}{2}} = C |\{u > 0\} \cap B_{1/2}|^{\frac{1}{2}} && \text{directly pulling out } u^+ \\
&\leq C(1-\delta)^{\frac{1}{2}} |B_{1/2}|^{\frac{1}{2}} && \text{use assumption (6.18)}
\end{aligned}$$

If our δ is sufficiently close to 1, then of course we can make RHS strictly smaller than 1, in which case we're done. But we wish to prove the result (6.19) for any $\delta > 0$. Hence we're stuck at δ small.

2. The clever idea is to again use truncation. Define

$$v_k := \frac{(u - t_k)^+}{1 - t_k} \qquad t_k := 1 - \frac{1}{2^k} \qquad \forall\, k \geq 0$$

Our v_k shall be thought of as lowering u dyadically and rescaling back via 2^k so that

$$\|v_k\|_{L^\infty(B_{3/4})} = 1 \tag{6.20}$$

Notice v_k remains a subsolution, so we throw v_k into our tryout

$$\begin{aligned}
\|v_k\|_{L^\infty(B_{1/4})} &\overset{(6.12)}{\leq} C \|v_k\|_{L^2(B_{1/2})} \\
&\leq C \|v_k\|_{L^\infty(B_{1/2})} |\{v_k > 0\} \cap B_{1/2}|^{\frac{1}{2}} \\
&\overset{(6.20)}{\leq} C |\{u > t_k\} \cap B_{1/2}|^{\frac{1}{2}}
\end{aligned}$$

How lovely would it be if we can make the RHS as small as we wish for k large enough! Indeed, we make the claim that

$$\forall\, \varepsilon > 0, \qquad \exists\, k(\varepsilon) \gg 1, \qquad \text{s.t.} \qquad |\{u > t_{k(\varepsilon)}\} \cap B_{1/2}| < \varepsilon \tag{6.21}$$

If so, let's pick $\varepsilon = \frac{1}{(2C)^2}$ so for $k = k(\varepsilon)$ as above

$$v_k = \frac{(u - t_k)^+}{1 - t_k} \leq C |\{u > t_k\} \cap B_{1/2}|^{\frac{1}{2}} \overset{(6.21)}{\leq} \frac{1}{2} \qquad \forall\, x \in B_{1/4}$$

Rearranging

$$\begin{aligned}
u &\leq t_k + \frac{1}{2}(1 - t_k) = 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} \\
&= 1 - \frac{1}{2^{k+1}} \qquad \forall\, x \in B_{1/4}
\end{aligned}$$

We pick $c(\delta) := \frac{1}{2^{k+1}}$ to conclude. But wait! Where's the dependence on δ? This piece of information shall only be made clear once we prove the claim.

3. So in this step we prove Claim (6.21). As usual, assume not. Then there exists $\varepsilon > 0$ s.t. for any $k \geq 0$, we have
$$|\{u > t_k\} \cap B_{1/2}| = |\{v_k > 0\} \cap B_{1/2}| \geq \varepsilon \tag{6.22}$$
Let's see where we're going in this contradiction. In the end we want to show that
$$|\{0 < v_k < \frac{1}{2}\} \cap B_{1/2}| \geq \beta \tag{6.23}$$
for some $\beta > 0$ that is independent of k! Why is this a problem? Well, all sets $\{0 < v_k < \frac{1}{2}\}$ are in fact disjoint
$$\{0 < v_k < \frac{1}{2}\} = \{0 < \frac{(u - t_k)^+}{1 - t_k} < \frac{1}{2}\} = \{t_k < u < t_{k+1}\}$$
hence summing over yields a contradiction
$$|\{0 < u < 1\} \cap B_{1/2}| \geq \sum_{k=0}^{\infty} |\{0 < v_k < \frac{1}{2}\} \cap B_{1/2}| \overset{(6.23)}{=} \infty$$
Now how do we prove (6.23)? (See [FRRO22] Lemma 3.15, this is known as the De Giorgi's Isoperimetric Inequality. Roughly this is quantitative version that H^1 function cannot have jump discontinuity)

(a) We first give a lower bound for LHS of (6.23)
$$|\{0 < v_k < \frac{1}{2}\} \cap B_{1/2}|$$
with ∇v_k. We use Caccioppoli (6.15) with cutoff choice $\eta = 1$ on $B_{1/2}$ and $\eta \in C_0^\infty(B_{3/4})$.
$$\begin{aligned}\int_{B_{1/2}} |\nabla v_k|^2 &\leq C \int_{B_{3/4}} v_k^2 \qquad \text{Caccioppoli (6.15)}\\ &\leq C \|v_k\|^2_{L^\infty(B_{3/4})} |\{v_k > 0\} \cap B_{3/4}| \overset{(6.20)}{\leq} C \qquad \text{uniformly in } k\end{aligned} \tag{6.24}$$
This is precisely where we need 1/2 to go a little bit inside. Now on the other hand
$$\begin{aligned}\int_{B_{1/2}} |\nabla v_k| \mathbb{1}_{\{0<v_k<\frac{1}{2}\}} &\leq \left(\int_{B_{1/2}} |\nabla v_k|^2\right)^{\frac{1}{2}} |\{0 < v_k < \frac{1}{2}\} \cap B_{1/2}|^{\frac{1}{2}} \qquad \text{use Hölder}\\ &\overset{(6.24)}{\leq} C|\{0 < v_k < \frac{1}{2}\} \cap B_{1/2}|^{\frac{1}{2}}\end{aligned}$$
Hence we take away
$$\left(\int_{B_{1/2}\cap\{0<v_k<\frac{1}{2}\}} |\nabla v_k|\right)^2 \leq C|\{0 < v_k < \frac{1}{2}\} \cap B_{1/2}| \tag{6.25}$$

(b) Next, we build a cutoff $\overline{v}_k$ s.t.
$$\overline{v}_k := \begin{cases} 0 & \{v_k \leq 0\} \\ v_k & \{0 < v_k < \frac{1}{2}\} \\ \frac{1}{2} & \{v_k \geq \frac{1}{2}\} \end{cases}$$
In this way we inherit the information from LHS of (6.25)
$$\begin{aligned}\int_{B_{1/2}\cap\{0<v_k<\frac{1}{2}\}} |\nabla v_k| &= \int_{B_{1/2}} |\nabla \overline{v}_k|\\ &\geq C \int_{B_{1/2}} |\overline{v}_k - \fint_{B_{1/2}} \overline{v}_k| \qquad \text{use Poincaré}\\ &\geq C \int_{B_{1/2}\cap\{v_k\leq 0\}} |\fint_{B_{1/2}} \overline{v}_k| + C \int_{B_{1/2}\cap\{v_k\geq\frac{1}{2}\}} |\frac{1}{2} - \fint_{B_{1/2}} \overline{v}_k|\\ &\geq C\frac{1}{4} \min\{|B_{1/2} \cap \{v_k \leq 0\}|, |B_{1/2} \cap \{v_k \geq \frac{1}{2}\}|\} \qquad \text{two cases, either } \fint_{B_{1/2}} \overline{v}_k \geq \frac{1}{4} \text{ or not}\end{aligned}$$

Now what are the measure for the portion of $\{v_k \le 0\}$ and $\{v_k \ge \frac{1}{2}\}$ in $B_{1/2}$ respectively?

$$
\begin{aligned}
|\{v_k \le 0\} \cap B_{1/2}| &= |\{\frac{(u-t_k)^+}{1-t_k} \le 0\} \cap B_{1/2}| = |\{u \le t_k\} \cap B_{1/2}| \\
&\ge \underbrace{|\{u \le 0\} \cap B_{1/2}| \ge \delta |B_{1/2}|}_{\text{use Assumption (6.18)}} \\
|\{v_k \ge \frac{1}{2}\} \cap B_{1/2}| &= |\{(u-t_k)^+ \ge \frac{1}{2^{k+1}}\} \cap B_{1/2}| \\
&\ge |\{u \ge 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}}\} \cap B_{1/2}| = |\{u \ge t_{k+1}\} \cap B_{1/2}| \\
&= \underbrace{|\{v_{k+1} \ge 0\} \cap B_{1/2}| \ge \varepsilon}_{\text{use Contradictory Assumption (6.22)}}
\end{aligned}
$$

Now define

$$\beta := C \min\{\delta, \varepsilon\} > 0$$

we have a uniform in k lower bound for LHS of (6.23). Notice the result depends on $\delta > 0$ in the sense that, if $\delta = 0$, our argument is inconclusive.

Finally we're done :)

□

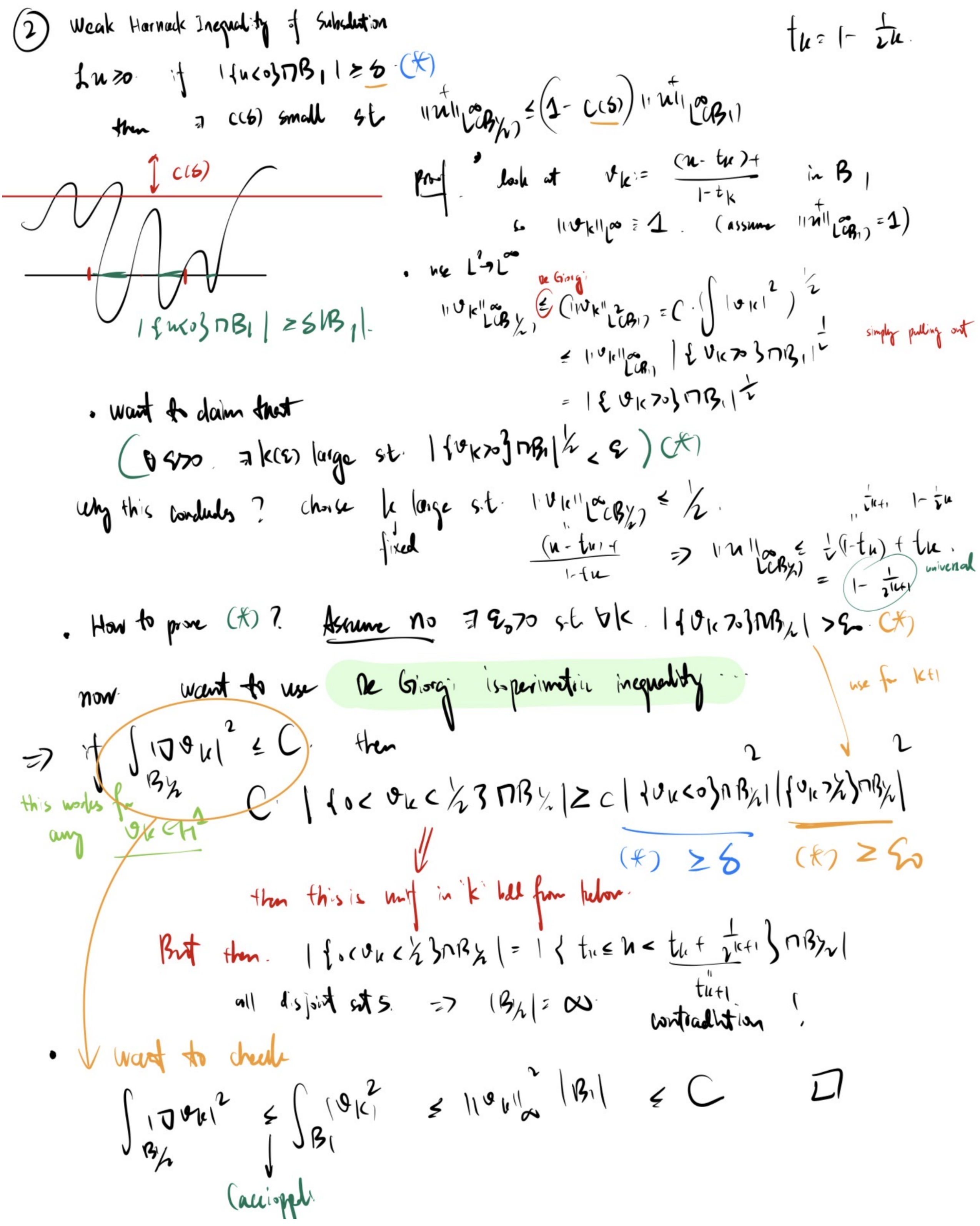


Figure 6.3: Weak Harnack for Subsolution Version 1

6.6.2.2 Oscillation Decay implies $C^{0,\alpha}$

Oscillation Decay

Corollary 6.6.1 (Oscillation Decay; [FRRO22] Proposition 3.13, [HL11] Theorem 4.10). *Assume* $u \in H^1(B_1)$ *solution*

$$\mathcal{L}u = 0 \qquad \text{in } B_1 \text{ in the weak sense}$$

Then for $\operatorname{osc}_{B_r} u := \sup_{B_r} u - \inf_{B_r} u$, *there exists* $\gamma = \gamma(n, \lambda, \Lambda) > 0$ *s.t.*

$$\operatorname*{osc}_{B_{1/4}} u \leq \gamma \operatorname*{osc}_{B_{1/2}} u \tag{6.26}$$

Proof. 1. First note by Local Boundedness Theorem 6.6.1, $u \in L^\infty(B_{3/4})$. Identifying u as equivalent modulo Lebesgue−a.e., it is eligible to consider two case: For a.e. $x \in B_{1/2}$, we either have

$$u(x) \geq \frac{1}{2}\big(\sup_{x\in B_{1/2}} u + \inf_{x\in B_{1/2}} u\big)$$

or

$$u(x) \leq \frac{1}{2}\big(\sup_{x\in B_{1/2}} u + \inf_{x\in B_{1/2}} u\big)$$

Let's be lazy and denote

$$\alpha_r := \sup_{x\in B_r} u, \qquad \beta_r := \inf_{x\in B_r} u \qquad \forall\ 0 < r < 1$$

Then we have two corresponding relations for both cases. In the first case

$$u \geq \frac{1}{2}(\alpha_{\frac{1}{2}} + \beta_{\frac{1}{2}}) \iff u - \alpha_{\frac{1}{2}} \geq -\frac{1}{2}(\alpha_{\frac{1}{2}} - \beta_{\frac{1}{2}}) \iff \frac{\alpha_{\frac{1}{2}} - u}{\alpha_{\frac{1}{2}} - \beta_{\frac{1}{2}}} \leq \frac{1}{2}$$

In the second case

$$u \leq \frac{1}{2}(\alpha_{\frac{1}{2}} + \beta_{\frac{1}{2}}) \iff u - \beta_{\frac{1}{2}} \leq \frac{1}{2}(\alpha_{\frac{1}{2}} - \beta_{\frac{1}{2}}) \iff \frac{u - \beta_{\frac{1}{2}}}{\alpha_{\frac{1}{2}} - \beta_{\frac{1}{2}}} \leq \frac{1}{2}$$

We go one step further and convert the relation in functions into relation in sets. In particular, we divide $B_{1/2}$ into two portions! We either have

$$|\{u \geq \frac{1}{2}(\alpha_{\frac{1}{2}} + \beta_{\frac{1}{2}})\} \cap B_{1/2}| = |\{\frac{\alpha_{\frac{1}{2}} - u}{\alpha_{\frac{1}{2}} - \beta_{\frac{1}{2}}} \leq \frac{1}{2}\} \cap B_{1/2}| \geq \frac{1}{2}|B_{1/2}| \tag{6.27}$$

or

$$|\{u \leq \frac{1}{2}(\alpha_{\frac{1}{2}} + \beta_{\frac{1}{2}})\} \cap B_{1/2}| = |\{\frac{u - \beta_{\frac{1}{2}}}{\alpha_{\frac{1}{2}} - \beta_{\frac{1}{2}}} \leq \frac{1}{2}\} \cap B_{1/2}| \geq \frac{1}{2}|B_{1/2}| \tag{6.28}$$

Why do we do that? Let's apply Weak Harnack Theorem 6.6.2.

2. If (6.27) holds, apply Theorem 6.6.2 to the function

$$w := \frac{\alpha_{\frac{1}{2}} - u}{\alpha_{\frac{1}{2}} - \beta_{\frac{1}{2}}} - \frac{1}{2}$$

Since u is a solution, u is supersolution hence w is subsolution. Also, notice

$$-\frac{1}{2} \leq w \leq \frac{1}{2} \qquad \forall\ x \in B_{1/2} \implies \left\|w^+\right\|_{L^\infty(B_{1/2})} \leq \frac{1}{2}$$

Now there exists $0 < c(\frac{1}{2}) < 1$ s.t.

$$\left\|\big(\frac{\alpha_{\frac{1}{2}} - u}{\alpha_{\frac{1}{2}} - \beta_{\frac{1}{2}}} - \frac{1}{2}\big)^+\right\|_{L^\infty(B_{1/4})} \leq \frac{1}{2} - \frac{1}{2}c(\frac{1}{2})$$

$$\alpha_{\frac{1}{2}} - u \leq \left(1 - \frac{1}{2}c(\frac{1}{2})\right)(\alpha_{\frac{1}{2}} - \beta_{\frac{1}{2}}) \qquad \forall\ x \in B_{1/4}$$

$$\alpha_{\frac{1}{4}} - \beta_{\frac{1}{4}} \leq \alpha_{\frac{1}{2}} - \beta_{\frac{1}{4}} \leq \left(1 - \frac{1}{2}c(\frac{1}{2})\right)(\alpha_{\frac{1}{2}} - \beta_{\frac{1}{2}})$$

$$\operatorname*{osc}_{B_{1/4}} u \leq \left(1 - \frac{1}{2}c(\frac{1}{2})\right) \operatorname*{osc}_{B_{1/2}} u$$

3. If alternatively (6.28) holds, apply Theorem 6.6.2 to the function

$$w := \frac{u - \beta_{\frac{1}{2}}}{\alpha_{\frac{1}{2}} - \beta_{\frac{1}{2}}} - \frac{1}{2}$$

For the same $0 < c(\frac{1}{2}) < 1$

$$\left\| (\frac{u-\beta_{\frac{1}{2}}}{\alpha_{\frac{1}{2}}-\beta_{\frac{1}{2}}} - \frac{1}{2})^+ \right\|_{L^\infty(B_{1/4})} \leq \frac{1}{2} - \frac{1}{2}c(\frac{1}{2})$$

$$u - \beta_{\frac{1}{2}} \leq \left(1 - \frac{1}{2}c(\frac{1}{2})\right)(\alpha_{\frac{1}{2}} - \beta_{\frac{1}{2}}) \qquad \forall\, x \in B_{1/4}$$

$$\alpha_{\frac{1}{4}} - \beta_{\frac{1}{4}} \leq \alpha_{\frac{1}{4}} - \beta_{\frac{1}{2}} \leq \left(1 - \frac{1}{2}c(\frac{1}{2})\right)(\alpha_{\frac{1}{2}} - \beta_{\frac{1}{2}})$$

$$\operatorname*{osc}_{B_{1/4}} u \leq \left(1 - \frac{1}{2}c(\frac{1}{2})\right) \operatorname*{osc}_{B_{1/2}} u$$

Conclude by defining

$$\gamma := 1 - \frac{1}{2}c(\frac{1}{2}) \in (0,1)$$

Notice indeed we need u as a solution for improvement from both sides to work!

□

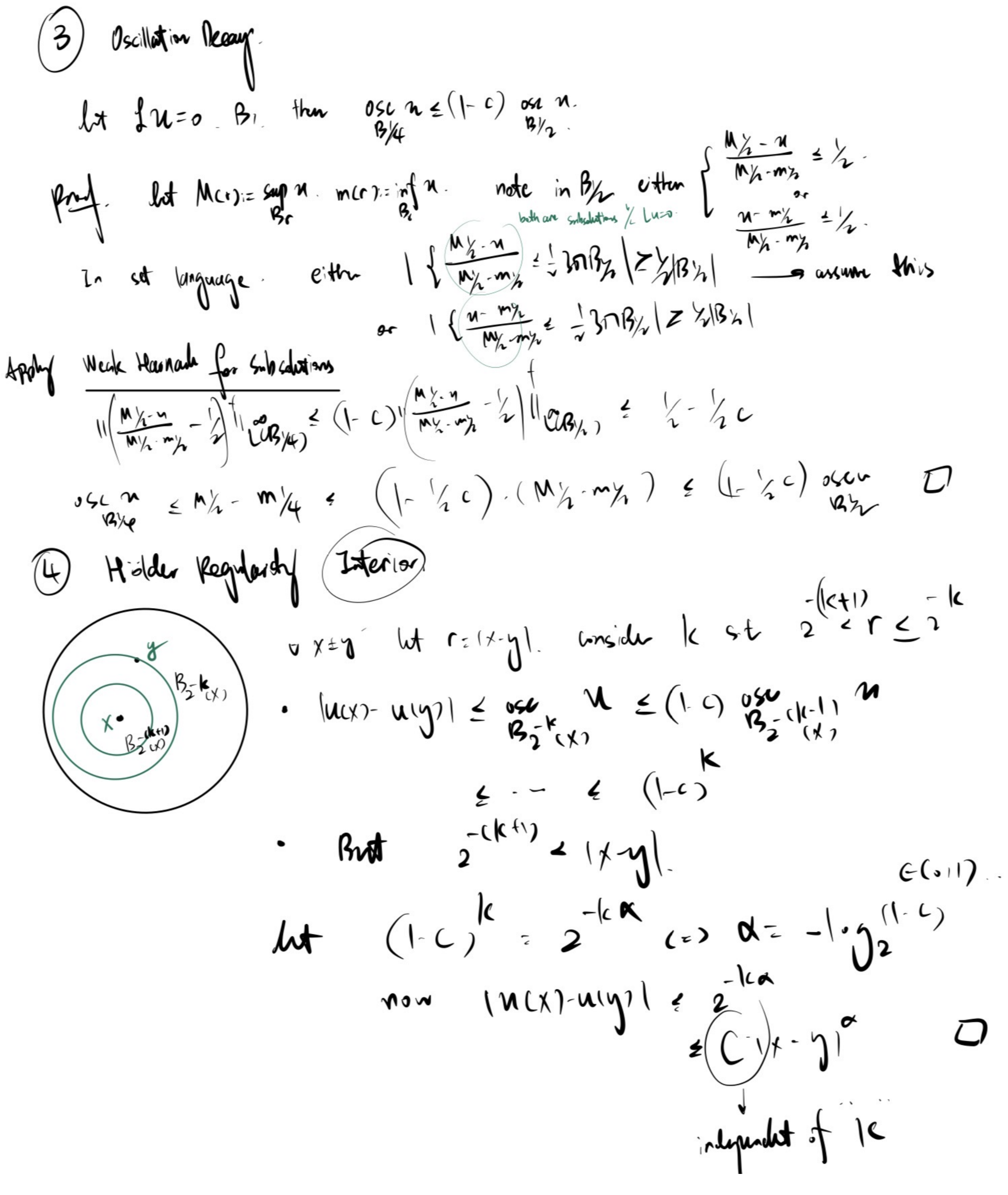


Figure 6.4: Oscillation Decay and Hölder Regularity

Oscillation Decay implies $C^{0,\alpha}$

Corollary 6.6.2 ($L^\infty \to C^\alpha$; [FRRO22] Corollary 2.7)**.** *Assume $u \in H^1(B_1)$ solution*

$$\mathcal{L}u = 0 \qquad \text{in } B_1 \text{ in the weak sense}$$

Then $u \in C^\alpha(B_{1/4})$ for some $\alpha = \alpha(n, \lambda, \Lambda) \in (0,1)$. In particular

$$\|u\|_{C^\alpha(B_{1/4})} \le C \|u\|_{L^\infty(B_{1/2})} \tag{6.29}$$

Proof. 1. Again, Theorem 6.6.1 gives $u \in L^\infty(B_{1/2})$. Leveraging linearity it suffices to assume

$$\|u\|_{L^\infty(B_{1/2})} \le \frac{1}{2}$$

Why $\frac{1}{2}$? Well then

$$\operatorname*{osc}_{B_{1/2}} u = \sup_{B_{1/2}} u - \inf_{B_{1/2}} u \leq 1$$

To aim for (6.29), our goal is to show there exists $\alpha \in (0,1)$ s.t. for some $C \geq 0$

$$\sup_{x,\, y \in B_{1/4},\, x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq C$$

2. Let's see how oscillation decay helps. Take any $y \in B_{1/4}$. Then notice for any $k \geq 2$

$$B_{2^{-k}}(y) \subset B_{1/2}$$

We consider arbitrary

$$x \in B_{2^{-k}}(y) \setminus B_{2^{-(k+1)}}(y)$$

Let's control the difference of function values!

$$\begin{aligned} |u(x) - u(y)| &\leq \operatorname*{osc}_{B_{2^{-k}}(y)} u \leq \gamma \operatorname*{osc}_{B_{2^{-(k-1)}}(y)} u && \text{use Oscillation Decay Corollary 6.6.1 and Rescaling} \\ &\leq \gamma^{k-2} \operatorname*{osc}_{B_{1/4}(y)} u \leq \gamma^{k-2} \operatorname*{osc}_{B_{1/2}} u \leq \gamma^{k-2} \end{aligned}$$

We rename

$$\gamma^{k-2} = 2^{(k-2)\log_2 \gamma} = C_0 2^{-\alpha k} \qquad \alpha := -\log_2(\gamma) \in (0,1), \qquad C_0 := \gamma^{-2}$$

and notice that for our choice of x and y

$$|x-y| \geq 2^{-(k+1)} \implies |x-y|^\alpha \geq C 2^{-\alpha k}$$

Thus for some $C = C(\gamma) = C(n, \lambda, \Lambda) > 0$

$$|u(x) - u(y)| \leq C|x-y|^\alpha \qquad \forall\, y \in B_{1/4}, \quad x \in B_{2^{-k}}(y) \setminus B_{2^{-(k+1)}}(y)$$

Divide to the LHS, taking supremum first in k, then in y to conclude.

□

6.7 Harnack Inequality

We establish Harnack Inequality for non-negative solutions ($u \geq 0$) to

$$\mathcal{L}u := \operatorname{div}(A(x)\nabla u(x)) \qquad \forall\, x \in B_1$$

for $A = (a_{ij})$ uniformly elliptic and $L^\infty(B_1)$.

Harnack Inequality

Theorem 6.7.1 (De Silva Analysis II 2025)**.** *Let $u \in H^1(B_1)$ solve*

$$\mathcal{L}u = 0, \qquad u \geq 0 \qquad B_1$$

Then there exists $C = C(n, \lambda, \Lambda) > 0$ s.t.

$$u \leq Cu(0) \qquad \forall\, x \in B_{1/2}$$

In order to gain the above full Harnack Inequality, one need weak Harnack.
For Version 1 we control the interior L^∞ norm, while for Version 2 we control the interior L^p norm.

6.7.1 Weak Harnack for Supersolution Version 1 (De Giorgi Oscillation, self-contained Proof)

We collect the counterpart to Theorem 6.6.2. In the supersolution case we assume for

1. $u \geq 0$ bounded from below. In fact this is also given by the Local Boundedness Theorem (6.12) applied to $-u$ for u supersolution, then using $(-u)^+$ is the negative part of u.
2. Some set that touches 1 of positive measure, so solution gets attracted upwards.

Theorem 6.7.2 (Weak Harnack for Supersolution Version 1)**.** *Assume $u \in H^1(B_1)$ supersolution*

$$\mathcal{L}u \leq 0 \qquad \text{in } B_1 \text{ in weak sense}$$

and $u \geq 0$ non-negative. Assume there exists $\delta > 0$ s.t.

$$|\{u \geq 1\} \cap B_{1/2}| \geq \delta |B_{1/2}| \tag{6.30}$$

Then the positive measure attracts our solution upwards, i.e., there exists $0 < c(n, \lambda, \Lambda, \delta) < 1$ s.t.

$$c(\delta) \leq \inf_{B_{1/4}} u \tag{6.31}$$

We could directly prove Theorem 6.7.2 using Log function that transforms from our supersolution back into subsolution, and then apply De Giorgi Local Boundedness 6.6.1.

Convex composition yields Subsolution We start with a lemma.

Lemma 6.7.1 ([HL11] Lemma 4.6)**.** *Let $\Phi \in C^{0,1}_{loc}(\mathbb{R})$ be convex. Let $\mathcal{L}u := \partial_i(a_{ij}\partial_j u)$ in B_1.*

1. *If $\mathcal{L}u \geq 0$ subsolution and $\Phi' \geq 0$, then $v = \Phi(u)$ is subsolution provided $v \in H^1_{loc}(B_1)$.*
2. *If $\mathcal{L}u \leq 0$ supersolution and $\Phi' \leq 0$, then $v = \Phi(u)$ is subsolution provided $v \in H^1_{loc}(B_1)$.*

Proof. First assume $\Phi \in C^2(\mathbb{R}^n)$, then $\Phi'' \geq 0$.

1. Assume $\Phi' \geq 0$ and u is subsolution. Then for any $\varphi \in C_0^\infty(B_1)$, $\varphi \geq 0$

$$\begin{aligned}\int_{B_1} a_{ij}\partial_i v \partial_j \varphi &= \int_{B_1} a_{ij}\Phi'(u)\partial_i u \partial_j \varphi \\ &= \int_{B_1} a_{ij}\partial_i u \partial_j(\Phi'(u)\varphi) - \underbrace{\int_{B_1} a_{ij}\varphi\Phi''(u)\partial_i u\partial_j u}_{\varphi \geq 0,\ \Phi \text{ convex, and } a_{ij} \text{ uniformly elliptic}} \\ &\leq \int_{B_1} a_{ij}\partial_i u \partial_j \underbrace{(\Phi'(u)\varphi)}_{\Phi' \geq 0 \text{ and } \varphi \geq 0} \leq 0 \qquad \text{using } u \text{ is subsolution}\end{aligned}$$

2. Assume $\Phi' \leq 0$ and u is supersolution. Then

$$\begin{aligned}\int_{B_1} a_{ij}\partial_i v \partial_j \varphi &= \int_{B_1} a_{ij}\Phi'(u)\partial_i u \partial_j \varphi \\ &= \int_{B_1} a_{ij}\partial_i u \partial_j(\Phi'(u)\varphi) - \underbrace{\int_{B_1} a_{ij}\varphi\Phi''(u)\partial_i u \partial_j u}_{\text{Exact same reason}} \\ &\leq \underbrace{\int_{B_1} a_{ij}\partial_i u \partial_j(\Phi'(u)\varphi) \leq 0}_{\Phi' \leq 0 \text{ and } u \text{ is supersolution}}\end{aligned}$$

In general convolve $\Phi_\varepsilon(s) := \Phi * \eta_\varepsilon$ and signs of derivatives are preserved. Since $\Phi \in C^{0,1}_{\mathrm{loc}}$, $\Phi'_\varepsilon \to \Phi'$ a.e. by Rademacher Theorem. Then by DCT we conclude. □

Sobole-Poincaré Lemma

Lemma 6.7.2 ([HL11] Lemma 4.8)**.** *For any $\varepsilon > 0$, there exists $C = C(\varepsilon, n) > 0$ s.t. for any $u \in H^1(B_1)$ s.t.*

$$|\{x \in B_1 \mid u = 0\}| \geq \varepsilon |B_1|$$

One has

$$\int_{B_1} u^2 \leq C \int_{B_1} |\nabla u|^2$$

The proof is directly contained in the proof for (6.36) below.

Proof of Weak Harnack Supersolution Version 1 using log

Proof using log *([HL11] Theorem 4.9).* 1. Let's discuss our outline. In our assumption $u \geq 0$. Since we wish to deal with log, let's take a sequence $\varepsilon > 0$ s.t. $u + \varepsilon \geq \varepsilon > 0$ and define

$$v_\varepsilon := (\log(u + \varepsilon))_- \tag{6.32}$$

Our Claim 1 is that for u a supersolution, v_ε (6.32) gives a subsolution for any ε. Notice v_ε is non-negative function. Then in a sequal of Claims to prove, we deduce

$$\begin{aligned}\left\|v_\varepsilon^+\right\|_{L^\infty(B_{1/4})} = \sup_{B_{1/4}} v_\varepsilon &\leq C(n, \lambda, \Lambda) \left\|v_\varepsilon\right\|_{L^2(B_{1/2})} && \text{Local Boundedness Theorem 6.6.1} \\ &\leq C(\delta, n, \lambda, \Lambda) \left\|\nabla v_\varepsilon\right\|_{L^2(B_{1/2})} && \text{Claim 2: Poincaré-Sobolev (6.36)} \qquad (6.33) \\ &\leq C(\delta, n, \lambda, \Lambda) && \text{Claim 3: Universal Log Bound (6.37)} \qquad (6.34)\end{aligned}$$

Then spelling out (6.32) we obtain

$$\begin{aligned}(\log(u + \varepsilon))_- \leq C(\delta, n, \lambda, \Lambda) = C \qquad & \forall\, x \in B_{1/4} \\ \log(u + \varepsilon) \geq -C & \\ u + \varepsilon \geq e^{-C} \qquad & \forall\, x \in B_{1/4}, \qquad \forall\, \varepsilon > 0 \\ \inf_{x \in B_{1/4}} u \geq e^{-C} =: c(\delta, n, \lambda, \Lambda) = c(\delta) \in (0, 1) &\end{aligned}$$

That's what we want in (6.31) :) Hence it suffices to prove Claim 1, 2, and 3.

2. In this step we prove Claim 1 ([HL11] Lemma 4.6). We observe some facts about $v_\varepsilon \equiv \Phi_\varepsilon(u)$.
   (a) Notice $\frac{d}{dt}\big|_{t=1} \log(t) = 1 \neq 0$. Hence taking negative part yields $\Phi_\varepsilon(t) = (\log(t + \varepsilon))_- \in C^{0,1}(\mathbb{R}_+)$ as optimal regularity.
   (b) $\log(t + \varepsilon)$ is increasing function in t. Taking negative part flips the sign, hence decreasing. In particular $\Phi'_\varepsilon(t) \leq 0$ for any $t \geq 0$.

   Due to only Lipschitz continuity of $\Phi_\varepsilon(t)$, we mollify $\Phi_\varepsilon * \eta_r(t) \in C^\infty(r, \infty)$ where η_r is standard mollifier rescaled to support in $(0, r)$. The good thing about mollification is that

$$\begin{aligned}(\Phi_\varepsilon * \eta_r)'(t) = \Phi'_\varepsilon * \eta_r(t) \leq 0 \qquad & \text{the derivative falls on whichever we like} \\ \Phi_\varepsilon * \eta_r(t) \to \Phi_\varepsilon \qquad & \text{a.e. as } r \to 0\end{aligned}$$

We denote $\Phi_{\varepsilon,r}(t) := \Phi_\varepsilon * \eta_r(t)$. Thus let's try to verify that $\Phi_{\varepsilon,r}(u)$ is our subsolution and send $r \to 0$ at the end. Indeed, for any $\varphi \in H_0^1(B_1)$

$$\int_{B_1} A(x)\nabla(\Phi_{\varepsilon,r}(u)) \cdot \nabla\varphi = \int_{B_1} A(x)\Phi'_{\varepsilon,r}(u)\nabla u \cdot \nabla\varphi \le 0$$

Here we used both $\Phi'_{\varepsilon,r}(u) \ge 0$ and u is our supersolution by assumption. Thus $\Phi_{\varepsilon,r}(u)$ is a subsolution for any r. Now why can we send $r \to 0$ and pass the limit under the integral using DCT? We have uniform estimate on the derivative, where we in fact used convexity

$$\begin{aligned} |\Phi'_{\varepsilon,r}(u)| &= |\Phi'_\varepsilon(u) * \eta_r| \le |\Phi'_\varepsilon(0)| \qquad \text{using } \Phi_\varepsilon \text{ is convex and } \Phi'_\varepsilon \le 0 \\ &\le \frac{1}{\varepsilon} \qquad \text{uniformly bounded in } r \text{ for each } \varepsilon > 0 \end{aligned}$$

Thus pass $r \to 0$ and conclude for any $\varphi \in H_0^1(B_1)$

$$\int_{B_1} A(x)\nabla(v_\varepsilon) \cdot \nabla\varphi = \int_{B_1} \Phi'_\varepsilon(u)\nabla u \cdot \varphi \le 0$$

3. In this step we prove Claim 2 (6.33) ([HL11] Lemma 4.8). Observe that assumption (6.30) translates to

$$|\{v_\varepsilon = 0\} \cap B_{1/2}| \ge |\{u \ge 1\} \cap B_{1/2}| \ge \delta |B_{1/2}| \tag{6.35}$$

We wish to prove a Poincaré-Sobolev Inequality for H^1 function with positive density of zero set. More precisely, we show that for any $\delta > 0$ s.t. (6.35) holds, there exists $C(\delta, n) > 0$ s.t.

$$\left(\int_{B_{1/2}} v_\varepsilon^2 dx\right)^{\frac{1}{2}} \le C(\delta, n)\left(\int_{B_{1/2}} |\nabla v_\varepsilon|^2 dx\right)^{\frac{1}{2}} \tag{6.36}$$

Assume not, i.e., there exists $\delta > 0$ s.t. for any $C > 0$, there exists some function $v \in H^1(B_{1/2})$ s.t. (6.35) holds but (6.36) fails. Then for the sequence $C_k = k$, we take a sequence of function $v_{k,\varepsilon} \in H^1(B_{1/2})$ s.t.

$$\|v_{k,\varepsilon}\|_{L^2(B_{1/2})} = 1, \qquad \left(\int_{B_{1/2}} |\nabla v_{k,\varepsilon}|^2 dx\right)^{\frac{1}{2}} \le \frac{1}{k} \to 0$$

A first observation is that $\|v_{k,\varepsilon}\|_{H^1(B_{1/2})} \le C$ is uniformly bounded in H^1 norm in k. Experts in Calculus of Variations immediately notice

(a) $H^1 \Subset L^2$ using Rellich and $\frac{2n}{n-2} > 2$. Hence we may assume $v_{k,\varepsilon} \to v_0$ strongly in L^2.

(b) H^1 Hilbert space is reflexive, hence using Banach-Alaoglu that uniformly bounded sequence in H^1 is precompact in weak(weak$*$) topology, we assume $v_{k,\varepsilon} \to v_0$ weakly in H^1.

In particular, following lower-semicontinuity of weak H^1 convergence

$$\left(\int_{B_{1/2}} |\nabla v_0|^2 dx\right)^{\frac{1}{2}} \le \lim_{k\to\infty} \left(\int_{B_{1/2}} |\nabla v_{k,\varepsilon}|^2 dx\right)^{\frac{1}{2}} = 0$$

hence $\nabla v_0 = 0$ a.e. in $B_{1/2}$, implying $v_0 \equiv c$ constant a.e. in $B_{1/2}$. Now how do we contradict (6.35)?

$$\begin{aligned} 0 = \lim_{k\to\infty} \int_{B_{1/2}} |v_{k,\varepsilon} - v_0|^2 &\ge \lim_{k\to\infty} \int_{B_{1/2} \cap \{v_{k,\varepsilon}=0\}} |v_{k,\varepsilon} - v_0|^2 \qquad \text{Restricting to set of interest} \\ &\ge \lim_{k\to\infty} \int_{B_{1/2} \cap \{v_{k,\varepsilon}=0\}} |v_0|^2 \ge |v_0|^2 \lim_{k\to\infty} |\{v_{k,\varepsilon} = 0\} \cap B_{1/2}| \\ &\ge |v_0|^2 \delta |B_{1/2}| > 0 \qquad \text{using our choice of } v_{k,\varepsilon} \text{ satisfies (6.35)} \end{aligned}$$

We therefore reach a contradiction and (6.36) follows.

4. In this step we prove Claim 3 (6.34). It suffices to prove for any $\varepsilon > 0$

$$\int_{B_{1/2}} |\nabla(\log(u + \varepsilon))|^2 \le C(n, \lambda, \Lambda) \tag{6.37}$$

To estimate for log, this step uses clever test function construction due to Moser. Let's for simplicity denote $u_\varepsilon := u + \varepsilon$. For any $\eta \in C_0^\infty(B_1)$ s.t. $\eta \equiv 1$ in $B_{1/2}$ and $|\nabla \eta| \leq 2$, we construct

$$\frac{\eta^2}{u_\varepsilon} \in H_0^1(B_1)$$

Since $u_\varepsilon = u + \varepsilon > 0$ is supersolution by assumption, we test against the above to obtain

$$\begin{aligned}
0 &\leq \int_{B_1} a_{ij}\partial_i u_\varepsilon \partial_j (\frac{\eta^2}{u_\varepsilon}) dx = \int_{B_1} a_{ij}\partial_i u_\varepsilon \left(-\frac{\eta^2}{u_\varepsilon^2}\partial_j u_\varepsilon + \frac{2\eta}{u_\varepsilon}\partial_j \eta \right) dx \\
&= -\int_{B_1} a_{ij}\eta^2 \partial_i(\log(u_\varepsilon))\partial_j(\log(u_\varepsilon)) dx + \int_{B_1} a_{ij}\partial_i(\log(u_\varepsilon)) 2\eta \partial_j \eta dx \qquad \text{using log} \\
&\leq -\int_{B_1} \lambda \eta^2 |\nabla(\log(u_\varepsilon))|^2 + \epsilon \int_{B_1} \Lambda |\nabla(\log(u_\varepsilon))|^2 \eta^2 + C(\epsilon)\int_{B_1} \Lambda |\nabla \eta|^2 \qquad \epsilon\text{-Young's Inequality}
\end{aligned}$$

Choosing $\epsilon \ll 1$ to absorb into LHS yields

$$\int_{B_1} \eta^2 |\nabla(\log(u_\varepsilon))|^2 \leq C(\lambda, \Lambda)\int_{B_1} |\nabla \eta|^2$$

Using $\eta \equiv 1$ in $B_{1/2}$ we recover (6.37). Notice indeed the bound is universal in ε.

□

6.7.2 Weak Harnack for Supersolution Version 1 (directly apply Theorem 6.6.2)

Version 1 (Supersolution) We record a direct proof of Theorem 6.7.2 using Theorem 6.6.2.

Proof using Theorem 6.6.2. As one could probably sense, what is some subsolution we can construct to apply our previous result? Of course

$$v := 1 - u$$

Since (6.30) is equivalent to

$$|\{0 \geq 1 - u\} \cap B_{1/2}| \geq \delta |B_{1/2}|$$

Applying Weak Harnack for Subsolution 6.6.2 yields

$$\left\|(1-u)^+\right\|_{L^\infty(B_{1/4})} \leq (1 - c(\delta)) \left\|(1-u)^+\right\|_{L^\infty(B_{3/4})}$$

Ah! Now we use $u \geq 0$ which forces $1 - u \leq 1$, and maximum over the LHS to conclude

$$1 - \inf_{B_{1/4}} u \leq 1 - c(\delta)$$
$$c(\delta) \leq \inf_{B_{1/4}} u$$

□

Version 1 Variant (Supersolution) We discuss variants of Weak Harnack Version 1. For Supersolution, this is actually the one we shall use.

Theorem 6.7.3 (Weak Harnack for Supersolution Version 1 variant; De Silva Analysis II 2025)**.** *Assume $u \in H^1(B_1)$ supersolution*

$$\mathcal{L}u \leq 0 \qquad \textit{in } B_1 \textit{ in weak sense,} \qquad u \geq 0 \qquad \textit{non-negative}$$

Assume 0 is Lebesgue point s.t. $u(0) = 1$ (this is to say, u is sufficiently small at the point 0!!!). Then for any $\delta > 0$, there exists $C(\delta) \gg 1$ s.t.

$$|\{u \geq C(\delta)\} \cap B_{1/2}| < \delta |B_{1/2}| \tag{6.38}$$

Compare the result with the Key argument from Krylov-Safonov (8.45).

Proof using Theorem 6.6.2. Assume for contradiction that there exists $\delta > 0$ s.t. for any C we have

$$|\{u \geq C\} \cap B_{1/2}| \geq \delta |B_{1/2}|$$

Let's again apply Weak Harnack for Subsolution 6.6.2 to $C - u$! So there exists $c(\delta) > 0$ s.t.

$$\left\|(C-u)^+\right\|_{L^\infty(B_{1/4})} \leq (1 - c(\delta)) \left\|(C-u)^+\right\|_{L^\infty(B_{3/4})}$$
$$C - 1 \leq (1 - c(\delta))C$$
$$C \leq \frac{1}{c(\delta)} < \infty$$

Oh no we have an upper bound for C contradicting arbitrariness. Note we need 0 to be Lebesgue point to pass from L^∞ to the actual pointwise value :) □

6.7.3 Weak Harnack for Supersolution Version 2

Theorem 6.7.4 (Weak Harnack for Supersolution Version 2; De Silva Analysis II 2025)**.** *Assume $u \in H^1(B_1)$ supersolution*

$$\mathcal{L}u \le 0, \qquad u \ge 0 \qquad B_1$$

Assume 0 is Lebesgue point s.t. $u(0) = 1$. Then there exists $p = p(n, \lambda, \Lambda) > 0$ and $C = C(n, \lambda, \Lambda)$ s.t.

$$\int_{B_{1/2}} u^p dx \le C \tag{6.39}$$

Compare this with Lemma 8.3.4, which also proves $L^{\varepsilon}_{\text{weak}}$ result for supersolutions, and essentially they're the same method... Actually they're just the same.

Proof. Let's see what one wants to control. Denote

$$a(t) = |\{u > t\} \cap B_{1/2}|$$

Then

$$\int_{B_{1/2}} u^p = p \int_0^{\infty} a(t) t^{p-1} dt$$

We would like to show for $t \ge t_0$ sufficiently large and for some $M > 0$ universal to be chosen

$$a(t) = |\{u > t\} \cap B_{1/2}| \le C t^{-M} \tag{6.40}$$

If so, by choosing for example $p = \frac{M}{2} > 0$, directly integrating we can conclude

$$\int_{B_{1/2}} u^p \le p \int_0^{t_0} |B_{1/2}| t^{p-1} dt + Cp \int_{t_0}^{\infty} t^{-M+p-1} dt \le C(t_0, M, p)$$

In fact, (6.40) **itself can be regarded as Weak Harnack that gives $L^M_{\textbf{weak}}$**

To prove the decay (6.40), **one want to show for $C \gg 1$ large and $t \ge t_0$ chosen universal**

$$a(Ct) \le \frac{1}{2} a(t) \tag{6.41}$$

Why does this suffice? One may iterate for t_0

$$a(C^k t_0) \le \frac{1}{2} a(C^{k-1} t_0) \le \cdots \le \frac{1}{2^k} a(t_0) \le \frac{1}{2^k} |B_{1/2}|$$

Now for any $s \ge t_0$, one may choose $k \in \mathbb{N}$ s.t.

$$C^k t_0 \le s < C^{k+1} t_0$$

so that

$$a(s) \le a(C^k t_0) \le \frac{1}{2^k} |B_{1/2}|$$

Now how do we bound $\frac{1}{2^k}$?

$$\begin{aligned} C^k &\le \frac{s}{t_0} < C^{k+1} \\ k &\le \log_C(\frac{s}{t_0}) < k+1 \\ 2^{-(k+1)} &< 2^{-\log_C(\frac{s}{t_0})} = (\frac{s}{t_0})^{-\log_C(2)} \\ 2^{-k} &\le 2 t_0^{\log_C 2} \cdot s^{-M} \end{aligned}$$

where

$$M = \log_C 2 > 0$$

Now we prove for (6.41).

1. First, one may choose $t_0 = t_0(n, \lambda, \Lambda)$ s.t.

$$a(t_0) = |\{u \geq t_0\} \cap B_{1/2}| \overset{(6.38)}{\leq} \frac{1}{2}|B_{1/2}| \tag{6.42}$$

   Why is this eligible? Because one has the Weak Harnack for Supersolution Version 1.

2. Now upon dividing u by large constant (particularly t_0), one may assume for $t = 1$ and prove that for $C \gg 1$ large to be chosen universal

$$a(C) \leq \frac{1}{2}a(1) \tag{6.43}$$

Now we prove for (6.43) **using Calderon-Zygmund Decomposition**. In particular we use the A-B Lemma 8.3.1. The result that one wish to obtain is essentially of the same form for its result (8.38)

$$|\{u > C\} \cap Q_1| \leq \frac{1}{2}|\{u > 1\} \cap Q_1|$$

once we replace balls with Cubes (rescaled to Q_1). We denote

$$A = \{u > C\} \cap Q_1, \qquad B = \{u > 1\} \cap Q_1$$

Ok, what do we need to check?

1. $|A| \leq \frac{1}{2}$. This is simply done using (6.42)

$$a(C) \leq a(t_0) \leq \frac{1}{2}|Q_1| = \frac{1}{2}$$

2. We want to check (8.37). Since we've picked a sequence of non-overlapping cubes $\{Q^j\}$ that satisfies

$$\frac{1}{2}|Q^j| < |A \cap Q^j| \tag{6.44}$$

   to cover A, then each of the predecessors $\tilde{Q}^j$ satisfies

$$\frac{1}{2}|\tilde{Q}^j| \geq |A \cap \tilde{Q}^j| \tag{6.45}$$

   To satisfy (8.37), it suffices to check all such $\tilde{Q}^j$ that satisfies (6.45) are included in B, in other words

$$\tilde{Q}_j \subseteq B = \{u > 1\} \cap Q_1 \qquad \forall\, \tilde{Q}^j \text{ s.t. (6.45) holds} \tag{6.46}$$

Finally we prove for (6.46). Assume not, so for any $C > 0$ and thus the dyadic cubes $\{Q^j\}$ chosen that satisfies (6.44) and (6.45), there exists some $\tilde{Q}^j$ that satisfies (6.45) But there exists also $\tilde{x} \in \tilde{Q}^j$ s.t.

$$u(\tilde{x}) \leq 1$$

Now fix some cube $Q^*(\tilde{x})$ centered at x and also covers one of the successors of $\tilde{Q}^j$, that we call Q^j (abuse of notation). Note Q^j always exists because $\tilde{Q}_j$ is chosen as a predecessor. There we have set inclusion (ok let's assume our solution satisfies the equation in Q_2)

$$Q^j \subseteq Q^*(\tilde{x}) \subseteq Q_2$$

Ok, but why do we want to do this? Because of $u(\tilde{x}) \leq 1$, one may apply Weak Harnack for Supersolution on this $Q^*(\tilde{x})$ so that for any $\delta > 0$, there always exists $C(\delta)$ large s.t.

$$|\{u > C(\delta)\} \cap Q^*(\tilde{x})| \overset{(6.38)}{\leq} \delta|Q^*(\tilde{x})|$$

But on the other hand

$$\begin{aligned} |\{u > C(\delta)\} \cap Q^*(\tilde{x})| &\geq |\{u > C(\delta)\} \cap Q^j| \\ &\overset{(6.44)}{\geq} \frac{1}{2}|Q^j| = \frac{1}{2^{jn+1}} \end{aligned}$$

Which says

$$\delta|Q^*(\tilde{x})| \geq \frac{1}{2^{jn+1}}$$

Notice this j is fixed from our contradictory assumption so it has no dependence on δ. Thus take $\delta \to 0$ to contradict.

□

6.7.4 Weak Harnack for Subsolution Version 2

Theorem 6.7.5 (Weak Harnack for Subsolution Version 2; De Silva Analysis II 2025)**.** *Assume $u \in H^1(B_1)$ is subsolution*

$$\mathcal{L}u \geq 0 \qquad B_1$$

Then given any $p > 0$, there exists $C = C(p, n, \lambda, \Lambda)$ s.t.

$$\sup_{B_{1/2}} u \leq C \left(\int_{B_1} (u^+)^p \right)^{\frac{1}{p}}$$

Proof. First we normalize to $\|u^+\|_{L^p} = 1$. In the following we write u for u^+. We also identify u with its essentially supremum representation a.e. And we work with $u(0)$, which we assume to be a Lebesgue point.

We assume for contradiction that

$$u(0) \gg 1$$

which shall be made precise later.

Now we claim there exists a sequence of points $\{y_k\} \subseteq B_1$ with $y_0 = 0$ s.t.

1. u grows at these y_k
$$u(y_{k+1}) \geq (1 + c_0)u(y_k) \tag{6.47}$$
for $c_0 > 0$ universal to be chosen.

2. y_k are sufficiently close to each other so they do not escape $B_{1/2}$
$$|y_{k+1} - y_k| \leq C_0 u(y_k)^{-\sigma} \tag{6.48}$$
for C_0 and $\sigma = \sigma(n, p)$ universal to be chosen.

Reach contradiction using the sequence. Morally we want the limiting point of $\{y_k\}$ (which exists upon extracting by subsequence via Bolzano-Weierstrass) to not exit $B_{1/4}$. To ensure this

$$\begin{aligned}\sum_{k=0}^{\infty} |y_{k+1} - y_k| &\overset{(6.48)}{\leq} C_0 \sum_{k=0}^{\infty} u(y_k)^{-\sigma} = C_0 \sum_{k=1}^{\infty} \left(\prod_{j=0}^{k-1} \frac{u(y_{j+1})}{u(y_j)} \right)^{-\sigma} u(y_0)^{-\sigma} + C_0 u(y_0)^{-\sigma} \\ &\overset{(6.47)}{\leq} \underbrace{C_0 \sum_{k=0}^{\infty} (1 + c_0)^{-k\sigma} u(y_0)^{-\sigma} \leq \frac{1}{4}}_{\text{want to ensure}}\end{aligned}$$

Hence choosing $u(y_0) = u(0)$ sufficiently large s.t. (for c_0, σ, C_0 to choose universal later)

$$u(0) \geq \max\{ \left(2C_0 \sum_{k=0}^{\infty} (1 + c_0)^{-k\sigma} \right)^{\frac{1}{\sigma}}, 2\}$$

We have $y_\infty \in \overline{B_{1/4}}$ necessarily. And

$$u(y_\infty) = \lim_{k \to \infty} u(y_k) = \infty$$

Define $\{y_k\}$ by induction. For the base we just take $y_0 = 0$. Now assume $\{y_k\}$ has been constructed. We claim it makes sense to consider r_k as the larges radius s.t.

$$\sup_{B_{r_k}(y_k)} u \leq (1 + c_0)u(y_k)$$

First of all using WLOG these are Lebesgue points... There exists some radius around y_k s.t. the above holds. Using Least upper bound property such r_k exists.

Now we let

$$v := (1 + c_0)u(y_k) - u$$

and we consider v in the ball $B_{r_k}(y_k)$, which is non-negative by definition. Notice such v is a supersolution. We're able to apply Weak Harnack for Supersolution Version 1 variant (6.38) so that for our choice of $\frac{1}{2}$ (looking at the complement)

$$|\{v \leq C(\frac{1}{2})v(y_k)\} \cap B_{r_k}(y_k)| \geq \frac{1}{2}|B_{r_k}(y_k)| \tag{6.49}$$

Why is this useful? Because

$$\begin{aligned} v &\le C(\frac{1}{2})v(y_k) \\ (1+c_0)u(y_k) - u &\le C(\frac{1}{2})c_0 u(y_k) \\ (1+c_0 - C(\frac{1}{2})c_0)u(y_k) &\le u \end{aligned}$$

If we now **choose** c_0 **small** so that

$$(1+c_0 - C(\frac{1}{2})c_0) \ge \frac{1}{2}$$

Then

$$\{v \le C(\frac{1}{2})v(y_k)\} \subseteq \{u \ge \frac{1}{2}u(y_k)\}$$

and therefore we take away

$$|\{u \ge \frac{1}{2}u(y_k)\} \cap B_{r_k}(y_k)| \overset{(6.49)}{\ge} \frac{1}{2}|B_{r_k}(y_k)| \tag{6.50}$$

We're going to use this along with our integrability assumption on u! Note

$$\begin{aligned} 1 &\ge \int_{B_{r_k}(y_k)} u^p \ge \int_{B_{r_k}(y_k)\cap\{u \ge \frac{1}{2}u(y_k)\}} u^p \\ &\overset{(6.50)}{\ge} 2^{-p}u(y_k)^p \frac{1}{2}|B_{r_k}(y_k)| \\ &= C(n,p)\cdot r_k^n u(y_k)^p \end{aligned}$$

Thus

$$r_k \le \frac{1}{C(n,p)^{\frac{1}{n}}} u(y_k)^{-\frac{p}{n}}$$

Choose our

$$C_0 = \frac{1}{C(n,p)^{\frac{1}{n}}}, \qquad \sigma = \frac{p}{n}$$

By definition of r_k, there exists some Lebesgue point y_{k+1} on $\partial B_{r_k}(y_k)$ s.t.

$$r_k = |y_{k+1} - y_k|$$

This concludes the proof. □

Chapter 7

Dirichlet Problem for Minimal Surface Equation

We discuss existence of classical solutions $u \in C^2(\overline{B_1})$ to the Dirichlet Problem

$$\begin{cases} \operatorname{div}(\nabla F(\nabla u)) = 0 & B_1 \\ u = \varphi & \partial B_1 \end{cases} \tag{7.1}$$

where $F \in C^{2,\alpha}(\mathbb{R}^n)$, $D^2F > 0$, and $\varphi \in C^{2,\alpha}$.

For example consider the minimal surface equation

$$F(p) = \sqrt{1+|p|^2}$$

Then

$$\partial_i F = \frac{p_i}{\sqrt{1+|p|^2}}$$
$$\partial_{ij} F = \frac{\delta_{ij}}{\sqrt{1+|p|^2}} - \frac{p_i p_j}{(1+|p|^2)^{\frac{3}{2}}}$$

and one can obtain

$$\nabla F(p) = \frac{p}{\sqrt{1+|p|^2}}$$

Hence the minimal surface equation takes the form

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$$

Why is $D^2F > 0$?

$$D^2F = \frac{1}{(1+|p|^2)^{\frac{3}{2}}}\left(\sqrt{1+|p|^2}I - pp^T\right)$$

so

$$v^T D^2F v = \frac{1}{(1+|p|^2)^{\frac{3}{2}}}\left(\sqrt{1+|p|^2}|v|^2 - |p\cdot v|^2\right) > 0 \qquad \forall\, v \neq 0$$

Thus D^2F is positive-definite at every $p \in \mathbb{R}^n$.

In particular we shall see how De-Giorgi-Nash-Moser is used.

7.1 A priori $C^{2,\alpha}$ global estimate

The goal is to prove the a priori estimate for minimal surface equations.

Theorem 7.1.1 (De Silva Analysis II Spring 2025). *Assume $u \in C^{2,\alpha}(\overline{B_1})$ solves (7.1). Then we have*

$$\|u\|_{C^{2,\alpha}(\overline{B_1})} \le C(n, \alpha, \|F\|_{C^{2,\alpha}}, \|\varphi\|_{C^{2,\alpha}}) \tag{7.2}$$

Proof. **Reduce to $C^{1,\alpha}$ Global Bound.**
We claim it suffices to control the goal $C^{1,\alpha}$ norm of u

$$\|u\|_{C^{1,\alpha}(\overline{B_1})} \leq C(F, \|\varphi\|_{C^{2,\alpha}}) \tag{7.3}$$

Why? Since we can pass the divergence inside if everything is smooth.

$$\operatorname{div}(\nabla F(\nabla u)) = F_{ij}(\nabla u)\partial_{ij}u = 0$$

Since $u \in C^{2,\alpha}$ by assumption we know $F_{ij}(\nabla u) \in C^{0,\alpha}$. Can we run the Global Schauder Estimate? Yet not! What is the $C^{0,\alpha}$ norm of the coefficient matrix ?

$$a_{ij} = F_{ij}(\nabla u)$$

If we look at our Global $C^{2,\alpha}$ estimate from Schauder Theory, we have

$$\|u\|_{C^{2,\alpha}(\overline{B_1})} \leq C(n, \|F_{ij}(\nabla u)\|_{C^{0,\alpha}}, \alpha, \|\varphi\|_{C^{2,\alpha}})$$

Ah! If we want our desired uniform bound on $C^{2,\alpha}$ as in (7.2), we exactly need the help of our goal

$$\|F_{ij}(\nabla u)\|_{C^{0,\alpha}} \leq \underbrace{\left\|D^2F\right\|_{C^{0,\alpha}} \|\nabla u\|_{C^{0,\alpha}}}_{\text{need both Hölder}} \overset{(7.3)}{\leq} C$$

Remark on Bootstrap
If we furthermore assume that $F \in C^{k,\alpha}$ for $k \geq 2$, then applying interior Schauder one obtain estimate

$$\|u\|_{C^{k+2,\alpha}(\overline{B_{1/2}})} \leq C\left(n, \|F\|_{C^{k+2,\alpha}}, \|u\|_{C^{k+1,\alpha}}\right)$$

If we further know $\varphi \in C^{k+2,\alpha}$ one use the global Schauder Estimate to gain

$$\|u\|_{C^{k+2,\alpha}(\overline{B_1})} \leq C\left(n, \|F\|_{C^{k+2,\alpha}}, \|u\|_{C^{k+1,\alpha}}, \|\varphi\|_{C^{k+2,\alpha}}\right)$$

Step 1: L^∞ bound for u
Using Maximum Principle for the elliptic equation

$$F_{ij}(\nabla u)\partial_{ij}u = 0$$

we know

$$\min_{\partial B_1}\varphi \leq u \leq \max_{\partial B_1}\varphi$$

Notice the novelty here is that, the coefficient matrix is only elliptic but not uniformly elliptic... But Weak Maximum Principle still holds! See (3.5).

Step 2: L^∞ bound for ∇u on ∂B_1
Let $x_0 \in \partial B_1$. Denote the inner unit normal ν_{x_0}.
Since φ is C^2 at each boundary point, there exists linear functions $\ell_{x_0}^+$, $\ell_{x_0}^-$ defined over $\overline{B_1}$ s.t.

$$\ell_{x_0}^- \leq \varphi \leq \ell_{x_0}^+ \qquad \partial B_1$$
$$\ell_{x_0}^-(x_0) = \varphi(x_0) = \ell_{x_0}^+(x_0)$$

Precisely, they're defined as

$$\begin{aligned}\ell_{x_0}^{\pm}(x) &:= \varphi(x_0) + \nabla\varphi(x_0)\cdot(x-x_0) \qquad \underbrace{\pm\mu(x-x_0)\cdot\nu_{x_0}}_{\text{linear perturbation in normal direction}} \\ &= \varphi(x) + O(|x-x_0|^2) \pm \mu(x-x_0)\cdot\nu_{x_0}\end{aligned}$$

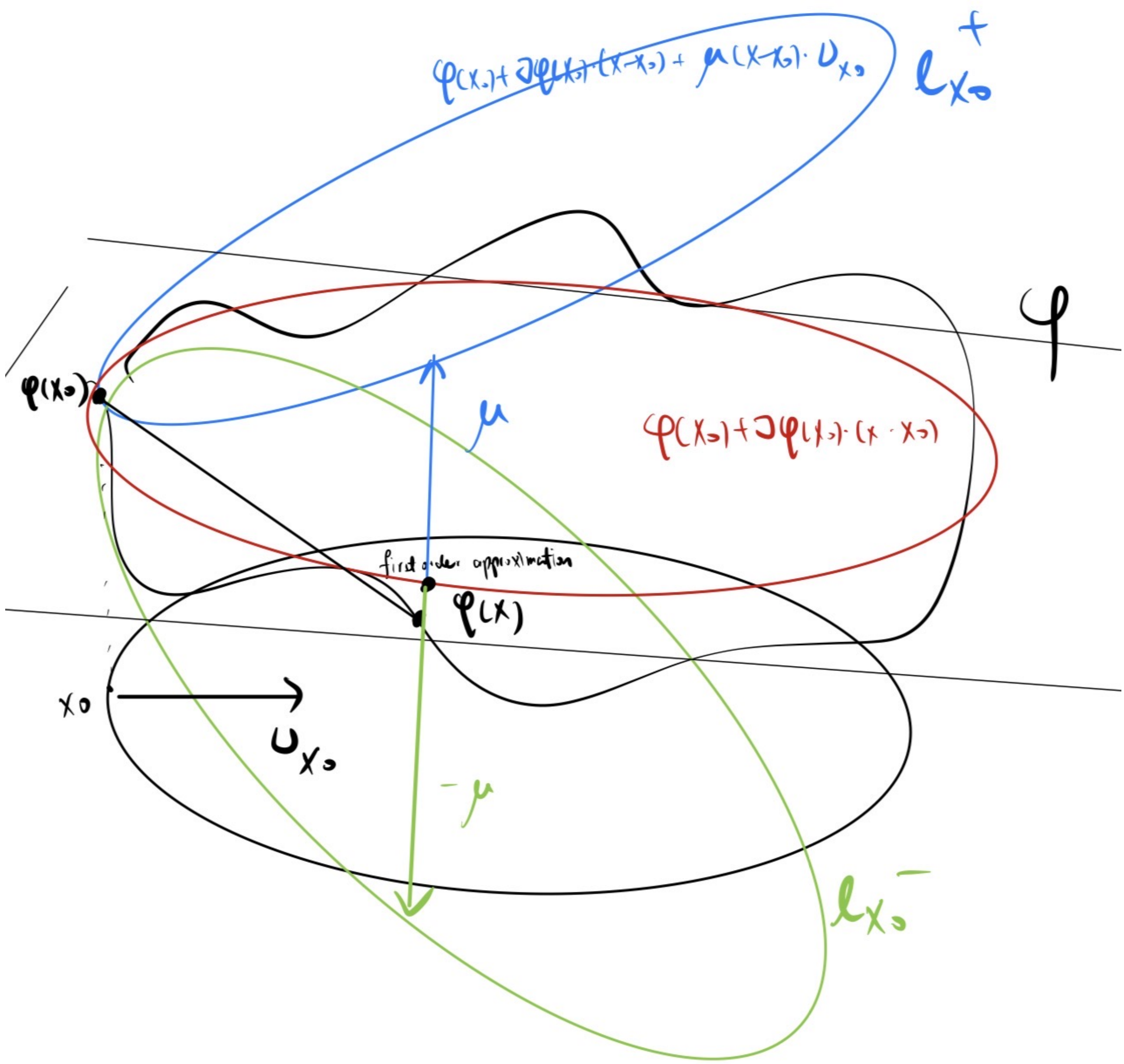


Figure 7.1: Picture for Step 2

We've chosen μ large so

$$\ell_{x_0}^- \leq \varphi = u \leq \ell_{x_0}^+ \qquad \partial B_1$$

Why can we do the trapping of φ on ∂B_1? Using our domain is convex, the linear term

$$\mu(x - x_0) \cdot \nu_{x_0}$$

is in fact quadratic in ∂B_1. This can be seen as follows: If $x_0 = 0$ and $\nu_{x_0} = (0, \cdots, 0, 1)$, then for $x \in \partial B_1 = \partial B_1((0, \ldots, 0, 1))$

$$1 = \sum_{i=1}^{n-1} x_i^2 + (x_n - 1)^2$$

$$|x|^2 = 2x_n = 2(x - x_0) \cdot \nu_{x_0}$$

Ok, now we've trapped our boundary data φ using $\ell_{x_0}^{\pm}$. Consider

$$u - \ell_{x_0}^{\pm} = u - \varphi(x_0) - \nabla\varphi(x_0) \cdot (x - x_0) \pm \mu(x - x_0) \cdot \nu_{x_0}$$

Now since $u - \ell_{x_0}^{\pm}$ **solve the equation with F translated**

$$F_{ij}(\nabla u - \nabla \ell_{x_0}^{\pm}) \partial_{ij}(u - \ell_{x_0}^{\pm}) = F_{ij}(\nabla u - \nabla \ell_{x_0}^{\pm}) \partial_{ij} u = 0$$

By using Maximum Principle applied

$$\ell^-_{x_0} \le u \le \ell^+_{x_0} \qquad B_1 \tag{7.4}$$

Now how does (7.4) help?

1. By taking our μ sufficiently large this controls the derivative in the ν_{x_0} direction

$$\ell^-_{x_0}(x_0 + t\nu_{x_0}) - u(x_0) \le u(x_0 + t\nu_{x_0}) - u(x_0) \le \ell^+_{x_0}(x_0 + t\nu_{x_0}) - u(x_0)$$
$$\nabla\varphi(x_0)\cdot\nu_{x_0} - \mu \le \frac{1}{t}u(x_0 + t\nu_{x_0}) - u(x_0) \le \nabla\varphi(x_0)\cdot\nu_{x_0} + \mu$$

   Sending $t \to 0$ to get the normal derivative is bounded

$$|\nabla u(x_0)\cdot\nu_{x_0}| \le C$$

2. For any other tangential direction, u satisfies boundary data φ, and they match precisely with the ones of boundary data, hence they're controlled.

Step 3: L^∞ bound for ∇u over $\overline{B_1}$
We differentiate in $\ell = 1, \cdots, n$ direction

$$\begin{aligned} 0 &= \partial_\ell(\operatorname{div}(\nabla F(\nabla u))) \\ &= \operatorname{div}(D^2F(\nabla u)\nabla\partial_\ell u) \end{aligned} \tag{7.5}$$

By weak maximum principle for elliptic divergence form operators (no need for uniformly ellitic!), we know

$$\sup_{x\in\overline{B_1}} \partial_\ell u(x) \le \sup_{x\in\partial B_1} \partial_\ell u(x) \le C$$

where the boundary L^∞ on ∇u we can bound from step 2.

Step 4: Interior $C^{1,\alpha}$ Estimate
Since we've shown

$$\|\nabla u\|_{L^\infty(\overline{B_1})} \le C(n, \|\varphi\|_{C^{2,\alpha}(\overline{B_1})})$$

Now the equation (7.5) is uniformly elliptic

$$\lambda I \le D^2F(\nabla u) \le \Lambda I$$

for certain λ, Λ that depends on n, φ, and F. Do we know any uniform $C^{0,\alpha}$ bound on this matrix? No! So we're in the case of uniformly elliptic operator with bounded measurable coefficients.

Thus in this step we apply De-Giorgi-Nash-Moser! The great theory gives us

$$\|\partial_\ell u\|_{C^{0,\alpha}(\overline{B_{1/2}})} \le C\,\|\partial_\ell u\|_{L^\infty}$$

Which is interior $C^{1,\alpha}$.

Step 5: Boundary $C^{0,\alpha}$ for Tangential Derivatives
We flatten out the boundary. First cover ∂B_1 with a finite number of balls, say $\{B^i\}_{i=1}^N$. For each of them (say B) flatten out the boundary by a C^2 diffeomorphism.

$$\begin{aligned} \psi : B\cap B_1(0) &\to B_1^+ \\ x &\mapsto y \end{aligned}$$

and denote $x = \psi^{-1}(y)$ as the inverse. How does the equation change?

$$u(x) = \tilde u(y) = \tilde u \circ \psi(x)$$

Then

$$\begin{aligned} \partial_i u(x) &= \partial_k \tilde u \partial_i \psi^k \\ \partial_{ij} u(x) &= \partial_{k\ell}\tilde u \partial_i\psi^k\partial_j\psi^\ell + \partial_k\tilde u\partial_{ij}\psi^k \end{aligned}$$

so

$$\nabla u(x) = D_x\psi(\psi^{-1}(y))\nabla\tilde u(y)$$

Recall u solves

$$\operatorname{div}(\nabla F(\nabla u)) = 0$$

so for any $v \in H_0^1(B)$

$$\begin{aligned} 0 &= \int_B \nabla v \cdot \nabla F(\nabla u) dx \\ &= \int_{B_1^+} D_x\psi(\psi^{-1}(y))\nabla\tilde{v}(y) \cdot \nabla F(D_x\psi(\psi^{-1}(y))\nabla\tilde{u}(y))|\det(D_y\psi^{-1})|dy \end{aligned}$$

In particular let's choose ψ to be a measure-preserving deformation so that

$$|\det(D_y\psi^{-1})| = 1$$

Hence $\tilde{u}$ solves in B_1^+ the equation

$$\begin{cases} \operatorname{div}(D(y)^T \nabla F(D(y)\nabla\tilde{u})) = 0 & B_1^+ \\ \tilde{u} = \tilde{\varphi} & \{y_n = 0\} \end{cases} \tag{7.6}$$

where

$$D(y) = D_x\psi(\psi^{-1}(y)) \in C^1$$

is a matrix with bounded inverse.

Now one may differentiate in the y_i direction for $i < n$ so

$$\begin{aligned} 0 &= \partial_i\partial_\ell(D^T_{\ell k}\partial_k F(D(y)\nabla\tilde{u})) = \partial_\ell(\partial_i(D^T_{\ell k})\partial_k F(D(y)\nabla\tilde{u}) + D^T_{\ell k}\partial_{kj}F(D(y)\nabla\tilde{u})\partial_i(D(y)_{jm}\partial_m\tilde{u})) \\ &= \partial_\ell(\underbrace{D^T_{\ell k}\partial_{kj}F(D(y)\nabla\tilde{u})D_{jm}}_{\text{call this } A_{\ell m}}\partial_{mi}\tilde{u}) + \partial_\ell\left(\partial_i(D^T_{\ell k})\partial_k F(D(y)\nabla\tilde{u}) + D^T_{\ell k}\partial_{kj}F(D(y)\nabla\tilde{u})\partial_i(D(y)_{jm})\partial_m\tilde{u}\right) \end{aligned}$$

our $\partial_i\tilde{u}$ solves

$$\begin{aligned} \operatorname{div}(A\nabla\partial_i\tilde{u}) &= \operatorname{div}(g) \qquad B_1^+ \\ \partial_i\tilde{u} &= \partial_i\tilde{\varphi} \qquad \{y_n = 0\} \end{aligned} \tag{7.7}$$

for

$$g^\ell = -(\partial_i(D^T_{\ell k})\partial_k F(D(y)\nabla\tilde{u}) + D^T_{\ell k}\partial_{kj}F(D(y)\nabla\tilde{u})\partial_i(D(y)_{jm})\partial_m\tilde{u}) \in L^\infty$$

Now the **boundary version of De-Giorgi-Nash-Moser** gives

$$\|\partial_i\tilde{u}\|_{C^{0,\alpha}(B^+_{1/2})} \le C\left(\|\partial_i\tilde{u}\|_{L^\infty(B_1^+)} + \|g\|_{L^\infty(B_1^+)} + \|\partial_i\tilde{\varphi}\|_{C^{0,\alpha}(B_1^+)}\right) \le C(\text{data})$$

Then transform back to $\partial_i u$.

Step 6: Boundary $C^{0,\alpha}$ for Normal Derivatives

We still work in the B_1^+ setting.

1. To achieve the boundary estimate, one claim it suffices to show

$$\|\partial_n\tilde{u}\|_{C^{0,\alpha}(B_{r/2}(y))} \le C \qquad \text{universal} \tag{7.8}$$

where for any $y \in B^+_{1/2}$, we particularly choose

$$r = \frac{1}{2}y_n$$

so the ball stays away from $\{y_n = 0\}$. Note we want a bound universal in $\tilde{u}$ and in y, therefore in r. To see this concludes the proof, for any consider $y' \in B'_{1/2}$ s.t. $y = (y', y_n)$. Then denote

$$y^k = (y', \frac{1}{2^k}y_n)$$

Now using continuity of $\partial_n\tilde{u}$ up to the boundary (by our a priori assumption)

$$\begin{aligned} |\partial_n\tilde{u}(y',0) - \partial_n\tilde{u}(y)| &= \lim_{k\to\infty}|\partial_n\tilde{u}(y) - \partial_n\tilde{u}(y', \frac{1}{2^k}y)| \\ &\le \sum_{k=0}^\infty |\partial_n\tilde{u}(y', \frac{1}{2^k}y_n) - \partial_n\tilde{u}(y', \frac{1}{2^{k+1}}y_n)| \\ &\overset{(7.8)}{\le} \sum_{k=0}^\infty C(\frac{1}{2^{k+1}})^\alpha|y_n|^\alpha \end{aligned}$$

Thus we have (for other direction we first connect to vertical then use overlapping balls)

$$\|\partial_n\tilde{u}\|_{C^{0,\alpha}(B^+_{1/2})} \le C(n,\alpha)$$

2. We claim to show for (7.8), it suffices to show that

$$\fint_{B_r(y)} |\nabla \partial_n \tilde{u}|^2 \leq C r^{2\alpha - 2} \tag{7.9}$$

for any $y \in B^+_{1/2}$ and $r = \frac{y_n}{2}$. Notice this does not really follow from the characterisation of Hölder Space via growth of local integral, since $r = \frac{y_n}{2}$ here are fixed by where y is.

Ok, why does this help? We claim this follows from De-Giorgi-Nash-Moser.

Let's go back to the original equation, so $\partial_n u$ are originally solutions to the equation (7.5). Now we can apply interior De-Giorgi-Nash-Moser. We would like to use Poincaré, so we work with

$$\partial_n u - \fint_{B_1} \partial_n u$$

Now let $v = \partial_n u$

$$\begin{aligned} [v]_{C^{0,\alpha}(\overline{B_{1/2}})} &= [v - \fint v]_{C^{0,\alpha}(\overline{B_{1/2}})} \\ &\underbrace{\overset{\text{De-Giorgi}}{\leq} \left\| v - \fint v \right\|_{L^2(B_1)}}_{\text{Back to original equation so no force}} \\ &\overset{\text{Poincaré}}{\leq} C \left\| \nabla v \right\|_{L^2} \end{aligned}$$

By rescaling to $\overline{v}(x) = v(rx)$ for $x \in B_1$ we rewrite the above as

$$\begin{aligned} r^{2\alpha} [v]^2_{C^{0,\alpha}(\overline{B_{r/2}})} &= [\overline{v}]^2_{C^{0,\alpha}(\overline{B_{1/2}})} \leq C \left\| \nabla \overline{v} \right\|^2_{L^2(B_1)} \\ &= C r^2 \int_{B_1} |\nabla v|^2 (rx) dx \leq C r^2 \fint_{B_r} |\nabla v|^2 dx \end{aligned}$$

so that

$$[v]_{C^{0,\alpha}(\overline{B_{r/2}})} \leq C r^{1-\alpha} \left(\fint_{B_r} |\nabla v|^2 dx \right)^{\frac{1}{2}} \overset{(7.9)}{\leq} C$$

where in the last step we used our assumption, and transformed to the B^+_1 model upon C^2 flattening out the boundary with bounded inverse. On the LHS, we also need to go back to B^+_1 under domain transformation.

3. In this last step we prove (7.9). From the equation of $\tilde{u}$ in B^+_1 (7.6), we pass the divergence inside

$$0 = \partial_\ell (D^T_{\ell k} \partial_k F(D(y)\nabla \tilde{u})) = \partial_\ell (D^T_{\ell k}) \partial_k F(D(y)\nabla\tilde{u}) + D^T_{\ell k} \partial_{kj} F(D(y)\nabla\tilde{u}) \partial_\ell (D_{jm} \partial_m \tilde{u})$$

so that

$$D^T_{\ell k} \partial_{kj} F(D(y)\nabla\tilde{u}) D_{jm} \partial_m \partial_\ell \tilde{u} = -D^T_{\ell k} \partial_{kj} F(D(y)\nabla\tilde{u}) (\partial_\ell D_{jm}) \partial_m \tilde{u} - \partial_\ell (D^T_{\ell k}) \partial_k F(D(y)\nabla\tilde{u})$$

In particular the LHS includes the term $\partial_{nn}\tilde{u}$. Then one throw all other terms to the RHS, and one gain

$$|\nabla \partial_n \tilde{u}|^2 \leq C \left(\sum_{i<n} |\nabla \partial_i \tilde{u}|^2 + 1 \right)$$

So it suffices to bound (7.9) for $i < n$

$$\fint_{B_r(y)} |\nabla \partial_i \tilde{u}|^2 \leq C \qquad \forall\ i < n$$

Ok why is this easier? Because my $\partial_i \tilde{u}$ solve the equation (7.7). In such form of the equation one may apply **Caccioppoli** to $\partial_i u - \partial_i u(y)$ at the fixed point y. Rescaling to unit ball, Caccioppoli writes (abuse of notation still denote $\partial_i \tilde{u}$)

$$\left\| \nabla \partial_i \tilde{u} \right\|_{L^2(B_{1/2})} = \left\| \nabla (\partial_i \tilde{u} - \partial_i \tilde{u}(y)) \right\|_{L^2(B_{1/2})} \overset{\text{Caccioppoli}}{\leq} C \left\| \partial_i \tilde{u} - \partial_i \tilde{u}(y) \right\|_{L^2(B_{3/4})} + C \left\| g \right\|_{L^\infty}$$

Rescaling back yields

$$r^2 \fint_{B_r(y)} |\nabla \partial_i \tilde{u}|^2 \leq C \fint_{B_{3r/2}(y)} |\partial_i \tilde{u} - \partial_i \tilde{u}(y)|^2 + Cr^2 \|g\|_{L^\infty}$$
$$\fint_{B_r(y)} |\nabla \partial_i \tilde{u}|^2 \leq C \frac{1}{r^2} \fint_{B_{3r/2}(y)} |\partial_i \tilde{u} - \partial_i \tilde{u}(y)|^2 + C \|g\|_{L^\infty}$$

We study the middle term

$$\frac{1}{r^2} \fint_{B_{3r/2}(y)} |\partial_i \tilde{u} - \partial_i \tilde{u}(y)|^2 \leq C \frac{1}{r^2} \fint_{B_{3r/2}(y)} [\partial_i \tilde{u}]^2_{C^{0,\alpha}} r^{2\alpha} \overset{\text{Step 4}}{\leq} Cr^{2\alpha-2}$$

Notice the term g gets eaten because for r small, $r^{2\alpha-2}$ is large.

This concludes the whole proof of a prior global estimate.

□

7.2 Method of Continuity

Implicit Function Theorem

Lemma 7.2.1 (Implicit Function Theorem)**.** *For X, Y infinite dimensional Banach Spaces*

$$T : X \to Y$$

with $T \in C^1$,

$$DT(x_0) : X \to Y$$

invertible at some $x_0 \in X$, then there exists a neighborhood of x_0 s.t.

$$B = B_r(x_0), \qquad T(B_r(x_0)) = V \qquad \text{open in } Y$$

so that $T|_B$ is invertible and C^1.

Proof with Method of Continuity

Theorem 7.2.1 (De Silva Analysis II 2025)**.** *For $F \in C^{2,\alpha}(\mathbb{R}^n)$, $D^2F > 0$ and boundary data $\varphi \in C^{2,\alpha}$ sufficiently nice. Then there exists unique $u \in C^2(B_1) \cap C^0(\overline{B_1})$ that solves (7.1)*

$$\begin{aligned} \operatorname{div}(\nabla F(\nabla u)) = 0 \qquad & B_1 \\ u = \varphi \qquad & \partial B_1 \end{aligned}$$

Moreover, $u \in C^{2,\alpha}(\overline{B_1})$ and

$$\|u\|_{C^{2,\alpha}(\overline{B_1})} \leq C(n, \alpha, \|F\|_{C^{2,\alpha}}, \|\varphi\|_{C^{2,\alpha}}) \tag{7.10}$$

If $F \in C^\infty$ then $u \in C^\infty(B_1)$. If $\varphi \in C^\infty$ as well then $u \in C^\infty(\overline{B_1})$.

Proof. We consider

$$\begin{cases} \operatorname{div}(\nabla F(\nabla u_t)) = 0 & B_1 \\ u_t = t\varphi & \partial B_1 \end{cases} \tag{DPt}$$

Consider

$$\mathcal{A} := \{t \in [0,1] \mid \text{(DPt) has a solution } u_t \in C^{2,\alpha}(\overline{B_1})\}$$

Clearly $0 \in \mathcal{A}$. We want for $\mathcal{A}$ to be both open and closed. Then this implies

$$\mathcal{A} = [0,1]$$

hence $1 \in \mathcal{A}$, and so (DPt) with $t = 1$ has a solution.

1. $\mathcal{A}$ is closed. To see $\mathcal{A}$ is closed we use our a priori estimates (7.2). Let $t_n \in \mathcal{A}$ s.t. $t_n \to t_* \in [0,1]$. We want to show $t_* \in \mathcal{A}$.

   For our sequence t_n chosen, we have uniform $C^{2,\alpha}$ estimate in t_n

   $$\|u_{t_n}\|_{C^{2,\alpha}(\overline{B_1})} \leq C(n, \alpha, \|F\|_{C^{2,\alpha}}, \|\varphi\|_{C^{2,\alpha}}) \qquad \forall\ n$$

   By Ascoli-Arzela, one has extract a convergent subsequence $\{u_{t_{n_j}}\}$ that converges in C^2 uniformly to a function $u_{t_*} \in C^{2,\alpha}(\overline{B_1})$. Due to C^2 uniform convergence, we see

   $$0 = \operatorname{div}(\nabla F(\nabla u_{t_n})) \to \operatorname{div}(\nabla F(\nabla u_{t_*}))$$

   and indeed u_{t_*} solves (DPt) at t_* with boundary data $t_*\varphi$.

2. $\mathcal{A}$ is open. For any $t \in \mathcal{A}$, we want to show there exists a neighborhood around t s.t. is still included in $\mathcal{A}$. We setup the map to which we apply the Implicit Function Theorem. Consider

   $$T : C^{2,\alpha}(\overline{B}_1) \to C^{\alpha}(\overline{B}_1) \times C^{2,\alpha}(\partial B_1)$$
   $$u \mapsto \left(\operatorname{div}(\nabla F(\nabla u)),\ u|_{\partial B_1}\right)$$

   First we assume $F \in C^4$. Then one may compute classically the differential of T

   $$DT(u)[v] = (\operatorname{div}(D^2F(\nabla u)\nabla v),\ v|_{\partial B_1})$$

   Indeed, for $u + \varepsilon v$

   $$\operatorname{div}(\nabla F(\nabla(u + \varepsilon v))) - \operatorname{div}(\nabla F(\nabla u)) = \operatorname{div}(D^2F(\nabla u)\varepsilon\nabla v) + o(\varepsilon)$$

   Now when is this $DT(u)$ a linear isomorphism? **This is to ask whether we can solve the Dirichlet problem (find unique solution** $v \in C^{2,\alpha}(\overline{B_1})$**) for any pair of** $(f, g) \in C^{0,\alpha}(\overline{B_1}) \times C^{2,\alpha}(\partial B_1)$

   $$\begin{cases} \operatorname{div}(D^2F(\nabla u)\nabla v) = f & B_1 \\ v = g & \partial B_1 \end{cases}$$

   Note our assumption is that $t \in \mathcal{A}$, so $u = u_t \in C^{2,\alpha}(\overline{B_1})$ solves (DPt). At u one may use our a priori estimate (7.2) so that

   $$\|u\|_{C^{2,\alpha}(B_1)} \leq C$$

   In particular the operator

   $$\lambda I \leq D^2F(\nabla u) \leq \Lambda I$$

   is uniformly elliptic. Moreover, our coefficient matrix $D^2F(\nabla u)$ is at least $C^{0,\alpha}$, so Schauder Theory for divergence form equation (6.9) applies. But recall this only gives us $v \in C^{1,\alpha}$. To attain $v \in C^{2,\alpha}$, we really need to pass the divergence inside so

   $$\operatorname{div}(D^2F(\nabla u)\nabla v) = \partial_\ell(F_{\ell k}(\nabla u)\partial_k v) = \underbrace{F_{\ell k m}(\nabla u)\partial_m\partial_\ell u \partial_k v}_{\text{treat as force } \textbf{using } F \in C^4} + F_{\ell k}(\nabla u)\partial_k\partial_\ell v$$

   Now v solves the non-divergence form equation

   $$F_{\ell k}(\nabla u)\partial_k\partial_\ell v = f - F_{\ell k m}(\nabla u)\partial_m\partial_\ell u\partial_k v$$

   with Hölder coefficient matrix. Now $v \in C^{2,\alpha}(\overline{B_1})$.

   Therefore for F smooth enough, $DT(u)$ for u_t with $t \in \mathcal{A}$ is invertible. Applying the Implicit Function Theorem, we know $\mathcal{A}$ is open.

   Then we remove $F \in C^4$.

   We approximate $F \in C^{2,\alpha}$ by $F_m \in C^4$. For each F_m the existence Theory is well-developed, thus one solve for $u_m \in C^{2,\alpha}(\overline{B_1})$ that satisfies the estimate (7.10). Because we have the uniform estimate we can pass to the limit upon Ascoli-Arzela and get $u \in C^{2,\alpha}(\overline{B_1})$. Since the equations converges (convergence in $C^{2,\alpha}$ for F_m is good enough), necessarily u solves the Dirichlet Problem (7.1).

□

Chapter 8

Viscosity Approach

8.1 Viscosity Solutions of Elliptic Equations

8.1.1 Preliminaries

Tangent Parabolas Before we begin, one introduce the basic test function candidate for second order equations: Parabola.

We say P is a paraboloid of opening $M > 0$ if

$$P(x) = \ell_0 + \ell(x) \pm \frac{M}{2}|x|^2$$

for $\ell_0 \in \mathbb{R}$ and ℓ linear function. P is convex for $+M$ and concave for $-M$.

Touching by Parabola from above/below Given two continuous functions $u, v \in C(\Omega)$. We say v touches u from above at x_0 in $A \subseteq \Omega$ if

$$\begin{cases} u(x) \leq v(x) & \forall\, x \in A \\ u(x_0) = v(x_0) & x_0 \in A \end{cases}$$

Definition 8.1.1. *Let $u \in C(\Omega)$ and $A \subseteq \Omega$ open.*

For any $x_0 \in A$, define

$$\overline{\Theta}(u, A)(x_0) := \inf\{M > 0 \mid \textit{there exists convex parabola with opening } M \textit{ that touches } u \textit{ from above at } x_0 \textit{ in } A\}$$
$$\underline{\Theta}(u, A)(x_0) := \inf\{M > 0 \mid \textit{there exists concave parabola with opening } M \textit{ that touches } u \textit{ from below at } x_0 \textit{ in } A\}$$

and consider

$$\Theta(u, A)(x_0) := \sup\{\overline{\Theta}(u, A)(x_0), \underline{\Theta}(u, A)(x_0)\} \leq \infty$$

Heuristically, how do we bound $\Theta(u, A)(x_0)$? If one is able to construct, say, a convex parabola

$$P(x_0 + h) = L(x_0 + h) + \frac{M}{2}|h|^2$$

with opening uniform in x_0, that bounds u via

$$L(x_0 + h) \leq u(x_0 + h) \leq L(x_0 + h) + \frac{M}{2}|h|^2$$

for h small, then using infimum,

$$\Theta(u, A)(x_0) \leq M$$

$C^{1,1}$ from above/below

Definition 8.1.2. *Given $x_0 \in \Omega$, we say u is $C^{1,1}$ by above at x_0 if there exists $x_0 \subseteq A \subseteq \Omega$ open set s.t.*

$$\overline{\Theta}(u, A)(x_0) < \infty$$

Similarly define u $C^{1,1}$ by below at x_0, $C^{1,1}$ at x_0.

Lemma 8.1.1. *That u is $C^{1,1}$ at x_0 implies u is differentiable at x_0.*

Proof. Let $B_r(x_0)$ for $r > 0$ small s.t. for some $M > 0$, and $a, b \in \mathbb{R}^n$

$$u(x) \leq u(x_0) + a \cdot (x - x_0) + \frac{M}{2}|x - x_0|^2 \qquad \forall\, x \in B_r(x_0)$$
$$u(x) \geq u(x_0) + b \cdot (x - x_0) - \frac{M}{2}|x - x_0|^2$$

Now for $x = x_0 + he$ for some $e \in \mathbb{S}^{n-1}$, $h > 0$

$$b \cdot e - \frac{M}{2}h \leq \frac{u(x_0 + he) - u(x_0)}{h} \leq a \cdot e + \frac{M}{2}h$$

Now sending $h \searrow 0$ yields

$$b \cdot e \leq a \cdot e$$

On the other hand consider $-e$. For $x = x_0 - he$ for $h > 0$

$$-b \cdot e - \frac{M}{2}h \leq \frac{u(x_0 - he) - u(x_0)}{h} \leq -a \cdot e + \frac{M}{2}h$$

sending again $h \searrow 0$ yields

$$b \cdot e \geq a \cdot e$$

Now this holds for any $e \in \mathbb{S}^{n-1}$ so $a = b \in \mathbb{R}^n$. Now we see again, for any $e \in \mathbb{S}^{n-1}$ and $h > 0$

$$\left|\frac{u(x_0 + he) - u(x_0) - a \cdot he}{h}\right| \leq \frac{M}{2}h$$
$$\lim_{h \to 0} \left|\frac{u(x_0 + he) - u(x_0) - a \cdot he}{h}\right| = 0$$

Thus u is differentiable at x_0 and in fact $a = \nabla u(x_0)$. ☐

Second Differential Quotients We define the second differential quotients of u at x_0

$$\Delta_h^2 u(x_0) := \frac{u(x_0 + h) + u(x_0 - h) - 2u(x_0)}{|h|^2}, \qquad h \in \mathbb{R}^n \qquad x_0 \pm h \in \Omega \tag{8.1}$$

1. Notice $\Delta_h^2 P = M$ for convex paraboloid, and $\Delta_h^2 P = -M$ for concave paraboloid.

   Proof. Let P be convex.

   $$\Delta_h^2 P(x) = \frac{P(x + h) + P(x - h) - 2P(x)}{|h|^2} = \frac{\ell(x + h) + \frac{M}{2}|x + h|^2 + \ell(x - h) + \frac{M}{2}|x - h|^2 - 2\ell(x) - M|x|^2}{|h|^2}$$
   $$= \frac{1}{|h|^2}\left(\frac{M}{2}(|x|^2 + 2x \cdot h + |h|^2 + |x|^2 - 2x \cdot h + |h|^2) - M|x|^2\right) = M$$

   ☐

2. For any $x_0 \in \Omega$, and $h \in \mathbb{R}^n$ s.t. $\overline{B}_{|h|}(x_0) \subset \Omega$,

   $$-\underline{\Theta}(u, B_{|h|}(x_0))(x_0) \leq \Delta_h^2 u(x_0) \leq \overline{\Theta}(u, B_{|h|}(x_0))(x_0) \tag{8.2}$$

   Proof. WLOG we prove $\Delta_h^2 u(x_0) \leq \overline{\Theta}(u, B_{|h|}(x_0))(x_0)$, and assume $\overline{\Theta}(u, B_{|h|}(x_0))(x_0) < \infty$. Then for any $\varepsilon > 0$, there exists convex paraboloid P that touches u by above at x_0 in $B_{|h|}(x_0)$ s.t.

   $$\Delta_h^2 P \leq \overline{\Theta}(u, B_{|h|}(x_0))(x_0) + \varepsilon$$

   But using the fact that P touches u by above at x_0, we have

   $$\begin{cases} u(x_0) = P(x_0) & \text{at } x_0 \\ u(x) \leq P(x) & \forall\, x \in B_{|h|}(x_0) \end{cases}$$

   Hence

   $$\Delta_h^2 u(x_0) = \frac{u(x_0 + h) + u(x_0 - h) - 2u(x_0)}{|h|^2} = \frac{u(x_0 + h) + u(x_0 - h) - 2P(x_0)}{|h|^2}$$
   $$\leq \frac{P(x_0 + h) + P(x_0 - h) - 2P(x_0)}{|h|^2} = \Delta_h^2 P$$
   $$\leq \overline{\Theta}(u, B_{|h|}(x_0))(x_0) + \varepsilon \qquad \forall\, \varepsilon > 0$$

   Take $\varepsilon \to 0$ on RHS to yield $\Delta_h^2 u(x_0) \leq \overline{\Theta}(u, B_{|h|}(x_0))(x_0)$. ☐

3. Let us note the Integration by Parts formula for the second differential quotient.

Lemma 8.1.2. *Let $u \in C(\Omega)$ and $\varphi \in C_0(\Omega)$. Then for $|h|$ sufficiently small s.t.* $\operatorname{supp}(\varphi) \pm h \subset \Omega$

$$\int_\Omega u(x)\Delta_h^2\varphi(x)\,dx = \int_\Omega \Delta_h^2 u(x)\varphi(x)\,dx$$

Proof.

$$\begin{aligned}
&\int_\Omega u(x)\frac{\varphi(x+h)+\varphi(x-h)-2\varphi(x)}{|h|^2}\\
&= \frac{1}{|h|^2}\left(\int_\Omega u(x)\varphi(x+h)\,dx + \int_\Omega u(x)\varphi(x-h)\,dx - 2\int_\Omega u(x)\varphi(x)\,dx\right)\\
&= \frac{1}{|h|^2}\left(\int_{\operatorname{supp}(\varphi)-h} u(x)\varphi(x+h)\,dx + \int_{\operatorname{supp}(\varphi)+h} u(x)\varphi(x-h)\,dx - 2\int_{\operatorname{supp}(\varphi)} u(x)\varphi(x)\,dx\right)\\
&= \frac{1}{|h|^2}\left(\int_{\operatorname{supp}(\varphi)} u(x-h)\varphi(x)\,dx + \int_{\operatorname{supp}(\varphi)} u(x+h)\varphi(x)\,dx - 2\int_{\operatorname{supp}(\varphi)} u(x)\varphi(x)\,dx\right)\\
&= \int_{\operatorname{supp}(\varphi)} \frac{u(x+h)+u(x-h)-2u(x)}{|h|^2}\varphi(x)\,dx\\
&= \int_\Omega \frac{u(x+h)+u(x-h)-2u(x)}{|h|^2}\varphi(x)\,dx = \int_\Omega \Delta_h^2 u(x)\varphi(x)\,dx
\end{aligned}$$

□

$W^{2,p}$ **bound via** $\Theta(u,\varepsilon)$

Proposition 8.1.1 ([CC95] Proposition 1.1)**.** *Let $p \in (1,\infty]$. Let $u \in C(\Omega)$. Then for any $\varepsilon > 0$ and*

$$\Theta(u,\varepsilon)(x) := \Theta(u, B_\varepsilon(x)\cap\Omega)(x) \qquad \forall\ x \in \Omega \tag{8.3}$$

One has estimate that $\|\Theta(u,\varepsilon)\|_{L^p(\Omega)} < \infty$ implies $D^2u \in L^p(\Omega)$ via

$$\left\|D^2u\right\|_{L^p(\Omega)} \le 2\,\|\Theta(u,\varepsilon)\|_{L^p(\Omega)} \tag{8.4}$$

Proof. First note it suffices to prove for any $\varphi \in C_0^\infty(\Omega)$ and $1 \le i,j \le n$

$$\left|\int_\Omega u\,\partial_i\partial_j\varphi\,dx\right| \le 2\,\|\Theta(u,\varepsilon)\|_{L^p(\Omega)}\,\|\varphi\|_{L^{p'}(\Omega)}$$

using Riesz Representation, where $\frac{1}{p}+\frac{1}{p'}=1$.

1. Let $\{e_i\}$ denote canonical basis of $\mathbb{R}^n$, and denote $\partial_i = \partial_{e_i}$. One has

$$\begin{aligned}
\partial^2_{e_i+e_j}\varphi &= \partial_{e_i+e_j,e_i+e_j}\varphi = (\partial_i+\partial_j)^2\varphi\\
&= (\partial_i^2 + 2\partial_i\partial_j + \partial_j^2)\varphi = \partial_i^2\varphi + 2\partial_i\partial_j\varphi + \partial_j^2\varphi\\
\partial_i\partial_j\varphi &= \frac{1}{2}\left(\partial^2_{e_i+e_j}\varphi - \partial_i^2\varphi - \partial_j^2\varphi\right)
\end{aligned}$$

On the other hand, let $v = \frac{e_i+e_j}{\sqrt{2}}$, then using rescaling argument

$$\begin{aligned}
\partial_v^2\varphi &= \partial^2_{\frac{1}{\sqrt{2}}(e_i+e_j)}\varphi = \partial_{\frac{1}{\sqrt{2}}(e_i+e_j)}\lim_{|h|\to 0}\frac{\varphi(x+|h|\frac{1}{\sqrt{2}}(e_i+e_j))-\varphi(x)}{|h|}\\
&= \partial_{\frac{1}{\sqrt{2}}(e_i+e_j)}\frac{1}{\sqrt{2}}\lim_{|h|\to 0}\frac{\varphi(x+|h|\frac{1}{\sqrt{2}}(e_i+e_j))-\varphi(x)}{\frac{|h|}{\sqrt{2}}}\\
&= \partial_{\frac{1}{\sqrt{2}}(e_i+e_j)}\frac{1}{\sqrt{2}}\partial_{e_i+e_j}\varphi = \frac{1}{2}\partial^2_{e_i+e_j}\varphi\\
\partial_i\partial_j\varphi &= \frac{1}{2}\left(2\partial_v^2\varphi - \partial_i^2\varphi - \partial_j^2\varphi\right)
\end{aligned}$$

Now the RHS consists only of a linear combination of second order directional derivatives with unit length directions in $\mathbb{R}^n$. Hence it suffices to estimate for arbitrary unit length direction (where we choose to be canonical e_i WLOG)

$$\left|\int_\Omega u\,\partial_i^2\varphi\,dx\right| \le \|\Theta(u,\varepsilon)\|_{L^p(\Omega)}\,\|\varphi\|_{L^{p'}(\Omega)}$$

2. Now for $\text{supp}(\varphi) \subset\subset \Omega$, the good thing about dealing with $\partial_i^2 \varphi$ is that the differential quotient takes exactly the form (8.1) with $h = \delta e_i$ for $\delta \to 0$ in the limit. Hence apply Integration by Parts

$$\int_\Omega u\, \partial_i^2 \varphi \, dx = \lim_{\delta \to 0} \int_\Omega u\, \Delta^2_{\delta e_i} \varphi \, dx = \lim_{\delta \to 0} \int_\Omega \Delta^2_{\delta e_i} u\, \varphi \, dx$$

Now for δ sufficiently small s.t. $\delta < \varepsilon$ and that $\text{supp}(\varphi) + \overline{B}_\delta(0) \subset \Omega$, one may apply (8.2)

$$\begin{aligned} \left| \int_\Omega u\, \partial_i^2 \varphi \, dx \right| \le \left| \lim_{\delta \to 0} \int_\Omega \Delta^2_{\delta e_i} u\, \varphi \, dx \right| &\le \left| \int_\Omega \Theta(u, \varepsilon)(x) \varphi(x) \, dx \right| \\ &\le \|\Theta(u,\varepsilon)\|_{L^p(\Omega)} \|\varphi\|_{L^{p'}(\Omega)} \end{aligned}$$

□

$C^{1,1}$ bound via $\Theta(u, \varepsilon)$

Proposition 8.1.2 ([CC95] Proposition 1.2)**.** *Let $u \in C(\Omega)$, and B a convex domain s.t. $\overline{B} \subset \Omega$. Then for any $\varepsilon > 0$ and $\Theta(u, \varepsilon)$ as in* (8.3)

$$\Theta(u,\varepsilon)(x) := \Theta(u, B_\varepsilon(x) \cap \Omega)(x) \qquad \forall\, x \in \overline{B}$$

One has estimate that $\|\Theta(u,\varepsilon)\|_{L^\infty(B)} < \infty$ implies $u \in C^{1,1}(\overline{B})$ via

$$|Du(x) - Du(y)| \le 2n \, \|\Theta(u,\varepsilon)\|_{L^\infty(B)} \, |x - y| \qquad \forall\, x, y \in \overline{B} \tag{8.5}$$

Proof. 1. Since by assumption $\Theta(u,\varepsilon)(x) < \infty$ for any $x \in \overline{B}$, we know u is differentiable at any $x \in \overline{B}$. Then using Proposition 8.1.1 with $p = \infty$ and on B

$$\left\|D^2 u\right\|_{L^\infty(B)} \le 2 \, \|\Theta(u,\varepsilon)\|_{L^\infty(B)}$$

Hence $D^2 u \in L^\infty(B)$ implies $\partial_i u \in W^{1,\infty}(B)$ for any $i = 1, \cdots, n$.

2. Using characterisation of $W^{1,\infty}$, we know $\partial_i u$ are Lipschitz continuous. Thus using B is convex, hence including linear combinations of $x, y \in \overline{B}$, we obtain

$$\begin{aligned} \partial_i u(x) - \partial_i u(y) &= \int_0^1 \frac{d}{dt} \partial_i u(tx + (1-t)y) \, dt \\ &= \sum_{j=1}^n \int_0^1 \partial_i \partial_j u(tx + (1-t)y)(x_j - y_j) \, dt \qquad \forall\, x, y \in \overline{B} \end{aligned}$$

Thus

$$\begin{aligned} |Du(x) - Du(y)|^2 &= \sum_{i=1}^n |\partial_i u(x) - \partial_i u(y)|^2 \le \sum_{i=1}^n |\sum_{j=1}^n \int_0^1 \partial_i \partial_j u(tx + (1-t)y)(x_j - y_j) \, dt|^2 \\ &\le \sum_{i=1}^n \left((\sum_{j=1}^n \|D\partial_j u\|^2_{L^\infty(B)})^{\frac{1}{2}} (\sum_{j=1}^n |x_j - y_j|^2)^{\frac{1}{2}} \right)^2 \\ &\le n \left(\sum_{j=1}^n \|D\partial_j u\|^2_{L^\infty(B)} \right) \left(\sum_{j=1}^n |x_j - y_j|^2 \right) \\ &\le n^2 \left\|D^2 u\right\|^2_{L^\infty(B)} |x - y|^2 \\ |Du(x) - Du(y)| &\le n \left\|D^2 u\right\|_{L^\infty(B)} |x - y| \le 2n \, \|\Theta(u,\varepsilon)\|_{L^\infty(B)} \, |x - y| \qquad \forall\, x, y \in \overline{B} \end{aligned}$$

□

Punctually Second Order Differentiable

Definition 8.1.3 ([CC95] Definition 1.4)**.** *We say $u \in C(\Omega)$ in Ω is punctually second order differentiable at $x_0 \in \Omega$ if*

there exists paraboloid P s.t.

$$u(x) = P(x) + o(|x - x_0|^2) \qquad x \to x_0$$

i.e.

$$\lim_{x\to x_0}\frac{|u(x)-P(x)|}{|x-x_0|^2}=0$$

In this case one may define

$$D^2u(x_0):=D^2P(x_0)$$

Note this is well-defined because such parabola is unique!

Let's say something about the polynomial P given by the definition of punctual second order differentiability.

1. Since u and P are both continuous,

$$u(x_0)=\lim_{x\to x_0}u(x)=\lim_{x\to x_0}P(x)+o(|x-x_0|^2)=P(x_0)$$

2. Assume P of the structure

$$P(x)=u(x_0)+a\cdot(x-x_0)+\frac{1}{2}M|x-x_0|^2$$

Then

$$\lim_{x\to x_0}\frac{|u(x)-u(x_0)-a\cdot(x-x_0)|}{|x-x_0|}=\lim_{x\to x_0}\frac{|\frac{1}{2}M|x-x_0|^2+o(|x-x_0|^2)|}{|x-x_0|}=0$$

Thus by definition u is differentiable at x_0 and people define

$$\nabla u(x_0):=a$$

Lemma 8.1.3. *u second order differentiable at x_0 implies u punctually second order differentiable at $x_0\in\Omega$. And $D^2u(x_0)$ agrees.*

Proof. Note using u second order differentiable at x_0, $\nabla u(x_0)$ and $D^2u(x_0)$ are unique and well-defined. Now let the parabola be

$$P(x):=u(x_0)+\nabla u(x_0)\cdot(x-x_0)+\frac{1}{2}(x-x_0)^TD^2u(x_0)(x-x_0)$$

We check for $x=x_0+he$ with $e\in\mathbb{S}^{n-1}$, $h>0$

$$\begin{aligned}\frac{|u(x)-P(x)|}{|x-x_0|^2}&=\frac{|u(x_0+he)-P(x_0+he)|}{|h|^2}\\&=\frac{|u(x_0+he)-u(x_0)-\nabla u(x_0)\cdot he+\frac{1}{2}he^TD^2u(x_0)he|}{h^2}\end{aligned}$$

Conclude via Taylor's Expansion. □

The following is the Alexandroff-Buselman-Feller Theorem.

Theorem 8.1.1 ([CC95] Theorem 1.5; [EG15] Theorem 6.4)**.** *Let u be convex function in ball B_d. Then u is punctually second order differentiable at a.e. $x_0\in B_d$.*

Using the Alexandroff Theorem, one may prove.

Proposition 8.1.3 ([CC95] Proposition 1.6)**.** *Let $u\in C(\Omega)$ in a convex domain Ω, and assume for some $K>0$ positive constant*

$$\underline{\Theta}(u,\Omega)(x)\le K\qquad\forall\ x\in\Omega\tag{8.6}$$

Then

$$u(x)+\frac{K}{2}|x|^2$$

is convex in Ω. In particular, u is punctually second order differentiable at a.e. $x\in\Omega$.

Proof. To show $w(x)=u(x)+\frac{K}{2}|x|^2$ is convex, it suffices to see that its second differential quotient has a sign. To do so, compute for any $x_0\pm h\in\Omega$ (here convexity of the domain ensure $x_0\in\Omega$)

$$\begin{aligned}\Delta_h^2w(x_0)&=\Delta_h^2(\frac{K}{2}|x_0|^2)+\Delta_h^2u(x_0)=K+\Delta_h^2u(x_0)\\&\overset{(8.2)}{\ge}K-\underline{\Theta}(u,\Omega)(x_0)\ge0\end{aligned}$$

This is to say

$$\frac{1}{2}(w(x_0+h)+w(x_0-h))\ge u(x_0)$$

Using arbitrariness of x_0 and h, and that w is continuous, this implies convexity of w in Ω. Finally, apply Alexandroff Theorem to conclude $u+\frac{K}{2}|x|^2$ is punctually second order differentiable, and subtracting by a parabola preserves the differentiability. □

8.1.2 Uniform Ellipticity

In this section we generalize the idea of ellipticity.

8.1.2.1 Space of $n \times n$ real symmetric matrices

Denote $n \times n$ symmetric matrices over the reals as

$$\mathrm{Sym}(n) := \{M \in \mathbb{R}^{n\times n} \mid M = M^T\}$$

One identify $\mathrm{Sym}(n)$ as vector space of dimension $\frac{n(n+1)}{2}$.

What are some facts about real symmetric $n \times n$ matrices?

1. All eigenvalues $\lambda_i(M)$ for $M \in \mathrm{Sym}(n)$ are real.
2. Eigenvectors corresponding to distinct eigenvalues are orthogonal. Thus there is an orthonormal basis of $\mathbb{R}^n$ that are eigenvectors of M.
3. Therefore, $M \in \mathrm{Sym}(n)$ is always orthogonally diagonalisable, i.e., there exists $\Lambda(M) = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ real diagonal matrix and Q orthogonal matrix (with orthonormal columns) s.t.

$$M = Q^T \Lambda Q, \qquad Q^T Q = I$$

Operator Norm $\|M\|$ In fact, it is a Banach space by defining the L^2 to L^2 operator norm (equal to the largest size of the eigenvalue)

$$\|M\| := \sup_{\substack{x\in\mathbb{R}^n \\ |x|=1}} |Mx| = \max_{1\le i\le n} |\lambda_i(M)| \qquad \forall\ M \in \mathrm{Sym}(n)$$

Proof of equality. For any $M \in \mathrm{Sym}(n)$, one may do spectral decomposition so that $M = Q^T\Lambda(M)Q$ where $\Lambda(M)$ is the diagonal matrix of eigenvalues, and $Q = (q_i)_{1\le i\le n}$ are the orthonormal basis w.r.t. $\Lambda(M)$. Since M is symmetric, the eigenvalues are real. Now for any $x \in \mathbb{R}^n$ with $|x| = 1$, one may represent x w.r.t. the basis Q, i.e.

$$x = \sum_{i=1}^n \alpha_i q_i, \qquad |\alpha|^2 = 1$$

Thus

$$Mx = \sum_{i=1}^n \alpha_i\lambda_i q_i$$

Since the basis $\{q_i\}$ are orthogonal,

$$|Mx|^2 = (\sum_{i=1}^n \alpha_i\lambda_i q_i)^2 = (\sum_{i=1}^n \alpha_i\lambda_i)^2 \le \max_{1\le i\le n} |\lambda_i(M)|^2 (\sum_{i=1}^n \alpha_i^2) = \max_{1\le i\le n} |\lambda_i(M)|^2$$

For the other side, let $x = q_k$ where $\lambda_k = \max_{1\le i\le n} |\lambda_i(M)|$. Now

$$|Mq_k| = |\lambda_k q_k| = \lambda_k = \max_{1\le i\le n} |\lambda_i(M)|$$

□

Frobenius Norm $\|M\|_{\textbf{Frobenius}}$ It is moreover a Hilbert space if one equip $\mathrm{Sym}(n)$ with inner product as matrix contraction

$$\langle M, N\rangle := \mathrm{Tr}(MN) = \sum_{i,j} M_{ij}N_{ij}$$

Let's check this.

Proof. Let $M, N, P \in \mathrm{Sym}(n)$ and $a, b \in \mathbb{R}$. For linearity, using linearity of the trace,

$$\langle aM + bN,\, P\rangle = \mathrm{Tr}\big((aM + bN)P\big) = a\,\mathrm{Tr}(MP) + b\,\mathrm{Tr}(NP) = a\langle M, P\rangle + b\langle N, P\rangle.$$

and similar for the second augment. For symmetry, since $\mathrm{Tr}(X) = \mathrm{Tr}(X^\top)$ for any square matrix X,

$$\langle M, N\rangle = \mathrm{Tr}(MN) = \mathrm{Tr}\big((MN)^\top\big) = \mathrm{Tr}(N^\top M^\top).$$

Because M, N are symmetric, $M^\top = M$, $N^\top = N$, hence

$$\langle M, N\rangle = \mathrm{Tr}(NM) = \langle N, M\rangle.$$

For positive-definiteness, let $M \in \mathrm{Sym}(n)$,

$$\langle M, M\rangle = \mathrm{Tr}(M^2) = \sum_{i=1}^{n}(M^2)_{ii} = \sum_{i,j=1}^{n} M_{ij}M_{ji}.$$

Since symmetry yields $M_{ij} = M_{ji}$, this equals

$$\langle M, M\rangle = \sum_{i=1}^{n} M_{ii}^2 + 2\sum_{i<j} M_{ij}^2 \ \geq 0.$$

Moreover, $\langle M, M\rangle = 0$ implies all entries $M_{ij} = 0$, so $M = 0$.

In fact, for symmetric matrices

$$\langle M, N\rangle = \mathrm{Tr}(MN) = \mathrm{Tr}(M^\top N) = \sum_{i,j} M_{ij}N_{ij},$$

which is the restriction of the Frobenius inner product. Since $\mathrm{Sym}(n)$ is finite-dimensional, completeness holds automatically, hence it is a Hilbert space. □

Notice the *(Frobenius) norm* induced by the inner produce writes

$$\|M\|_{\text{Frobenius}} = (\mathrm{Tr}(M^2))^{\frac{1}{2}} = (\sum_{i,\,j=1}^{n} M_{ij}^2)^{\frac{1}{2}}$$

What is the relationship between $\|M\|_{\text{Frobenius}}$ and the operator norm $\|M\|$?

First note for $\lambda_i(M)$ denoting the real eigenvalues of M

$$\|M\|_{\text{Frobenius}} = \sqrt{\sum_{i=1}^{n} \lambda_i(M)^2}$$

Proof. Let $M = Q^T\Lambda Q$ be spectral decomposition where $Q = (q_i)_{1\leq i\leq n}$ be orthonormal basis corresponding to eigenvalues Λ. Then using diagonalization

$$\|M\|_{\text{Frobenius}}^2 = \mathrm{Tr}(M^2) = \mathrm{Tr}(Q^T\Lambda QQ^T\Lambda Q) = \mathrm{Tr}(Q^T\Lambda^2 Q) = \sum_{k=1}^{n} \lambda_k^2$$

□

Thus one has relationship

$$\|M\| \leq \|M\|_{\text{Frobenius}} \leq \sqrt{n}\,\|M\| \tag{8.7}$$

Consequently the topology induced by the two norm structures are the same, and the norms are equivalent.

Supremum Norm $\|M\|_\infty$ Let's also consider the sup norm

$$\|M\|_\infty := \sup_{i,j}|M_{ij}|$$

One has the following relations

$$\|M\|_\infty \leq \|M\| \leq n\,\|M\|_\infty$$
$$\|M\|_\infty \leq \|M\|_{\text{Frobenius}} \leq n\,\|M\|_\infty$$

Take the example $M =$ all ones so

$$\|M\|_\infty = 1$$

But

$$\|M\|_{\text{Frobenius}} = \left(\sum_{i,j} M_{ij}^2\right)^{\frac{1}{2}} = (n^2 \times 1)^{\frac{1}{2}} = n$$
$$\|M\| = \max\{n, 0, \cdots, 0\} = n$$

Decomposition Theorem Let's introduce a decomposition theorem for $M \in \mathrm{Sym}(n)$ into positive and negative parts.

Proposition 8.1.4 (Positive/negative parts of a symmetric matrix)**.** *Let $M \in \mathrm{Sym}(n)$ be real symmetric. Then there exist unique $M^+, M^- \in \mathrm{Sym}(n)$ such that*

$$M = M^+ - M^-, \qquad M^\pm \geq 0, \qquad M^+M^- = 0. \tag{8.8}$$

Moreover, M^+ and M^- are functions of M (they commute with M), and their eigen-structure is determined by that of M: if $Mv_i = \lambda_i v_i$ with $\{v_i\}$ orthonormal, then

$$M^+ v_i = \max(\lambda_i, 0)\, v_i, \qquad M^- v_i = \max(-\lambda_i, 0)\, v_i.$$

Proof. 1. Existence. By the spectral theorem for self-adjoint operators, in our case for $M \in \mathrm{Sym}(n)$, there is an orthogonal Q and a diagonal $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ with

$$M = Q\Lambda Q^\top$$

Define diagonal matrices

$$\Lambda^+ := \mathrm{diag}\big(\max(\lambda_i, 0)\big), \qquad \Lambda^- := \mathrm{diag}\big(\max(-\lambda_i, 0)\big),$$

and set

$$M^+ := Q\Lambda^+Q^\top, \qquad M^- := Q\Lambda^-Q^\top.$$

Clearly $M^\pm \geq 0$, $M^+ - M^- = Q(\Lambda^+ - \Lambda^-)Q^\top = Q\Lambda Q^\top = M$, and $M^+M^- = Q(\Lambda^+\Lambda^-)Q^\top = 0$ since each diagonal product $\max(\lambda_i, 0)\max(-\lambda_i, 0) = 0$.

2. Uniqueness. Suppose $M = A - B$ with $A, B \geq 0$ and $AB = 0$. Because A, B are symmetric, $(AB)^\top = BA = 0$ as well. Hence

$$M^2 = (A - B)^2 = A^2 + B^2 \qquad \text{(cross terms vanish)}.$$

Let $N := A + B \geq 0$. Then

$$N^2 = A^2 + AB + BA + B^2 = A^2 + B^2 = M^2.$$

Thus N is a positive semidefinite square root of M^2. By uniqueness of the positive semidefinite square root, $N = |M| := (M^2)^{1/2}$. Therefore

$$A = \tfrac{1}{2}(N + M) = \tfrac{1}{2}(|M| + M) = M^+, \qquad B = \tfrac{1}{2}(N - M) = \tfrac{1}{2}(|M| - M) = M^-,$$

which proves uniqueness.

3. Eigenvalues. In the eigenbasis of M, $M^\pm$ are diagonal with entries $\max(\lambda_i, 0)$ and $\max(-\lambda_i, 0)$, respectively. Hence $M^\pm$ share the same eigenvectors as M, and

$$\mathrm{Spec}(M^+) = \{\max(\lambda_i, 0)\}_i, \qquad \mathrm{Spec}(M^-) = \{\max(-\lambda_i, 0)\}_i.$$

In particular, $\mathrm{rank}(M^+)$ equals the number of positive eigenvalues of M (counted with multiplicity), $\mathrm{rank}(M^-)$ equals the number of negative eigenvalues, $|M| = M^+ + M^-$, and $M^+M^- = M^-M^+ = 0$.

□

Example 8.1.1 ([Moo12] Section 8.1)**.** *Let's look at $n = 2$.*

1. *Notice the matrices*

$$M_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad M_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad M_3 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

form a basis for Sym(2).

Proof. For any $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ one align

$$\begin{aligned} a &= x_1 + x_3 \\ b &= x_2 \\ c &= -x_1 + x_3 \end{aligned}$$

so that

$$M = \frac{a - c}{2} M_1 + bM_2 + \frac{a + c}{2} M_3$$

and thus $\{M_1, M_2, M_3\}$ spans Sym(2). That they're independent is trivial. □

2. *Now consider a linear isomorphism*

$$\begin{aligned} T : \mathrm{Sym}(2) &\to \mathbb{R}^3 \\ M_1 &\mapsto \sqrt{2}e_1 \\ M_2 &\mapsto \sqrt{2}e_2 \\ M_3 &\mapsto \sqrt{2}e_3 \end{aligned}$$

If equip Sym(2) *with the inner product structure, one can check that T is isometry.*

Proof. For $M = \sum_i x_i M_i = \begin{pmatrix} x_1 + x_3 & x_2 \\ x_2 & -x_1 + x_3 \end{pmatrix}$ and $N = \sum_i y_i M_i = \begin{pmatrix} y_1 + y_3 & y_2 \\ y_2 & -y_1 + y_3 \end{pmatrix}$, note

$$MN = \begin{pmatrix} (x_1 + x_3)(y_1 + y_3) + x_2 y_2 & (x_1 + x_3)y_2 + x_2(-y_1 + y_3) \\ x_2(y_1 + y_3) + (-x_1 + x_3)y_2 & x_2 y_2 + (-x_1 + x_3)(-y_1 + y_3) \end{pmatrix}$$

Now

$$\begin{aligned} \langle M, N \rangle &= \mathrm{Tr}(MN) \\ &= (x_1 + x_3)(y_1 + y_3) + x_2 y_2 + x_2 y_2 + (-x_1 + x_3)(-y_1 + y_3) \\ &= 2\,(x_1 y_1 + x_2 y_2 + x_3 y_3). \end{aligned}$$

Meanwhile,

$$\langle T(M), T(N) \rangle_{\mathbb{R}^3} = (\sqrt{2}x_1,\, \sqrt{2}x_2,\, \sqrt{2}x_3) \cdot (\sqrt{2}y_1,\, \sqrt{2}y_2,\, \sqrt{2}y_3) = 2\,(x_1 y_1 + x_2 y_2 + x_3 y_3).$$

Thus $\langle T(M), T(N) \rangle_{\mathbb{R}^3} = \langle M, N \rangle$ for all M, N, i.e., T is an isometry. □

3. *Ok, let's pause and see how we understand the eigenvalues of M in $\mathbb{R}^3$.*

   (a) *For any $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, its eigenvalues are computed as*

   $$(a - \lambda)(c - \lambda) - b^2 = \lambda^2 - (a + c)\lambda + ac - b^2 = 0 \implies \lambda_\pm = \frac{a + c \pm \sqrt{(a - c)^2 + 4b^2}}{2}$$

   In terms of $M = x_1 M_1 + x_2 M_2 + x_3 M_3$, these are

   $$\lambda_\pm = \frac{2x_3 \pm \sqrt{4x_1^2 + 4x_2^2}}{2} = x_3 \pm \sqrt{x_1^2 + x_2^2}$$

   Note $T(M) = (\sqrt{2}x_1, \sqrt{2}x_2, \sqrt{2}x_3)$, whose vertical component has height $\sqrt{2}x_3$ and horizontal components have length $\sqrt{2x_1^2 + 2x_2^2}$.

   (b) *In view of the formula for eigenvalues, one can compute the distance between $T(M)$ and the positive and negative cones with slope* 1

   $$z = \pm\sqrt{x^2 + y^2}$$

   Let $p = \sqrt{2}\,(x_1, x_2, x_3) \in \mathbb{R}^3$ and let $\mathcal{C}_\pm = \{(x, y, z) :\, z = \pm\sqrt{x^2 + y^2}\}$ be the two cones. By rotational symmetry about the z–axis, the distance between p and $\mathcal{C}_\pm$ equals the distance in the (r, z)-plane from the point $(\sqrt{2x_1^2 + 2x_2^2}, \sqrt{2}x_3)$ to the union of lines $z = \pm r$.
   In the (r, z)-plane, the (perpendicular) distance from$(\sqrt{2x_1^2 + 2x_2^2}, \sqrt{2}x_3)$ to $z = r$ is

   $$\mathrm{dist}\big(\sqrt{2}(x_1, x_2, x_3),\, \mathcal{C}_+\big) = \frac{|\sqrt{2x_1^2 + 2x_2^2} - \sqrt{2}x_3|}{\sqrt{2}} = |\lambda_-|$$

   and to $z = -r$ is

   $$\mathrm{dist}\big(\sqrt{2}(x_1, x_2, x_3),\, \mathcal{C}_-\big) = \frac{|\sqrt{2x_1^2 + 2x_2^2} + \sqrt{2}x_3|}{\sqrt{2}} = |\lambda_+|$$

4. *As in Proposition 8.1.4, for any $M \in$ Sym(2) one may decompose $M = M^+ - M^-$. Since T is isometry, it preserves the inner structure, hence*

$$\langle T(M^+), T(M^-) \rangle_{\mathbb{R}^3} = \langle M^+, M^- \rangle = \mathrm{Tr}(M^+ M^-) \overset{(8.8)}{=} 0$$

Now

(a) *If $\lambda_+ > 0 > \lambda_-$, then for M^+ it has eigenvalues $\lambda_+, 0$, while M^- has eigenvalues $-\lambda_-, 0$. Thus in the picture, M^+ lies on the line $z = r$, while M^- lies on the line $z = -r$. This is because*

i. *$T(M)$ lies in the plane spanned by $T(M^+)$ and $T(M^-)$.*

ii. *$T(M^+)$ and $T(M^-)$ are orthogonal.*

and together they force M^+ and M^- to both lie on the (r, z)-plane. Moreover

$$\left\|M^+\right\| = \lambda_+ \qquad \left\|M^-\right\| = -\lambda_-$$

(b) *If $\lambda_+ \geq \lambda_- > 0$ then M^+ has eigenvalues λ_+, λ_- while $M^- = 0$ is zero matrix. In this case $M^+ = M$, and M lies strictly on top of the positive cone $\mathcal{C}_+$. In particular when $\lambda_+ = \lambda_-$, M points upwards in the z axis. Similarly if $0 > \lambda_+ \geq \lambda_-$ then $M^- = -M$, and M lies strictly beneath the negative cone $\mathcal{C}_-$. In particular when $\lambda_+ = \lambda_-$, M points downwards in the z axis.*

(c) *If $\lambda_+ > \lambda_- = 0$, then $M = M^+$ and M lies on the positive cone $\mathcal{C}_+$. If $0 = \lambda_+ > \lambda_-$ then $M = -M^-$ and M lies on the negative cone $\mathcal{C}_-$.*

(d) *In the degenerate case both $\lambda_+ = \lambda_- = 0$, M is simply the zero vector.*

8.1.2.2 Uniform Ellipticity

Consider an operator as function on symmetric matrices

$$\mathcal{F} : \mathrm{Sym}(n) \to \mathbb{R}$$

What do we mean by saying $\mathcal{F}$ is uniformly elliptic? Let's first take a look at the definition.

Definition 8.1.4 (Uniform Ellipticity; [CC95] Definition 2.1)**.** *$\mathcal{F}$ is called uniformly elliptic if there exists elliptic constants $0 < \lambda \leq \Lambda$ s.t.*

$$\lambda \|N\| \leq \mathcal{F}(M+N) - \mathcal{F}(M) \leq \Lambda \|N\| \qquad \forall\ M, N \in \mathrm{Sym}(n), \qquad N \geq 0 \tag{8.9}$$

Notice by decomposition $N = N^+ - N^-$ as in (8.8) one obtain an equivalence criteria

Lemma 8.1.4 ([CC95] Lemma 2.2)**.** *$\mathcal{F}$ is uniformly elliptic iff*

$$\lambda \left\|N^+\right\| - \Lambda \left\|N^-\right\| \leq \mathcal{F}(M+N) - \mathcal{F}(M) \leq \Lambda \left\|N^+\right\| - \lambda \left\|N^-\right\| \qquad \forall\ M, N \in \mathrm{Sym}(n) \tag{8.10}$$

Proof. $\Longrightarrow$. Take any $N \in \mathrm{Sym}(n)$ and decompose $N = N^+ - N^-$. (8.9) yields

$$\begin{aligned}\lambda \left\|N^+\right\| &\leq \mathcal{F}((M - N^-) + N^+) - \mathcal{F}(M - N^-) \leq \Lambda \left\|N^+\right\| \\ \lambda \left\|N^-\right\| &\leq \mathcal{F}((M - N^-) + N^-) - \mathcal{F}(M - N^-) \leq \Lambda \left\|N^-\right\|\end{aligned}$$

Rearranging yields

$$\lambda \left\|N^+\right\| - \Lambda \left\|N^-\right\| \leq \mathcal{F}(M+N) - F(M) \leq \Lambda \left\|N^+\right\| - \lambda \left\|N^-\right\|$$

In particular one obtain (8.10). $\Longleftarrow$. Now assume $N \in \mathrm{Sym}(n)$ with $N \geq 0$, N^- vanishes and we recover (8.9). ☐

We make two immediate observations that

1. $\mathcal{F}(M)$ is increasing function in $M \in \mathrm{Sym}(n)$. This is usually know as ellipticity.

2. $\mathcal{F}(M)$ is Lipschitz function in $M \in \mathrm{Sym}(n)$.

Example 8.1.2 ([Moo12] Section 8.1)**.** *Recall the case for $n = 2$. Fix our $M \in \mathrm{Sym}(n)$, and we consider the variable $N \in \mathrm{Sym}(n)$. Assume one has no contribution in $\mathcal{F}$ upon adding N, i.e.*

$$\mathcal{F}(M+N) - \mathcal{F}(M) = 0 \tag{8.11}$$

In particular, ***this is achieved via requiring $M + N$ and M to lies in the same level set $\mathcal{F}^{-1}(a)$.***

Then

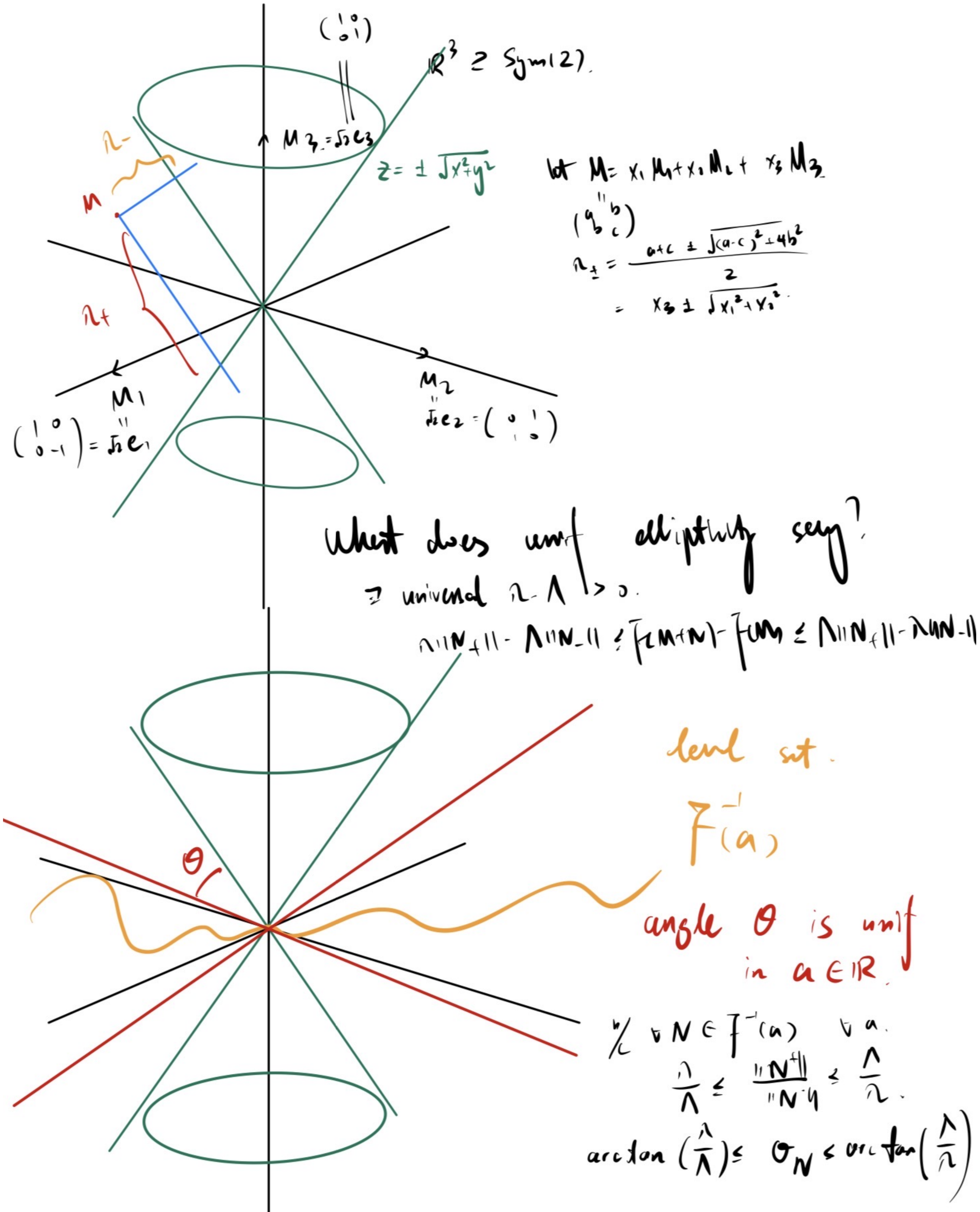


Figure 8.1: Visualizing Uniform Ellipticity

1. *If* $\lambda_+(N) > 0 > \lambda_-(N)$, (8.10) *reads*

$$\frac{\lambda}{\Lambda} \leq \frac{\|N^+\|}{\|N^-\|} = \frac{\lambda_+(N)}{\lambda_-(N)} \leq \frac{\Lambda}{\lambda}$$

upon division. Consider the angle $\theta_N \in (0, \frac{\pi}{2})$ *sitting in between with one side* N^+ *and pointing clockwise s.t.*

$$\tan(\theta_N) = \frac{\|N^+\|}{\|N^-\|}$$

Then the above inequality reads using monotonicity

$$\arctan(\frac{\lambda}{\Lambda}) \leq \theta_N \leq \arctan(\frac{\Lambda}{\lambda}) \qquad \forall\ N \in \mathrm{Sym}(n) \qquad \textit{s.t. } (8.11) \textit{ holds and } \lambda_+(N) > 0 > \lambda_-(N)$$

In particular, no matter how small $\|N\|$ is, the size of the angle θ_N has uniform bounds. If one further understand N as elements of the level set $\mathcal{F}^{-1}(a)$ that varies as $a \in \mathbb{R}$, and consider the cone of wider opening (with degree $\theta := \frac{\pi}{4} + \min\{\arctan(\frac{\lambda}{\Lambda}), \frac{\pi}{4}\}$). Then the cone centered at 0 with opening θ fully traps the level sets $\mathcal{F}^{-1}(a)$ uniformly in $a \in \mathbb{R}$.

2. *If $\lambda_+(N) \geq \lambda_-(N) > 0$, or $0 > \lambda_+(N) \geq \lambda_-(N)$, or $\lambda_+ > \lambda_- = 0$, or $0 = \lambda_+ > \lambda_-$, (8.10) holds trivially since one only has either the positive or negative part. The degenerate case $\lambda_+ = \lambda_- = 0$ holds as well.*

Example 8.1.3. *One may consider a most trivial example of a uniformly elliptic operator.*

$$\mathcal{L}u := a_{ij}\partial_{ij}u \qquad A = (a_{ij}) \textit{ symmetric with eigenvalues in } [\lambda, \Lambda]$$

In fact the operator norm of A writes $\|A\| = \Lambda$.

Let's check such

$$\mathcal{F}_A(N) := \langle A, N \rangle$$

is indeed uniformly elliptic with constants λ, $n\Lambda$

$$\lambda \|N\| \leq \langle A, N \rangle \leq n\Lambda \|N\| \qquad \forall\ N \geq 0 \tag{8.12}$$

Proof. Notice since $\mathcal{F}_A$ is linear in its argument,

$$\mathcal{F}_A(M + N) - \mathcal{F}_A(M) = \mathcal{F}_A(N)$$

Now using Cauchy-Schwarz

$$\begin{aligned} \mathcal{F}_A(N) = \langle A, N \rangle &\leq \|A\|_{\text{Frobenius}} \|N\|_{\text{Frobenius}} \\ &\overset{(8.7)}{\leq} \sqrt{n}\,\|A\|\,\sqrt{n}\,\|N\| = n\Lambda \|N\| \end{aligned}$$

On the other hand, since $A - \lambda I$ is semi-positive definite, $A - \lambda I \geq 0$. Since $N \geq 0$, the product remains non-negative

$$(A - \lambda I)N \geq 0$$

Now using the trace of non-negative matrix is non-negative

$$\begin{aligned} \mathcal{F}_A(N) = \langle A, N \rangle = \mathrm{Tr}(AN) &= \mathrm{Tr}((A - \lambda I)N) + \lambda \mathrm{Tr}(IN) \\ &\geq 0 + \lambda \|N\| \end{aligned}$$

□

8.1.3 Viscosity Solution

Viscosity Solution for Laplace Let's start with the simple definition for the Laplace.

Definition 8.1.5. *Let $f \in C(B_1)$.*

We say $u \in C(B_1)$ is a viscosity supersolution to $\Delta u = f$, i.e.

$$\Delta u \leq f \qquad \text{in the viscosity sense}$$

if for any $\varphi \in C^2(B_1)$ and $x_0 \in B_1$ s.t. $u - \varphi$ achieves a local minimum, necessarily

$$\Delta\varphi(x_0) \leq f(x_0)$$

We say $u \in C(B_1)$ is viscosity subsolution to $\Delta u = f$, i.e.

$$\Delta u \geq f \qquad \text{in the viscosity sense}$$

if for any $\varphi \in C^2(B_1)$ and $x_0 \in B_1$ s.t. $u - \varphi$ achieves a local maximum, necessarily

$$\Delta\varphi(x_0) \geq f(x_0)$$

We say $u \in C(B_1)$ is a viscosity solution to $\Delta u = f$ if u is both supersolution and subsolution.

Lemma 8.1.5. *Let $u \in C(\overline{B_1})$ be viscosity solution to $\Delta u = 0$. Then u is a classical solution to Laplace Equation.*

Proof. Consider v as harmonic replacement of u in B_1. For any $\delta > 0$ define

$$v_\delta(x) := v(x) + \delta(1 - |x|^2)$$

Then simple computation yields

$$\Delta v_\delta = \Delta v - 2n\delta < 0 \qquad \forall\, x \in B_1$$

This is to say, using u is subsolution, $u - v_\delta$ cannot achieve local maximum anywhere in B_1. Hence using that v is harmonic replacement

$$\sup_{B_1} u - v_\delta \leq \sup_{\partial B_1} u - v_\delta = 0$$

Hence

$$u(x) \leq v(x) + \delta(1 - |x|^2) \qquad \forall\, x \in B_1 \qquad \forall\, \delta > 0$$

Sending $\delta \to 0$ yields

$$u \leq v \qquad B_1$$

On the other hand define

$$v'_\delta(x) := v(x) + \delta(|x|^2 - 1)$$

gives

$$\Delta v'_\delta = \Delta v + 2n\delta > 0$$

Hence using u is supersolution, $u - v'_\delta$ cannot achieve local minimum in B_1, therefore

$$\inf_{B_1} u - v'_\delta \geq \inf_{\partial B_1} u - v'_\delta = 0$$

Thus

$$u \geq v + \delta(|x|^2 - 1) \qquad \forall\, x \in B_1, \qquad \forall\, \delta > 0$$

sending $\delta \to 0$ yields the result. □

Viscosity Solution for uniformly elliptic operator Now for a general definition.

Definition 8.1.6 ([CC95] Definition 2.3)**.** *Consider uniformly elliptic operator $\mathcal{F}$ with elliptic constants $0 < \lambda \leq \Lambda$. Let $f \in C(\Omega)$.*

We say a function $u \in C(\Omega)$ is viscosity supersolution to $\mathcal{F}(D^2u) = f$, i.e.

$$\mathcal{F}(D^2u) \leq f \qquad \text{in the viscosity sense}$$

if for any $\varphi \in C^2(\Omega)$ and $x_0 \in \Omega$ s.t. $u - \varphi$ attains local minimum, necessarily

$$\mathcal{F}(D^2\varphi(x_0)) \leq f(x_0)$$

We say $u \in C(\Omega)$ is viscosity subsolution to $\mathcal{F}(D^2u) = f$, i.e.

$$\mathcal{F}(D^2u) \geq f \qquad \text{in the viscosity sense}$$

if for any $\varphi \in C^2(\Omega)$ and $x_0 \in \Omega$ s.t. $u - \varphi$ attains local maximum, necessarily

$$\mathcal{F}(D^2\varphi(x_0)) \geq f(x_0)$$

We say $u \in C(\Omega)$ is viscosity solution if u is both supersolution and subsolution.

One may replace $u-\varphi$ having maximum or minimum by touching from above or below. Moreover, using $\mathcal{F}$ is uniformly elliptic, one may replace φ with quadratic polynomials.

Lemma 8.1.6 ([CC95] Proposition 2.4)**.** *The following are equivalent.*

1. *u is viscosity subsolution as in Definition 8.1.6*
2. *iff for any $\varphi \in C^2(\Omega)$ s.t. $u(x_0) = \varphi(x_0)$, $u \leq \varphi$ locally near x_0, necessarily*
$$\mathcal{F}(D^2\varphi(x_0)) \geq f(x_0)$$
3. *iff for any P quadratic polynomial in Ω s.t. $u(x_0) = P(x_0)$, $u \leq P$ locally near x_0, necessarily*
$$\mathcal{F}(D^2P(x_0)) \geq f(x_0)$$

Proof. (1) implies (2). Let $u-\varphi$ achieve local maximum, i.e. at some $x_0 \in \Omega$

$$u - \varphi \leq u(x_0) - \varphi(x_0) \qquad \text{for certain } B_r(x_0) \subseteq \Omega$$

Then

$$\begin{cases} u \leq \varphi + u(x_0) - \varphi(x_0) & B_r(x_0) \\ u(x_0) = \varphi(x_0 + u(x_0) - \varphi(x_0) & x_0 \end{cases}$$

so the function $\varphi + u(x_0) - \varphi(x_0)$ locally touches u from above. Indeed

$$\Delta\varphi(x_0) = \Delta(\varphi + u(x_0) - \varphi(x_0))(x_0)$$

(2) implies (1). On the other hand, let

$$\begin{cases} u \leq \varphi & B_r(x_0) \\ u(x_0) = \varphi(x_0) & x_0 \end{cases}$$

Then the function

$$u - \varphi \leq u(x_0) - \varphi(x_0) = 0 \qquad B_r(x_0)$$

so $u-\varphi$ achieves local maximum at x_0.

(2) implies (3). Since P as quadratic polynomial is itself C^2 function.

(3) implies (2). For any $\varphi \in C^2(\Omega)$ s.t. φ touches u from above at x_0. Define

$$P := u(x_0) + \nabla\varphi(x_0)\cdot(x-x_0) + \frac{1}{2}(x-x_0)^T D^2\varphi(x_0)(x-x_0) + \frac{\varepsilon}{2}|x-x_0|^2$$

So that

$$D^2P(x_0) = D^2\varphi(x_0) + \varepsilon I$$

Picking $|x-x_0|$ sufficiently small one ensure $P \geq u$ locally near x_0. Thus

$$\mathcal{F}(D^2\varphi(x_0) + \varepsilon I) \geq f(x_0) \qquad \forall\, \varepsilon > 0$$

Using $\mathcal{F}$ is continuous (as result of uniform ellipticity) in $\mathrm{Sym}(n)$, one send $\varepsilon \to 0$ and deduce

$$\mathcal{F}(D^2\varphi(x_0)) \geq f(x_0)$$

□

Viscosity Solution and Classical Solution The following justifies why people wish for Punctually Second order differentiability.

Lemma 8.1.7 ([CC95] Lemma 2.5)**.** *Let u be viscosity subsolution*

$$\mathcal{F}(D^2u) \geq f \qquad \Omega$$

and assume u is punctually second order differentiable at $x_0 \in \Omega$.

Then at the point x_0

$$\mathcal{F}(D^2u(x_0)) \geq f(x_0)$$

Proof. Since u is punctually second order differentiable at $x_0 \in \Omega$, there exists P parabola s.t.

$$\lim_{x \to x_0} \frac{|u(x) - P(x)|}{|x - x_0|^2} = 0$$

and we've defined $D^2u(x_0) = D^2P(x_0)$. Now consider the polynomial

$$P_\varepsilon(x) = P(x) + \frac{\varepsilon}{2}|x - x_0|^2$$

We claim P_ε touches u from above at x_0. It is trivial that

$$u(x_0) = P_\varepsilon(x_0)$$

Now the two-sided control gives for any $\eta > 0$ there exists $\delta = \delta(x_0, \eta) > 0$ s.t.

$$|u(x) - P(x)| \leq \eta |x - x_0|^2 \qquad \forall\ |x - x_0| < \delta$$

Now choose $\eta = \frac{\varepsilon}{2}$ so

$$u(x) \leq P(x) + \frac{\varepsilon}{2}|x - x_0|^2 = P_\varepsilon(x) \qquad \forall\ |x - x_0| < \delta(x_0, \varepsilon)$$

Thus using u is viscosity subsolution one obtain

$$\mathcal{F}(D^2P_\varepsilon(x_0)) = \mathcal{F}(D^2P(x_0) + \varepsilon I) \geq f(x_0) \qquad \forall\ \varepsilon > 0$$

Using continuity of $\mathcal{F}$ and definition that $D^2u(x_0) = D^2P(x_0)$ to conclude. □

Corollary 8.1.1 ([CC95] Corollary 2.6)**.** *Let $u \in C^2(\Omega)$. Then u is viscosity subsolution in Ω iff u is classical*

$$\mathcal{F}(D^2u(x)) \geq f(x) \qquad \forall\ x \in \Omega$$

Proof. $\Longleftarrow$. For u classical solution, ellipticity of $\mathcal{F}$ implies monotonicity in $\Delta^2 u$. Let $\varphi \in C^2(\Omega)$ touch u from above at some $x_0 \in \Omega$, i.e. $u - \varphi$ has a local maximum 0 at the interior x_0. Then $\nabla(u - \varphi)(x_0) = 0$ and $D^2(u - \varphi)(x_0) \leq 0$, hence

$$D^2u(x_0) \leq D^2\varphi(x_0) \qquad \text{as symmetric matrices}$$

Now the ellipticity condition of $\mathcal{F}$ forces

$$f(x_0) \leq \mathcal{F}(D^2u(x_0)) \leq \mathcal{F}(D^2\varphi(x_0))$$

Thus u is viscosity subsolution.

$\Longrightarrow$. For u viscosity subsolution that is C^2, Lemma 8.1.7 gives $\mathcal{F}(D^2u(x)) \geq f(x)$ for any $x \in \Omega$. □

Construction for Viscosity Solutions One may naturally ask: given some viscosity solution, how do we construct new viscosity solutions out of it?

Closed under taking max/min For u, v viscosity subsolutions, $\sup\{u, v\}$ remains subsolution. For u, v viscosity supersolutions, $\inf\{u, v\}$ remains supersolution. ([CC95] Proposition 2.7)

Proof. For subsolutions, test functions φ touching $\sup\{u, v\}$ from above satisfies $\varphi \geq u$ and $\varphi \geq v$ near x_0. For supersolutions, φ touch from below. □

Extension of Supersolution Moreover, one may concatenate two solutions together. Let's say we have two supersolutions to the same operator, but with domain inclusion $\Omega' \subseteq \Omega$, and different forces. If the one defined on Ω lies on top of the other in $\partial\Omega'$, then one can build a new supersolution.

Lemma 8.1.8 ([CC95] Proposition 2.8)**.** *Let $f, g \in C(\Omega)$.*

Let u be supersolution with force f in Ω and v supersolution with force g in $\Omega' \subseteq \Omega$. Assume

$$u \leq v \qquad \Omega \cap \partial\Omega'$$

Then define

$$w := \begin{cases} u & \Omega \setminus \Omega' \\ \inf\{u, v\} & \overline{\Omega'} \cap \Omega \end{cases}, \qquad h := \begin{cases} f & \Omega \setminus \Omega' \\ \sup\{f, g\} & \overline{\Omega'} \cap \Omega \end{cases}$$

One obtain

$$\mathcal{F}(D^2w) \leq h$$

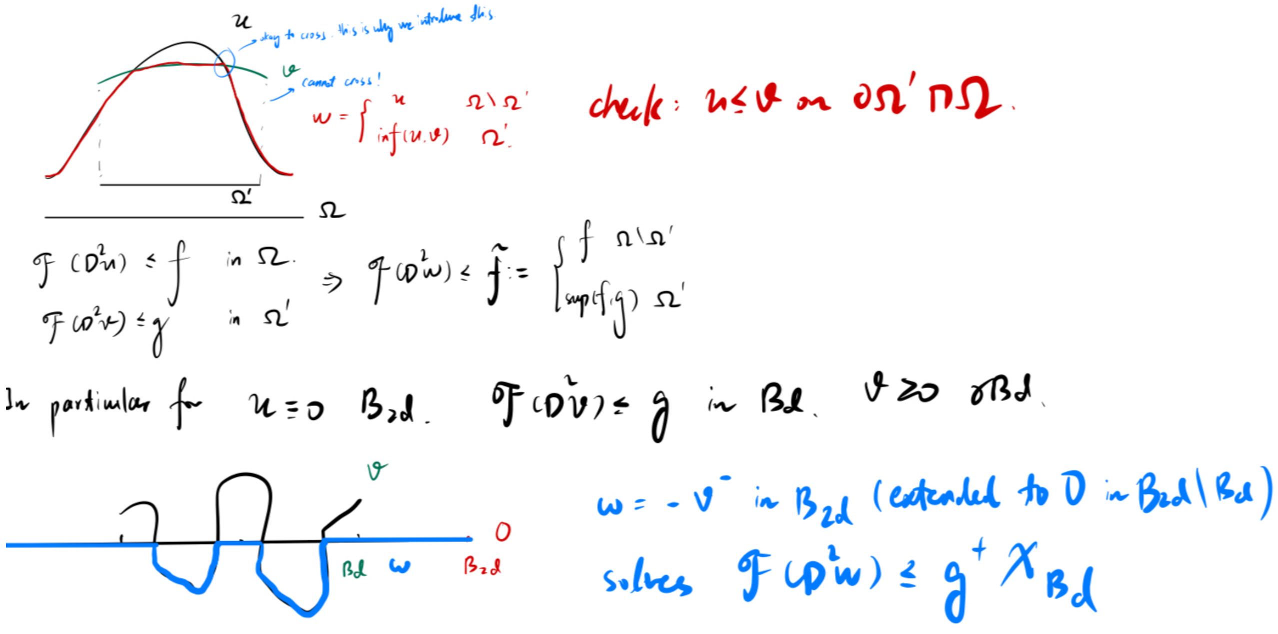


Figure 8.2: concatenation of viscosity supersolutions

Proof. The region $\Omega \setminus \Omega'$ follows from u. Let $\varphi \in C^2$ s.t. $w(x_0) = \varphi(x_0)$, $w \geq \varphi$ locally near $x_0 \in \overline{\Omega'} \cap \Omega$. If $w(x_0) = u(x_0)$ the result follows from

$$\mathcal{F}(D^2\varphi(x_0)) \leq f(x_0) \leq h(x_0)$$

If $w(x_0) = v(x_0) < u(x_0)$, then necessarily $x_0 \in \Omega$ (important), and hence the result follows from

$$\mathcal{F}(D^2\varphi(x_0)) \leq g(x_0) \leq h(x_0)$$

□

Closed under uniform limits The family of viscosity solutions is closed under uniform limits.

Lemma 8.1.9 ([CC95] Proposition 2.9)**.** *Let $\{\mathcal{F}_k\}$ be sequence of uniformly elliptic operators with same elliptic constants. Let $\{u_k\} \subseteq C(\Omega)$ s.t.*

$$\mathcal{F}_k(D^2u_k) \geq f$$

where $f \in C(\Omega)$.

Assume $\mathcal{F}_k$ and u_k converges locally uniformly to $\mathcal{F}$ and u
Then

$$\mathcal{F}(D^2u) \geq f$$

Notice that here we do not assume dependence of $\mathcal{F}_k$ on $x \in \Omega$. If, however, $\mathcal{F}_k$ does depend on x, we need $\mathcal{F}_k$ to be continuous in x, so that one can pass to the limit $x_k \to x_0$ in the step (8.14).

$$\begin{aligned} \mathcal{F}_k(D^2P_\varepsilon(x_k), x_k) &= \mathcal{F}_k(D^2P(x_k) + \varepsilon I, x_k) \geq f(x_k) \\ \mathcal{F}(D^2P_\varepsilon(x_0) + \varepsilon I, x_0) &\geq f(x_0) \qquad x_k \to x_0 \end{aligned} \tag{8.13}$$

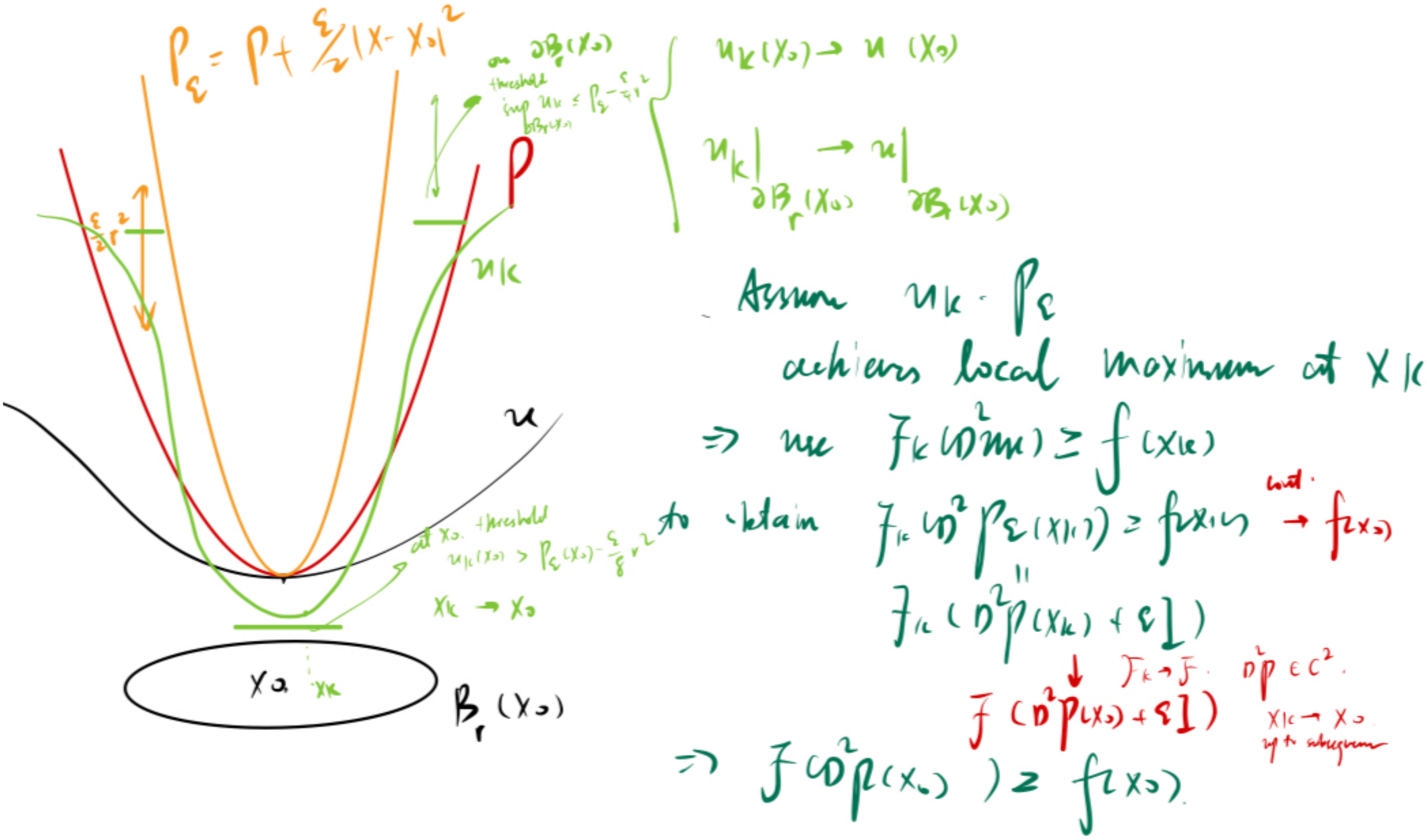


Figure 8.3: Closed Under Uniform Limits

Proof. Let P be quadratic polynomial touching u from above at x_0. There exists $r > 0$ s.t.

$$P \geq u \qquad B_r(x_0)$$

Consider

$$P_\varepsilon = P + \frac{\varepsilon}{2}|x - x_0|^2$$

Now

$$\sup_{\partial B_r(x_0)} u - P_\varepsilon \leq -\frac{\varepsilon}{2}r^2$$

$$u(x_0) = P_\varepsilon(x_0)$$

Now using $u_k \to u$ locally uniformly, for $k \geq k_0$ sufficiently large

$$\sup_{\partial B_r(x_0)} u_k - P_\varepsilon \leq -\frac{\varepsilon}{4}r^2, \qquad u_k(x_0) - P_\varepsilon(x_0) > -\frac{\varepsilon}{8}r^2$$

In particular, this is to say $u_k - P_\varepsilon$ achieves local maximum in $B_r(x_0)$ at some point, say $x_k \in B_r(x_0)$. By Bolzano-Weierstrass there exists $x_k \to x_* \in \overline{B_r(x_0)}$ up to subsequence. But recall

$$u - P_\varepsilon \qquad \text{has strict maximum at } x_0$$

Then necessarily $x_* = x_0$. On the other hand using

$$\mathcal{F}_k(D^2 u_k) \geq f$$

one obtain

$$\mathcal{F}_k(D^2 P_\varepsilon(x_k)) = \mathcal{F}_k(D^2 P(x_k) + \varepsilon I) \geq f(x_k) \tag{8.14}$$

Using P is C^2, that $\mathcal{F}_k$ locally uniformly converges to $\mathcal{F}$, and continuity of f at x_0, one pass to the limit

$$\mathcal{F}(D^2 P(x_0) + \varepsilon I) \geq f(x_0)$$

Thus taking $\varepsilon \to 0$ gives

$$\mathcal{F}(D^2 P(x_0)) \geq f(x_0)$$

□

8.1.4 Pucci Extremal Operators and Solution Class $\mathcal{S}$

Pucci Extremal Operator We remark that $\mathcal{M}^{\pm}$ are uniformly elliptic operators with no dependence on x!!!

Definition 8.1.7 (Pucci Extremal Operator)**.** *For* $0 < \lambda \leq \Lambda$.

$$\begin{aligned} \mathcal{M}^{-}(\cdot, \lambda, \Lambda) &: \mathrm{Sym}(n) \mapsto \mathbb{R} \\ & M \mapsto \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i \\ \mathcal{M}^{+}(\cdot, \lambda, \Lambda) &: \mathrm{Sym}(n) \mapsto \mathbb{R} \\ & M \mapsto \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i \end{aligned}$$

where $e_i = e_i(M)$ *are the* n *real eigenvalues of the symmetric matrix* M*.*

Usually, one plug in $M = D^2 P$ as Hessian of some paraboloid, hence the n eigenvalues are easily computable. In particular

Lemma 8.1.10. *For any* $M \in \mathrm{Sym}(n)$

$$\Lambda \left\| M^{+} \right\| - \lambda \left\| M^{-} \right\| \leq \mathcal{M}^{+}(M, \frac{\lambda}{n}, \Lambda) \tag{8.15}$$

Proof. We decompose $M = M^{+} - M^{-}$ where for e_i denoting eigenvalues of M

$$\mathrm{Tr}(M^{+}) = \sum_{e_i > 0} e_i, \qquad \mathrm{Tr}(M^{-}) = -\sum_{e_i < 0} e_i$$

Now

$$\mathcal{M}^{+}(M, \frac{\lambda}{n}, \Lambda) = \Lambda \sum_{e_i > 0} e_i + \frac{\lambda}{n} \sum_{e_i < 0} e_i = \Lambda \mathrm{Tr}(M^{+}) - \frac{\lambda}{n} \mathrm{Tr}(M^{-})$$

where

$$\Lambda \mathrm{Tr}(M^{+}) \geq \Lambda \left\| M^{+} \right\| \qquad \text{the largest positive } e_i \text{ is bounded by sum of } e_i > 0$$

and

$$\begin{aligned} -\mathrm{Tr}(M^{-}) &\geq -n \left\| M^{-} \right\| \qquad \text{the sum of } e_i < 0 \text{ is bounded by } n \text{ times the smallest } e_i \\ -\frac{\lambda}{n} \mathrm{Tr}(M^{-}) &\geq -\lambda \left\| M^{-} \right\| \end{aligned}$$

□

Characterisation of Pucci Operators One has immediate **characterisation of the Pucci operators**.

Lemma 8.1.11. *Denote*

$$\mathcal{A}_{\lambda, \Lambda} := \{ A \in \mathrm{Sym}(n) \mid \lambda |\xi|^2 \leq A_{ij} \xi_i \xi_j \leq \Lambda |\xi|^2 \}$$

Then

$$\begin{aligned} \mathcal{M}^{-}(M) &= \inf_{A \in \mathcal{A}_{\lambda, \Lambda}} \mathrm{Tr}(AM) = \inf_{A \in \mathcal{A}_{\lambda, \Lambda}} A_{ij} M_{ij} \qquad \forall\, M \in \mathrm{Sym}(n) \\ \mathcal{M}^{+}(M) &= \sup_{A \in \mathcal{A}_{\lambda, \Lambda}} \mathrm{Tr}(AM) = \sup_{A \in \mathcal{A}_{\lambda, \Lambda}} A_{ij} M_{ij} \end{aligned}$$

Proof. Let $M \in \mathrm{Sym}(n)$ and write its spectral decomposition $M = O^T D O$, where $O \in O(n)$ and $D = \mathrm{diag}(e_1, \ldots, e_n)$ with $e_i = e_i(M)$. For any $A \in \mathcal{A}_{\lambda, \Lambda}$ set $B := OAO^T$. Then $B \in \mathcal{A}_{\lambda, \Lambda}$ (since $\mathcal{A}_{\lambda, \Lambda}$ is invariant under orthogonal conjugation) and

$$\mathrm{Tr}(AM) = \mathrm{Tr}(A\, O^T D O) = \mathrm{Tr}(OAO^T D) = \mathrm{Tr}(BD).$$

Using $D = \sum_{i=1}^{n} e_i\, v_i \otimes v_i$ with $\{v_i\}$ the standard basis, we get

$$\mathrm{Tr}(BD) = \sum_{i=1}^{n} e_i \langle B v_i, v_i \rangle.$$

Because $B \in \mathcal{A}_{\lambda,\Lambda}$, for every unit vector v one has $\lambda \le \langle Bv, v\rangle \le \Lambda$. Hence

$$\sum_{e_i>0} e_i \langle Bv_i, v_i\rangle + \sum_{e_i<0} e_i \langle Bv_i, v_i\rangle \;\ge\; \lambda \sum_{e_i>0} e_i \;+\; \Lambda \sum_{e_i<0} e_i,$$

and similarly

$$\mathrm{Tr}(BD) \;\le\; \Lambda \sum_{e_i>0} e_i \;+\; \lambda \sum_{e_i<0} e_i.$$

These yield the bounds

$$\inf_{A\in\mathcal{A}_{\lambda,\Lambda}} \mathrm{Tr}(AM) \;\ge\; \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i = \mathcal{M}^-(M),$$
$$\sup_{A\in\mathcal{A}_{\lambda,\Lambda}} \mathrm{Tr}(AM) \;\le\; \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i = \mathcal{M}^+(M).$$

To prove attainability (and hence equality), choose A diagonal in the eigenbasis of M:

$$A = O^T \operatorname{diag}(a_1, \dots, a_n)\, O, \quad a_i = \begin{cases} \lambda, & e_i > 0, \\ \Lambda, & e_i < 0, \end{cases}$$

for the infimum case. Then $A \in \mathcal{A}_{\lambda,\Lambda}$ and

$$\mathrm{Tr}(AM) = \mathrm{Tr}(\operatorname{diag}(a_i)\, D) = \sum_{i=1}^n a_i e_i = \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i = \mathcal{M}^-(M).$$

Likewise, for the supremum take

$$a_i = \begin{cases} \Lambda, & e_i > 0, \\ \lambda, & e_i < 0, \end{cases}$$

□

Properties of Pucci Operators We list some basic properties of Pucci Operators.

1. For any M, $N \in \mathrm{Sym}(n)$

$$\mathcal{M}^+(M) + \mathcal{M}^-(N) \le \mathcal{M}^+(M+N) \le \mathcal{M}^+(M) + \mathcal{M}^+(N)$$
$$\mathcal{M}^-(M) + \mathcal{M}^-(N) \le \mathcal{M}^-(M+N) \le \mathcal{M}^-(M) + \mathcal{M}^+(N)$$

Solution Class $\overline{\mathcal{S}}$, $\underline{\mathcal{S}}$ Now we define the solution class to Extremal Pucci operators.

Definition 8.1.8 ([CC95] Definition 2.11)**.** *Let $f \in C(\Omega)$. Let $0 < \lambda \le \Lambda$.*
We say a function $u \in \overline{\mathcal{S}}(\lambda, \Lambda, f)$ if

$$\mathcal{M}^-(D^2u, \lambda, \Lambda) \le f \qquad \textit{in the viscosity sense}$$

We say $u \in \underline{\mathcal{S}}(\lambda, \Lambda, f)$ if

$$\mathcal{M}^+(D^2u, \lambda, \Lambda) \ge f \qquad \textit{in the viscosity sense}$$

Let

$$\mathcal{S}(\lambda, \Lambda, f) = \overline{\mathcal{S}}(\lambda, \Lambda, f) \cap \underline{\mathcal{S}}(\lambda, \Lambda, f)$$
$$\mathcal{S}^*(\lambda, \Lambda, f) = \overline{\mathcal{S}}(\lambda, \Lambda, |f|) \cap \underline{\mathcal{S}}(\lambda, \Lambda, -|f|)$$

Heuristically, for any $u \in \overline{S}$, to use this, one construct P quadratic polynomial s.t. P touches u from below at $x_0 \in \Omega$. With this, one have

$$\mathcal{M}^-(D^2P(x_0), \lambda, \Lambda) = \lambda \sum_{e_i>0} e_i(D^2P(x_0)) + \Lambda \sum_{e_i<0} e_i(D^2P(x_0)) \le f(x_0)$$

Now if we're able to encode important information (things we want to bound) in the eigenvalues of $D^2P(x_0)$, one get the bound via $f(x_0)$.

For $u \in \underline{\mathcal{S}}$, construct P parabola that touches u from above at $x_0 \in \Omega$, and use

$$\mathcal{M}^+(D^2P(x_0)) = \Lambda \sum_{e_i>0} e_i(D^2P(x_0)) + \lambda \sum_{e_i<0} e_i(D^2P(x_0)) \ge f(x_0)$$

Moreover, that $u \in \overline{S}$ implies (via Lemma 8.1.11) there exists a_{ij} uniformly elliptic ($\lambda|\xi|^2 \le a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2$) s.t.

$$a_{ij}\partial_{ij}u \le f$$

Remark 8.1.1. *Morally, what this class represents, is that the solutions are in fact viscosity subsolutions or supersolutions to linearized equations*

$$a_{ij}(x)\partial_{ij}u(x) = f(x)$$

with bounded measurable coefficients.

QUESTION: Why do I want to later write my estimates using the class $\mathcal{S}$ instead of for some fixed a_{ij}?

Morally Caffarelli want to write for linearized equations. BUT Why does he not write for some fixed a_{ij}?

This is because a_{ij} are bounded measurable, not continuous!!

Ok in fact they could make sense in the viscosity sense, but then one cannot pass to the limit. Say

$$\mathcal{F}(D^2u_k, x) := a_{ij}(x)\partial_{ij}u_k \geq 0 \qquad \forall\ k$$

And $u_k \to u$ uniformly. If a_{ij} are continuous one may pass to the limit. If however a_{ij} are not, if they're just measurable, which means the operator $\mathcal{F}(\cdot, x)$ is not continuous in the x variable, you cannot pass to the limit. This is because your point x_k is changing, and a_{ij} is not changing continuously w.r.t. x_k.

This is essentially the reason as stated in (8.13).

Now the class $\mathcal{M}^{\pm}$ includes the limit of all these continuous a_{ij} and then removes the dependence on x.

You want to make sure you can put any a_{ij} in the class. So it does not depend on x. So if I move from point to point I can pass to the limit.

Properties for $\mathcal{S}$ Class We state a bunch ([CC95] Lemma 2.12).

1. For $\lambda' \leq \lambda \leq \Lambda \leq \Lambda'$

$$\underline{\mathcal{S}}(\lambda, \Lambda, f) \subseteq \underline{\mathcal{S}}(\lambda', \Lambda', f)$$

   Same for $\overline{\mathcal{S}}$, $\mathcal{S}$, $\mathcal{S}^*$.

2. For $u \in \underline{\mathcal{S}}(\lambda, \Lambda, f)$, $-u \in \overline{\mathcal{S}}(\lambda, \Lambda, -f)$.

   Proof. For any P parabola that touches $-u$ from below at $x_0 \in \Omega$, $-P$ touches u from above at x_0, thus one has (for e_i denoting eigenvalues of $D^2P(x_0)$ and $\tilde{e}_i = -e_i$ denoting eigenvalues of $-D^2P(x_0)$)

$$\begin{aligned}\mathcal{M}^-(D^2P(x_0)) &= \lambda \sum_{e_i>0} e_i(D^2P(x_0)) + \Lambda \sum_{e_i<0} e_i(D^2P(x_0)) \\ &= -\left(\Lambda \sum_{\tilde{e}_i>0} \tilde{e}_i(-D^2P(x_0)) + \lambda \sum_{\tilde{e}_i<0} \tilde{e}_i(D^2P(x_0))\right) = -\mathcal{M}^+(-D^2P(x_0)) \\ &\leq -f(x_0)\end{aligned}$$

   Thus $-u \in \overline{\mathcal{S}}(\lambda, \Lambda, -f)$. □

3. *Rescaling.* For any $\alpha > 0$, $r > 0$, $u \in \underline{\mathcal{S}}(\lambda, \Lambda, f)$, the rescaled function

$$\tilde{u}(y) := \alpha u(\frac{y}{r}) \qquad \forall\ y \in r\Omega$$

   solves

$$\tilde{u} \in \underline{\mathcal{S}}(\lambda, \Lambda, \frac{\alpha}{r^2}f(\frac{y}{r})) \tag{8.16}$$

   Proof. For any $\tilde{P}$ parabola that touches $\tilde{u}$ from above at $y_0 \in r\Omega$, the rescaled parabola

$$P(y) = \frac{1}{\alpha}\tilde{P}(ry)$$

   touches u from above at $\frac{1}{r}y_0 \in \Omega$. Note

$$D^2P(y) = D^2(\frac{1}{\alpha}\tilde{P}(ry)) = \frac{1}{\alpha}r^2D^2\tilde{P}(ry)$$

   Thus

$$\mathcal{M}^+(D^2\tilde{P}(y_0)) = \frac{\alpha}{r^2}\mathcal{M}^+(D^2P(\frac{1}{r}y_0)) \geq \frac{\alpha}{r^2}f(\frac{1}{r}y_0)$$

   □

4. For $u \in \underline{\mathcal{S}}(\lambda, \Lambda, f)$, $\phi \in C^2(\Omega)$ s.t.

$$\mathcal{M}^+(D^2\phi(x)) \leq g(x) \qquad \forall\, x \in \Omega$$

One has

$$u - \phi \in \underline{\mathcal{S}}(\lambda, \Lambda, f - g) \tag{8.17}$$

This is to say, for $u - \phi$ to remain a subsolution, ϕ needs to be C^2, and a supersolution to $\mathcal{M}^+$.

Proof. For any P parabola that touches $u - \phi$ from above at $x_0 \in \Omega$, $P + \phi$ touches u from above at x_0, thus

$$\mathcal{M}^+(D^2(P + \phi), x_0) \geq f(x_0)$$

Now

$$f(x_0) \leq \mathcal{M}^+(D^2P(x_0)) + \mathcal{M}^+(D^2\phi(x_0)) \leq \mathcal{M}^+(D^2P(x_0)) + g(x_0)$$
$$\mathcal{M}^+(D^2P(x_0)) \geq f(x_0) - g(x_0)$$

□

Notice that, however, viscosity solutions are not closed under addition!

Construction for Solutions to $\overline{S}$, $\underline{S}$

Lemma 8.1.12 ([CC95] Remark 2). *Let $f \in L^\infty(\Omega)$.*

1. *u, $v \in \underline{\mathcal{S}}(f)$, $\sup\{u, v\} \in \underline{\mathcal{S}}(f)$; u, $v \in \overline{\mathcal{S}}(f)$, $\inf\{u, v\} \in \overline{\mathcal{S}}(f)$.*
2. *In particular $u \in \underline{\mathcal{S}}(f)$ implies $u_+ \in \underline{\mathcal{S}}(f)$; $u \in \overline{\mathcal{S}}(f)$ implies $-u_- = \inf\{u, 0\} \in \overline{\mathcal{S}}(f)$.*
3. *If $f \in C(\Omega)$, $\overline{\mathcal{S}}(f)$, $\underline{\mathcal{S}}(f)$ and $\mathcal{S}(f)$ are closed under local uniform limits.*

The following proposition justifies that $\mathcal{S}$ is a strong enough solution class to consider.

Proposition 8.1.5 ([CC95] Proposition 2.13). *Let $\mathcal{F}$ be uniformly elliptic operator with elliptic constants $0 < \lambda \leq \Lambda$. Assume*

$$\mathcal{F}(D^2u) \geq f$$

Then for any $\phi \in C^2(\Omega)$

$$u - \phi \in \underline{\mathcal{S}}(\frac{\lambda}{n}, \Lambda, f - \mathcal{F}(D^2\phi))$$

Similarly, if $\mathcal{F}(D^2u) \leq f$, then

$$u - \phi \in \overline{\mathcal{S}}(\frac{\lambda}{n}, \Lambda, f - \mathcal{F}(D^2\phi))$$

Proof. We prove for subsolutions. Let $\varphi \in C^2(\Omega)$ touch $u - \phi$ from above at x_0. This is to say $\phi + \varphi$ touch u from above at x_0. Using u as subsolution, denote e_i as eigenvalues of $D^2\varphi(x_0)$

$$\begin{aligned} f(x_0) &\leq \mathcal{F}(D^2(\varphi + \phi)(x_0)) \leq \mathcal{F}(D^2\phi(x_0)) + \Lambda \left\|(D^2\varphi(x_0))_+\right\| - \lambda \left\|(D^2\varphi(x_0))_-\right\| \\ &\leq \mathcal{F}(D^2\phi(x_0)) + \Lambda \sum_{e_i>0} e_i + \lambda \frac{1}{n} \sum_{e_i<0} e_i \\ &= \mathcal{F}(D^2\phi(x_0)) + \mathcal{M}^+(D^2\varphi(x_0), \frac{\lambda}{n}, \Lambda) \end{aligned}$$

where we've in particular used

$$\left\|(D^2\varphi(x_0))_+\right\| = \max_{e_i>0} e_i \leq \sum_{e_i>0} e_i$$
$$\left\|(D^2\varphi(x_0))_-\right\| = \max_{e_i<0} (-e_i) \geq \frac{1}{n} \sum_{e_i<0} (-e_i)$$

□

In particular, given any $\mathcal{F}(D^2u) \geq f$, u itself solves

$$u \in \underline{\mathcal{S}}(\frac{\lambda}{n}, \Lambda, f - \mathcal{F}(0,x))$$

If $\mathcal{F}(D^2u) \leq f$, then

$$u \in \overline{\mathcal{S}}(\frac{\lambda}{n}, \Lambda, f - \mathcal{F}(0,x))$$

Roughly speaking $\mathcal{S}(\lambda, \Lambda, f)$ is the class of all weak solutions in the viscosity sense to all linear uniformly elliptic equations in nondivergence form

$$a_{ij}\partial_{ij}u = f$$

with ellipticity constants λ, Λ and right hand side f.

8.1.5 Examples of Fully Nonlinear Elliptic Equations

Let $\mathcal{F}$ be a function in $\mathrm{Sym}(n) \times \Omega$ of class C^1.

One may extend $\mathcal{F}$ to the whole space of $n \times n$ real matrices via taking its symmetric part

$$\mathcal{F}(A, x) := \mathcal{F}(\frac{1}{2}(A + A^T), x)$$

so that $\mathcal{F}$ is a function of $n \times n$ variables a_{ij}, and in x.

We denote

$$\mathcal{F}_{ij}(A, x) := \frac{\partial \mathcal{F}}{\partial a_{ij}}(A, x)$$

If M, N are symmetric, then

$$D\mathcal{F}(M, x)N = \mathcal{F}_{ij}(M, x)N_{ij}$$

does not depend on the extension of $\mathcal{F}$.

Lemma 8.1.13. *If $\mathcal{F}$ is uniformly elliptic with constants λ, Λ, then*

$$\lambda|\xi|^2 \leq \mathcal{F}_{ij}(M, x)\xi_i\xi_j \leq \Lambda|\xi|^2 \qquad \forall\ M \in \mathrm{Sym}(n),\ \forall\ x \in \Omega,\ \forall\ \xi \in \mathbb{R}^n \tag{8.18}$$

On the other hand, (8.18) implies for any $M \in \mathrm{Sym}(n)$ fixed

$$\mathcal{F}_{ij}(A) := \mathcal{F}_{ij}(M, x)A_{ij}$$

is uniformly elliptic operator with elliptic constants λ, $n\Lambda$.

Proof. Let $\xi \in \mathbb{R}^n$. Let $M \in \mathrm{Sym}(n)$. We differentiate in $N = \xi \otimes \xi \geq 0$ direction, i.e.,

$$\begin{aligned}
\lambda t \|N\| \leq \mathcal{F}(M + tN) - \mathcal{F}(M) \leq \Lambda t \|N\| \\
\lambda \|N\| \leq \frac{\mathcal{F}(M + tN) - \mathcal{F}(M)}{t} \leq \Lambda \|N\| \\
\lambda \|N\| \leq \mathcal{F}_{ij}(M)N_{ij} \leq \Lambda \|N\|
\end{aligned}$$

Now since $\xi \otimes \xi$ is single rank symmetric matrix with one non-zero eigenvalue $|\xi|^2$

$$\|N\| = \|\xi \otimes \xi\| = |\xi|^2$$

This concludes (8.18). On the other hand, check (8.12). □

8.2 ABP Estimate: L^∞ Estimate

Supporting Hyperplane A function $L : \mathbb{R}^n \to \mathbb{R}$ is *affine* if

$$L(x) = \ell_0 + \ell(x) \qquad \text{for } \ell \text{ linear function on } \mathbb{R}^n \text{ and } \ell_0 \in \mathbb{R}$$

Let w be function defined in $A \subseteq \mathbb{R}^n$ open set and $x_0 \in A$. An affine function L is *supporting hyperplane for w at x_0 in A* if

$$L(x_0) = w(x_0), \qquad L(x) \leq w(x) \qquad \forall\, x \in A$$

Lemma 8.2.1. *If $A \subseteq \mathbb{R}^n$ is convex open set, and w is convex function defined on A, then for any $x_0 \in A$ there exits a supporting hyperplane for w at x_0 in A (but is not unique!).*

Proof. For any $x_0 \in A$, consider the convex open set

$$\{(x, y) \in A \times \mathbb{R} \mid y > w(x)\}$$

and the convex closed set $\{x_0\}$ (singleton). Now they have empty intersection, thus by Hahn Banach ([Bre11] Theorem 1.6) there exists a closed hyperplane L that separates the two sets. □

Convex Envelope

Definition 8.2.1 (Convex Envelope)**.** *Let $A \subseteq \mathbb{R}^n$ be convex open set. Let $v \in C(A)$ be continuous function. We define the convex envelope of v in A as*

$$\begin{aligned} \Gamma_v(x) &:= \sup\{w(x) \mid w \leq v \textit{ in } A,\ w \textit{ convex in } A\} && (8.19)\\ &= \sup\{L(x) \mid L \leq v \textit{ in } A,\ L \textit{ is affine}\} \qquad \forall\, x \in A && (8.20)\end{aligned}$$

Proof that (8.19) = (8.20). For any $x \in A$, for any $\varepsilon > 0$, there exists $w \leq v$ in A, w convex s.t.

$$\Gamma_v(x) - \varepsilon < w(x)$$

Since w convex in A, by Lemma 8.2.1, there exists a supporting hyperplane L for w at the point x, in particular

$$\Gamma_v(x) - \varepsilon < w(x) = L(x), \qquad L \leq w \leq v \qquad A$$

Thus $\Gamma_v(x) \leq$ (8.20). On the other hand due to set inclusion, $\Gamma_v(x) \geq$ (8.20). □

Immediately note that

1. Since the supremum of a family of convex functions remains convex, the function Γ_v is convex in A.

2. Since a convex function defined in an open set is continuous, the function Γ_v is continuous.

Given $v \in C(A)$, we call

$$\{x \in A \mid v(x) = \Gamma_v(x)\}$$

the lower contact set of v.

The ABP Estimate We demonstrate the classical Alexandroff-Bakelman-Pucci Estimate adapted to viscosity solutions.

Theorem 8.2.1 ([CC95] Theorem 3.2)**.** *Let $u \in \overline{S}(\lambda, \Lambda, f)$ in $B_d \subseteq \mathbb{R}^n$, with $f \in C(B_d) \cap L^\infty(B_d)$. Assume $u \in C(\overline{B_d})$ with $u \geq 0$ on ∂B_d. Then for $C = C(n, \lambda, \Lambda) > 0$*

$$\sup_{B_d} u^- \leq C \cdot d \cdot \left(\int_{B_d \cap \{u = \Gamma_u\}} (f^+)^n \right)^{\frac{1}{n}} \tag{8.21}$$

where Γ_u is the convex envelope for $-u^-$ in B_{2d} (where we extend u by 0 outside B_d, so $-u^- \in C(B_{2d})$).

The key is to show for u which could be very singular, Γ_u is in fact $C^{1,1}$, so one may apply the Classical ABP (Lemma 8.2.2).

8.2.1 Geometric Lemma

Let's recall the geometric essence of classical ABP.

Lemma 8.2.2 ([CC95] Lemma 3.4)**.** *Let $u \in C(\overline{B_d})$ s.t. $u \geq 0$ on ∂B_d. Consider $-u^- \leq 0$ in B_{2d} (extended to 0 outside B_d), and let Γ_u be the convex envelope of $-u^-$ in B_{2d}.*

Assume that *$\Gamma_u \in C^{1,1}(\overline{B_d})$.*

Then for $C = C(n) > 0$

$$\sup_{B_d} u^- \leq C \cdot d \cdot \left(\int_A \det(D^2 \Gamma_u) \right)^{\frac{1}{n}} \tag{8.22}$$

for $A \subseteq B_d$, $|B_d \setminus A| = 0$ and Γ_u second order differentiable on A.

Proof. Assume $u^- \not\equiv 0$, so there exits $x_0 \in B_d$ s.t.

$$M := \sup_{B_d} u^- = u^-(x_0) > 0$$

The goal is to prove

$$B_{\frac{M}{3d}}(0) \subseteq \nabla \Gamma_u(B_d) \tag{8.23}$$

If so, consider the measure so

$$M^n \leq C d^n |\nabla \Gamma_u(B_d)|$$

Since $\nabla \Gamma_u$ is Lipschtiz, by Rademacher there exists $A \subseteq B_d$, $|B_d \setminus A| = 0$ s.t. $D^2 \Gamma_u$ is defined on A, and the area formula holds

$$|\nabla \Gamma_u(B_d)| \leq \int_A |\det(D^2 \Gamma_u)|$$

Because Γ_u is convex one may remove the absolute value. Now

$$M^n \leq C d^n \int_A \det(D^2 \Gamma_u)$$

and (8.22) follows.

Proof of (8.23).

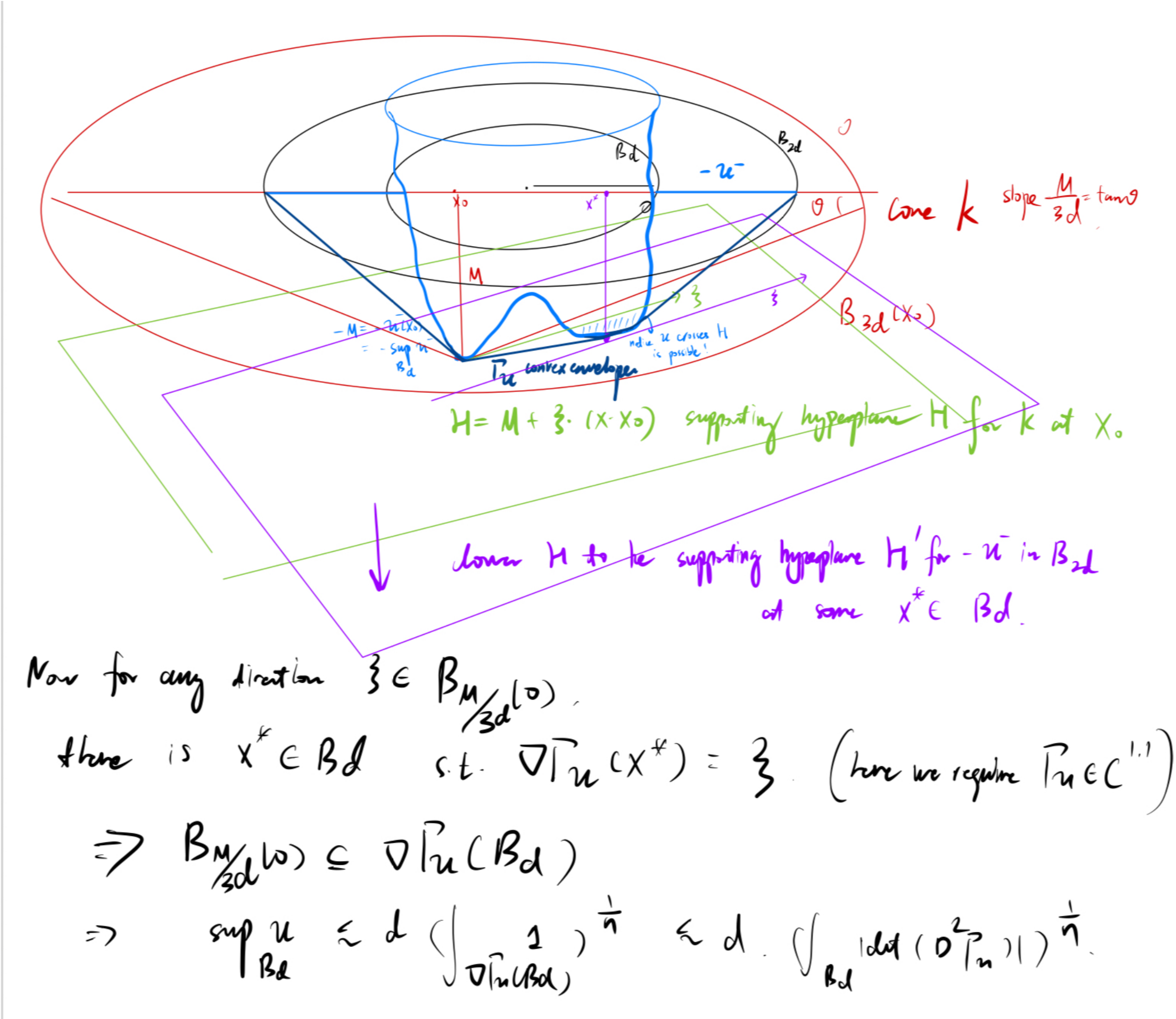


Figure 8.4: Proof of (8.23)

Take k the cone with $-M$ as vertex and $\partial B_{3d}(x_0)$ as base. Take

$$\xi \in B_{M/3d}(0)$$

and define a hyperplane $H = -M + \xi \cdot (x - x_0)$. Since the slope is strictly smaller than $M/3d$, and touches with k at x_0, H is a supporting hyperplane for k at x_0.

We want to construct another supporting hyperplane H' for Γ_u, at possibility another point $\tilde{x}$, in B_d with the same direction ξ. If so, since $\nabla\Gamma_u$ is continuous, one necessarily has

$$\xi = \nabla\Gamma_u(\tilde{x}) \implies B_{M/3d}(0) \subseteq \nabla\Gamma_u(B_d)$$

and we're happy.

If H is already a supporting hyperplane for Γ_u in B_d, simply take $H' = H$. If not, notice the function (this is where we use convexity!)

$$\Gamma_u - H \qquad \text{is convex function in the non-empty convex open set } \{H > \Gamma_u\} \cap B_{2d}$$

Thus there exists $\tilde{x} \in \{H > \Gamma_u\} \cap B_{2d}$ s.t. $\Gamma_u(\tilde{x}) - H(\tilde{x}) = -t$ achieves unique minimizer. Now we slide H downwards until it touches with Γ_u at this point $\tilde{x}$, which we call H'. In particular

$$H' = H - t = -M + \xi \cdot (x - x_0) - t$$

We claim

1. H' is supporting hyperplane for Γ_u at $\tilde{x}$. Indeed $\Gamma_u \geq H'$ in B_{2d}, and $\Gamma_u(\tilde{x}) = H'(\tilde{x})$ by construction.

2. $\tilde{x} \in B_d$. In fact we prove that the touch must be with $-u^-$, for if not, the value for Γ_u is achieved via a supporting hyperplane connecting to the nearest point of $-u^-(x)$. But this plane must have different slope compared to that of H', otherwise either one can move a bit lower along such direction contradicting H' is supporting hyperplane for Γ_u, or the slope is greater than $\frac{M}{3d}$ (due to construction of $B_{2d} \subsetneq B_{3d}(x_0)$) hence cannot be achieved by plane with ξ.

□

8.2.2 Key Lemma: $C^{1,1}$ bound at contact points

The Key Lemma concerns regularity of the convex envelope Γ_u at contact points.

This is to say, using u as supersolution, one may always touch the convex function (which we'll essentially take to be the convex envelope Γ_u) that lies and touches u from above, by a convex parabola $\|f^+\|_\infty |x|^2$ from above.

Lemma 8.2.3 ([CC95] Lemma 3.3)**.** *Let $u \in \overline{S}(\lambda, \Lambda, f)$ in $B_\delta \subseteq \mathbb{R}^n$, with $f \in L^\infty(B_\delta)$.*

Assume $\varphi \in C(B_\delta)$ is convex function s.t.

$$\begin{cases} 0 \le \varphi \le u & B_\delta \\ \varphi(0) = u(0) & \text{at the origin } 0 \end{cases}$$

Then there exists universal $\nu = \nu(n, \lambda, \Lambda) \in (0,1)$ s.t. for $C = C(n, \lambda, \Lambda) > 0$

$$\varphi(x) \le C \sup_{B_\delta} f^+ \cdot |x|^2 \qquad \forall\ x \in B_{\nu\delta} \tag{8.24}$$

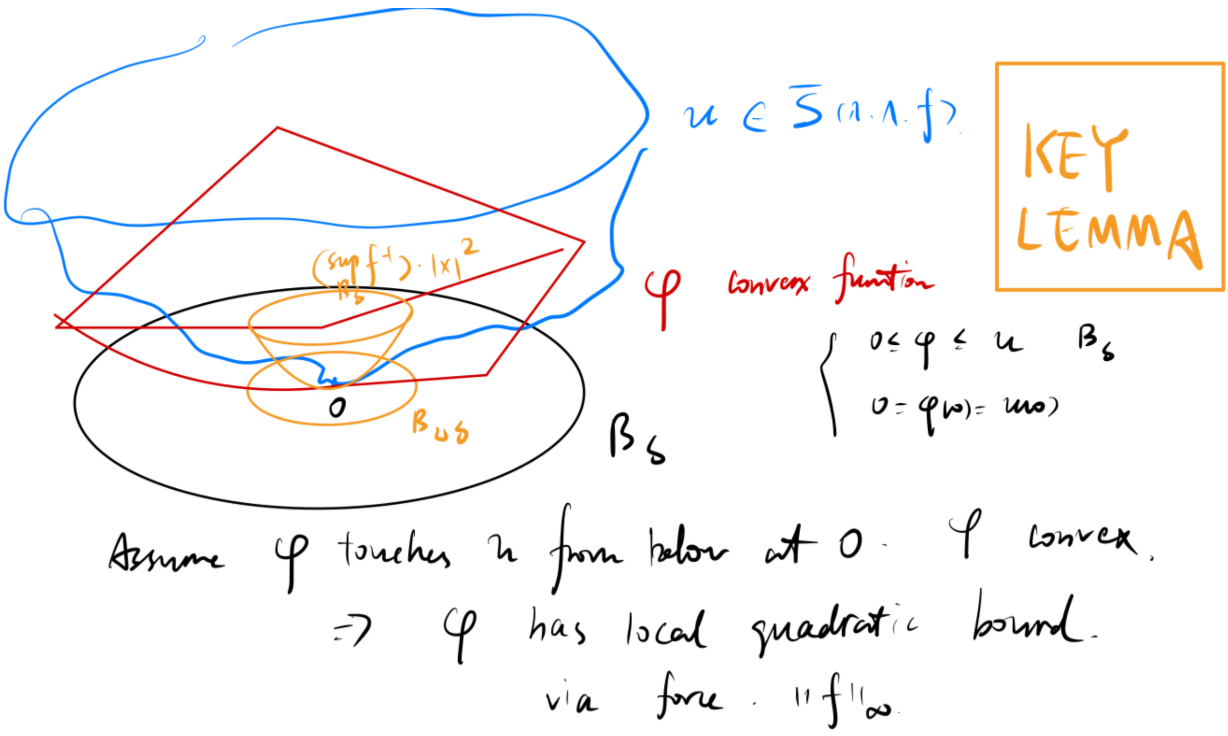


Figure 8.5: Illustration of (8.24)

Proof. First let

$$0 < r \le \frac{\delta}{4} \tag{8.25}$$

Define

$$\overline{C} := \frac{1}{r^2} \sup_{B_r} \varphi$$

Taking r sufficiently small universal, one would like to give a bound on $\overline{C}$.

Using φ is convex, its maximum over $\overline{B_r}$ is achieved on the boundary, thus there exists $x_0 \in \partial B_r$ s.t.

$$\varphi(x_0) = \sup_{B_r} \varphi = \overline{C} r^2$$

WLOG put $x_0 = (0, \cdots, 0, r)$. Consider the tangent plane H to B_r at x_0 so $H = \{x_n = r\}$. Since the closed convex set (as sublevel set of a convex function)

$$\{x \in B_\delta \mid \varphi(x) \le \overline{C}r^2\}$$

contains B_r, and touches at $x_0 \in \partial B_r$, along the set $H \cap B_\delta$, x_0 is in fact the minimum achieved, i.e.,

$$\varphi(x_0) = \overline{C}r^2 \le \varphi(x) \qquad \forall\, x \in B_\delta \cap H \tag{8.26}$$

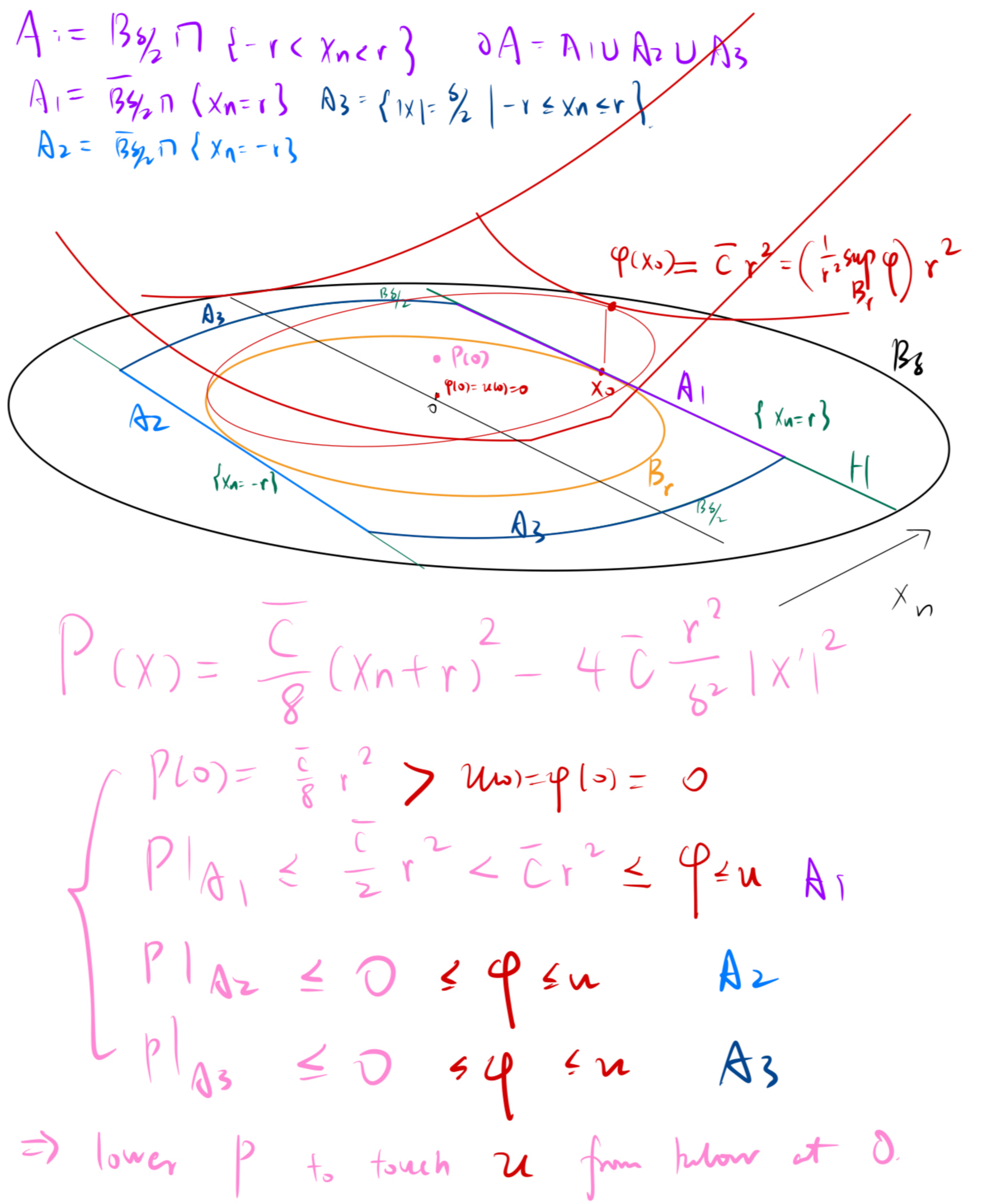


Figure 8.6: Detailed Proof of (8.24)

We construct a region A where we perform a touch using paraboloid from below. Define

$$\begin{aligned} A &:= B_{\delta/2} \cap \{-r < x_n < r\} \\ \partial A &:= A_1 \cup A_2 \cup A_3 \\ A_1 &:= B_{\delta/2} \cap H \\ A_2 &:= B_{\delta/2} \cap \{x_n = -r\} \\ A_3 &:= \partial B_{\delta/2} \cap \{-r < x_n < r\} \end{aligned}$$

Essentially we want to use a paraboloid P to touch u from below. But since $\varphi \leq u$ lies below, one want to ensure (via our given function φ)

$$\begin{aligned} P \leq \varphi \leq u \qquad & \partial A = A_1 \cup A_2 \cup A_3 \\ P(0) > \varphi(0) = u(0) & \end{aligned}$$

Then lowering P properly one may touch u from below using this P at possibly some point $y \in A$.

Now what we have for φ on these regions?

$$\begin{aligned} \varphi \overset{(8.26)}{\geq} \overline{C} r^2 \qquad & A_1 \subseteq B_\delta \cap H \\ \varphi \geq 0 \qquad & A_2 \cup A_3 \\ \varphi(0) = 0 & \end{aligned}$$

Thus to construct the paraboloid P, one need to ensure

$$\begin{aligned} P \leq \overline{C} r^2 \qquad & A_1 \\ P \leq 0 \qquad & A_2 \cup A_3 \\ P(0) > 0 & \end{aligned}$$

Construction of touching paraboloid P. Remember we want to bound $\overline{C}$. Define

$$P(x) := \frac{\overline{C}}{8}(x_n + r)^2 - 4\overline{C}\frac{r^2}{\delta^2}|x'|^2$$

We check that

$$\begin{aligned} P(0) &= \frac{\overline{C}}{8} r^2 > 0 \\ P &= \frac{\overline{C}}{2} r^2 - 4\overline{C}\frac{r^2}{\delta^2}|x'|^2 \leq \overline{C} r^2 \qquad A_1 \\ P &= -4\overline{C}\frac{r^2}{\delta^2}|x'|^2 \leq 0 \qquad A_2 \end{aligned}$$

The region A_3 is bit delicate. For any $x \in A_3$

$$\begin{aligned} \frac{\delta^2}{4} &= |x'|^2 + x_n^2 \leq |x'|^2 + r^2 \overset{(8.25)}{\leq} |x'|^2 + \frac{\delta^2}{16} \\ \frac{3}{16} &\leq \frac{|x'|^2}{\delta^2} \\ \frac{3}{4} r^2 &\leq 4r^2 \frac{|x'|^2}{\delta^2} \end{aligned}$$

Thus

$$\begin{aligned} \frac{\overline{C}}{8}(x_n + r)^2 \leq \frac{\overline{C}}{2} r^2 < \frac{3}{4}\overline{C} r^2 \leq 4\overline{C} r^2 \frac{|x'|^2}{\delta^2} \\ P < 0 \qquad A_3 \end{aligned}$$

Let's Touch! using $u \in \overline{S}(\lambda, \Lambda, f)$ and that (up to vertical translation) P touches u by below at $y \in A$, one obtain

$$\mathcal{M}^-(D^2 P(y), \lambda, \Lambda) \leq f(y) \leq \sup_{B_\delta} f^+$$

We compute eigenvalues for D^2P.

$$\begin{aligned}
\partial_i P(y) &= -8\overline{C}\frac{r^2}{\delta^2}y_i \qquad i<n \\
\partial_n P(y) &= \frac{\overline{C}}{4}(y_n+r) \\
\partial_{ii} P(y) &= -8\overline{C}\frac{r^2}{\delta^2} \qquad i<n \\
\partial_{nn} P(y) &= \frac{\overline{C}}{4} \\
\partial_{ij} P(y) &= 0 \qquad \forall\ i\neq j
\end{aligned}$$

Since this D^2P is diagonal matrix, one compute

$$\begin{aligned}
\mathcal{M}^-(D^2P(y)) &= \lambda\frac{\overline{C}}{4} - (n-1)\Lambda 8\overline{C}\frac{r^2}{\delta^2} \\
&= \left(\frac{\lambda}{4} - 8(n-1)\Lambda\frac{r^2}{\delta^2}\right)\overline{C} \le \sup_{B_\delta} f^+
\end{aligned}$$

We can play with this ratio $\frac{r}{\delta}$! Let's say we want

$$\begin{aligned}
\frac{\lambda}{8} &\le \frac{\lambda}{4} - 8(n-1)\Lambda\frac{r^2}{\delta^2} \\
\frac{r}{\delta} &\le \frac{1}{8}\sqrt{\frac{\lambda}{(n-1)\Lambda}}
\end{aligned}$$

Thus taking

$$\nu := \min\{\frac{1}{8}\sqrt{\frac{\lambda}{(n-1)\Lambda}}, \frac{1}{4}\} \in (0,1)$$

yields

$$\overline{C} = \frac{1}{r^2}\sup_{B_r}\varphi \le \frac{8}{\lambda}\sup_{B_\delta} f^+ \qquad \forall\ r\le \nu\delta$$

□

8.2.3 $C^{1,1}$ bound at non-contact points

Assuming there is universal bound on openings of convex parabolas at contact points $\{u=\Gamma_u\}$ (which is done in Key Lemma 8.2.3), the geometry of Γ_u allows us to further bound the openings over the whole region $\overline{B_d}$.

Here, again, one does not need the equation. Notice that in particular, our only assumption is (8.27), i.e., at each contact point $x\in\{u=\Gamma_u\}$, there is convex parabola of universal opening K that touches u from above at x.

We used our Key Lemma 8.2.3 to achieve this, yes. But this doesn't mean it's the only way!

Lemma 8.2.4 ([CC95] Lemma 3.5)**.** *Let $u\in C(\overline{B_d})$ s.t. $u\ge 0$ on ∂B_d, and Γ_u be convex envelope of $-u^-$ in B_{2d}.*

Let $K>0$ and $\varepsilon\in(0,d)$ be universal constants. Assume that for any $x_0\in\overline{B_d}\cap\{u=\Gamma_u\}$, there exists a convex parabola of opening K that touches Γ_u from above at x_0 in $B_\varepsilon(x_0)$, i.e.

$$\overline{\Theta}(\Gamma_u, B_\varepsilon(x_0))(x_0) \le K \qquad \forall\ x_0\in\overline{B_d}\cap\{u=\Gamma_u\} \tag{8.27}$$

Then $\Gamma_u\in C^{1,1}(\overline{B}_d)$, and there exists $A\subseteq B_d$, $|B_d\setminus A|=0$ and Γ_u is second order differentiable at A, and for $C=C(n)>0$

$$\sup_{B_d} u^- \le C(n)\cdot d\cdot\left(\int_{A\cap\{u=\Gamma_u\}}\det(D^2\Gamma_u)\right)^{\frac{1}{n}} \tag{8.28}$$

Proof. One need to show that $\Gamma_u\in C^{1,1}(\overline{B_d})$ so Lemma 8.2.2 applies. Furthermore, one need to ensure

$$\det(D^2\Gamma_u(x)) = 0 \qquad \text{a.e. } x\in B_d\setminus\{u=\Gamma_u\} \tag{8.29}$$

so that (8.28) is obtained from (8.22).

In **the main agenda to prove** $\Gamma_u \in C^{1,1}(\overline{B_d})$, in view of (8.5), one would like to obtain some universal parameter $\tilde{\varepsilon}$ and $\tilde{K}$ so that

$$\Theta(\Gamma_u, \tilde{\varepsilon})(x_0) \le \tilde{K} \qquad \forall\ x_0 \in B_d$$

Note our assumption (8.27) (really this is result of our Key Lemma) already gives the bound on the contact set $\{u = \Gamma_u\}$. Therefore the task reduces to show

$$\Theta(\Gamma_u, B_{\tilde{\varepsilon}}(x_0))(x_0) \le \tilde{K} \qquad \forall\ x_0 \in B_d \setminus \{u = \Gamma_u\} \tag{8.30}$$

for $\tilde{\varepsilon}$, $\tilde{K}$ to be determined.

Step 1: Study points $x_0 \in \overline{B_d} \setminus \{u = \Gamma_u\}$. Let L be supporting hyperplane for Γ_u at x_0 in $\overline{B_{2d}}$.

The goal is to represent the point x_0 outside the contact set, as convex combinations of points within the contact set. Moreover, one has lower bound on some of the convex combination coefficient, which we can use later!

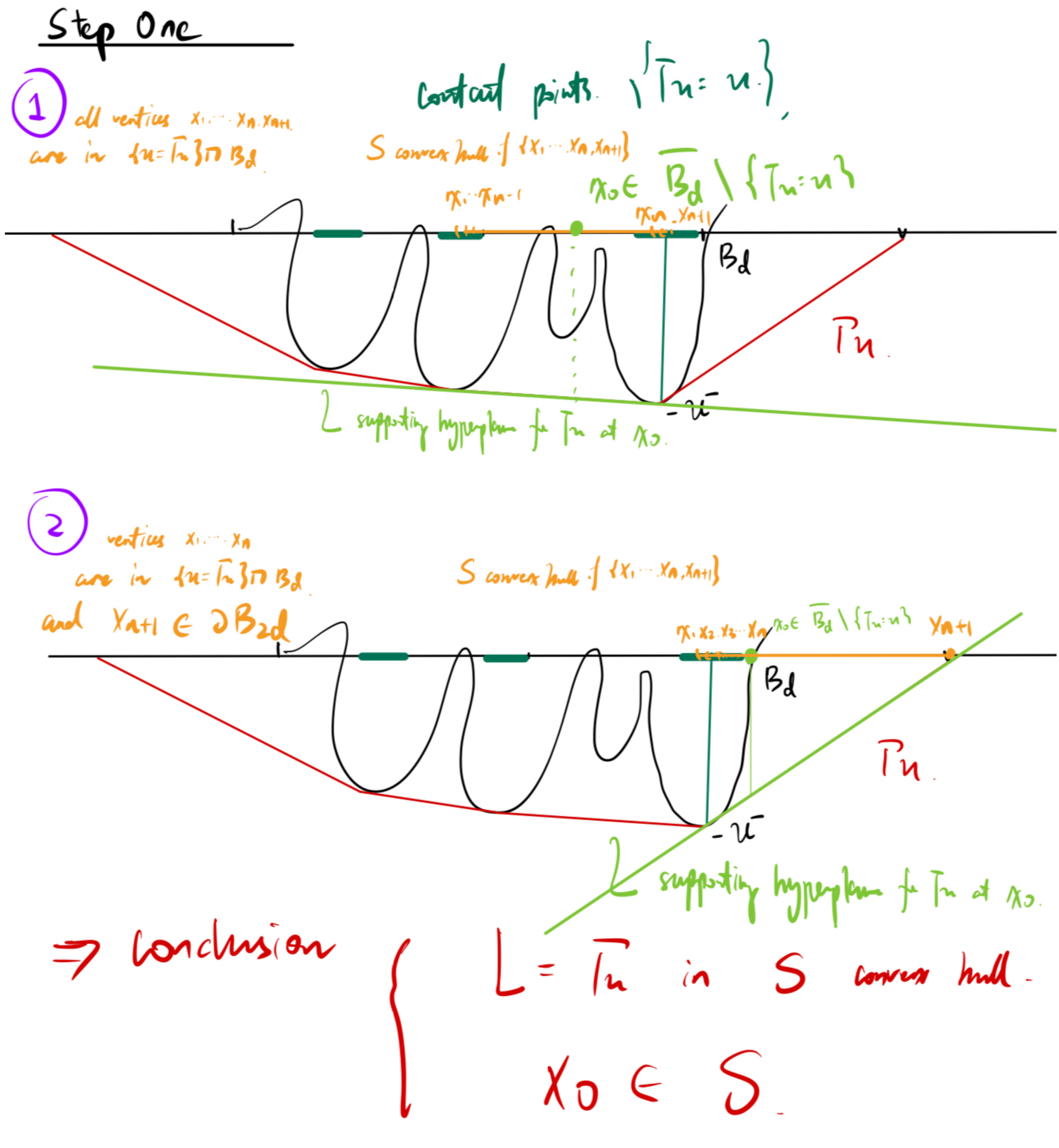


Figure 8.7: Step 1 for Lemma 8.2.4

(a) One want to show there exists $n+1$ points

$$\{x_1, \cdots, x_n, x_{n+1}\}$$

where either all points $\{x_i\} \subseteq B_d \cap \{u = \Gamma_u\}$ or there is only one distinct point $x_{n+1} \in \partial B_{2d}$, s.t.

$$x_0 \in S := \{\text{convex hull of the set } \{x_1, \cdots, x_{n+1}\}\}$$

and

$$L = \Gamma_u \qquad S \tag{8.31}$$

(b) There exists $\lambda_i \geq 0$, $\sum_{i=1}^{n+1} \lambda_i = 1$ s.t.

$$x_0 = \sum_{i=1}^{n+1} \lambda_i x_i$$

and there exists at least one i s.t. $x_i \in \{u = \Gamma_u\} \cap B_d$ and

$$\lambda_i \geq \frac{1}{3n} \tag{8.32}$$

Notice (8.31) implies that for each $x_0 \in B_d \setminus \{u = \Gamma_u\}$, there is a line segment (as subset of S) through which Γ_u is affine. Thus $D^2\Gamma_u(x_0) = 0$ for a.e. $x_0 \in B_d \setminus \{u = \Gamma_u\}$ where Γ_u is second-order differentiable. This is (8.29).

Proof of Step 1 (a). Using (8.20)

$$\Gamma_u(x) = \sup\{\tilde{L}(x) \mid \tilde{L} \leq -u^- \text{ in } \overline{B_{2d}},\ \tilde{L} \text{ is affine}\}$$

L is the hyperplane that realizes the supremum at x_0. Thus there exists a least one contact point of L and $-u^-$ in $\overline{B_{2d}}$. Thus the closed convex hull

$$C = \{\text{closed convex hull of } \{x \in \overline{B_{2d}} \mid L(x) = -u^-(x)\}\} \neq \varnothing$$

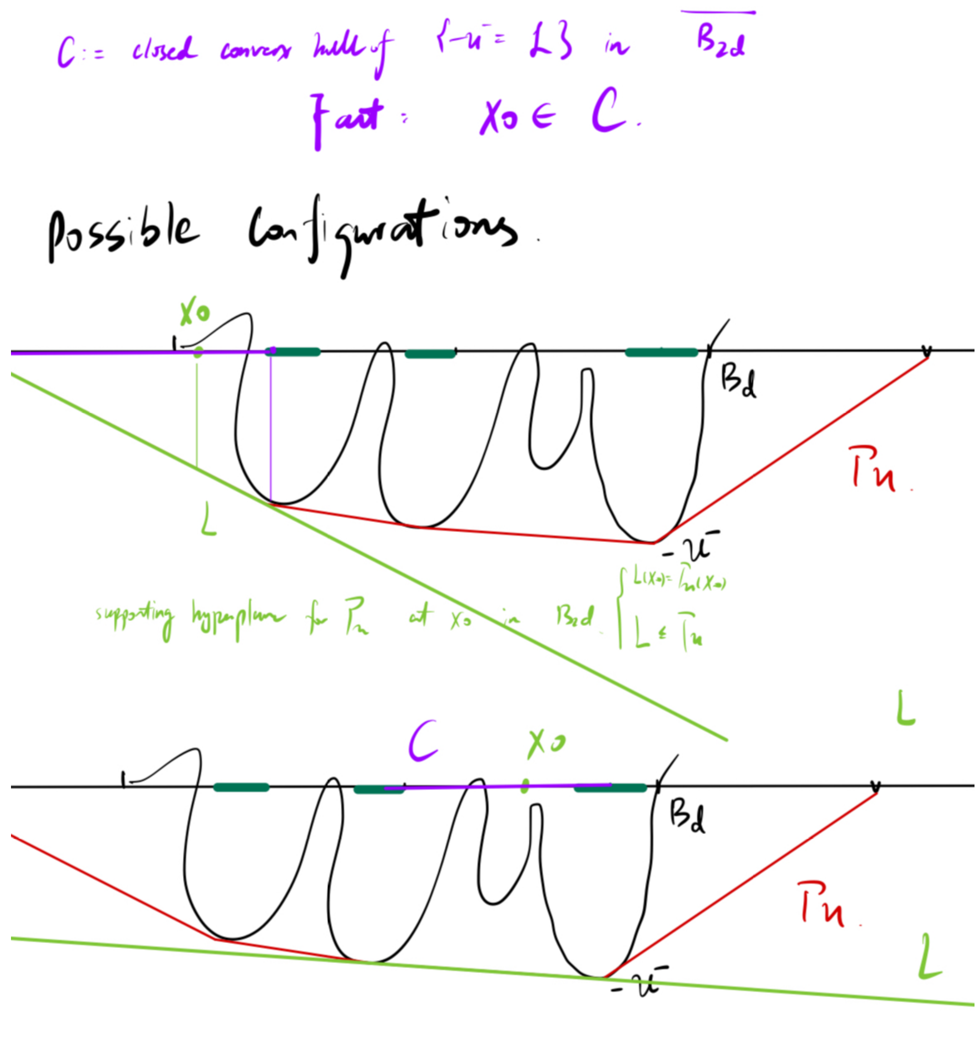


Figure 8.8: Step 1 $x_0 \in C$ possible configurations

We claim necessarily $x_0 \in C$. Assume not, then there exists affine function ℓ s.t.

$$\ell(x_0) > 0 \qquad \ell(C) < 0$$

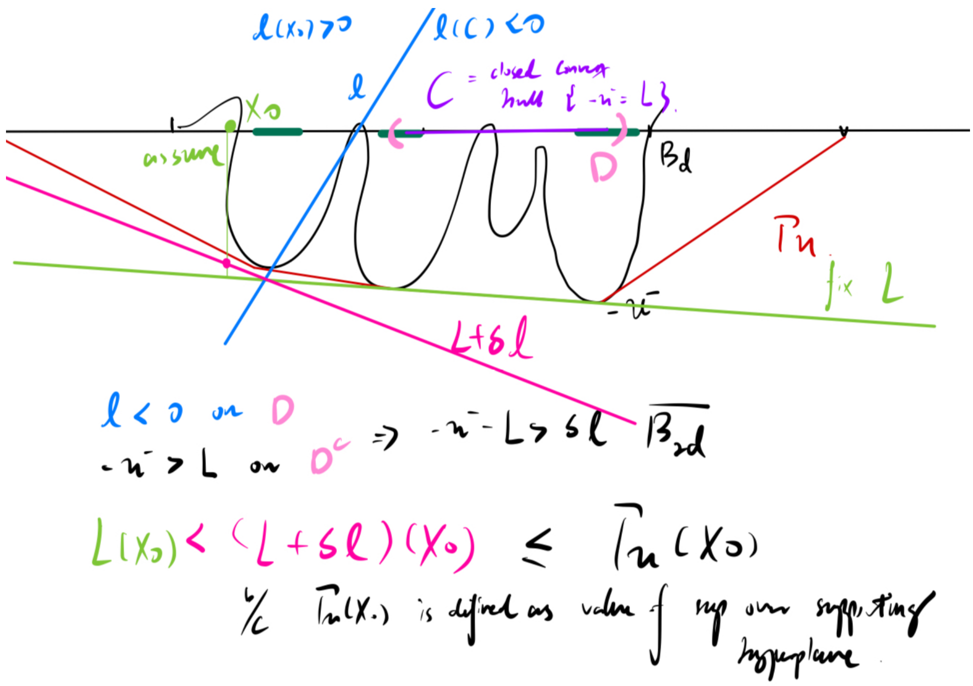


Figure 8.9: Step 1 $x_0 \in C$ detailed argument

Now there is open neighborhood D around C s.t.

$$\begin{aligned} \ell < 0 &\qquad D \\ -u^- \geq L &\qquad \overline{B_{2d}} \\ -u^- > L &\qquad D^c \end{aligned}$$

Thus one may choose $\delta > 0$ small universal s.t.

$$-u^- \geq L + \delta\ell \qquad \overline{B_{2d}}$$

This is to say, $L + \delta\ell$ belongs to the family of affine functions that lies below $-u^-$ in $\overline{B_{2d}}$. Then in particular at x_0 (in view of the definition for convex envelope (8.20) as supremum among such family at a point)

$$\begin{aligned} \Gamma_u(x_0) &\geq L(x_0) + \delta\ell(x_0) \\ &> L(x_0) = \Gamma_u(x_0) \qquad \text{by assumption, } L \text{ is supporting hyperplane for } \Gamma_u \text{ at } x_0 \end{aligned}$$

This is contradiction.

Now using $x_0 \in C$ as convex hull in $\mathbb{R}^n$, there exists $\{x_1, \cdots, x_{n+1}\} \subseteq \{x \in \overline{B_{2d}} \mid L(x) = -u^-(x)\}$ s.t. (Carathéodary Theorem)

$$x_0 = \sum_{i=1}^{n+1} \lambda_i x_i, \qquad \sum_{i=1}^{n+1} \lambda_i = 1, \qquad \lambda_i \geq 0$$

Since L is an admissible supporting hyperplane for (8.20), that $L(x) = -u^-(x)$ implies $L(x) = \Gamma_u(x)$. Now for convex combination of $x_1, \cdots, x_{n+1} \in \{L = \Gamma_u\}$, it remains in $\{L = \Gamma_u\}$. This is to say for S defined as the convex hull for $\{x_1, \cdots, x_{n+1}\}$

$$S \subseteq \{L = \Gamma_u\}$$

and this is (8.31).

It remains to check where x_0 really lies in.

1. Assume there's two distinct points among $\{x_1, \cdots, x_{n+1}\}$ that lies on ∂B_{2d}, say $x_n \neq x_{n+1}$, then since our supporting hyperplane L needs to satisfy

$$\begin{aligned} L(x_n) = L(x_{n+1}) &= 0 \\ L \leq -u^- = 0 &\qquad \partial B_{2d} \end{aligned}$$

The only possibility for $x_n \neq x_{n+1}$ is that $L \equiv 0$, i.e., $\Gamma_u \equiv 0$.

2. Assume there's some point $x_i \in B_{2d} \setminus B_d$, so

$$L(x_i) = -u^-(x_i) = 0$$

Then to ensure

$$L \le -u^-$$

this again forces $L \equiv 0$.

Proof of Step 1 (b). First assume all $\{x_i\}_{1\le i\le n+1} \subseteq B_d \cap \{u = \Gamma_u\}$, then $\lambda_i \ge \frac{1}{n+1} > \frac{1}{3n}$ for at least one i simply due to

$$\sum_{j=1}^{n+1} \lambda_i = 1, \qquad \lambda_j \ge 0$$

Then assume some $x_{n+1} \in \partial B_{2d}$, and we want to prove that λ_{n+1} cannot be the one exceeding $\frac{1}{3n}$ while all others didn't. In particular, assume $\lambda_i < \frac{1}{3n}$ for all $i \le n$, so $\lambda_{n+1} > \frac{2}{3}$. Thus

$$\begin{aligned} x_0 &= \lambda_{n+1} x_{n+1} + \sum_{i=1}^{n} \lambda_i x_i \\ |x_0| &\ge \lambda_{n+1}|x_{n+1}| - \sum_{i=1}^{n} \lambda_i |x_i| \\ &> \frac{2}{3} 2d - \sum_{i=1}^{n} \frac{1}{3n} d = d \end{aligned}$$

and contradiction.

Step 2: Proof of (8.30). Now for our $x_0 \in B_d \setminus \{u = \Gamma_u\}$ arbitrarily chosen, and L the supporting hyperplane for Γ_u at x_0 in $\overline{B_{2d}}$ (whose existence is due to Lemma 8.2.1), one may perform Step 1. In particular, one may pick out

$$x_1 \in B_d \cap \{u = \Gamma_u\}, \qquad \lambda_1 \ge \frac{1}{3n}$$

For any $|h| < d$, we write

$$x_0 + h = \sum_{i=1}^{n+1} \lambda_i x_i + h = \lambda_1 (x_1 + \frac{h}{\lambda_1}) + \sum_{i=2}^{n+1} \lambda_i x_i$$

To bound $\Theta(\Gamma_u, \tilde{\varepsilon})(x_0) \le \tilde{K}$, one expect to choose $|h|$ sufficiently small and pick universal $\tilde{K}$ s.t.

$$L(x_0 + h) \le \Gamma_u(x_0 + h) \le L(x_0 + h) + \tilde{K} \frac{|h|^2}{2} \qquad \forall\ |h| < \tilde{\varepsilon} \tag{8.33}$$

Let's do this. Using Γ_u is convex and L is supporting hyperplane for Γ_u

$$\begin{aligned} L(x_0 + h) \le \Gamma_u(x_0 + h) &= \Gamma_u(\lambda_1(x_1 + \frac{h}{\lambda_1}) + \sum_{i=2}^{n+1} \lambda_i x_i) \\ &\le \lambda_1 \Gamma_u(x_1 + \frac{h}{\lambda_1}) + \sum_{i=2}^{n+1} \lambda_i \Gamma_u(x_i) \end{aligned} \tag{8.34}$$

But $x_1 \in B_d \cap \{u = \Gamma_u\}$**, and on there we can use our assumption** (8.27)**!** This can be achieved as long as we make $\frac{h}{\lambda_1}$ small universal, i.e.

$$|\frac{h}{\lambda_1}| < \varepsilon$$

Can we do this? Recall we have made (8.32)! If we define

$$\tilde{\varepsilon} := \frac{1}{3n} \varepsilon$$

Then

$$\frac{|h|}{\lambda_1} \le 3n|h| < \varepsilon \qquad \forall\ |h| < \tilde{\varepsilon}$$

Therefore using (8.27) applied to x_1

$$\Gamma_u(x_1 + \frac{h}{\lambda_1}) \le L(x_1 + \frac{h}{\lambda_1}) + \frac{K}{2}|\frac{h}{\lambda_1}|^2$$

Now plugging back into (8.34) gives

$$\begin{aligned}
L(x_0+h) &\le \Gamma_u(x_0+h) \\
&\le \lambda_1 \left(L(x_1 + \frac{h}{\lambda_1}) + \frac{K}{2}|\frac{h}{\lambda_1}|^2 \right) + \sum_{i=2}^{n+1} \lambda_i \Gamma_u(x_i) \\
&= \lambda_1 L(x_1) + L(h) + \frac{K}{2\lambda_1}|h|^2 + \sum_{i=2}^{n+1} \lambda_i L(x_i) \qquad \text{using } x_i \in \{L = \Gamma_u\} \ (8.31) \\
&= L(x_0+h) + \frac{K}{2\lambda_1}|h|^2 \\
&\le L(x_0+h) + \frac{3nK}{2}|h|^2 \qquad \forall\ |h| < \tilde{\varepsilon}
\end{aligned}$$

We conclude (8.33) by defining $\tilde{K} = 3nK$. □

8.2.4 Proof of ABP Estimate Theorem 8.2.1

In this subsection we put together the Proof for ABP. The goal is to apply Lemma 8.2.4. To do so first we need to recover (8.27) from Lemma 8.2.3.

The first task is: given our solution $u \in \overline{S}(\lambda, \Lambda, f)$ in B_d with $u \ge 0$ on ∂B_d, **what does $-u^-$, extended to 0 in $B_{2d} \setminus B_d$ solve?**

Recall $u \in \overline{S}(\lambda, \Lambda, f)$ is to say

$$\mathcal{M}^-(D^2u, \lambda, \Lambda) \le f \qquad \text{in the viscosity sense}$$

Using Lemma 8.1.8, this can be viewed as concatenating the solution $0 \in \overline{S}(\lambda, \Lambda, 0)$ in B_{2d} with u in B_{2d}. Since

$$0 = u^- \qquad \partial B_d$$

The function

$$-u^- = \begin{cases} 0 & B_{2d} \setminus B_d \\ \inf\{u, 0\} & \overline{B_d} \end{cases}$$

together with the concatenated force

$$f^+ \chi_{B_d} = \begin{cases} 0 & B_{2d} \setminus B_d \\ \max\{f, 0\} & \overline{B_d} \end{cases}$$

solves

$$\mathcal{M}^-(D^2(-u^-), \lambda, \Lambda) \le f^+ \chi_{B_d}$$

Thus $-u^- \in \overline{S}(\lambda, \Lambda, f^+\chi_{B_d})$ in B_{2d}.

Now take $x_0 \in \overline{B_d} \cap \{u = \Gamma_u\}$ and L supporting hyperplane to Γ_u at x_0. Since L is affine, $\Gamma_u - L$ is convex in B_{2d}, and $-u^- - L \in \overline{S}(\lambda, \Lambda, f^+\chi_{B_d})$, then for $0 < \delta < d$ s.t.

$$x_0 \in B_\delta(x_0) \subseteq B_{2d}$$

One apply Lemma 8.2.3 to

$$\begin{aligned}
&0 \le \Gamma_u - L \le -u^- - L \qquad B_\delta(x_0) \\
&0 = (\Gamma_u - L)(x_0) = (-u^- - L)(x_0) \qquad \text{here we used the assumption } x_0 \in \overline{B_d} \cap \{u = \Gamma_u\}
\end{aligned}$$

Thus (8.24) reads

$$(\Gamma_u - L)(x) \le C(n) \cdot \sup_{B_\delta(x_0)} f^+\chi_{B_d} \cdot |x - x_0|^2 \qquad \forall\ x \in B_{\nu\delta}(x_0)$$

$$L(x) \le \Gamma_u(x) \le L(x) + C(n) \cdot \sup_{B_\delta(x_0)} f^+\chi_{B_d} \cdot |x - x_0|^2$$

Therefore assumption (8.27) is achieved. Apply Lemma 8.2.4 so (8.28) holds.

Also, using continuity of f at each x_0, taking $\delta \to 0$ yields

$$\det(D^2\Gamma_u(x_0)) \le Cf^+(x_0)^n \qquad \text{a.e. } \ x_0 \in B_d \cap \{u = \Gamma_u\}$$

This concludes Proof of Theorem 8.2.1.

8.2.5 Maximum Principle

Corollary 8.2.1 ([CC95] Theorem 3.6). *Let $\Omega \subseteq \mathbb{R}^n$ be bounded domain. Let $f \in C(\Omega) \cap L^\infty(\Omega)$.*
Assume $u \in \overline{S}(\lambda, \Lambda, f)$ in Ω, $u \in C(\overline{\Omega})$ and $u \geq 0$ on $\partial\Omega$.
Then for C universal

$$\sup_{\Omega} u^- \leq C \mathrm{diam}(\Omega) \left\| f^+ \right\|_{L^n(\Omega \cap \{u = \Gamma_u\})}$$

where Γ_u is convex envelope of $-u^-$ in B_{2d} s.t. $\Omega \subseteq B_d$, and u extended to 0 outside Ω.

Corollary 8.2.2 ([CC95] Corollary 3.7). *Assume $u \in C(\overline{\Omega})$.*

1. *If $u \in \overline{S}(\lambda, \Lambda, 0)$ with $u \geq 0$ on $\partial\Omega$, then*
$$u \geq 0 \qquad \Omega$$

2. *If $u \in \underline{S}(\lambda, \Lambda, 0)$ with $u \leq 0$ on $\partial\Omega$, then*
$$u \leq 0 \qquad \Omega$$

8.3 Krylov Safonov: Harnack Inequality, and $C^{0,\alpha}$ Regularity

We denote $Q_\ell(x_0)$ as open unit cube in $\mathbb{R}^n$ centered at x_0

$$Q_\ell(x_0) = \prod_{i=1}^{n}(x_0^i - \frac{\ell}{2}, x_0^i + \frac{\ell}{2})$$

and $Q_\ell = Q_\ell(0)$.

Barrier We define a useful Barrier.

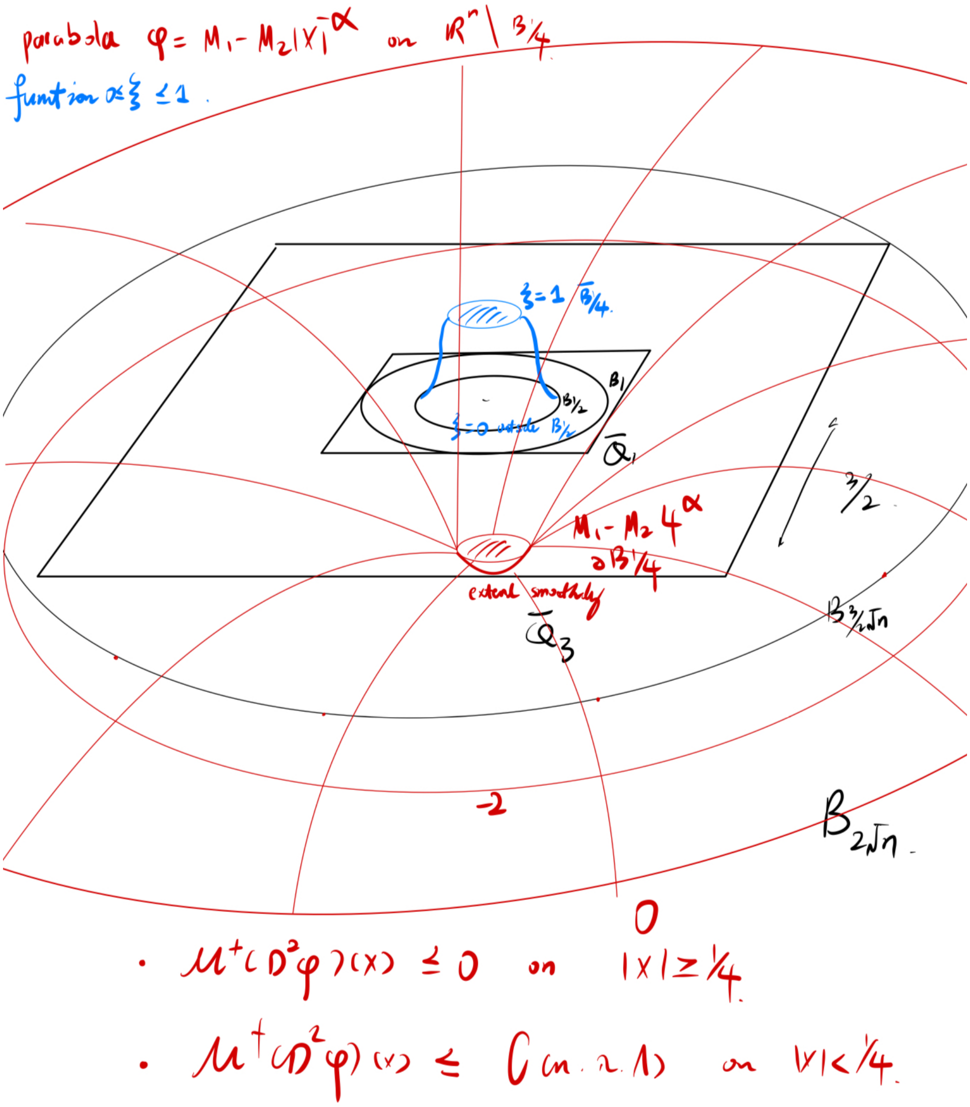


Figure 8.10: Barrier Function

For $\alpha \geq 1$ to be chosen, define

$$\varphi(x) = M_1 - M_2|x|^{-\alpha} \qquad \mathbb{R}^n \setminus B_{1/4}$$

Choose M_1 and M_2 so that

$$\varphi|_{\partial B_{2\sqrt{n}}} \equiv 0 \qquad \varphi|_{\partial B_{3\sqrt{n}/2}} \equiv -2$$

Now extend φ smoothly to all of $\mathbb{R}^n$ s.t. (depending on α)

$$\varphi \leq -2 \qquad Q_3$$

Now we determine what α is. Compute

$$\partial_i \varphi = M_2 \alpha x_i |x|^{-(\alpha+2)}$$
$$\partial_{ij}\varphi = M_2\alpha\delta_{ij}|x|^{-(\alpha+2)} - M_2\alpha(\alpha+2)x_i x_j |x|^{-(\alpha+4)}$$

Now let's see what the eigenvalues for $D^2\varphi$ is

$$D^2\varphi = M_2\alpha\left(|x|^{-(\alpha+2)}I - \frac{(\alpha+2)}{|x|^{\alpha+4}}xx^T\right)$$

Take $\frac{x}{|x|}$ radial unit vector so

$$\begin{aligned} D^2\varphi(x)\frac{x}{|x|} &= M_2\alpha\left(\frac{x}{|x|^{\alpha+3}} - \frac{(\alpha+2)}{|x|^{\alpha+4}}xx^T\frac{x}{|x|}\right) \\ (D^2\varphi(x)\frac{x}{|x|})_i &= M_2\alpha\left(\frac{x_i}{|x|^{\alpha+3}} - \frac{(\alpha+2)}{|x|^{\alpha+4}}\frac{x_i\sum_j x_j^2}{|x|}\right) \\ &= -M_2\alpha\frac{(\alpha+1)}{|x|^{\alpha+2}}\frac{x_i}{|x|} \end{aligned}$$

Thus $\frac{x}{|x|}$ is an eigenvector with eigenvalue $-M_2\alpha\frac{(\alpha+1)}{|x|^{\alpha+2}}$. Now for any other direction $u \perp \frac{x}{|x|}$, note

$$xx^Tu = 0$$

Thus

$$D^2\varphi(x)u = M_2\alpha|x|^{-(\alpha+2)}u$$

for $M_2\alpha|x|^{-(\alpha+2)}$ is eigenvalue of multiplicity $n-1$.

Consequently, the $\mathcal{M}^+(D^2\varphi)$ takes the form for any $|x| > \frac{1}{4}$

$$\begin{aligned} \mathcal{M}^+(D^2\varphi)(x) &= \Lambda(n-1)M_2\alpha|x|^{-(\alpha+2)} - \lambda M_2\alpha\frac{(\alpha+1)}{|x|^{\alpha+2}} \\ &= M_2\alpha|x|^{-(\alpha+2)}\left(\Lambda(n-1) - \lambda(\alpha+1)\right) \overset{\text{we want}}{\leq} 0 \end{aligned}$$

where for the last inequality to be obtained, one require

$$\frac{\Lambda(n-1)}{\lambda} - 1 \leq \alpha$$

On the other hand, one has (via smooth extension)

$$\mathcal{M}^+(D^2\varphi)(x) \leq C(n,\lambda,\Lambda) \qquad \forall\, x \in \overline{B}_{1/4}$$

Now take $0 \leq \xi \leq 1$ smooth function so that

$$\xi = 1 \quad \overline{B}_{1/4}, \qquad \xi = 0 \quad \mathbb{R}^n \setminus B_{1/2}$$

one obtain

$$\mathcal{M}^+(D^2\varphi,\lambda,\Lambda) \leq C\xi \qquad \forall\, x \in \mathbb{R}^n$$

Collecting what one needs, we obtain

$$\begin{aligned} \varphi &\geq 0 && \mathbb{R}^n \setminus B_{2\sqrt{n}} \\ \varphi &\leq -2 && Q_3 \\ \varphi &\geq -M && \mathbb{R}^n \\ \mathcal{M}^+(D^2\varphi,\lambda,\Lambda) &\leq C\xi && \forall\, x \in \mathbb{R}^n \end{aligned} \tag{8.35}$$

***A*-*B* Lemma** One apply Calderon-Zygmund to obtain an important A-B Lemma.

Let $A \subseteq Q_1$ the unit cube. Let $0 < \delta < 1$ denote the portion that A can at most take in Q_1, i.e.

$$|A| \leq \delta$$

Now apply Calderon Zygmund Decomposition (5.22) to the function χ_A. One may collect the sequence of non-overlapping dyadic cubes

$$\{Q^j\}$$

s.t.

$$A \subseteq \bigcup_j Q^j \quad \text{up to measure zero}, \qquad \delta < \frac{|Q^j \cap A|}{|Q^j|} \qquad \forall\, j \tag{8.36}$$

For each Q_j, assign its predecessor $\tilde{Q}_j$.

Now suppose one has another set B that contains A in Q_1. One ask: when can A take at most the same portion in B?

$$|A| \leq \delta|B|$$

Lemma 8.3.1 ([CC95] Lemma 4.2)**.** *Let $A \subseteq B \subseteq Q_1$. Let $0 < \delta < 1$.*
Assume

1. $|A| \leq \delta$
2. *B satisfies*

$$\forall\ Q^j \textit{ dyadic cube s.t. } \delta < \frac{|Q^j \cap A|}{|Q^j|}, \qquad \tilde{Q}^j \subseteq B \tag{8.37}$$

Then

$$|A| \leq \delta |B| \tag{8.38}$$

Proof. Apply Calderon Zygmund Decomposition so one get (8.36). Consider the family of predecessors $\{\tilde{Q}^j\}$ of cubes Q^j and relabel them so they're pairwise disjoint. One has

$$A \subseteq \bigcup_j Q^j \subseteq \bigcup_j \tilde{Q}^j$$

Moreover, since Q^j are the ones chosen (8.36), using Assumption on B (8.37), we know

$$\tilde{Q}^j \subseteq B \qquad \forall\ j$$

Thus

$$A \subseteq \bigcup_j Q^j \subseteq \bigcup_j \tilde{Q}^j \subseteq B$$

Since predecessors are the cubes one did not pick, one has

$$\frac{|\tilde{Q}^j \cap A|}{|\tilde{Q}^j|} \leq \delta \qquad \forall\ j$$

What's good about predecessors? Since predecessors are the cubes one did not pick, one has

$$\frac{|\tilde{Q}^j \cap A|}{|\tilde{Q}^j|} \leq \delta \qquad \forall\ j$$

Thus using pairwise disjoint

$$|A| \leq \sum_j |\tilde{Q}^i \cap A| \leq \sum_j \delta |\tilde{Q}^j| \leq \delta |B|$$

□

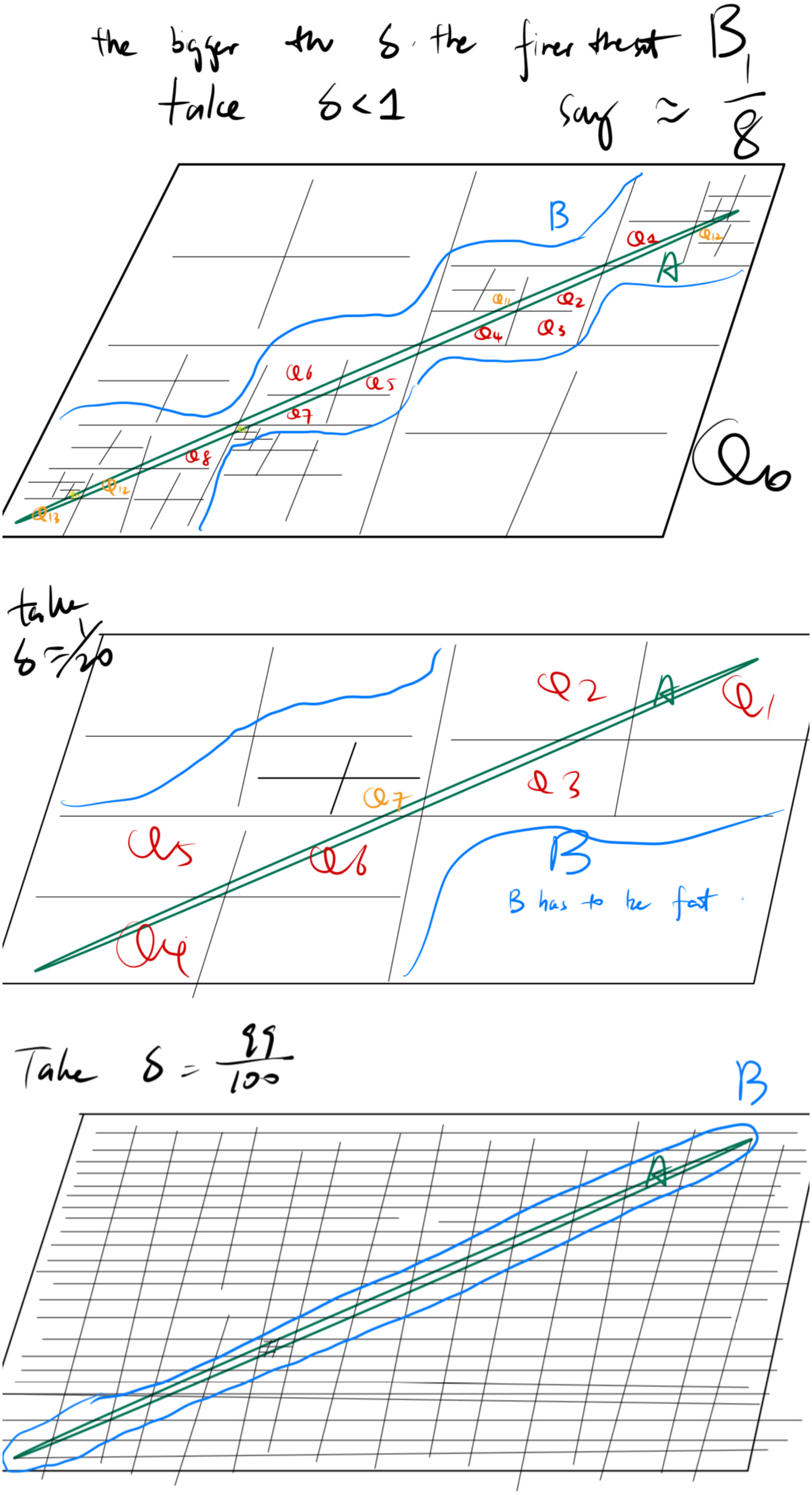


Figure 8.11: A-B Lemma Possible Configurations

Harnack Inequality The goal of this section is to show the Harnack Inequality. Recall

$$\mathcal{S}^*(\lambda,\Lambda,f)=\overline{\mathcal{S}}(\lambda,\Lambda,|f|)\cap\underline{\mathcal{S}}(\lambda,\Lambda,-|f|)\supseteq\mathcal{S}(\lambda,\Lambda,f)$$

Theorem 8.3.1 ([CC95] Theorem 4.3)**.** *Let $u\in S^*(\lambda,\Lambda,f)$ in Q_1. Assume $f\in C(Q_1)\cap L^\infty(Q_1)$. Also assume*

$$u\geq 0 \qquad Q_1$$

Then for $C>0$ universal

$$\sup_{Q_{1/2}} u\leq C\left(\inf_{Q_{1/2}} u+\|f\|_{L^n(Q_1)}\right) \tag{8.39}$$

We claim it suffices to prove the rescaled Theorem.

Lemma 8.3.2 ([CC95] Lemma 4.4)**.** *Let $u\in S^*(\lambda,\Lambda,f)$ in $Q_{4\sqrt{n}}$, $u\in C(\overline{Q}_{4\sqrt{n}})$. Assume $f\in C(Q_{4\sqrt{n}})\cap L^\infty(Q_{4\sqrt{n}})$. Assume*

$$u\geq 0 \qquad Q_{4\sqrt{n}}$$

Assume for ε_0 universal

$$\inf_{Q_{1/4}} u\leq 1, \qquad \|f\|_{L^n(Q_{4\sqrt{n}})}\leq\varepsilon_0$$

Then for $C>0$ universal constant

$$\sup_{Q_{1/4}} u\leq C \tag{8.40}$$

Indeed, upon rescaling, for any $\delta>0$

$$\frac{1}{\left(\inf\limits_{Q_{1/4}} u+\frac{1}{\varepsilon_0}\|f\|_{L^n(Q_{4\sqrt{n}})}+\delta\right)}u\in\mathcal{S}^*\Big(\frac{1}{\left(\inf\limits_{Q_{1/4}} u+\frac{1}{\varepsilon_0}\|f\|_{L^n(Q_{4\sqrt{n}})}+\delta\right)}f\Big)$$

then taking $\delta\to 0$ yields the estimate in $Q_{1/4}$. To go to $Q_{1/2}$ one apply covering procedure.

Throughout the rest, we assume $f\in C(Q_{4\sqrt{n}})\cap L^\infty(Q_{4\sqrt{n}})$.

Ideas for the Proof Now, the idea to prove (8.40) is to use the distribution function of u in Q_1

$$\lambda_u(t):=|\{u>t\}\cap Q_1|$$

Harnack's Inequality is equivalent to say there exists $C>0$ universal s.t.

$$\lambda_u(t)\equiv 0 \qquad \forall\, t\geq C$$

In the first step, one prove a power decay for supersolutions as in (8.46)

$$\lambda_u(t)=|\{u>t\}\cap Q_1|\leq d\cdot t^{-\varepsilon} \qquad t\geq 1$$

using ABP estimate, and then A-B Lemma.

In the second step, one apply the previous power decay to C_1-C_2u where u is a subsolution. Then conclude using a contradictory Carleson's Estimate.

8.3.1 Key Lemma: Weak Harnack Inequality for Supersolutions

Lemma 8.3.3 ([CC95] Lemma 4.5)**.** *There exists universal constants $\varepsilon_0>0$, $M>1$, $\mu\in(0,1)$ s.t. if $u\in\overline{\mathcal{S}}(\lambda,\Lambda,|f|)$, $u\in C(\overline{Q}_{4\sqrt{n}})$ and f satisfies*

$$u\geq 0 \qquad Q_{4\sqrt{n}} \tag{8.41}$$

$$\inf_{Q_3} u\leq 1 \tag{8.42}$$

$$\|f\|_{L^n(Q_{4\sqrt{n}})}\leq\varepsilon_0 \tag{8.43}$$

Then

$$|\{u\leq M\}\cap Q_1|>\mu \tag{8.44}$$

or equivalently

$$|\{u>M\}\cap Q_1|<1-\mu \tag{8.45}$$

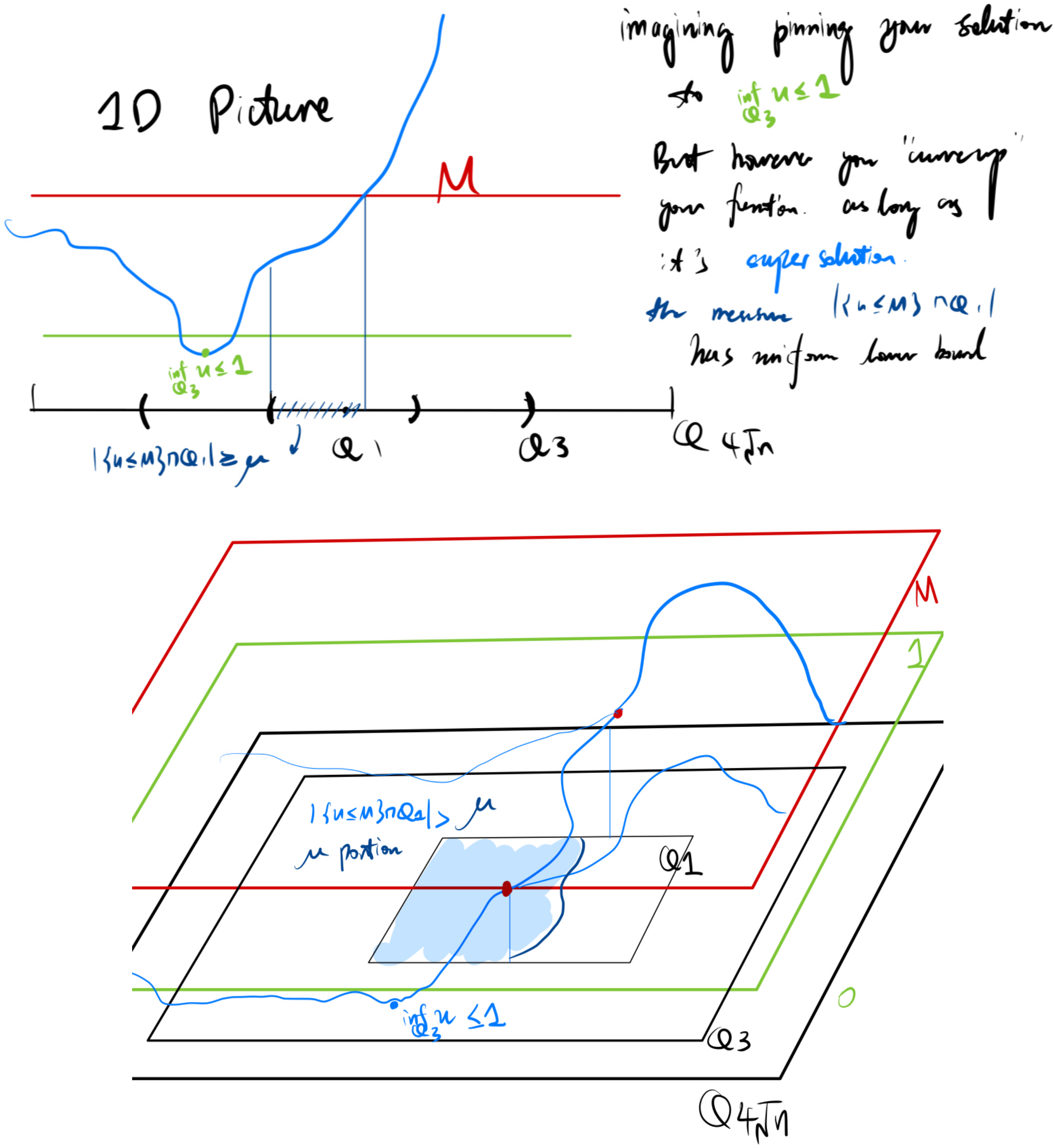


Figure 8.12: Weak Harnack Inequality for Supersolutions [CC95] Lemma 4.5

Proof. One want to add to u perturbation so one may apply ABP. Let φ be as in (8.35) with $0 \le \xi \le 1$ cutoff chosen.

We define perturbation w. Define

$$w := u + \varphi$$

Using

$$\mathcal{M}^+(D^2\varphi)(x) \le C\xi(x) \qquad \forall\ x \in \mathbb{R}^n$$

and (8.17) (to u a supersolution) one get

$$w = u + \varphi \in \overline{S}(\lambda, \Lambda, |f| + C\xi)$$

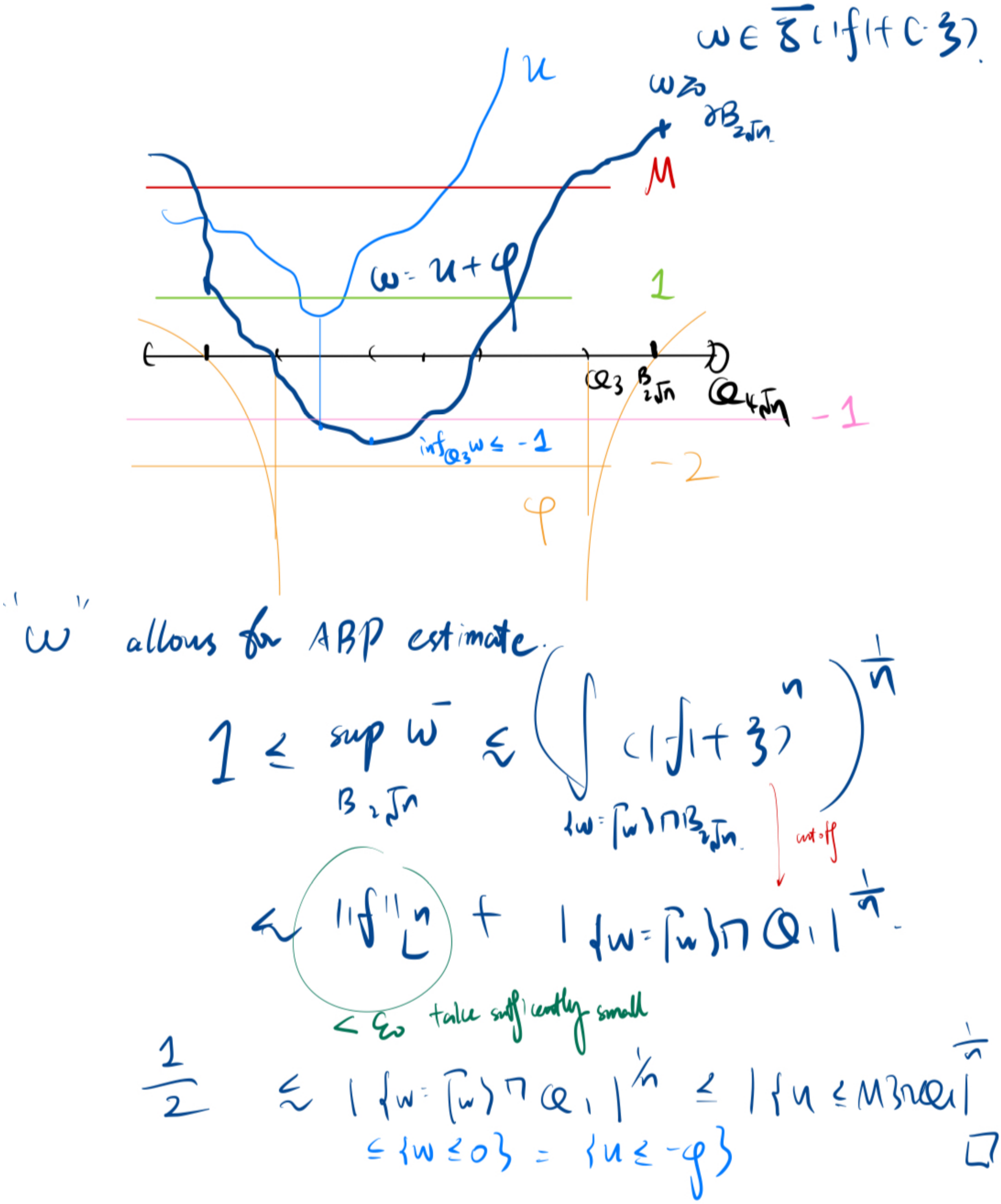


Figure 8.13: Proof for Weak Harnack Inequality for Supersolutions [CC95] Lemma 4.5

Apply ABP to w. Check that

$$w \geq u \geq 0 \qquad \partial B_{2\sqrt{n}}$$

$$\inf_{Q_3} w \leq \inf_{Q_3} u - 2 \overset{(8.42)}{\leq} -1$$

Thus apply ABP (8.21) to w in the region $B_{2\sqrt{n}}$. Now Γ_w denotes the convex envelope of w^- in $B_{2\sqrt{n}}$, in particular, $\Gamma_w \leq 0$. For universal $C > 0$

$$
\begin{aligned}
1 &\leq \sup_{B_{2\sqrt{n}}} w^- \leq C \left(\int_{B_{2\sqrt{n}} \cap \{w = \Gamma_w\}} (|f| + \xi)^n \right)^{\frac{1}{n}} \\
1 &\leq C \left(\|f\|_{L^n(Q_{4\sqrt{n}})} + |\{w = \Gamma_w\} \cap Q_1|^{\frac{1}{n}} \right) \qquad \text{using } \operatorname{supp}(\xi) \subseteq Q_1 \text{ and } \xi \in [0,1] \\
&\overset{(8.43)}{\leq} C\varepsilon_0 + C|\{w = \Gamma_w\} \cap Q_1|^{\frac{1}{n}}
\end{aligned}
$$

Now choosing ε_0 small depending on $C > 0$ to obtain

$$
\frac{1}{2} \leq |\{w = \Gamma_w\} \cap Q_1|^{\frac{1}{n}}
$$

Use information on Convex Envelope. Now using $\Gamma_w \leq 0$, that

$$
x \in \{w = \Gamma_w\} \subseteq \{w \leq 0\} = \{u \leq -\varphi\}
$$

Taking M as in the definition for φ (8.35) one conclude

$$
\frac{1}{2} \leq C|\{u \leq M\} \cap Q_1|^{\frac{1}{n}}
$$

Choose $\mu \in (0,1)$ accordingly. Notice our choice of universal μ now depends on M.

□

8.3.2 $L^{\varepsilon}_{\text{weak}}$: Power Decay of Distribution Function for Supersolution

Lemma 8.3.4 ([CC95] Lemma 4.6)**.** *There exists universal constants d, $\varepsilon > 0$ s.t. if $u \in \overline{\mathcal{S}}(\lambda, \Lambda, |f|)$ satisfies the assumptions as in Lemma 8.3.3, then*

$$
|\{u \geq t\} \cap Q_1| \leq d \cdot t^{-\varepsilon} \qquad \forall\, t > 1 \tag{8.46}
$$

Proof. The hope is to do induction and rescaling. We want to show for the same $M > 1$ and $\mu \in (0,1)$ as in Lemma 8.3.3, one has

$$
|\{u > M^k\} \cap Q_1| \leq (1-\mu)^k \qquad \forall\, k \geq 1 \tag{8.47}
$$

Note $k = 0$ trivially holds.

Why (8.47) **suffices**? For any $t > 1$, there exists $k \geq 1$ s.t. $M^{k-1} < t \leq M^k$. Now

$$
(k-1) < \log_M t \leq k
$$

Thus

$$
\begin{aligned}
|\{u \geq t\} \cap Q_1| &\leq |\{u > M^{k-1}\} \cap Q_1| \leq (1-\mu)^{k-1} = \frac{1}{1-\mu}(1-\mu)^k \\
&\leq \frac{1}{1-\mu}(1-\mu)^{\log_M(t)} = \frac{1}{1-\mu} t^{\log_M(1-\mu)} = d \cdot t^{-\varepsilon} \qquad \text{for } \varepsilon \in (0,1) \text{ and } d > 0 \text{ universal}
\end{aligned}
$$

Induction Argument Setup. The base case $k = 1$ is achieved in (8.45). Assume for $k - 1$, and denote

$$
A = \{u > M^k\} \cap Q_1, \qquad B = \{u > M^{k-1}\} \cap Q_1
$$

(8.47) will be shown if one prove

$$
|A| \leq (1-\mu)|B| \tag{8.48}
$$

One will use the A-B Lemma (8.38). To check the assumptions, notice again given by (8.44)

$$
|A| \leq |\{u > M\} \cap Q_1| \leq 1 - \mu
$$

and indeed $A \subseteq B \subseteq Q_1$. It suffices to check the set B satisfies: for any $Q = Q_{\frac{1}{2^i}}(x_0)$ dyadic cube, if

$$
|A \cap Q| > (1-\mu)|Q| \tag{8.49}
$$

Then its predecessor $\tilde{Q} \in B$.

Check A-B Lemma assumption (8.37). Assume for contradiction that for some $x_0 \in Q_1$, and some i, the dyadic cube $Q = Q_{\frac{1}{2^i}}(x_0)$ satisfies (8.49) but $\tilde{Q} \not\subseteq B$. Then there is

$$
\tilde{x} \in \tilde{Q} \qquad \text{s.t.} \qquad u(\tilde{x}) < M^{k-1}
$$

How do we use this pointwise information on u? Ah! We want this to be the point that hits infimum below 1 so one may again apply the Weak Harnack Inequality 8.3.3.

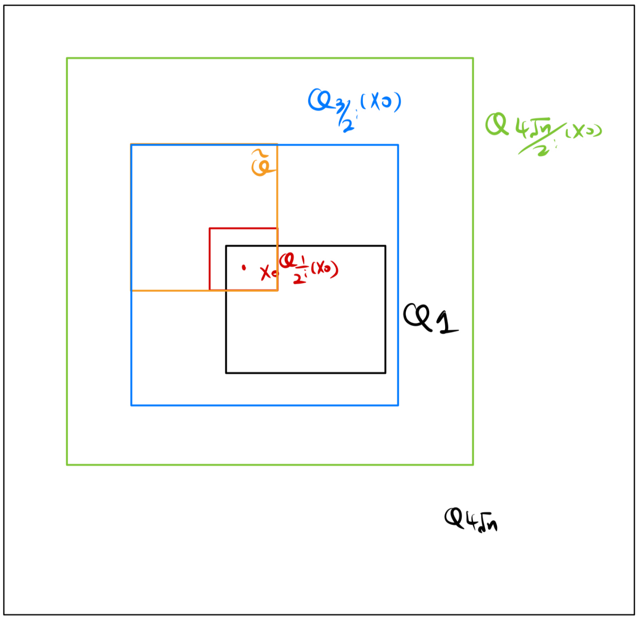


Figure 8.14: Domain for Rescaling in [CC95] Lemma 4.5

To do so, we **rescale**. Recall $Q = Q_{\frac{1}{2^i}}(x_0)$, we define

$$x := x_0 + \frac{1}{2^i}y, \qquad \tilde{u}(y) := \frac{1}{M^{k-1}}u(x_0 + \frac{1}{2^i}y)$$

What does $\tilde{u}$ solve? Note u is defined in

$$Q_{4\sqrt{n}/2^i}(x_0) \subseteq Q_{4\sqrt{n}} \qquad \text{for } x_0 \in Q_1$$

Now using (8.16)

$$\tilde{u} \in \overline{\mathcal{S}}(\lambda, \Lambda, \tilde{f}) \qquad Q_{4\sqrt{n}}, \qquad \tilde{f}(y) := \frac{1}{M^{k-1}2^{2i}}f(x_0 + \frac{1}{2^i}y)$$

and

$$\left\| \frac{1}{M^{k-1}2^{2i}}f(x_0 + \frac{1}{2^i}y) \right\|_{L^n(Q_{4\sqrt{n}})} = \frac{1}{M^{k-1}2^{2i}}\left(\int_{Q_{4\sqrt{n}}} f(x_0 + \frac{1}{2^i}y)^n dy \right)^{\frac{1}{n}} = \frac{1}{M^{k-1}2^{2i}}\left(2^{in}\int_{Q_{\frac{4\sqrt{n}}{2^i}}(x_0)} f(x)^n dx \right)^{\frac{1}{n}}$$

$$= \frac{1}{M^{k-1}2^i}\|f\|_{L^n(Q_{4\sqrt{n}/2^i}(x_0))} < \varepsilon_0 \qquad \text{using assumption (8.43)}$$

In particular, $\tilde{Q} \subseteq Q_{3/2^i}(x_0) \subseteq Q_{4\sqrt{n}/2^i}(x_0)$, so the 'bad point' $\tilde{x}$ lies in there. Now

$$\begin{aligned} \tilde{u} \geq 0 &\qquad Q_{4\sqrt{n}} \\ \inf_{Q_3}\tilde{u} \leq 1 &\qquad \text{using } \tilde{y} = 2^i(\tilde{x} - x_0) \in Q_3 \\ \left\|\tilde{f}\right\|_{L^n(Q_{4\sqrt{n}})} \leq \varepsilon_0 & \end{aligned}$$

So the result (8.44) writes

$$|\{\tilde{u} \leq M\} \cap Q_1| > \mu$$

But the LHS is

$$\int_{Q_1} \chi_{\{\tilde{u}\leq M\}}(y)dy = 2^{in}\int_{Q_{\frac{1}{2^i}}(x_0)} \chi_{\{u\leq M^k\}}(x)dx = \frac{1}{|Q|}|\{u \leq M^k\} \cap Q| > \mu$$

Now combining with assumption for Q (8.49) this is contradiction.

Consequently, applying A-B Lemma, (8.48) is shown.

□

8.3.3 Construction of Blow-Up sequence: Power Decay of Distribution Function for $C_1 - C_2 u$ for Subsolution

The key idea for Weak Harnack lies as follows: Assume supersolution u is small at some point. Then by rescaling, using this point of small value, one ensure

$$\inf \tilde{u} \leq 1$$

from which one can apply the Weak Harnack to get

$$|\{\tilde{u} \leq M\} \cap Q_1| > \mu$$

The question is: why is this useful? In the previous proof, assumption (8.49) gives the reverse measure bound

$$|\{\tilde{u} > M\} \cap Q_1| > 1 - \mu$$

whose sum adds up to the contradiction

$$|Q| > 1$$

Therefore u cannot be small anywhere.

Idea of the Proof The upshot is, one want to also assume for u a subsolution, so the Weak Harnack applies for $C_1 - C_2 u$, which is a supersolution. This happens if one can ensure $C_1 - C_2 u$ is small at some point, equivalent to u large at some point.

Now **assume u 'relatively small' everywhere, except u large at some point**. From the result (8.46) one has for u supersolution (with $\tilde{Q} \subseteq Q_1$ small)

$$|\{u \geq t_1\} \cap \tilde{Q}| < |\{u \geq t_1\} \cap Q_1| \leq d \cdot t_1^{-\varepsilon}$$

for any t_1 in our favor. Now apply the same to $C_1 - C_2 \tilde{u}$ a rescaled(in domain) supersolution as well (one need to rescale since this is pointwise big assumption!)

$$\begin{aligned} d \cdot t_2^{-\varepsilon} &\geq |\{C_1 - C_2 \tilde{u} > t_2\} \cap Q_1| = |\{\tilde{u} < \frac{C_1 - t_2}{C_2}\} \cap Q_1| \\ &= \tilde{C}^n |\{u < \frac{C_1 - t_2}{C_2}\} \cap \tilde{Q}| \end{aligned}$$

Now if let

$$t_1 = \frac{C_1 - t_2}{C_2}$$

using the same logic as above

$$|\tilde{Q}| = \tilde{C}^{-n} |Q_1| \leq d \cdot t_1^{-\varepsilon} + \tilde{C}^{-n} d \cdot (C_1 - C_2 t_1)^{-\varepsilon}$$

Now with multiple degrees of freedom on the RHS, one may make it strictly smaller than LHS so we reach contradiction. The contradiction tells us that: since we assume u large at a point, u **cannot be relatively too small everywhere**.

Quantifying Parameters We quantify the above behavior via the following. Assume $x_0 \in \overline{Q_{1/4}}$ some point, and there u is big

$$u(x_0) \gg 1, \qquad \text{precisely, take } \nu^{j-1} M_0$$

where $M_0 \gg 1$ is large universal constant that one wants to choose. Due to technical reasons, one denote $\nu = \frac{M_0}{M_0 - \frac{1}{2}} > 1$ as an iteration on 'largeness'. The behavior of u as a solution suggests that, for a neighborhood close enough to x_0

$$Q_{\ell_j}(x_0) \subseteq Q_1$$

The value of u cannot be too small. By playing with the choice

$$\ell_j = \sigma \cdot C(M_0, \nu, n, \varepsilon, j) \qquad \text{for } \sigma > 0 \text{ universal}$$

One wish to ensure that

$$\sup_{Q_{\ell_j}(x_0)} u \geq \nu^j M_0$$

Now this is fantastic for future purposes, since the iteration itself runs and explodes within the region $B_{1/2}$, contradicting continuity. Thus one cannot assume u arbitrarily big at a point to start with, and one has uniform bound on the sup (8.40)!

Proof In this section we prove precisely the following Lemma.

Lemma 8.3.5 ([CC95] Lemma 4.7)**.** *Let* $u \in \underline{S}(-|f|)$ *in* $Q_{4\sqrt{n}}$*. Let* f *satisfy* (8.43) *and* u *satisfy* (8.46) *(here this is an assumption).*

Then there are universal constants $M_0 > 1$ (8.58) (8.60), $\sigma > 0$ (8.59) *s.t. for* $\varepsilon > 0$ *as in* (8.46) *and*

$$\nu := \frac{M_0}{M_0 - \frac{1}{2}} > 1$$

whenever for some $j \geq 1$ *and* $|x_0|_\infty \leq \frac{1}{4}$, ***the value of*** u ***is large***

$$u(x_0) \geq \nu^{j-1} M_0 \tag{8.50}$$

Then on the cube $Q_{\ell_j}(x_0) \subseteq Q_1$ *with side length*

$$\ell_j = \sigma M_0^{-\frac{\varepsilon}{n}} \nu^{-\frac{\varepsilon j}{n}} \tag{8.51}$$

one has ***lower bound on the supremum over*** $Q_{\ell_j}(x_0)$

$$\sup_{Q_{\ell_j}(x_0)} u \geq \nu^j M_0 \tag{8.52}$$

Proof. Assume for contradiction that

$$\sup_{Q_{\ell_j}(x_0)} u < \nu^j M_0 \tag{8.53}$$

Apply Weak Harnack to u**.** Let us work on $Q_{\frac{\ell_j}{4\sqrt{n}}}(x_0) \subseteq Q_{\ell_j}(x_0) \subseteq Q_1$. Since u is assumed to satisfy (8.46) (which should be a result of supersolution!! However if any additional condition leads to the power decay of distribution function, we're also happy. See Theorem 8.3.3)

$$|\{u \geq \frac{\nu^j M_0}{2}\} \cap Q_{\frac{\ell_j}{4\sqrt{n}}}(x_0)| \leq |\{u \geq \frac{\nu^j M_0}{2}\} \cap Q_1| \overset{(8.46)}{\leq} d \cdot \left(\frac{\nu^j M_0}{2}\right)^{-\varepsilon} \tag{8.54}$$

Notice here one requires

$$\begin{aligned} \frac{\nu^j M_0}{2} &> 1 \qquad \forall\ j \geq 1 \\ \nu M_0 = \frac{M_0^2}{M_0 - \frac{1}{2}} &> 2 \\ (M_0 - 1)^2 > 0 &\qquad \text{which is valid due to } M_0 > 1 \end{aligned}$$

Apply Weak Harnack to $C_1 - C_2 u$**.** One first conduct bijection between domains

$$x = x_0 + \frac{\ell_j}{4\sqrt{n}} y, \qquad y \in Q_1 \subseteq Q_{4\sqrt{n}}, \qquad x \in Q_{\frac{\ell_j}{4\sqrt{n}}}(x_0) \subseteq Q_{\ell_j}(x_0)$$

Let us rescale for now

$$v(y) := C_1 - C_2 u(x_0 + \frac{\ell_j}{4\sqrt{n}} y), \qquad \forall\ y \in Q_{4\sqrt{n}}$$

Assume the Weak Harnack for v **applies (This we check later)!** So the result reads (applied to M_0)

$$|\{v(y) > M_0\} \cap Q_1| \leq d \cdot M_0^{-\varepsilon}$$

Note the LHS expands as

$$\begin{aligned} d \cdot M_0^{-\varepsilon} \geq |\{v(y) > M_0\} \cap Q_1| &= |\{u(x_0 + \frac{\ell_j}{4\sqrt{n}} y) < \frac{C_1 - M_0}{C_2}\} \cap Q_1| \\ &= (\frac{\ell_j}{4\sqrt{n}})^{-n} |\{u(x) < \frac{C_1 - M_0}{C_2}\} \cap Q_{\frac{\ell_j}{4\sqrt{n}}}(x_0)| \end{aligned}$$

$$|\{u(x) < \frac{C_1 - M_0}{C_2}\} \cap Q_{\frac{\ell_j}{4\sqrt{n}}}(x_0)| \leq \left(\frac{\ell_j}{4\sqrt{n}}\right)^n d \cdot M_0^{-\varepsilon} \tag{8.55}$$

One would like this to be meaningful, by which we mean matching the LHS of (8.55) with (8.54). One need to solve

$$\frac{C_1 - M_0}{C_2} = \frac{\nu^j M_0}{2}$$
$$M_0\left(C_2\frac{\nu^j}{2}+1\right) = C_1$$

If we take

$$C_1 = \frac{\nu}{\nu - 1}, \qquad C_2 = \frac{1}{\nu^{j-1}(\nu-1)M_0} \tag{8.56}$$

One indeed ensure (such smart choice)

$$M_0\left(C_2\frac{\nu^j}{2}+1\right) = \frac{\nu}{2(\nu-1)} + M_0 = 2M_0 = \frac{\nu}{\nu-1} = C_1$$

Using the result of Weak Harnack to Contradict. Now we arrive at

$$\begin{aligned} |Q_{\frac{\ell}{4\sqrt{n}}}(x_0)| &= |\{u(x) < \frac{C_1 - M_0}{C_2}\} \cap Q_{\frac{\ell_j}{4\sqrt{n}}}(x_0)| + |\{u \geq \frac{\nu^j M_0}{2}\} \cap Q_{\frac{\ell_j}{4\sqrt{n}}}(x_0)| \\ \left(\frac{\ell_j}{4\sqrt{n}}\right)^n &\overset{(8.54),(8.55)}{\leq} \left(\frac{\ell_j}{4\sqrt{n}}\right)^n d\cdot M_0^{-\varepsilon} + d\cdot\left(\frac{\nu^j M_0}{2}\right)^{-\varepsilon} \end{aligned} \tag{8.57}$$

The goal is to pick M_0 and σ universal s.t. (8.57) fail. One first choose

$$d\cdot M_0^{-\varepsilon} \leq \frac{1}{2} \iff M_0 \geq (2d)^{\frac{1}{\varepsilon}} \tag{8.58}$$

to arrive at

$$\frac{1}{2}\left(\frac{\ell_j}{4\sqrt{n}}\right)^n \leq d\cdot\left(\frac{\nu^j M_0}{2}\right)^{-\varepsilon}$$

Now by taking

$$\ell_j = \sigma M_0^{-\frac{\varepsilon}{n}}\nu^{-\frac{\varepsilon j}{n}}$$

One obtain from the above

$$\frac{1}{2}\left(\frac{\ell_j}{4\sqrt{n}}\right)^n = \frac{1}{2(4\sqrt{n})^n}\sigma^n M_0^{-\varepsilon}\nu^{-\varepsilon j} \leq d\cdot\left(\frac{\nu^j M_0}{2}\right)^{-\varepsilon}$$
$$\sigma \leq (d\cdot 2^{\varepsilon+1+2n})^{\frac{1}{n}}\sqrt{n}$$

But recall one hasn't chosen the σ yet. If we choose σ universal s.t.

$$\sigma > (d\cdot 2^{\varepsilon+1+2n})^{\frac{1}{n}}\sqrt{n} \tag{8.59}$$

Then the above leads to a contradiction.

Check Eligibility to Apply Weak Harnack. To apply Lemma 8.3.4 one need to check v is eligible. First note $u \in \underline{S}(\lambda,\Lambda,-|f|)$ in $Q_{\ell_j}(x_0)$, thus the rescaling (8.16) solves

$$v = C_1 - C_2 u(x_0 + \frac{\ell_j}{4\sqrt{n}}y) \in \overline{S}\left(\lambda,\Lambda, C_2(\frac{\ell_j}{4\sqrt{n}})^2|f(x_0+\frac{\ell_j}{4\sqrt{n}}y)|\right) \cap C(\overline{Q_{4\sqrt{n}}})$$

One has $v \geq 0$ in $Q_{4\sqrt{n}}$ due to the contradictory assumption (8.53)

$$u(x) \leq \frac{C_1}{C_2} \overset{(8.56)}{=} \nu^j M_0 \qquad \forall\, x \in Q_{\ell_j}(x_0)$$

Moreover, the large value at x_0 (8.50) ensures that $\inf_{Q_3} v \leq 1$

$$v(0) = C_1 - C_2 u(x_0) \leq 1$$
$$u(x_0) \geq \frac{C_1 - 1}{C_2} \overset{(8.56)}{=} \nu^{j-1}M_0$$

Finally

$$\left\|C_2(\frac{\ell_j}{4\sqrt{n}})^2|f(x_0+\frac{\ell_j}{4\sqrt{n}}y)|\right\|_{L^n(\overline{Q_{4\sqrt{n}}})} = C_2(\frac{\ell_j}{4\sqrt{n}})^2\left((\frac{\ell_j}{4\sqrt{n}})^{-n}\int_{Q_{\ell_j}(x_0)}|f(x)|^n dx\right)^{\frac{1}{n}}$$
$$= C_2(\frac{\ell_j}{4\sqrt{n}})\,\|f\|_{L^n(Q_{\ell_j}(x_0))} \leq C_2\frac{\ell_j}{4\sqrt{n}}\varepsilon_0$$

Now if one in addition require that

$$\sigma M_0^{-\frac{\varepsilon}{n}} \leq \frac{1}{2} \iff M_0 \geq (2\sigma)^{\frac{n}{\varepsilon}} \tag{8.60}$$

one obtain

$$\begin{aligned} C_2\frac{\ell_j}{4\sqrt{n}} &= \frac{1}{4\sqrt{n}\nu^{j-1}(\nu-1)M_0}\sigma M_0^{-\frac{\varepsilon}{n}}\nu^{-\frac{\varepsilon j}{n}} \\ &\leq \frac{\nu^{-\frac{\varepsilon j}{n}}}{8\sqrt{n}\nu^{j-1}(\nu-1)M_0} = \frac{\nu^{-\frac{\varepsilon j}{n}}}{4\sqrt{n}\nu^j} \qquad \text{using } \frac{\nu}{2(\nu-1)} = M_0 \\ &\leq 1 \end{aligned}$$

Hence all conditions (8.41), (8.42), (8.43) are satisfied. Apply (8.44) to conclude. □

8.3.4 Proof of Harnack Inequality 8.3.2

In this section we put together the proof of Harnack. Take

$$u \in \mathcal{S}^*(\lambda,\Lambda,f) = \overline{\mathcal{S}}(\lambda,\Lambda,|f|)\cap\underline{\mathcal{S}}(\lambda,\Lambda-|f|) \qquad Q_{4\sqrt{n}}$$

and $f \in L^\infty(Q_{4\sqrt{n}})\cap C(Q_{4\sqrt{n}})$. Assume that

$$\begin{aligned} u &\geq 0 \qquad Q_{4\sqrt{n}} \\ \inf_{Q_{1/4}} u &\leq 1 \\ \|f\|_{L^n(Q_{4\sqrt{n}})} &\leq \varepsilon_0 \end{aligned}$$

Recall the definition of ℓ_j (8.51).

$$\ell_j = \sigma M_0^{-\frac{\varepsilon}{n}}\nu^{-\frac{\varepsilon j}{n}} = \sigma\cdot\underbrace{(\nu^j M_0)^{-\frac{\varepsilon}{n}}}_{\text{size of } u(x_{j+1})^{-\frac{\varepsilon}{n}}}$$

We consider $j_0 \in \mathbb{N}$ universal, large s.t.

$$\sum_{j\geq j_0}\ell_j \leq \frac{1}{4} \tag{8.61}$$

The goal is to show

$$\sup_{Q_{1/4}} u \leq \nu^{j_0-1}M_0 \tag{8.62}$$

To show our target (8.62). Assume not, so there exists $|x_{j_0}| \leq \frac{1}{8}$ ($Q_{1/4}$ centered at 0) s.t.

$$u(x_{j_0}) > \nu^{j_0-1}M_0$$

Note this is assumption (8.50) for Lemma 8.3.5 (the pointwise 'largeness') with $j = j_0$. Now applying the result (8.52) one obtain (somewhere 'larger' around x_{j_0} with at most distance ℓ_{j_0} away)

$$\sup_{Q_{\ell_{j_0}}(x_{j_0})} u \geq \nu^{j_0}M_0$$

Now this implies existence of x_{j_0+1} s.t.

$$|x_{j_0+1}-x_{j_0}| \leq \frac{1}{2}\ell_{j_0}, \qquad u(x_{j_0+1}) \geq \nu^{j_0}M_0$$

But one is able to **iterate this process (thanks to Lemma 8.3.5), and would obtain a sequence** $\{x_j\}_{j\geq j_0}$ s.t.

$$u(x_j) \geq \nu^{j-1}M_0, \qquad |x_{j+1}-x_j| \leq \frac{1}{2}\ell_j \qquad \forall\, j \geq j_0$$

But where does x_j go to? Note

$$\begin{aligned}|x_j|_\infty &\le |x_{j_0}| + \sum_{i=j_0}^{j-1} |x_{i+1} - x_i|_\infty \\ &< \frac{1}{8} + \frac{1}{2} \sum_{j \ge j_0} \ell_j \overset{(8.61)}{\le} \frac{1}{4} \qquad \forall\ j \ge j_0\end{aligned}$$

So by Bolzano Weierstrass, up to subsequences, there exists a limiting point $|x_\infty| \le \frac{1}{4}$ s.t.

$$u(x_\infty) = \lim_{j\to\infty} \nu^{j-1} M_0 = \infty$$

But this contradicts continuity of u in the interior. Thus (8.62) holds and the proof is complete.

8.3.5 Weak Harnack for Supersolution (L^p) and Local Maximum Principle for Subsolution

Notice the full Harnack deals with inf over sup bounds. Assuming onesideness, either one may be replaced by certain L^p norm, by which we refer to a local integrability property.

Weak Harnack for Supersolution

Theorem 8.3.2 ([CC95] Theorem 4.8 (i)). *Let $u \in \overline{\mathcal{S}}(\lambda, \Lambda, f)$ in Q_1 and $u \ge 0$. $f \in L^\infty(Q_1) \cap C(Q_1)$.*
Then for some $p_0 > 0$ and $C > 0$ universal

$$\|u\|_{L^{p_0}(Q_{1/4})} \le C \left(\inf_{Q_{1/2}} u + \|f\|_{L^n(Q_1)} \right) \tag{8.63}$$

Proof. Assume $u \in \overline{\mathcal{S}}(f) \subseteq \overline{\mathcal{S}}(|f|)$ in $Q_{4\sqrt{n}}$, $\inf_{Q_3} u \le 1$, $u \ge 0$, and $\|f\|_{L^n(Q_{4\sqrt{n}})} \le \varepsilon_0$. Then using the power decay for distribution function (8.46) (essentially Weak L^ε) to obtain for $p_0 = \frac{\varepsilon}{2}$

$$\begin{aligned}\int_{Q_1} u^{p_0} &= p_0 \int_0^\infty t^{p_0-1} |\{u > t\} \cap Q_1| dt = p_0 \int_0^1 t^{p_0-1} |\{u > t\} \cap Q_1| dt + p_0 \int_1^\infty t^{p_0-1} |\{u > t\} \cap Q_1| dt \\ &\overset{(8.46)}{\le} p_0 |Q_1| \int_0^1 t^{\frac{\varepsilon}{2}-1} dt + p_0 d \int_1^\infty t^{-\frac{\varepsilon}{2}-1} dt \le C(d, \varepsilon, n)\end{aligned}$$

Now upon rescaling

$$\|u\|_{L^{p_0}(Q_1)} \le C \left(\inf_{Q_3} u + \|f\|_{L^n(Q_{4\sqrt{n}})} \right) \qquad \forall\ u \in \overline{\mathcal{S}}, \qquad u \ge 0$$

Now by covering one recover (8.63). □

As a consequence one has a strong maximum principle for Supersolutions.

Corollary 8.3.1 ([CC95] Proposition 4.9). *Let $u \in \overline{\mathcal{S}}(\lambda, \Lambda, 0)$ in $\Omega \subseteq \mathbb{R}^n$.*
Then if $u \ge 0$ on Ω, and $u(x_0) = 0$ for some $x_0 \in \Omega$, one has $u \equiv 0$ on Ω.

In particular using Proposition 8.1.5, whenever $F(0, x) = 0$ one has the Strong Maximum Principle for $\mathcal{F}(D^2 u) \le 0$.

Weak Harnack for Subsolution

Theorem 8.3.3 ([CC95] Theorem 4.8 (ii)). *Let $u \in \underline{\mathcal{S}}(\lambda, \Lambda, f)$ in Q_1. $f \in L^\infty(Q_1) \cap C(Q_1)$.*
Then for any $p > 0$, there exists a constant $C(p) = C(n, \lambda, \Lambda, p) > 0$ s.t.

$$\sup_{Q_{1/2}} u \le C(p) \left(\|u^+\|_{L^p(Q_{3/4})} + \|f\|_{L^n(Q_1)} \right) \tag{8.64}$$

Proof. **Assume** $u \in \underline{\mathcal{S}}(f) \subseteq \underline{\mathcal{S}}(-|f|)$ **in** $Q_{4\sqrt{n}}$, $\|f\|_{L^n(Q_{4\sqrt{n}})} \le \varepsilon_0$, **and** $u^+ \in L^\varepsilon(Q_1)$ with

$$\|u^+\|_{L^\varepsilon(Q_1)} \le d^{\frac{1}{\varepsilon}}$$

Then compute for any $t > 1$

$$|\{u \geq t\} \cap Q_1| = \int_{Q_1} \chi_{\{u \geq t\}} dt \leq \int_{Q_1} \frac{u^{+\varepsilon}}{t^\varepsilon} \leq d \cdot t^{-\varepsilon}$$

We recover (8.46)**!** Then, since u itself is subsolution, one satisfy assumptions to run directly Lemma 8.3.5 and then the proof for Lemma 8.3.2 to conclude

$$\sup_{Q_{1/4}} u \leq C$$

Rescaling back to recover the result for $p = \varepsilon$.

$$\sup_{Q_{1/4}} u \leq C \left(\|u^+\|_{L^\varepsilon(Q_1)} + \|f\|_{L^n(Q_{4\sqrt{n}})} \right)$$

Now **for general** $p > \varepsilon$, since on finite measure space Q_1 higher exponents embed into lower, we know using interpolation

$$\int_{Q_1} (u^+)^\varepsilon \leq \left\|(u^+)^\varepsilon\right\|_{L^{\frac{p}{\varepsilon}}(Q_1)} \cdot |Q_1|^{\frac{p-\varepsilon}{p}} = C(n, \varepsilon, p) \left(\int_{Q_1} (u^+)^p \right)^{\frac{\varepsilon}{p}}$$

where we're using the Hölder conjugates

$$\frac{1}{\frac{p}{\varepsilon}} + \frac{1}{\frac{p}{p-\varepsilon}} = 1$$

For $0 < p < \varepsilon$, one may conduct

$$\int_{Q_1} (u^+)^\varepsilon \leq \int_{Q_1} (u^+)^{\varepsilon - p} \cdot (u^+)^p \leq (\sup_{Q_1} u^+)^{\varepsilon - p} \int_{Q_1} (u^+)^p$$

so plugging into the previous result reads

$$\begin{aligned}
\sup_{Q_{1/4}} u &\leq C \left((\sup_{Q_1} u)^{\frac{\varepsilon - p}{\varepsilon}} \|u^+\|^{\frac{p}{\varepsilon}}_{L^p(Q_1)} + \|f\|_{L^n(Q_{4\sqrt{n}})} \right) \\
&\leq \frac{1}{2} \sup_{Q_1} u + C(n, \varepsilon, p) \left(\|u^+\|_{L^p(Q_1)} + \|f\|_{L^n(Q_{4\sqrt{n}})} \right)
\end{aligned}$$

One conduct rescaling for u defined in $Q_{8\sqrt{n}r}$ to

$$\begin{aligned}
\tilde{u}(y) &= u(ry) \\
\tilde{f}(y) &= r^2 f(ry)
\end{aligned}$$

Using

$$\begin{aligned}
\int_{Q_1} ((\tilde{u}(y))^+)^p dy &\leq \int_{B_1} ((\tilde{u}(y))^+)^p dy = \frac{1}{r^n} \int_{B_r} ((u(x))^+)^p dx \leq \frac{1}{r^n} \int_{Q_{2r}} ((u(x))^+)^p dx \\
\|\tilde{u}\|_{L^p(Q_1)} &\leq \frac{1}{r^{\frac{n}{p}}} \|u\|_{L^p(Q_{2r})} \\
\int_{Q_{4\sqrt{n}}} (\tilde{f}(y))^n dy &\leq \int_{B_{4\sqrt{n}}} (\tilde{f}(y))^n dy = r^n \int_{B_{4\sqrt{n}r}} f(x)^n dx \leq r^n \int_{Q_{8\sqrt{n}r}} f(x)^n dx \\
\left\|\tilde{f}\right\|_{L^n(Q_{4\sqrt{n}})} &\leq r \|f\|_{L^n(Q_{8\sqrt{n}r})}
\end{aligned}$$

so that for any $r \leq 1$

$$\sup_{Q_{r/4}} u \leq \frac{1}{2} \sup_{Q_r} u + C(n, \varepsilon, p) \left(r^{-\frac{n}{p}} \|u\|_{L^p(Q_{2r})} + \|f\|_{L^n(Q_{8\sqrt{n}r})} \right)$$

Let's denote for any $r \leq 1$

$$\begin{aligned}
M(r) &= \sup_{Q_r} u \\
A &= C \|u\|_{L^p(Q_2)} \\
B &= \|f\|_{L^n(Q_{8\sqrt{n}})}
\end{aligned}$$

therefore the above writes

$$M(\frac{1}{4}r) \leq \frac{1}{2}M(r) + Ar^{-\frac{n}{p}} + B \qquad \forall\, r \leq 1$$

The term $r^{-\frac{n}{p}}$ is notorious.

Instead if one does not restrict to a fixed scale $\frac{1}{4}$, and run again whatever is above, one get

$$M(t) \leq \frac{1}{2}M(s) + A(s-t)^{-\frac{n}{p}} + B$$

The one has to conclude following [HL11] Lemma 4.3 to obtain

$$M(t) \leq C(n,p)(\frac{A}{(s-t)^{\frac{n}{p}}} + B)$$

Otherwise apply use Lemma 1.8.5.

□

8.3.6 C^α Regularity

Oscillation Decay and Interior C^α Regularity

Proposition 8.3.1 ([CC95] Proposition 4.10). *Let $u \in \mathcal{S}^*(\lambda, \Lambda, f)$ in Q_1.*

Then there exists $\mu < 1$ universal s.t.

$$\operatorname*{osc}_{Q_{1/2}} u \leq \mu \operatorname*{osc}_{Q_1} u + 2\,\|f\|_{L^n(Q_1)}$$

Thus there exists $\alpha \in (0,1)$ and $C > 0$ universal s.t.

$$\|u\|_{C^{0,\alpha}(\overline{Q}_{1/2})} \leq C\left(\|u\|_{L^\infty(Q_1)} + \|f\|_{L^n(Q_1)}\right) \tag{8.65}$$

Proof. Denote

$$M_r = \sup_{Q_r} u, \qquad m_r = \inf_{Q_r} u, \qquad \operatorname*{osc}_{Q_r} u = M_r - m_r$$

Then apply the Harnack Inequality (8.39) to $u - m_1$ and $M_1 - u$ in Q_1 to obtain

$$M_{1/2} - m_1 \leq C\left(m_{1/2} - m_1 + \|f\|_{L^n(Q_1)}\right)$$
$$M_1 - m_{1/2} \leq C\left(M_1 - M_{1/2} + \|f\|_{L^n(Q_1)}\right)$$

Summing up both inequalities one obtain

$$\operatorname*{osc}_{Q_1} u + \operatorname*{osc}_{Q_{1/2}} u \leq C\left(\operatorname*{osc}_{Q_1} u - \operatorname*{osc}_{Q_{1/2}} u + 2\,\|f\|_{L^n(Q_1)}\right)$$
$$(1+C)\operatorname*{osc}_{Q_{1/2}} u \leq (C-1)\operatorname*{osc}_{Q_1} u + 2C\,\|f\|_{L^n(Q_1)}$$
$$\operatorname*{osc}_{Q_{1/2}} u \leq \frac{C-1}{C+1}\operatorname*{osc}_{Q_1} u + \frac{2C}{C+1}\,\|f\|_{L^n(Q_1)}$$

Hölder Continuity follows from the classical approach (see [FRRO22] Corollary 2.12). □

Now by covering, one achieve the result on balls, say, of the form

$$[u]_{C^{0,\alpha}(\overline{B_{1/2}})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^n(B_1)}\right)$$

Now how does rescaling work? Suppose we want estimate on the ball $B_r(x_0)$, then

$$\tilde{u}(y) := u(x_0 + ry)$$
$$\tilde{f}(y) := r^2 f(x_0 + ry)$$

defines the function in B_1. Then apply the estimate to $\tilde{u}$ yields

$$\frac{|\tilde{u}(y_1) - \tilde{u}(y_2)|}{|y_1 - y_2|^\alpha} = r^\alpha \frac{|u(x_0 + ry_1) - u(x_0 + ry_2)|}{|x_0 + ry_1 - (x_0 + ry_2)|^\alpha} = r^\alpha \frac{u(x_1) - u(x_2)}{|x_1 - x_2|^\alpha}$$
$$r^\alpha \frac{u(x_1) - u(x_2)}{|x_1 - x_2|^\alpha} \leq C\left(\|\tilde{u}\|_{L^\infty(B_1)} + \left\|\tilde{f}\right\|_{L^n(B_1)}\right)$$
$$\leq C\left(\|u\|_{L^\infty(B_r(x_0))} + r^2 \cdot r^{-1}\,\|f\|_{L^n(B_r(x_0))}\right)$$
$$r^\alpha [u]_{C^{0,\alpha}(\overline{B_{r/2}(x_0)})} \leq C\left(\|u\|_{L^\infty(B_r(x_0))} + r\,\|f\|_{L^n(B_r(x_0))}\right)$$

Compactness

Proposition 8.3.2 ([CC95] Proposition 4.11)**.** *Let $\{\mathcal{F}_k\}$ be sequence of uniformly elliptic operators with ellipticity constants λ, Λ. Let $\{u_k\}_{k\geq 1} \subseteq C(\Omega)$ be viscosity solutions*

$$\mathcal{F}_k(D^2u_k, x) = f(x) \qquad \Omega$$

for $f \in C(\Omega)$.

Assume $\{\mathcal{F}_k\}$ converges uniformly on $\mathrm{Sym}(n)\times\Omega$ to $\mathcal{F}$, and $\{u_k\}$ are uniformly bounded on compact subsets of Ω.

Then up to subsequence, there exists $u \in C(\Omega)$ s.t. $\{u_k\}$ converges uniformly to u on compact subsets of Ω. Moreover

$$\mathcal{F}(D^2u(x), x) = f(x) \qquad \text{in the viscosity sense in } \Omega$$

Proof. We first show $u_k \to u$ locally uniformly via Ascoli-Arzela. It suffices to show $\{u_k\}$ are locally uniformly equi-continuous.

Since u_k are viscosity solutions to $\mathcal{F}_k(D^2u_k) = f$, by Proposition 8.1.8

$$u_k \in \mathcal{S}(\frac{\lambda}{n}, \Lambda, f - \mathcal{F}_k(0,x))$$

Now, for any $\Omega' \Subset \Omega'' \Subset \Omega$, apply Hölder Estimate

$$\begin{aligned}|u_k(x) - u_k(y)| &\leq \|u_k\|_{C^{0,\alpha}(\overline{\Omega'})}\, |x-y|^{\alpha} \\ &\overset{(8.65)}{\leq} C\left(\|u_k\|_{L^\infty(\Omega'')} + \|f - \mathcal{F}_k(0,x)\|_{L^n(\Omega'')}\right)|x-y|^{\alpha} \\ &\leq C(n, \Omega', \Omega'', f, \mathcal{F})|x-y|^{\alpha} \qquad \text{uniformly in } k\end{aligned}$$

where we've used the fact that $\mathcal{F}_k$ converges locally uniformly in x to $\mathcal{F}$, and that u_k are uniformly bounded.

Thus up to subsequence $u_k \to u$ locally uniformly. Since u as a uniform limit is continuous on any compact subdomain of Ω, $u \in C(\Omega)$.

Since $f \in C(\Omega)$, one conclude using Proposition 8.1.9. □

Boundary C^α Regularity

Proposition 8.3.3 ([CC95] Proposition 4.12)**.** *Let $u \in \underline{\mathcal{S}}(\lambda, \Lambda, 0)$ in B_1.*

Assume for $0 < \beta < 1$, $u \in C(\overline{B_1})$ satisfied Hölder Boundary condition

$$u|_{\partial B_1} = \varphi \in C^{0,\beta}(\partial B_1)$$

Then for any $x_0 \in \partial B_1$, u is $C^{0,\frac{\beta}{2}}$ Hölder continuous at x_0 with

$$\sup_{x\in\overline{B_1}} \frac{|u(x) - u(x_0)|}{|x-x_0|^{\frac{\beta}{2}}} \leq 2^{\beta/2} \sup_{x\in\partial B_1} \frac{|\varphi(x)-\varphi(x_0)|}{|x-x_0|^{\beta}} \tag{8.66}$$

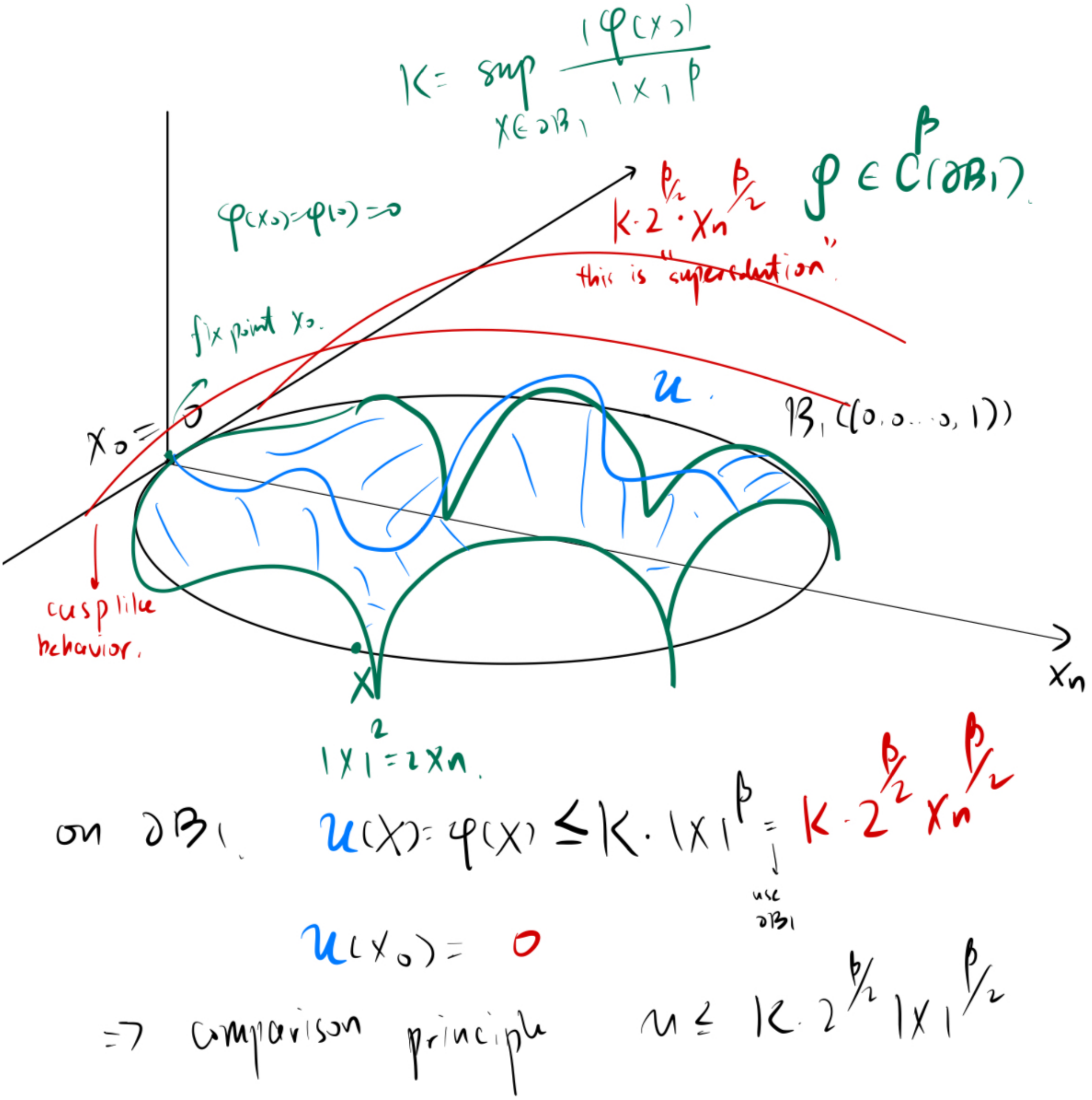


Figure 8.15: Boundary Hölder Regularity

Proof. Let $x_0 = 0$ and fix $B_1 = B_1(0, \cdots, 0, 1)$. shift $\varphi(x_0) = 0$. Denote $K = \sup_{x\in\partial B_1} \frac{|\varphi(x)|}{|x|^\beta}$. Now for any point $x \in \partial B_1$, one has

$$\sum_{i=1}^{n-1} x_i^2 + (x_n - 1)^2 = 1$$

$$|x|^2 = 2x_n$$

and in particular

$$u(x) = \varphi(x) \le K|x|^\beta = 2^{\frac{\beta}{2}} K x_n^{\frac{\beta}{2}} \qquad \forall\ x \in \partial B_1$$

On the other hand $u(x_0) = \varphi(x_0) = 0$. One think of defining a barrier function

$$v(x) = 2^{\frac{\beta}{2}} K x_n^{\frac{\beta}{2}}$$

Compute

$$\partial_{nn} v(x) = 2^{\frac{\beta}{2}} K \frac{\beta}{2} \frac{\beta - 2}{2} x_n^{\frac{\beta}{2} - 2} < 0 \qquad \forall\ x \in B_1$$

Thus

$$\mathcal{M}^+(D^2 v(x)) = \lambda 2^{\frac{\beta}{2}} K \frac{\beta}{2}\frac{\beta-2}{2} x_n^{\frac{\beta}{2}-2} < 0$$

so this is supersolution to Extremal Pucci Operator. Then a shift down by v

$$u - v \in \underline{\mathcal{S}}(\lambda, \Lambda, 0)$$

remains a subsolution (8.17) as well. Now we use this fact. Since subsolution satisfies Corollary 8.2.2 Weak Maximum Principle, that $u - v \le 0$ on ∂B_1 implies

$$u \le v = 2^{\frac{\beta}{2}} K x_n^{\frac{\beta}{2}} \le 2^{\frac{\beta}{2}} K |x|^{\frac{\beta}{2}}$$

so in particular (8.66) is satisfied. □

Global Hölder Regularity

Proposition 8.3.4 ([CC95] Proposition 4.13)**.** *Let $u \in \mathcal{S}(\lambda, \Lambda, 0)$ in B_1. Assume that $u \in C(\overline{B_1})$ and for $0 < \beta < 1$*

$$u|_{\partial B_1} = \varphi \in C^{0,\beta}(\partial B_1)$$

Then for $\gamma = \min(\frac{\beta}{2}, \alpha)$ with α in Proposition 8.3.1.

$$\|u\|_{C^{0,\gamma}(\overline{B_1})} \le C \|\varphi\|_{C^{0,\beta}(\partial B_1)}$$

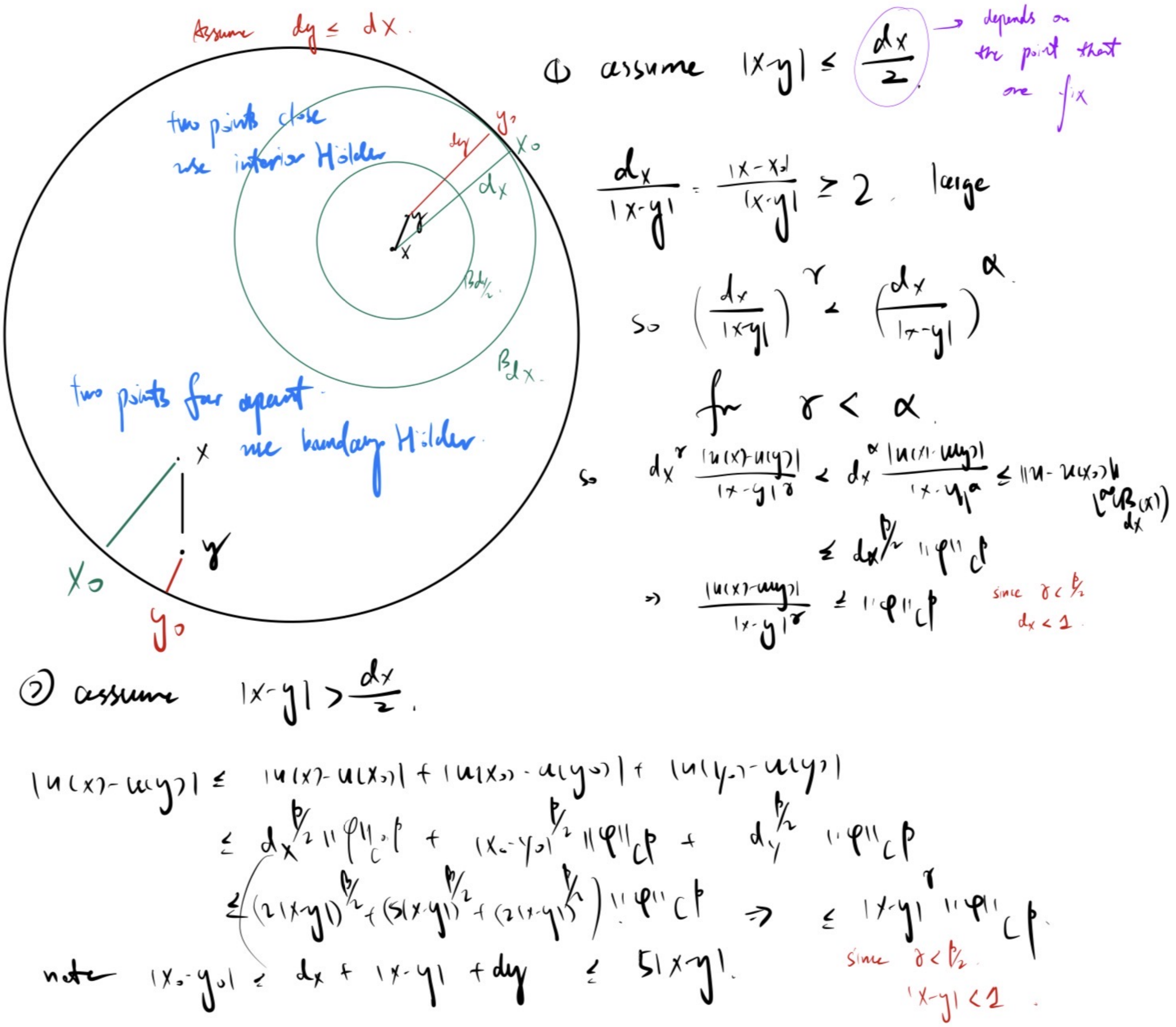


Figure 8.16: Global Hölder Regularity

Proof. For any x, $y \in B_1$, consider x_0, $y_0 \in \partial B_1$ that achieves

$$\operatorname{dist}(x, \partial B_1) = d_x = |x - x_0|, \qquad \operatorname{dist}(y, \partial B_1) = d_y = |y - y_0|$$

We decide on the ball on which one can apply the estimate. Assume that $d_y \leq d_x$.

If $|x-y| \leq \frac{d_x}{2}$**,** x, y **are close**. One would like to apply interior Hölder estimate in the ball $B_{d_x}(x)$ (hence there's rescaling!). Since in this case

$$\frac{d_x}{|x-y|} \geq 2 \implies (\frac{d_x}{|x-y|})^\gamma \leq (\frac{d_x}{|x-y|})^\alpha$$

Therefore

$$(\frac{d_x}{|x-y|})^\gamma |u(x) - u(y)| \leq (\frac{d_x}{|x-y|})^\alpha |u(x) - u(y)| \overset{(8.65)}{\leq} C \left\| u - u(x_0) \right\|_{L^\infty(B_{d_x}(x))} \qquad \text{using interior Hölder}$$
$$\overset{(8.66)}{\leq} C d_x^{\frac{\beta}{2}} \left\| \varphi \right\|_{C^{0,\beta}(\overline{B_1})} \qquad \text{using Boundary Hölder}$$

Now since $d_x < 1$, $\gamma \leq \frac{\beta}{2}$, one has $d_x^{\frac{\beta}{2}} \geq d_x^\gamma$. Thus

$$\frac{|u(x) - u(y)|}{|x-y|^\gamma} \leq C \left\| \varphi \right\|_{C^{0,\beta}(\overline{B_1})} \qquad \forall\ |x-y| \leq \frac{d_x}{2}$$

If $|x-y| > \frac{d_x}{2}$**,** x, y **are far apart**. Then consider

$$\begin{aligned}
|u(x) - u(y)| &\leq |u(x) - u(x_0)| + |u(x_0) - u(y_0)| + |u(y_0) - u(y)| \\
&\overset{(8.66)}{\leq} C \left(d_x^{\frac{\beta}{2}} + |x_0 - y_0|^{\frac{\beta}{2}} + d_y^{\frac{\beta}{2}} \right) \left\| \varphi \right\|_{C^{0,\beta}(\partial B_1)} \\
&\leq C \left(2^{\frac{\beta}{2}} |x-y|^{\frac{\beta}{2}} + 5^{\frac{\beta}{2}} |x-y|^{\frac{\beta}{2}} + 2^{\frac{\beta}{2}} |x-y|^{\frac{\beta}{2}} \right) \left\| \varphi \right\|_{C^{0,\beta}(\partial B_1)} \\
&\leq C(\beta) |x-y|^{\frac{\beta}{2}} \left\| \varphi \right\|_{C^{0,\beta}(\partial B_1)} \\
&\leq C(\beta) |x-y|^\gamma \left\| \varphi \right\|_{C^{0,\beta}(\partial B_1)} \qquad \forall\ |x-y| > \frac{d_x}{2}
\end{aligned}$$

□

8.4 Jensen's Approximation, Uniqueness of Solutions and $C^{1,\alpha}$ Regularity

Let's suppose $\mathcal{F}$ uniformly elliptic and does not depend on $x \in \Omega$.

8.4.1 Jensen's Approximation

Upper/Lower ε-envelope Let $u \in C(\Omega)$ and H be open set s.t. $\overline{H} \subseteq \Omega$.

For any $\varepsilon > 0$, define the *upper ε-envelope of u w.r.t. H* as

$$u^\varepsilon(x_0) := \sup\{u(x) + \varepsilon - \frac{1}{\varepsilon}|x - x_0|^2 \mid x \in \overline{H}\} \qquad \forall\ x_0 \in H$$

This is the sup-convolution of a lift of the graph $u + \varepsilon$ with the concave parabola $-\frac{1}{\varepsilon}|x|^2$.

Define the *lower ε-envelope of u w.r.t. H* as

$$u_\varepsilon(x_0) := \inf\{u(x) - \varepsilon + \frac{1}{\varepsilon}|x - x_0|^2 \mid x \in \overline{H}\}$$

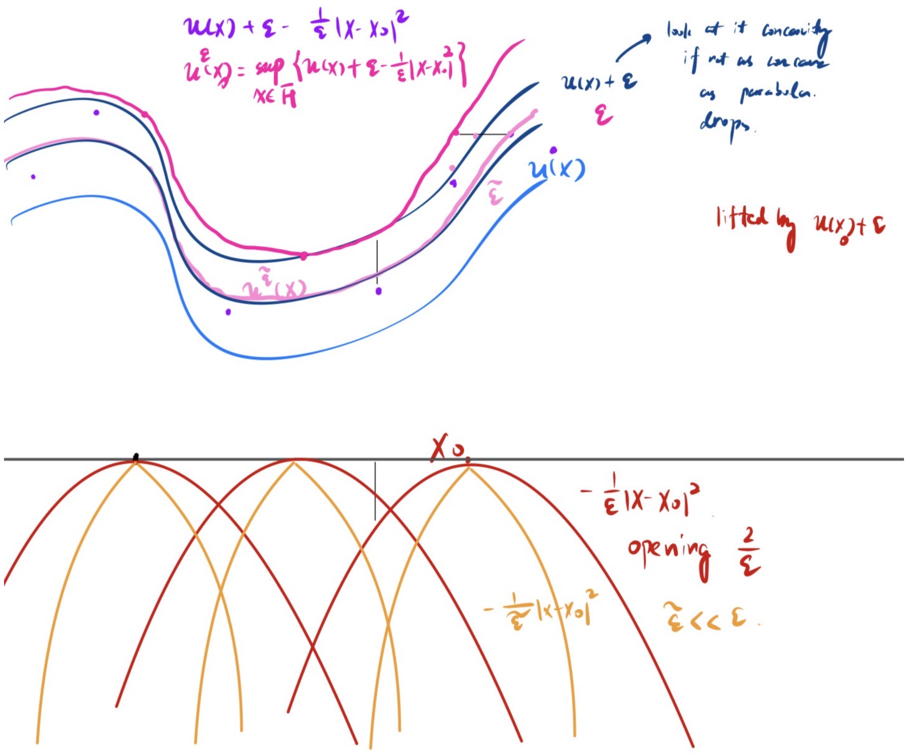


Figure 8.17: upper ε-envelope of u

One has properties of ε-envelopes.

Lemma 8.4.1 ([CC95] Lemma 5.2)**.** *1. For any $x_0 \in H$, there exists some $x_0^* \in \overline{H}$ s.t. achieves the value of the convex upper envelope*

$$u^\varepsilon(x_0) = u(x_0^*) + \varepsilon - \frac{1}{\varepsilon}|x_0^* - x_0|^2 \tag{8.67}$$

Proof. For fixed $x_0 \in H$, the function

$$x \mapsto u(x) + \varepsilon - \frac{1}{\varepsilon}|x - x_0|^2$$

over $\overline{H}$ is a continuous function. Thus there exists $x_0^* \in \overline{H}$ s.t. the map achieves supremum, which by definition, equals $u^\varepsilon(x_0)$. □

2. For any $x_0 \in H$

$$u^\varepsilon(x_0) \geq u(x_0) + \varepsilon \tag{8.68}$$

Proof. $x_0 \in H \subseteq \overline{H}$ is one possible point and thus

$$u(x_0) + \varepsilon - \cancel{\frac{1}{\varepsilon}|x_0 - x_0|^2} \le \sup\{u(x) + \varepsilon - \frac{1}{\varepsilon}|x - x_0|^2 \mid x \in \overline{H}\} = u^\varepsilon(x_0)$$

□

3. *For any* x_0, $x_1 \in H$

$$|u^\varepsilon(x_0) - u^\varepsilon(x_1)| \le \frac{3}{\varepsilon}\mathrm{diam}(H)|x_0 - x_1| \tag{8.69}$$

Proof. For any $x \in \overline{H}$

$$\begin{aligned}
u^\varepsilon(x_0) &\ge u(x) + \varepsilon - \frac{1}{\varepsilon}|x - x_0|^2 \\
&\ge u(x) + \varepsilon - \frac{1}{\varepsilon}\left(|x - x_1|^2 + 2(x - x_1)\cdot(x_1 - x_0) + |x_1 - x_0|^2\right) \\
&\ge u(x) + \varepsilon - \frac{1}{\varepsilon}|x - x_1|^2 - \frac{2}{\varepsilon}|x - x_1||x_1 - x_0| - \frac{1}{\varepsilon}|x_1 - x_0|^2 \\
&= u^\varepsilon(x_1) - \frac{1}{\varepsilon}2\mathrm{diam}(H)|x_1 - x_0| - \frac{1}{\varepsilon}|x_1 - x_0|^2 \\
u^\varepsilon(x_1) &\le u^\varepsilon(x_0) + \frac{3}{\varepsilon}\mathrm{diam}(H)|x_1 - x_0|
\end{aligned}$$

□

4. *For any* $0 < \varepsilon < \varepsilon'$

$$u^\varepsilon(x_0) \le u^{\varepsilon'}(x_0)$$

Proof. For any $x \in \overline{H}$, and for $x_0 \in H$ fixed

$$\begin{aligned}
&u(x) + \varepsilon - \frac{1}{\varepsilon}|x - x_0|^2 - \left(u(x) + \varepsilon' - \frac{1}{\varepsilon'}|x - x_0|^2\right) \\
&= \varepsilon - \varepsilon' + \left(\frac{1}{\varepsilon'} - \frac{1}{\varepsilon}\right)|x - x_0|^2 < 0
\end{aligned}$$

Now take supremum respectively. □

5. *For any* $x_0 \in H$ *and* x_0^* *as in* (8.67)

$$|x_0 - x_0^*|^2 \le \varepsilon \operatorname*{osc}_H u \tag{8.70}$$

Proof.

$$|x_0^* - x_0|^2 \overset{(8.67)}{=} \varepsilon(u(x_0^*) + \varepsilon - u^\varepsilon(x_0)) \overset{(8.68)}{\le} \varepsilon(u(x_0^*) - u(x_0)) \le \varepsilon \operatorname*{osc}_H u$$

□

6. *For any* $x_0 \in H$ *and* x_0^* *as in* (8.67)

$$0 < u^\varepsilon(x_0) - u(x_0) \le u(x_0^*) - u(x_0) + \varepsilon \tag{8.71}$$

Proof. The first strict inequality is due to the (8.68). The second is due to (8.67). □

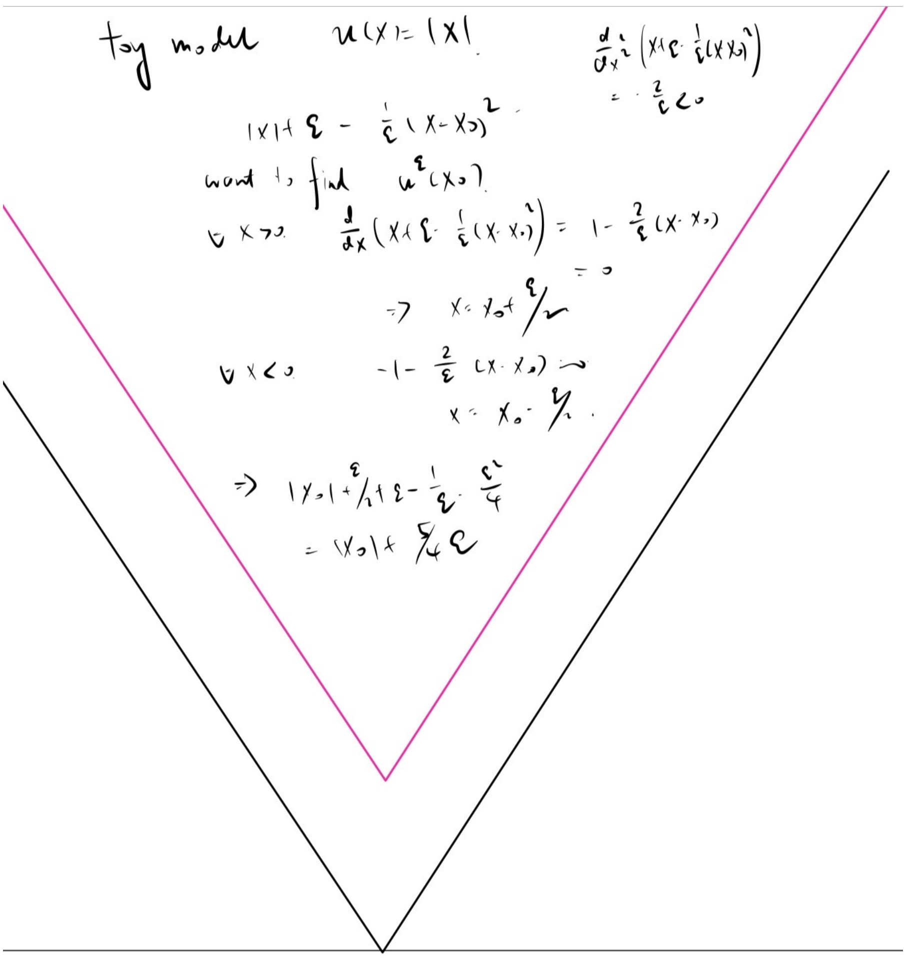


Figure 8.18: upper ε-envelope of $u = |x|$

Uniform Convergence

Proposition 8.4.1 ([CC95] Theorem 5.1 (i)). *Let $u \in C(\Omega)$, H open s.t. $\overline{H} \subseteq \Omega$, and u^ε be upper ε-envelope of u in H.*

Then $u^\varepsilon \in C(H)$, and $u^\varepsilon \to u$ uniformly in H as $\varepsilon \to 0$.

Proof. Using (8.69) we know u^ε are continuous in H. To show uniform convergence, for any $x_0 \in H$, using (8.71)

$$0 < u^\varepsilon(x_0) - u(x_0) \leq u(x_0^*) - u(x_0) + \varepsilon$$

Using uniform continuity of u in $\overline{H}$, for any $\eta > 0$ we know there exists $\delta = \delta(\eta) > 0$ s.t. for ε small

$$|x_0^* - x_0|^2 \overset{(8.70)}{\leq} \varepsilon \operatorname*{osc}_H u < \delta \implies |u(x_0^*) - u(x_0)| < \eta$$

Thus for any $\eta > 0$, one may choose $\varepsilon = \varepsilon(\eta, \delta) > 0$ small so

$$0 < u^\varepsilon(x_0) - u(x_0) \leq u(x_0^*) - u(x_0) + \varepsilon \leq \eta + \varepsilon \qquad \forall\, x_0 \in H$$

Since independent of x_0 this is uniform convergence as one pass $\eta \to 0$. □

u^ε is punctually second order differentiable

Proposition 8.4.2 ([CC95] Theorem 5.1 (ii)). *For any $x_0 \in H$, there is a concave parabola of opening $\frac{2}{\varepsilon}$ that touches u^ε from below at $x_0 \in H$, in other words*

$$\underline{\Theta}(u^\varepsilon, H)(x_0) \leq \frac{2}{\varepsilon}$$

In particular, this means u^ε is $C^{1,1}$ from below in H. Using Proposition 8.1.3 $u^\varepsilon + \frac{1}{\varepsilon}|x|^2$ is convex in H, thus u^ε is punctually second order differentiable a.e. in Ω (only a.e. due to Alexandroff Theorem)!

Proof. Fix $x_0 \in H$, the consider x_0^* as in (8.67). Define the parabola

$$P(x) := u(x_0^*) + \varepsilon - \frac{1}{\varepsilon}|x_0^* - x|^2$$

so that it satisfies

$$\begin{cases} P(x) \le u^\varepsilon(x) & H \\ P(x_0) = u^\varepsilon(x_0) & x_0 \in H \end{cases}$$

The first line follows from, for fixed $x \in H$

$$P(x) = u(x_0^*) + \varepsilon - \frac{1}{\varepsilon}|x_0^* - x|^2 \le \sup\{u(y) + \varepsilon - \frac{1}{\varepsilon}|y - x|^2 \mid y \in H\} = u^\varepsilon(x)$$

and the second line is due to (8.67). □

u^ε is viscosity subsolution

Proposition 8.4.3 ([CC95] Theorem 5.1 (iii))**.** *Let u be viscosity subsolution to*

$$\mathcal{F}(D^2u) \ge 0 \qquad \Omega$$

Let H_1 be open set, $\overline{H_1} \subseteq H$ where u^ε is defined.

Then there exists $\varepsilon_0 = \varepsilon_0(u, H, H_1)$ s.t. u^ε is viscosity subsolution

$$\mathcal{F}(D^2u^\varepsilon) \ge 0 \qquad H_1 \qquad \textit{in the viscosity sense} \tag{8.72}$$

Moreover, $\mathcal{F}(D^2u^\varepsilon)(x) \ge 0$ a.e. $x \in H_1$.

Proof. For any $x_0 \in H_1$, and let P be the parabola that touches u^ε from above at x_0.

Establishing the point we work with. For $x_0 \in H_1$ consider $x_0^* \in \overline{H}$ as in (8.67). But if $x_0^* \in \partial H$ we're unable to evaluate u^ε there (this is why we need H_1). Hence, in view of (8.70)

$$|x_0 - x_0^*| \le \varepsilon \underset{H}{\operatorname{osc}} u$$

by choosing $\varepsilon < \varepsilon_0$ where ε_0 depends on $\underset{H}{\operatorname{osc}} u$, one can ensure

$$x_0 \in H_1 \implies x_0^* \in H$$

Now choose $x \in H$ sufficiently close to x_0^* s.t.

$$x - x_0^* + x_0 \in H$$

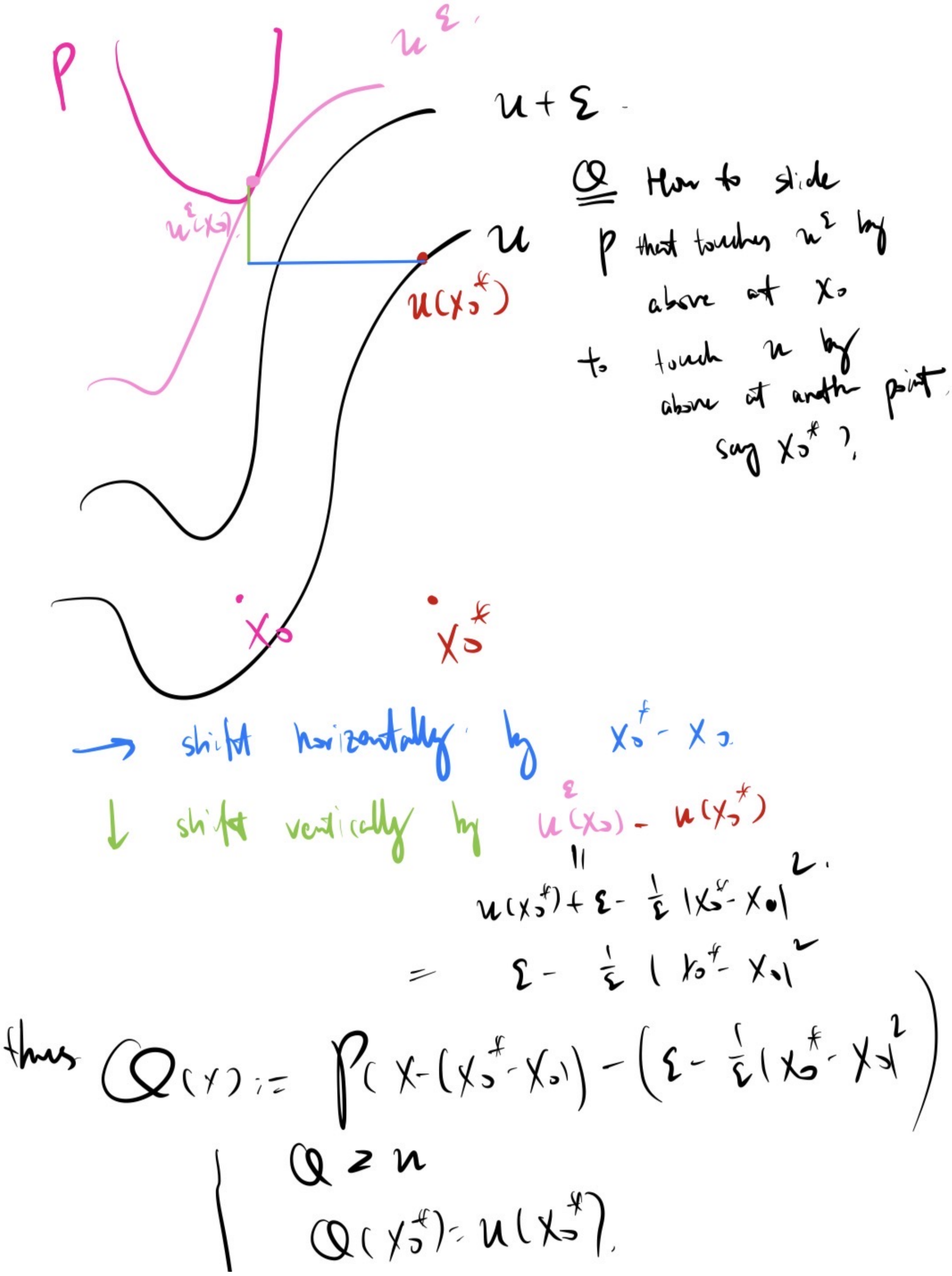


Figure 8.19: Construction of polynomial Q

Construct Q s.t. $u \le Q$. We evaluate our u^ε at this point $x - x_0^* + x_0$ and compare with the value at x (since by definition u^ε is supremum)

$$\begin{aligned} u^\varepsilon(x - x_0^* + x_0) &= \sup\{u(y) + \varepsilon - \frac{1}{\varepsilon}|y - (x - x_0^* + x_0)|^2 \mid y \in \overline{H}\} \\ &\ge u(x) + \varepsilon - \frac{1}{\varepsilon}|x - x + x_0^* - x_0|^2 \\ u(x) &\le u^\varepsilon(x - x_0^* + x_0) - \varepsilon + \frac{1}{\varepsilon}|x_0 - x_0^*|^2 \end{aligned}$$

Using P touches u^ε from above at x_0, we know

$$u(x) \le P(x - x_0^* + x_0) - \varepsilon + \frac{1}{\varepsilon}|x_0 - x_0^*|^2 \qquad \text{for } x \text{ sufficiently close to } x_0^*, \text{ so } x - x_0^* + x_0 \text{ close to } x_0$$

Now the above RHS is parabola in x, we define

$$Q(x) := P(x - x_0^* + x_0) - \varepsilon + \frac{1}{\varepsilon}|x_0 - x_0^*|^2$$

Establish touching $u(x_0^*) = Q(x_0^*)$ Note at x_0^*,

$$\begin{aligned} Q(x_0^*) &= P(x_0) - \varepsilon + \frac{1}{\varepsilon}|x_0 - x_0^*|^2 \overset{P \text{ touch } u^\varepsilon \text{ at } x_0}{=} u^\varepsilon(x_0) - \varepsilon + \frac{1}{\varepsilon}|x_0 - x_0^*|^2 \\ &\overset{(8.67)}{=} u(x_0^*) + \varepsilon - \frac{1}{\varepsilon}|x_0^* - x_0|^2 - \varepsilon + \frac{1}{\varepsilon}|x_0 - x_0^*|^2 \\ &= u(x_0^*) \end{aligned}$$

Thus Q touches u from above at x_0^*.

Let's use this! Since u is viscosity subsolution

$$0 \leq \mathcal{F}(D^2 Q(x_0^*)) = \mathcal{F}(D^2 P(x_0))$$

Thus u^ε is viscosity subsolution in H_1. Now using Lemma 8.1.7, one obtain pointwise information (since u^ε is punctually second order differentiable)

$$\mathcal{F}(D^2 u^\varepsilon(x_0)) \geq f(x_0)$$

□

8.4.2 Uniqueness of Solution

Recall we've not covered whether $\underline{\mathcal{S}}$ are closed under addition. In fact, no. But their difference could solve the Extremal Pucci Equation.

Key Theorem

Theorem 8.4.1 ([CC95] Theorem 5.3)**.** *Let $\mathcal{F}$ be uniformly elliptic with elliptic constants λ, Λ. Let u be viscosity subsolution*

$$\mathcal{F}(D^2 u) \geq 0 \qquad \Omega$$

and v be viscosity supersolution

$$\mathcal{F}(D^2 v) \leq 0 \qquad \Omega$$

Then

$$u - v \in \underline{\mathcal{S}}(\frac{\lambda}{n}, \Lambda) \qquad \Omega$$

Proof. In view of (8.72), we take H_1 and H open subsets of Ω s.t. $\overline{H_1} \subseteq H \subseteq \overline{H} \subseteq \Omega$.

It suffices to show $u^\varepsilon - v_\varepsilon \in \underline{\mathcal{S}}(\frac{\lambda}{n}, \Lambda)$ **in** H_1. Why this suffices? Using uniform convergence Proposition 8.4.1, that u^ε and v_ε are respectively subsolution and supersolutions in H_1 by (8.72), and that viscosity solutions are closed under local uniform limits Proposition 8.1.9, the result follows by arbitrariness of H_1.

Now let P parabola touch $u^\varepsilon - v_\varepsilon$ from above at $x_0 \in \overline{B_r(x_0)} \subseteq H_1$. We want to show that

$$\mathcal{M}^+(D^2 P(x_0)) \geq 0 \tag{8.73}$$

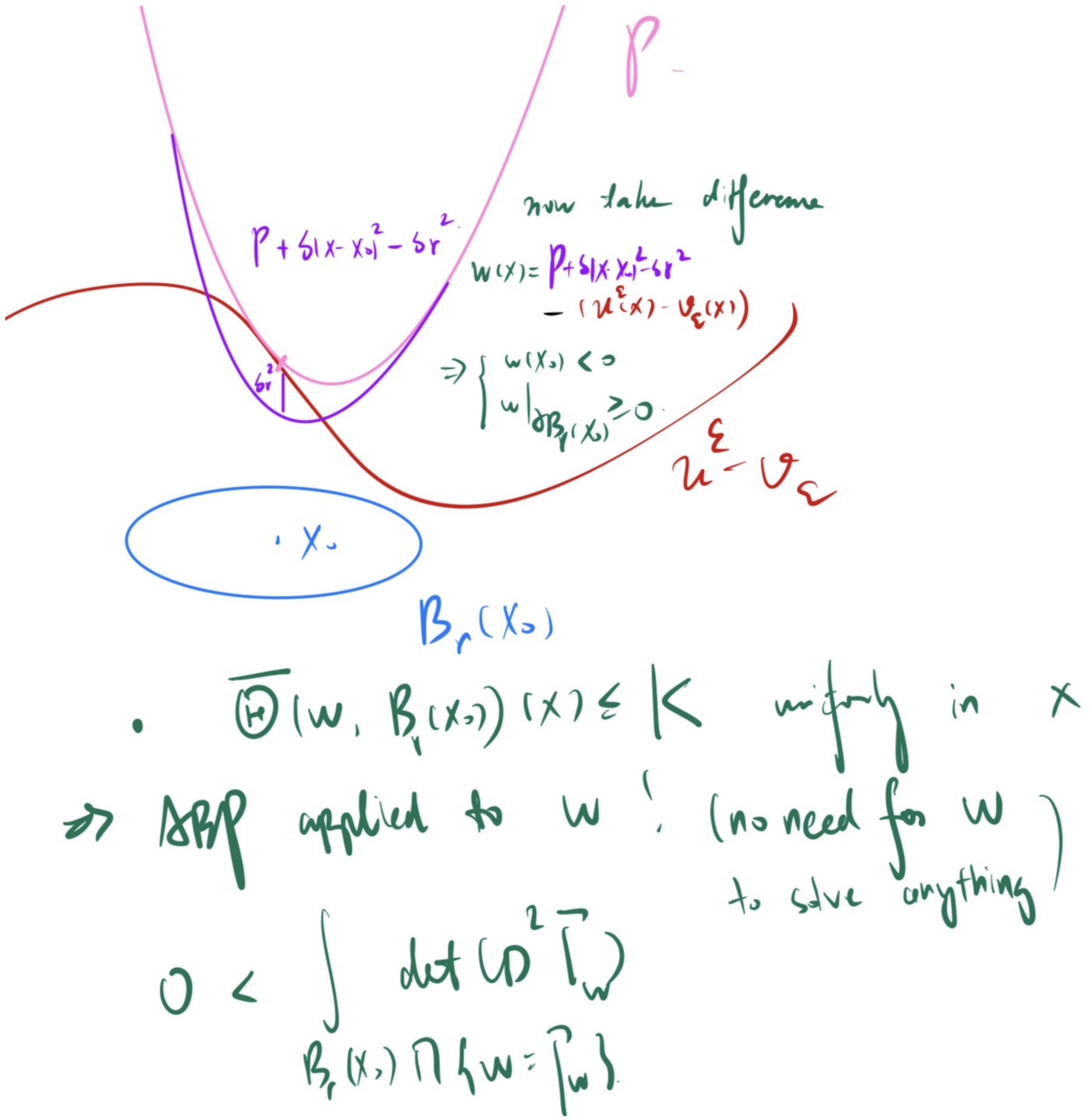


Figure 8.20: w to show (8.73)

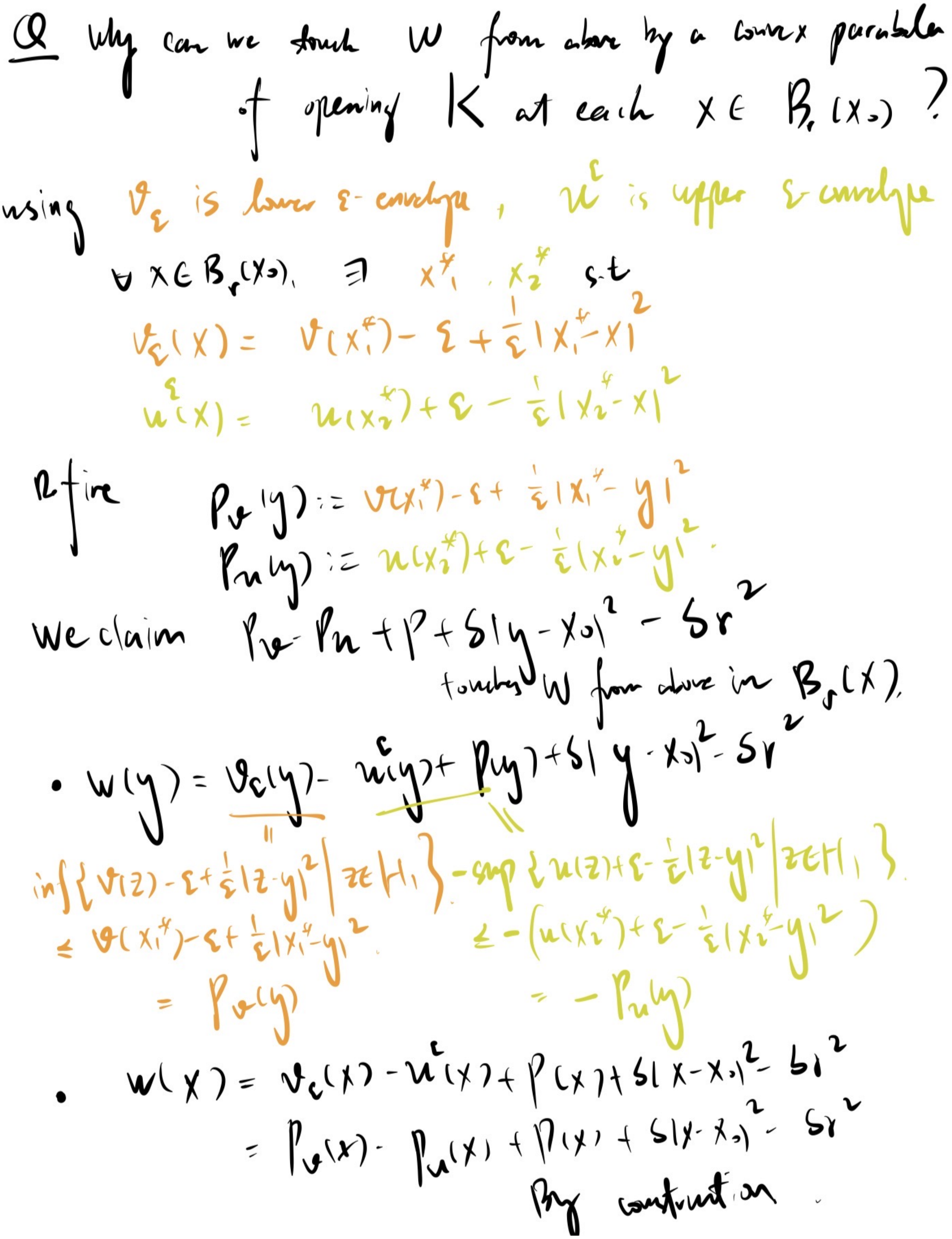
Construction of w and apply ABP. We perturb down P a little bit while fixing its boundary value on $\partial B_r(x_0)$, then take its difference with $u^\varepsilon - v_\varepsilon$.

$$w(x) := P(x) + \delta|x - x_0|^2 - \delta r^2 - (u^\varepsilon(x) - v_\varepsilon(x)) \qquad \forall\, x \in B_r(x_0)$$

Then using $P(x_0) = u^\varepsilon(x_0) - v_\varepsilon(x_0)$, one see immediately for $\delta > 0$

$$\begin{cases} w \geq 0 & \partial B_r(x_0) \\ w(x_0) < 0 & x_0 \in B_r(x_0) \end{cases}$$

Now we use the important fact of Proposition 8.4.2. We claim for any $x \in B_r(x_0)$, there exists a convex parabola P_x that touches w from above in the ball $B_r(x)$. Why?



Q Why can we touch w from above by a convex parabola of opening K at each $x \in B_r(x_0)$?

using v_ε is lower ε-envelope, u^ε is upper ε-envelope.

$\forall x \in B_r(x_0)$, $\exists$ x_1^*, x_2^* s.t.

$v_\varepsilon(x) = v(x_1^*) - \varepsilon + \frac{1}{\varepsilon}|x_1^* - x|^2$

$u^\varepsilon(x) = u(x_2^*) + \varepsilon - \frac{1}{\varepsilon}|x_2^* - x|^2$

Define $P_v(y) := v(x_1^*) - \varepsilon + \frac{1}{\varepsilon}|x_1^* - y|^2$

$P_u(y) := u(x_2^*) + \varepsilon - \frac{1}{\varepsilon}|x_2^* - y|^2$.

We claim $P_v - P_u + P + \delta|y - x_0|^2 - \delta r^2$ touches w from above in $B_r(x)$.

- $w(y) = v_\varepsilon(y) - u^\varepsilon(y) + P(y) + \delta|y - x_0|^2 - \delta r^2$

$\inf\{v(z) - \varepsilon + \frac{1}{\varepsilon}|z-y|^2 \mid z \in H_1\} - \sup\{u(z) + \varepsilon - \frac{1}{\varepsilon}|z-y|^2 \mid z \in H_1\}$

$\leq v(x_1^*) - \varepsilon + \frac{1}{\varepsilon}|x_1^* - y|^2 = P_v(y)$ $\quad\leq -(u(x_2^*) + \varepsilon - \frac{1}{\varepsilon}|x_2^* - y|^2) = -P_u(y)$

- $w(x) = v_\varepsilon(x) - u^\varepsilon(x) + P(x) + \delta|x - x_0|^2 - \delta r^2$

$= P_v(x) - P_u(x) + P(x) + \delta|x - x_0|^2 - \delta r^2$ By construction.

Figure 8.21: Why one can touch w from above by convex parabola at each $x \in B_r(x_0)$

Since v_ε is lower ε-envelope and u^ε is upper ε-envelope, for given $x \in B_r(x_0)$ fixed, there exists x_1^*, $x_2^* \in \overline{H}$ (since the envelopes are defined in H) s.t.

$$v_\varepsilon(x) = v(x_1^*) - \varepsilon + \frac{1}{\varepsilon}|x_1^* - x|^2$$
$$u^\varepsilon(x) = u(x_2^*) + \varepsilon - \frac{1}{\varepsilon}|x_2^* - x|^2$$

Using x we've fixed x_1^*, x_2^*, and then using definition of the envelopes we define

$$P_v(y) := v(x_1^*) - \varepsilon + \frac{1}{\varepsilon}|x_1^* - y|^2$$
$$P_u(y) := u(x_2^*) + \varepsilon - \frac{1}{\varepsilon}|x_2^* - y|^2$$

so that

$$v_\varepsilon(y) = \inf\{v(z) - \varepsilon + \frac{1}{\varepsilon}|z-y|^2 \mid z \in \overline{H}\} \leq v(x_1^*) - \varepsilon + \frac{1}{\varepsilon}|x_1^* - y|^2 = P_v(y)$$
$$u^\varepsilon(y) = \sup\{u(z) + \varepsilon - \frac{1}{\varepsilon}|z-y|^2 \mid z \in \overline{H}\} \geq u(x_2^*) + \varepsilon - \frac{1}{\varepsilon}|x_2^* - y|^2 = P_u(y)$$

Thus upon defining

$$P_x(y) := P_v(y) - P_u(y) + P(y) + \delta|y - x_0|^2 - \delta r^2$$

one check for r sufficiently small depending on H_1, H

$$\begin{aligned} w(y) &= P(y) + \delta|y-x_0|^2 - \delta r^2 - (u^\varepsilon(y) - v_\varepsilon(y)) \\ &\leq P(y) + P_v(y) - P_u(y) + \delta|y-x_0|^2 - \delta r^2 = P_x(y) \qquad \forall\, y \in B_r(x) \\ w(x) &= P_x(x) \end{aligned}$$

Thus P_x touches w from above at x in $B_r(x)$.

In particular, since D^2P_x only depends on D^2P and ε, it is uniform in y. Thus there exists K universal constant s.t.

$$\overline{\Theta}(w, B_r(x))(x) \leq K \qquad \forall\, x \in B_r(x_0)$$

Ah! But now using this, one meets the assumption (8.27) for ABP Lemma 8.2.4. Therefore apply the estimate in $B_r(x_0)$ gives

$$\sup_{B_r(x_0))} w^- \leq C(n,r)\left(\int_{B_r(x_0)\cap\{w=\Gamma_w\}} \det(D^2\Gamma_u)\right)^{\frac{1}{n}}$$

In particular, since there exists point x_0 s.t. $w(x_0) < 0$ is strict

$$0 < \int_{B_r(x_0)\cap\{w=\Gamma_w\}} \det(D^2\Gamma_u) \tag{8.74}$$

On the other hand, since u^ε and v_ε are a.e. punctually second order differentiable in $B_r(x_0)$ via Proposition 8.4.2, and since u^ε, v_ε remain viscosity sub/supersolutions (8.72), there exists $A \subseteq B_r(x_0)$ s.t. $|B_r(x_0)\backslash A| = 0$

$$\mathcal{F}(D^2u^\varepsilon(x)) \geq 0 \qquad \forall\, x \in A \tag{8.75}$$
$$\mathcal{F}(D^2v_\varepsilon(x)) \leq 0 \qquad \forall\, x \in A \tag{8.76}$$

Using pointwise information $x_1 \in \{w = \Gamma_w\} \cap A$.

Also, since Γ_w is convex and $\Gamma_w \leq w$, and now w is punctually second order differentiable in A, we get

$$D^2w(x) \geq 0 \qquad \forall\, x \in A \cap \{w = \Gamma_w\} \tag{8.77}$$

Now what about this set? Thanks to our ABP result (8.74) and $w(x_0) < 0$, the set can't have measure zero

$$|A \cap \{w = \Gamma_w\}| \neq 0$$

Thus there must exist some point $x_1 \in A \cap \{w = \Gamma_w\}$ s.t. one can use all pointwise information above. In particular (note D^2P at all points agree)

$$\begin{aligned} 0 &\overset{(8.75)}{\leq} \mathcal{F}(D^2u^\varepsilon(x_1)) \overset{\text{definition of } w}{=} \mathcal{F}(D^2v_\varepsilon(x_1) + D^2P - D^2w(x_1) + 2\delta I) \\ &\overset{(8.77)}{\leq} \mathcal{F}(D^2v_\varepsilon(x_1) + D^2P + 2\delta I) \qquad \text{and using } \mathcal{F} \text{ elliptic} \\ &\leq \mathcal{F}(D^2v_\varepsilon(x_1) + D^2P) + 2\Lambda\delta \qquad \text{using } \mathcal{F} \text{ uniformly elliptic} \\ &\leq \mathcal{F}(D^2v_\varepsilon(x_1)) + \Lambda\left\|(D^2P)^+\right\| - \lambda\left\|(D^2P)^-\right\| + 2\Lambda\delta \qquad \text{using } \mathcal{F} \text{ uniformly elliptic} \\ &\overset{(8.77)}{\leq} \Lambda\left\|(D^2P)^+\right\| - \lambda\left\|(D^2P)^-\right\| + 2\Lambda\delta \\ &\overset{(8.15)}{\leq} \mathcal{M}^+(D^2P, \frac{\lambda}{n}, \Lambda) + 2\Lambda\delta \end{aligned}$$

Send $\delta \to 0$ to conclude. Note a key here is one cannot use any subadditivity of $\mathcal{F}$! □

Uniqueness for Dirichlet Problem

Theorem 8.4.2 ([CC95] Corollary 5.4)**.** *Let $\mathcal{F}$ be uniformly elliptic, $\varphi \in C(\partial\Omega)$. The Dirichlet Problem*

$$\begin{cases} \mathcal{F}(D^2u) = 0 & \Omega \\ u = \varphi & \partial\Omega \end{cases}$$

has at most one viscosity solution $u \in C(\overline{\Omega})$.

Proof. Assume not, say there exists u_1, $u_2 \in C(\overline{\Omega})$ viscosity solutions. Then using Theorem 8.4.1

$$u_1 - u_2 \in \mathcal{S}(\frac{\lambda}{n}, \Lambda, 0) \qquad \Omega$$

But $u_1 - u_2 = 0$ on $\partial\Omega$. Now using Weak Maximum Principle Corollary 8.2.2 on both sides

$$u_1 = u_2 \qquad \Omega$$

□

Notice if the Dirichlet Problem already has a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$, then automatically uniqueness follows (so no need for Jensen Approximation). Indeed if v is another viscosity solution, applying Proposition 8.1.5

$$v - u \in \mathcal{S}(\frac{\lambda}{n}, \Lambda, 0)$$

Now Weak Maximum Principle Corollary 8.2.2 again concludes uniqueness.

8.4.3 $C^{1,\alpha}$ Regularity for $\mathcal{F}(D^2u) = 0$

We give $C^{1,\alpha}$ estimates for solutions of

$$\mathcal{F}(D^2u) = 0$$

Translational Difference in Sym(n) For any $h > 0$ denote

$$\Omega_h = \{x \in \Omega \mid d(x, \partial\Omega) > h\}$$

Proposition 8.4.4 ([CC95] Proposition 5.5)**.** *Let u be viscosity solution*

$$\mathcal{F}(D^2u) = 0 \qquad \Omega$$

Then for any $e \in \mathbb{S}^{n-1}$, $h > 0$

$$u(x + he) - u(x) \in \mathcal{S}(\frac{\lambda}{n}, \Lambda) \qquad \Omega_h \tag{8.78}$$

Proof. First note translations $v(x) = u(x + he)$ remain solutions to $\mathcal{F}(D^2v) = 0$. Now apply Theorem 8.4.1 from both directions. □

Difference Quotient Estimate In the following we discuss a technical Lemma, heuristically saying a uniform C^α bound in β-difference quotients improve the regularity to $C^{\alpha+\beta}$.

Lemma 8.4.2 ([CC95] Lemma 5.6)**.** *Let $0 < \alpha < 1$, $0 < \beta \leq 1$ and $K > 0$. Assume*

$$u \in L^\infty([-1, 1]), \qquad \|u\|_{L^\infty([-1,1])} \leq K$$

Define for $h \in \mathbb{R}$, $0 < |h| \leq 1$ the β-difference quotient

$$v_{\beta,h}(x) := \frac{u(x+h) - u(x)}{|h|^\beta}, \qquad \forall\ x \in I_h := \begin{cases} [-1, 1-h] & h > 0 \\ [-1-h, 1] & h < 0 \end{cases} \tag{8.79}$$

Assume that the β-difference quotient satisfies $C^{0,\alpha}$ uniform bound

$$v_{\beta,h} \in C^{0,\alpha}(I_h), \qquad \|v_{\beta,h}\|_{C^{0,\alpha}(I_h)} \leq K \qquad \forall\ 0 < |h| \leq 1 \tag{8.80}$$

Then there exists $C = C(\alpha + \beta) > 0$ constants s.t.

1. *If $\alpha + \beta < 1$*

$$u \in C^{0,\alpha+\beta}[-1, 1], \qquad \|u\|_{C^{0,\alpha+\beta}[-1,1]} \leq CK$$

2. *If* $\alpha+\beta\geq 1$

$$u\in C^{0,1}[-1,1],\qquad \|u\|_{C^{0,1}[-1,1]}\leq CK$$

Proof. **Define the technical setup**. We bound

$$|u(x+\varepsilon)-u(x)|\qquad \forall\ -1\leq x\leq 0,\ \varepsilon>0,\qquad x+\varepsilon\leq 1$$

In particular, for x and ε fixed, we choose $i\geq 0$ integer s.t.

$$x+2^i\varepsilon\leq 1<x+2^{i+1}\varepsilon$$

and define $\tau_0=2^i\varepsilon$. Now

$$-1\leq x\leq x+\tau_0\leq 1$$

and

$$\frac{1}{2}\leq\tau_0\leq 2$$

This is due to

$$\begin{aligned} 2^i\varepsilon-1\leq x+2^i\varepsilon\leq 1 &\qquad 1<x+2^{i+1}\varepsilon\leq 2\cdot 2^i\varepsilon\\ \tau_0\leq 2 &\qquad \frac{1}{2}\leq\tau_0\end{aligned}$$

Now for this τ_0 fixed by our choice of ε, i, we define

$$w(\tau):=u(x+\tau)-u(x),\qquad \forall\ 0<\tau\leq\tau_0$$

The precise goal is to bound $w(\varepsilon)$ via constant times ε up to the correct power.

Use our assumption $C^{0,\alpha}$ bound on β-difference quotient. Note for any $0<\tau\leq\tau_0$, one transform the discrete second difference quotient into difference of β-difference quotients, on which we can apply $C^{0,\alpha}$ uniform bound

$$\begin{aligned}|w(\tau)-2w(\frac{\tau}{2})| &= |u(x+\tau)-u(x)-2(u(x+\frac{\tau}{2})-u(x))| = |u(x+\tau)-2u(x+\frac{\tau}{2})+u(x)|\\ &= (\frac{\tau}{2})^\beta\Big|\frac{1}{(\frac{\tau}{2})^\beta}\Big(u(x+\tau)-u(x+\frac{\tau}{2})\Big)-\frac{1}{(\frac{\tau}{2})^\beta}\Big(u(x+\frac{\tau}{2})-u(x)\Big)\Big|\\ &\overset{(8.79)}{=} (\frac{\tau}{2})^\beta|v_{\beta,\frac{\tau}{2}}(x+\frac{\tau}{2})-v_{\beta,\frac{\tau}{2}}(x)|\\ &\overset{(8.80)}{\leq} K(\frac{\tau}{2})^\beta(\frac{\tau}{2})^\alpha\leq CK\tau^{\alpha+\beta}\end{aligned}\tag{8.81}$$

Iterate. One may iterate (8.81) from τ_0 all the way down to $\frac{\tau_0}{2^i}=\varepsilon$. For universal $C=C(\alpha+\beta)$

$$\begin{aligned}|w(\tau_0)-2w(\frac{\tau_0}{2})| &\leq CK\tau_0^{\alpha+\beta}\\ |w(\frac{\tau_0}{2})-2w(\frac{\tau_0}{2^2})| &\leq CK(\frac{\tau_0}{2})^{\alpha+\beta}\\ \cdots|w(\frac{\tau_0}{2^{i-1}})-2w(\frac{\tau_0}{2^i})| &\leq CK(\frac{\tau_0}{2^{i-1}})^{\alpha+\beta}\end{aligned}$$

The information we care are w at τ_0 (we have $\frac{1}{2}\leq\tau_0\leq 2$) and the $w(\varepsilon)=w(\frac{\tau_0}{2^i})$ that one want to control. We notice

$$|2^{j-1}w(\frac{\tau_0}{2^{j-1}})-2^jw(\frac{\tau_0}{2^j})|\leq CK2^{j-1}(\frac{\tau_0}{2^{j-1}})^{\alpha+\beta}=CK2^{(j-1)(1-(\alpha+\beta))}\tau_0^{\alpha+\beta}\qquad\forall\ 1\leq j\leq i\tag{8.82}$$

Now one may sum up

$$\begin{aligned}|w(\tau_0)-2^iw(\varepsilon)| = |w(\tau_0)-2^iw(\frac{\tau_0}{2^i})| &\leq\sum_{j=1}^i|2^{j-1}w(\frac{\tau_0}{2^{j-1}})-2^jw(\frac{\tau_0}{2^j})|\\ &\overset{(8.82)}{\leq}\sum_{j=1}^i CK2^{(j-1)(1-(\alpha+\beta))}\tau_0^{\alpha+\beta}\\ &= CK\sum_{j=1}^i 2^{(j-1)(1-(\alpha+\beta))}\tau_0^{\alpha+\beta}\end{aligned}$$

Finally

$$\begin{aligned}
|w(\varepsilon)| &\le 2^{-i}|w(\tau_0)| + 2^{-i}|2^i w(\varepsilon) - w(\tau_0)| \\
&\le \tau_0^{-1}\varepsilon 2K + \tau_0^{-1}\varepsilon CK \sum_{j=1}^{i} 2^{(j-1)(1-(\alpha+\beta))}\tau_0^{\alpha+\beta} \qquad \text{using } 2^{-i} = \tau_0^{-1}\varepsilon \\
&\le 4K\varepsilon + 4CK\varepsilon \sum_{j=0}^{i-1} 2^{j(1-(\alpha+\beta))}\tau_0^{\alpha+\beta-1} \qquad \text{using } \tau_0 \ge 1/2 \text{ for only the first term}
\end{aligned}$$

If $\alpha+\beta<1$, then the sum writes

$$\sum_{j=0}^{i-1} 2^{j(1-(\alpha+\beta))} = \frac{1-(\varepsilon^{-1}\tau_0)^{1-(\alpha+\beta)}}{1-2^{1-(\alpha+\beta)}} = \frac{1}{2^{1-(\alpha+\beta)}-1}\left((\varepsilon^{-1}\tau_0)^{1-(\alpha+\beta)} - 1\right)$$

so that

$$\sum_{j=0}^{i-1} 2^{j(1-(\alpha+\beta))}\tau_0^{\alpha+\beta-1} \le \frac{1}{2^{1-(\alpha+\beta)}-1}\left(\varepsilon^{(\alpha+\beta)-1} - \tau_0^{\alpha+\beta-1}\right)$$

and $\varepsilon^{\alpha+\beta}$ dominates

$$|w(\varepsilon)| \le 4C_1K\varepsilon + 4C_2K\varepsilon^{\alpha+\beta} \le C_3K\varepsilon^{\alpha+\beta}$$

If on the other hand, $\alpha+\beta\ge 1$, then simply using $\tau_0 \le 2$

$$|w(\varepsilon)| \le C_4K\varepsilon$$

□

$C^{1,\alpha}$ **Estimate** Using our previous result on $C^{0,\alpha}$ Regularity for $\mathcal{S}^*$ Proposition 8.3.1 and the previous Lemma 8.4.2, one may improve to $C^{1,\alpha}$ Regularity.

The key step, is again, that translations, hence difference quotients are solutions to $\mathcal{S}$. Once we have this, $C^{0,\alpha}$ applies to β-difference quotients, whereas $\|u\|_{C^{0,\alpha}} \le CK$ because $u\in\mathcal{S}$ itself. Now apply to $\beta=\alpha$, one get a chain $\|u\|_{C^{k\alpha}} \le CK$. Consider the termination condition, and $C^{1,\alpha}$ is the best one can hit for under this machinery.

Corollary 8.4.1 ([CC95] Corollary 5.7)**.** *Let u be viscosity solution to $\mathcal{F}(D^2u)=0$ in B_1.*
Then $u\in C^{1,\alpha}(\overline{B_{1/2}})$ for $0<\alpha<1$, $C>0$ universal, and

$$\|u\|_{C^{1,\alpha}(\overline{B_{1/2}})} \le C\left(\|u\|_{L^\infty(B_1)} + |\mathcal{F}(0)|\right) \tag{8.83}$$

Proof. **Use** (8.78) **for setup**. Fix any $e\in\mathbb{S}^{n-1}$ and $0<h<1/8$. Now, using zero force, for any $\beta\in(0,1]$ (to choose later!)

$$v_\beta(x) := \frac{1}{h^\beta}\left(u(x+he)-u(x)\right) \in \mathcal{S}(\frac{\lambda}{n},\Lambda) \qquad B_{7/8} \tag{8.84}$$

Question: How to use this fact? Recall one already obtained $C^{0,\alpha}$ interior estimates for solution in $\mathcal{S}^*$ (8.65), where $\alpha\in(0,1)$ is universal. Apply the result to v_β to get

$$\|v_\beta\|_{C^{0,\alpha}(\overline{B_r})} \le C(s,r)\,\|v_\beta\|_{L^\infty(B_{\frac{r+s}{2}})} \le C(r,s)\,\|u\|_{C^\beta(\overline{B_s})} \tag{8.85}$$

For any $r<s\le 7/8$, and $0<h<\frac{s-r}{2}$. Note for $\beta=1$ it is $C^{0,1}$ norm on the RHS. In particular, the numbers r, s are chosen so that

$$r+h < r+\frac{s-r}{2} = \frac{s+r}{2}, \qquad \frac{s+r}{2}+h < \frac{s+r}{2}+\frac{s-r}{2} = s \le 7/8$$

This is of the form $C^{0,\alpha}$ estimate for β-difference quotient. We're so tempted to use our Lemma 8.4.2, but we can't for now, since the RHS u defines v_β. The Question is: How to make the RHS universal?

Base Step on C^α. We want to initiate an iteration using (8.85). The main reason is that, for u a solution to $\mathcal{F}(D^2u)=0$, from Proposition 8.1.5 we know

$$u\in\mathcal{S}(\frac{\lambda}{n},\Lambda,-F(0))$$

Now u itself is applicable for (8.65). One obtain

$$\|u\|_{C^{0,\alpha}(\overline{B_{7/8}})} \leq C\left(\|u\|_{L^\infty(B_1)} + |\mathcal{F}(0)|\right) =: CK \tag{8.86}$$

Thus choosing the β above as α, one may replace RHS in (8.85) via LHS of (8.86). Let $r = r_1 < s = 7/8$ so the estimate writes

$$\|v_\alpha\|_{C^{0,\alpha}(\overline{B_{r_1}})} \leq C(r_1)\,\|u\|_{C^{0,\alpha}(\overline{B_{7/8}})} \leq C(r_1)K \tag{8.87}$$

for $0 < h < \frac{7/8 - r_1}{2}$.

Induction on $C^{k\alpha}$. Now, thanks to our previous $C^{0,\alpha}$ result! The estimate (8.87) establishes $C^{0,\alpha}$ estimate on α-difference quotient of v, and most importantly, it is uniform in h and e.

Thus applying Lemma 8.4.2 gives

$$\|u\|_{C^{0,2\alpha}(\overline{B_{r_2}})} \leq C(r_1, r_2)K \qquad r_2 < r_1$$

Keep iterating one get $u \in C^{0,3\alpha}(\overline{B_{r_3}})$. Now, consider the universal integer i s.t. $i\alpha < 1$ but $(i+1)\alpha \geq 1$. We keep doing the above until $(i+1)\alpha$ is achieved so that

$$\|u\|_{C^{0,1}(\overline{B_{3/4}})} \leq CK$$

Here we may choose $3/4$ universal since we didn't pick $r_2, r_3, \cdots$, and now we do.

Termination. Now we want to use (8.85) for $\beta = 1$. This is doable now, because we already established our Lipschitz bound on u. Thus

$$\|v_1\|_{C^{0,\alpha}(\overline{B_{1/2}})} \leq C\,\|u\|_{C^{0,1}(\overline{B_{3/4}})} \leq CK \qquad \forall\ e \in \mathbb{S}^{n-1}, \qquad 0 < h < 1/8$$

Since v_1 is difference quotient of u for h and e, we conclude $u \in C^{1,\alpha}(\overline{B_{1/2}})$. In particular

$$\|u\|_{C^{1,\alpha}(\overline{B_{1/2}})} \leq CK$$

□

8.4.4 Application to Concave Equations

Definition 8.4.1 (Concave)**.** *The operator*

$$\mathcal{F} : \mathrm{Sym}(n) \to \mathbb{R}$$

is concave if $\mathcal{F}$ is concave in the space of symmetric matrices, i.e.,

$$\mathcal{F}(\alpha M + (1-\alpha)N) \geq \alpha\mathcal{F}(M) + (1-\alpha)\mathcal{F}(N) \qquad \forall\ M,\, N \in \mathrm{Sym}(n),\ \forall\ 0 \leq \alpha \leq 1$$

Theorem 8.4.3 ([CC95] Theorem 5.8)**.** *Let $\mathcal{F}$ be concave uniformly elliptic operator. Let u, v be viscosity subsolutions in Ω*

$$\mathcal{F}(D^2u) \geq 0, \qquad \mathcal{F}(D^2v) \geq 0 \qquad \Omega$$

Then $\frac{1}{2}(u+v)$ is viscosity subsolution in Ω

$$\mathcal{F}(\frac{1}{2}(D^2u + D^2v)) \geq 0$$

Proof. **It suffices to show for $u^\varepsilon + v^\varepsilon$ remains viscosity subsolution to** $\mathcal{F}(\frac{1}{2}(D^2u^\varepsilon + D^2v^\varepsilon)) \geq 0$. This is again due to uniform convergence, that u^ε and v^ε are viscosity subsolutions in a possibly smaller domain, and $\mathcal{F}$ closed under uniform limits.

Let P be parabola that touches $u^\varepsilon + v^\varepsilon$ from above at x_0, one wish to show that

$$\mathcal{F}(D^2P) \geq 0$$

To do this, again consider perturbing

$$w(x) := P(x) + \delta|x - x_0|^2 - \delta r^2 - \frac{1}{2}(u^\varepsilon(x) + v^\varepsilon(x))$$

and going over the whole ABP machinery, so that there exists $x_1 \in A \cap B_r(x_0)$ where u^ε, v^ε and w are punctually second order differentiable,

$$D^2u^\varepsilon(x_1) \geq 0, \qquad D^2v^\varepsilon(x_1) \geq 0, \qquad D^2w(x_1) \geq 0$$

so that

$$\begin{aligned}0 &\le D^2(P(x) + \delta|x - x_0|^2 - \delta r^2 - \frac{1}{2}(u^\varepsilon(x) + v^\varepsilon(x)))(x_1)\\ &\le D^2P + 2\delta I - D^2(\frac{1}{2}(u^\varepsilon(x_1) + v^\varepsilon(x_1)))\end{aligned}$$

Apply $\mathcal{F}$ to both sides

$$\mathcal{F}(D^2(\frac{1}{2}(u^\varepsilon(x_1) + v^\varepsilon(x_1)))) \le \mathcal{F}(D^2P) + 2\delta\Lambda$$

Now **using concavity of** $\mathcal{F}$

$$\mathcal{F}(D^2(\frac{1}{2}(u^\varepsilon(x_1) + v^\varepsilon(x_1)))) \ge \frac{1}{2}\mathcal{F}(D^2u^\varepsilon(x_1)) + \frac{1}{2}\mathcal{F}(D^2v^\varepsilon(x_1)) \ge 0$$

Thus

$$0 \le \mathcal{F}(D^2P) + 2\delta\Lambda$$

pass $\delta \to 0$ to conclude. □

Corollary 8.4.2 ([CC95] Corollary 5.9)**.** *Let $\mathcal{F}$ be concave and suppose in the viscosity sense*

$$\mathcal{F}(D^2u) = 0 \qquad \Omega$$

Then for any $e \in \mathbb{S}^{n-1}$ and $h > 0$, the second order difference quotient

$$\frac{1}{h^2}(u(x + he) + u(x - he) - 2u(x)) \in \underline{\mathcal{S}}(\frac{\lambda}{n}, \Lambda) \qquad \Omega \tag{8.88}$$

Proof. We rewrite

$$\frac{1}{h^2}(u(x + he) + u(x - he) - 2u(x)) = \frac{1}{h^2}(u(x + he) + u(x - he)) - \frac{2}{h^2}u(x)$$

Since both $v(x) = u(x \pm he)$ are viscosity solutions, from Theorem 8.4.3 $\frac{1}{2}(u(x + he) + u(x - he))$ is viscosity solution to $\mathcal{F}(\cdot) = 0$. Now apply Theorem 8.4.1 with $\frac{1}{2}(u(x + he) + u(x - he)$ subsolution to $\mathcal{F} \ge 0$ and u supersolution $\mathcal{F} \le 0$, we know their difference belongs to $\underline{\mathcal{S}}$. □

Corollary 8.4.3 ([CC95] Corollary 5.10)**.** *Let $\mathcal{F}$ be concave and assume $u \in C^2(\Omega)$ is a solution to*

$$\mathcal{F}(D^2u) = 0 \qquad \Omega$$

Then for any $e \in \mathbb{S}^{n-1}$

$$\partial_{ee}u = \frac{\partial^2}{\partial e \partial e}u \in \underline{\mathcal{S}}(\frac{\lambda}{n}, \Lambda) \qquad \Omega$$

Proof. Use (8.88), and that $u \in C^2$ so one has uniform convergence, and $\underline{\mathcal{S}}$ is closed under uniform convergence. □

8.5 Evans-Krylov: $C^{2,\alpha}$ Interior Estimates for Concave Equations

In this section we give proof for the famous Evans-Krylov Interior $C^{2,\alpha}$ Estimate via $C^{1,1}$ norm.

Theorem 8.5.1 ([CC95] Theorem 6.1)**.** *Let $\mathcal{F}$ be concave uniformly elliptic operator. Let $u \in C^2(B_1)$ solve*

$$\mathcal{F}(D^2u) = 0 \qquad B_1$$

Then $u \in C^{2,\alpha}(\overline{B_{1/2}})$ and there exists $\alpha \in (0,1)$ and $C = C(n, \lambda, \Lambda) > 0$ universal s.t.

$$\|u\|_{C^{2,\alpha}(\overline{B_{1/2}})} \leq C\, \|u\|_{C^{1,1}(\overline{B_{3/4}})} \tag{8.89}$$

8.5.1 Preparation

Reduction to proof of Oscillation Lemma 8.5.1 We claim it suffices to prove

Lemma 8.5.1. *Given assumptions in Theorem 8.5.1. There exists a universal constant $0 < \delta_0 < 1$ s.t. if $\operatorname{diam}(D^2u(B_1)) = 2$, then*

$$\operatorname{diam}(D^2u(B_{\delta_0})) \leq 1 \tag{8.90}$$

We note one may **indeed assume** $\operatorname{diam}(D^2u) = 1$ since for $\mathcal{F}(D^2u) = 0$,

$$\tilde{u} := \frac{1}{t}u$$

solves the equation

$$\tilde{F}(D^2\tilde{u}) := \frac{1}{t}\mathcal{F}(tD^2\tilde{u}) = 0$$

where $\tilde{F}$ have the same elliptic constants as $\mathcal{F}$.

Proof. For P touching $\tilde{u}$ from above at x_0

$$\mathcal{F}(D^2P(x_0)) \geq 0$$

Note tP therefore touches $t\tilde{u} = u$ from above at x_0, thus

$$\tilde{F}(D^2P) = \frac{1}{t}\mathcal{F}(D^2tP(x_0)) \geq 0$$

Hence $\tilde{u}$ solves $\tilde{\mathcal{F}}(D^2\tilde{u}) = 0$. Due to rescaling $\frac{1}{t}$ in the front, $\tilde{\mathcal{F}}$ and $\mathcal{F}$ have same elliptic constants. □

Now take $t := \frac{\operatorname{diam}(D^2u)}{2}$ and we work with $\tilde{\mathcal{F}}$ along with $\tilde{u}$ instead.
We record the important iteration lemma.

Lemma 8.5.2 ([GT01] Lemma 8.23)**.** *Let $w : (0, R_0] \to [0, \infty)$ be non-decreasing function s.t. for $0 < \gamma, \tau < 1$ and σ non-decreasing*

$$w(\tau R) \leq \gamma w(R) + \sigma(R) \qquad \forall\ R \leq R_0 \tag{8.91}$$

Then there exists $C = C(\gamma, \tau) > 0$ and for any $\mu \in (0,1)$, there exists $\alpha = \alpha(\gamma, \tau, \mu) > 0$ s.t.

$$w(R) \leq C\left((\frac{R}{R_0})^{\alpha} w(R_0) + \sigma(R^{\mu}R_0^{1-\mu}) \right) \qquad \forall\ R \leq R_0 \tag{8.92}$$

In our case, we take $R_0 = 1$

$$\begin{aligned} w(R) &:= \operatorname{diam}(D^2u(B_R)) \\ \tau &= \delta_0 \\ \gamma &= \frac{1}{2} \\ \sigma &= 0 \end{aligned}$$

Why can we infer from (8.90) the following fact?

$$\operatorname{diam}(D^2u(B_{\delta_0 R})) \leq \frac{1}{2}\operatorname{diam}(D^2u(B_R)) \qquad \forall\ R \leq 1$$

Proof. By rescaling $u_R(x) := u(Rx)$ for $x \in B_1$, we know u_R solves the equation

$$\mathcal{F}_R(D^2u_R) := R^2\mathcal{F}(\frac{1}{R^2}D^2u_R) = 0 \qquad B_1$$

where $\mathcal{F}_R$ and $\mathcal{F}$ have same elliptic constants. Now that Lemma 8.5.1 is satisfied implies

$$\mathrm{diam}(D^2u_R(B_1)) = 2 = R^2\mathrm{diam}(D^2u(B_R)) \implies \mathrm{diam}(D^2u_R(B_{\delta_0})) = R^2\mathrm{diam}(D^2u(B_{\delta_0 R})) \leq 1$$

which is to say

$$w(\delta_0 R) \leq \frac{1}{2}w(R) \qquad \forall\ R \leq 1$$

□

Now what does the Lemma 8.5.2 infer?

$$\mathrm{diam}(D^2u(B_R)) \leq CR^\alpha \mathrm{diam}(D^2u(B_1)) \qquad \forall\ R \leq 1$$

But this is exactly a bound on $[D^2u]_{C^{0,\alpha}(\overline{B_R})}$, which in turn proves (8.89).

Remark that on the RHS, one need $\mathrm{diam}(D^2u)$, this is essentially $\left\|D^2u\right\|_\infty$, or equivalently, $\|u\|_{C^{1,1}}$.

Proof of Hölder Iteration Lemma We provide proof for the key Lemma that gains our Hölder Regularity.

Proof of Lemma 8.5.2. Fix initially $R_1 \leq R_0$. Now for any $R \leq R_1$

$$w(\tau R) \leq \gamma w(R) + \sigma(R_1)$$

We iterate so for positive m

$$\begin{aligned} w(\tau^m R_1) &\leq \gamma^m w(R_1) + \sigma(R_1)\sum_{i=0}^{m-1}\gamma^i \\ &\leq \gamma^m w(R_0) + \frac{\sigma(R_1)}{1-\gamma} \end{aligned} \tag{8.93}$$

Now for any $R \leq R_1$, choose m s.t.

$$\begin{aligned} \tau^m R_1 &< R \leq \tau^{m-1}R_1 \\ m\log\tau &< \log(\frac{R}{R_1}) \leq (m-1)\log\tau \\ (m-1) &\leq \log_\tau(\frac{R}{R_1}) < m \end{aligned}$$

so that

$$\begin{aligned} w(R) &\leq w(\tau^{m-1}R_1) \\ &\overset{(8.93)}{\leq} \underbrace{\gamma^{m-1}w(R_0)}_{\text{write as } \frac{1}{\gamma}\gamma^m} + \frac{\sigma(R_1)}{1-\gamma} \\ &\leq \frac{1}{\gamma}\gamma^{\log_\tau(\frac{R}{R_1})}w(R_0) + \frac{\sigma(R_1)}{1-\gamma} \\ &= \frac{1}{\gamma}(\frac{R}{R_1})^{\log_\tau(\gamma)}w(R_0) + \frac{\sigma(R_1)}{1-\gamma} \end{aligned}$$

Now let $R_1 = R_0^{1-\mu}R^\mu$ so that, denoting

$$\alpha = (1-\mu)\log_\tau(\gamma) > 0$$

one conclude

$$\begin{aligned} w(R) &\leq \frac{1}{\gamma}(\frac{R}{R_0^{1-\mu}R^\mu})^{\log_\tau(\gamma)}w(R_0) + \frac{\sigma(R_0^{1-\mu}R^\mu)}{1-\gamma} \\ &= \frac{1}{\gamma}(\frac{R}{R_0})^{(1-\mu)\log_\tau(\gamma)}w(R_0) + \frac{\sigma(R_0^{1-\mu}R^\mu)}{1-\gamma} \end{aligned}$$

□

Tool for Proof of Lemma 8.5.1: Weak Harnack We record a power decay of the distribution function of supersolutions, which follows from the weak Harnack Inequality Lemma 8.3.4 directly upon rescaling.

Lemma 8.5.3 ([CC95] Lemma 6.3)**.** *Let $v \in \overline{\mathcal{S}}(\lambda, \Lambda, 0)$ in B_1 and $v \geq 0$ on B_1. Then there exists C, $\delta > 0$ universal constants s.t.*

$$C|\{v \geq 1\} \cap B_{1/4}|^{\delta} \leq \inf_{B_{1/2}} v \tag{8.94}$$

Tool for Proof of Lemma 8.5.1: Positive and Negative parts for matrices in the same level set are comparable We also record an important feature of uniform ellipticity. Here we do not need concavity of $\mathcal{F}$.

Lemma 8.5.4 ([CC95] Lemma 6.4)**.** *Let $\mathcal{F}$ be uniformly elliptic. There exists $c_0 = \frac{\lambda}{\Lambda+\lambda}$ constant that only depends on the ellipticity constants s.t. for any $F(M_1) = F(M_2)$, one has*

$$c_0 \|M_2 - M_1\| \leq \left\|(M_2 - M_1)^+\right\| \tag{8.95}$$

$$= \sup_{\substack{|e|=1 \\ e\in\mathbb{R}^n}} (e^T(M_2 - M_1)e)^+ \tag{8.96}$$

Proof. We compute

$$\begin{aligned}
0 = \mathcal{F}(M_2) - \mathcal{F}(M_1) &\overset{(8.10)}{\leq} \Lambda \left\|(M_2 - M_1)^+\right\| - \lambda \left\|(M_2 - M_1)^-\right\| \\
&\leq (\Lambda + \lambda) \left\|(M_2 - M_1)^+\right\| - \lambda \left(\left\|(M_2 - M_1)^+\right\| + \left\|(M_2 - M_1)^-\right\|\right) \\
&\leq (\Lambda + \lambda) \left\|(M_2 - M_1)^+\right\| - \lambda \|M_2 - M_1\| \\
c_0 \|M_2 - M_1\| &\leq \left\|(M_2 - M_1)^+\right\|
\end{aligned}$$

Now for $M_2 - M_1 \in \mathrm{Sym}(n)$, one may spectral decompose

$$M_2 - M_1 = ODO^T$$

for D diagonal matrix and O orthogonal matrix. Now $(M_2 - M_1)^+$ as the positive part of eigenvalues corresponds to positive diagonal entries of D, thus using operator norm is measured by the supremum among the eigenvalues

$$\left\|(M_2 - M_1)^+\right\| = \sup_{|e|=1} (e^T De)^+ = \sup_{|e|=1} (e^T(M_2 - M_1)e)^+$$

□

8.5.2 Key Lemma: Universal Reduction on number of covering balls

Lemma 8.5.5 ([CC95] Lemma 6.5)**.** *Given assumptions in Theorem 8.5.1. There exists $\varepsilon_0 > 0$ universal s.t. if*

$$1 < \mathrm{diam}(D^2u(B_1)) \leq 2$$

and for some $\varepsilon < \varepsilon_0$, assume $D^2u(B_1) \subseteq \mathrm{Sym}(n)$ is covered by $N \geq 1$ balls B^i

$$D^2u(B_1) \subseteq \bigcup_{i=1}^{N} B^i \qquad \text{of radius } \varepsilon \leq \varepsilon_0 \text{ in } \mathrm{Sym}(n)$$

Then $D^2u(B_{1/2})$ is covered by $N-1$ balls among $B^1, \cdots, B^N$.

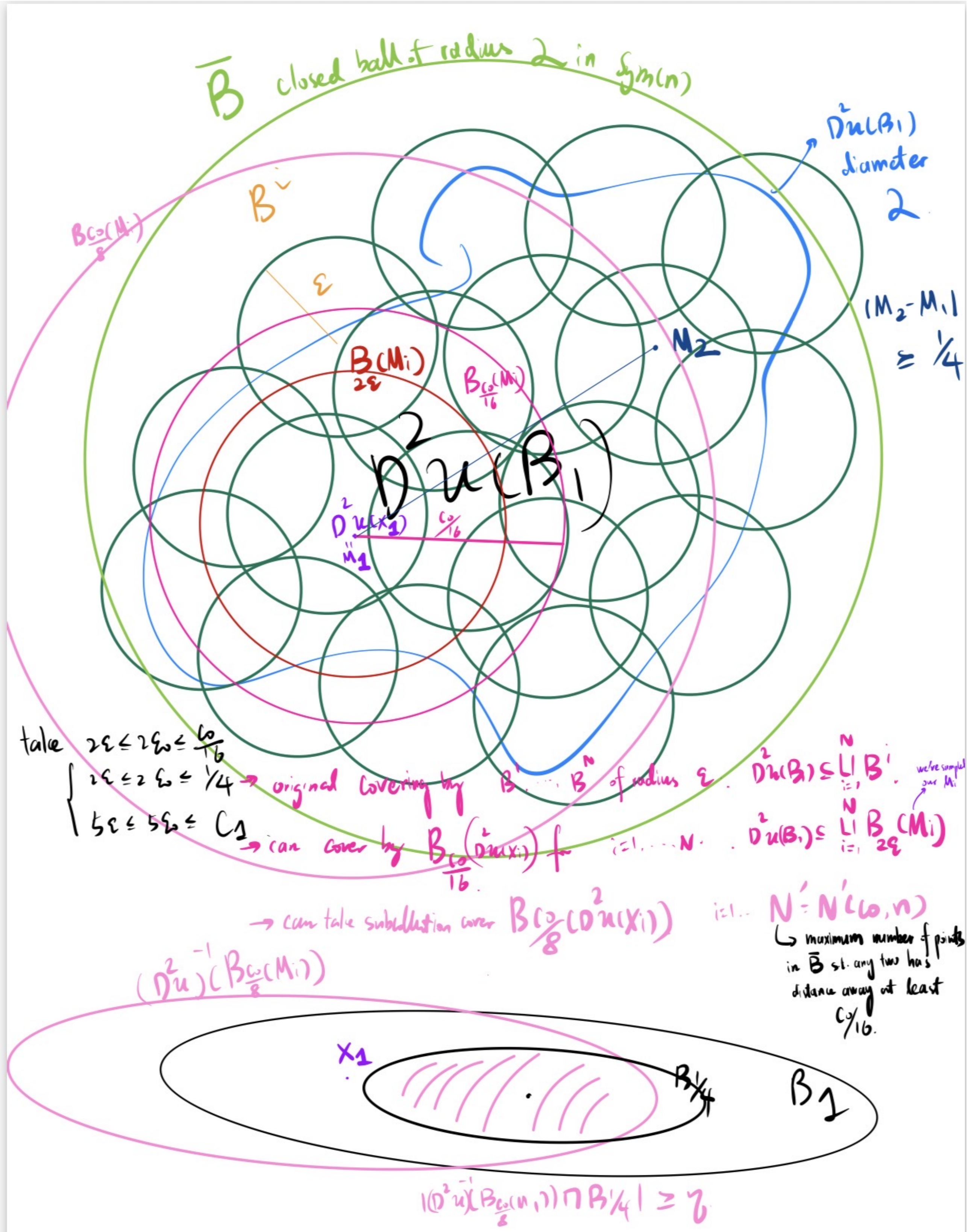


Figure 8.22: Overview of Lemma 8.5.5

Proof. **Setup**. Choose $x_i \in B_1$ for $i = 1, \cdots, N$ and denote $M_i := D^2u(x_i)$, so that

$$B^i \subseteq B_{2\varepsilon}(M_i) \qquad i = 1, \cdots, N$$

From now on one work with the covering

$$D^2u(B_1) \subseteq \bigcup_{i=1}^{N} B_{2\varepsilon}(M_i)$$

Setup Start with $D^2u(B_1) \subseteq \bigsqcup_{i=1}^{N} B^i$.

- Pick $X_i \in B_1$, $1 \le i \le N$, denote $M_i := D^2u(X_i)$

  s.t. $B^i \subseteq B_{2\varepsilon}(M_i)$ $\forall\, i = 1, \dots, N$

notice. $D^2u(B_1) \subseteq \bigsqcup_{i=1}^{N} B^i \subseteq \bigsqcup_{i=1}^{N} B_{2\varepsilon}(M_i)$ remains a cover!

- let $2\varepsilon < 2\varepsilon_0 \le \frac{c_0}{16}$ so $D^2u(B_1) \subseteq \bigsqcup_{i=1}^{N} B_{\frac{c_0}{16}}(M_i)$

  "c_0" for later to match with key tool ① and key setup computation

- 

  radius 2. let $N' = N'(n, c_0)$ be maximum number of points in $\bar{B}$ that has distance at least $\frac{c_0}{16}$ away from each other

  How to understand this N'? → universal b/c $\bar{B}$ radius 2 universal

Claim Covering reduced! $D^2u(B_1) \subseteq \bigsqcup_{i=1}^{N'} B_{\frac{c_0}{8}}(M_i)$ upon relabelling

using original covering. $D^2u(B_1) \subseteq \bigsqcup_{i=1}^{N} B_{\frac{c_0}{16}}(M_i)$

this mean $\forall\, M_i\ \exists\, M_j$ s.t. $\|M_i - M_j\| < \frac{c_0}{8}$.

then we sort of want to keep "M_j" s.t. are $\frac{c_0}{16}$ away from each other

Once $\|M_i - M_j\| < \frac{c_0}{16}$ then once we replace radius with $2 \cdot \frac{c_0}{16} = \frac{c_0}{8}$ for one. this completely covers the other.

So the algorithm goes: once $\|M_i - M_j\| < \frac{c_0}{16}$, one of them must be removed.

But want to keep as many left

we select the ones based on $\|M_i - M_j\| \ge \frac{c_0}{16}$.

$\Rightarrow$ N' many left at most.

↳ important: this number is universal

- $\exists\, M_i$, say M_1 s.t. key setup

$$\left|(D^2u)^{-1}\left(B_{\frac{c_0}{8}}(M_1)\right) \cap B_{1/4}\right| \ge \eta > 0$$

η universal b/c N' universal.

We choose

$$2\varepsilon < 2\varepsilon_0 \le \frac{c_0}{16}$$

where $c_0 = \frac{\lambda}{\Lambda+\lambda}$ is the constant in Lemma 8.5.4.

In the following we want to extract a covering with a universal amount of elements! This is quite difficult. First cover

$$D^2u(B_1) \subseteq \bigcup_{i=1}^{N} B_{\frac{c_0}{16}}(M_i)$$

Notice $\operatorname{diam}(D^2u) \le 2$ means all points $\{M_j\} \subseteq \overline{B}$ where $\overline{B}$ is a ball of radius 2 in the space $\operatorname{Sym}(n)$. Now let

$$N' := \text{Maximum number of points in } \overline{B} \text{ s.t. each point has distance at least } \frac{c_0}{16} \text{ away from each other}$$

By selecting $N' = N'(n, c_0)$ many M_j among the original ones, one may ensure, upon relabeling and doubling the radius

$$D^2u(B_1) \subseteq \bigcup_{i=1}^{N'} B_{\frac{c_0}{8}}(M_i)$$

Now such N' is universal, and one may pick M_1 and $\eta = \eta(N', n) > 0$ s.t. the preimage has nontrivial measure overlapping $B_{1/4}$ is the domain

$$|(D^2u)^{-1}\left(B_{\frac{c_0}{8}}(M_1)\right) \cap B_{1/4}| \ge \eta > 0 \tag{8.97}$$

Key Tool 1.

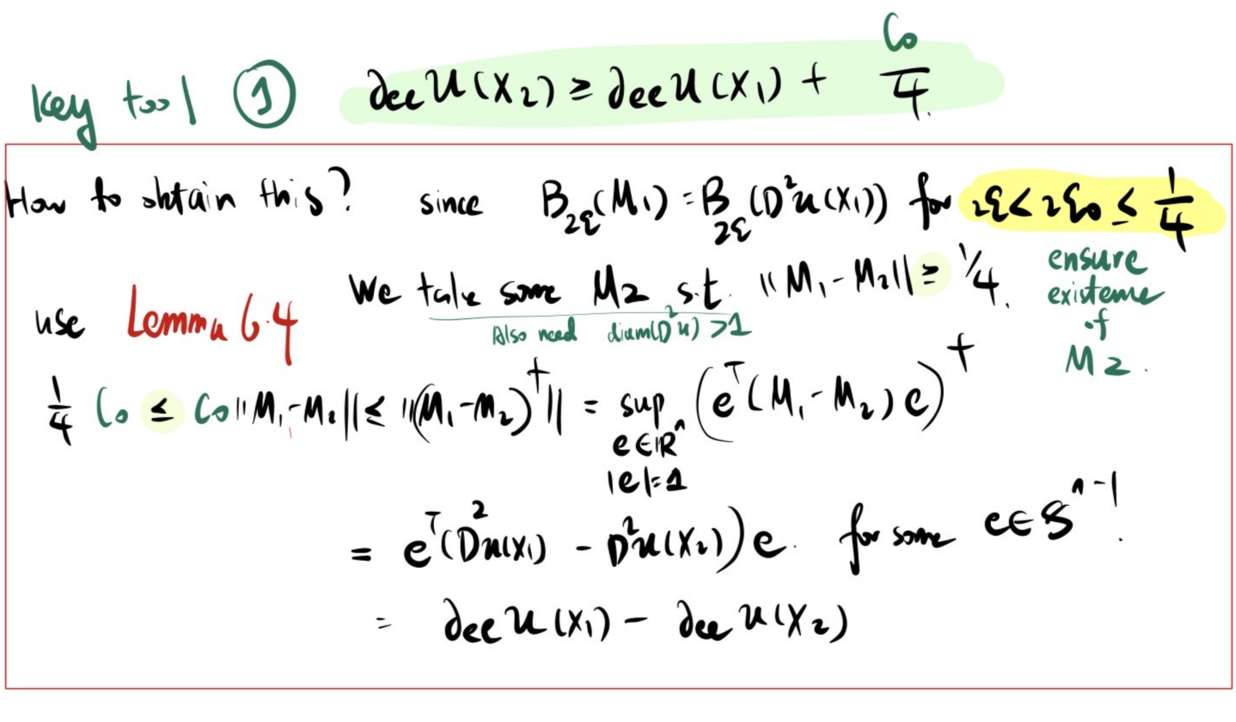


Figure 8.24: Lemma 8.5.5 Key Tool 1

We further ensure

$$2\varepsilon < 2\varepsilon_0 \le \frac{1}{4}$$

Using $\operatorname{diam}(D^2u) > 1$, we know for M_1 given as in (8.97), there exists M_2 among our collection s.t.

$$\|M_1 - M_2\| \ge \frac{1}{4} > 2\varepsilon$$

Now using Lemma 8.5.4

$$\frac{1}{4}c_0 \le c_0 \left\|M_1 - M_2\right\| \overset{(8.95)}{\le} \left\|(M_1 - M_2)^+\right\| \overset{(8.96)}{=} \sup_{|e|=1, e\in\mathbb{R}^n} (e^T(M_1 - M_2)e)^+$$
$$= \sup_{|e|=1, e\in\mathbb{R}^n} (e^T(D^2u(x_1) - D^2u(x_2))e)^+$$
$$\le \partial_{ee}u(x_1) - \partial_{ee}u(x_2) \qquad \text{for some } e \in \mathbb{S}^{n-1}$$

where we used definition for $M_2 = D^2u(x_2)$. Thus

$$\partial_{ee}u(x_2) \ge \partial_{ee}u(x_1) + \frac{c_0}{4} \tag{8.98}$$

Key Tool 2.
Denote $K := \sup_{B_1}\partial_{ee}u$. Then let

$$v := K - \partial_{ee}u$$

Recall from Corollary 8.4.3 that for u a solutions to

$$\mathcal{F}(D^2u) = 0 \qquad B_1$$

while $\mathcal{F}$ is concave, we know $\partial_{ee}u \in \underline{\mathcal{S}}(\frac{\lambda}{n}, \Lambda, 0)$. Thus $v \in \overline{\mathcal{S}}(\frac{\lambda}{n}, \Lambda, 0)$ is supersolution. Also notice

$$v \ge 0 \qquad B_1$$

We're so tempted to apply Weak Harnack to v! In fact here we directly can, but let's see why we want to do so.

Recall from (8.97) we have the universal bound for the set.

$$S := (D^2u)^{-1}\left(B_{\frac{c_0}{8}}(M_1)\right) \cap B_{1/4} = \{y \in B_{1/4} \mid |D^2u(y) - D^2u(x_1)| < \frac{c_0}{8}\}$$

Let's study which set contains this.

Notice we already knew

$$\sup_{B_1}\partial_{ee}u - \partial_{ee}u(x_1) \ge \partial_{ee}u(x_2) - \partial_{ee}u(x_1) \overset{(8.98)}{\ge} \frac{c_0}{4}$$

Hence focusing on the set S above, for any $y \in S$

$$v(y) = \sup_{B_1}\partial_{ee}u - \partial_{ee}u(y) = \sup_{B_1}\partial_{ee}u - \partial_{ee}u(x_1) + \partial_{ee}u(x_1) - \partial_{ee}u(y)$$
$$\overset{(8.98),\ y\in S}{\ge} \frac{c_0}{4} - \frac{c_0}{8} = \frac{c_0}{8}$$

In other words

$$S \subseteq B_{1/4} \cap \{v \ge \frac{c_0}{8}\}$$

Then using our (8.97) on such set yields

$$\eta \le |S| \le |B_{1/4} \cap \{v \ge \frac{c_0}{8}\}|$$

Ah. Now we see why we want to apply the Weak Harnack. This is for the sake of bounding v from below by a universal constant. Let's do so via (8.94). For some possibly different $\delta = \delta(\frac{\lambda}{n}, \Lambda) > 0$

$$\frac{8}{c_0}\inf_{B_{1/2}} v = \frac{8}{c_0}\inf_{B_{1/2}} K - \partial_{ee}u \ge C|B_{1/4} \cap \{\frac{8}{c_0}v \ge 1\}|^\delta \ge \eta^\delta$$

We obtain for $C_1 > 0$ universal

$$\inf_{B_{1/2}} K - \partial_{ee}u \ge C_1 \tag{8.99}$$

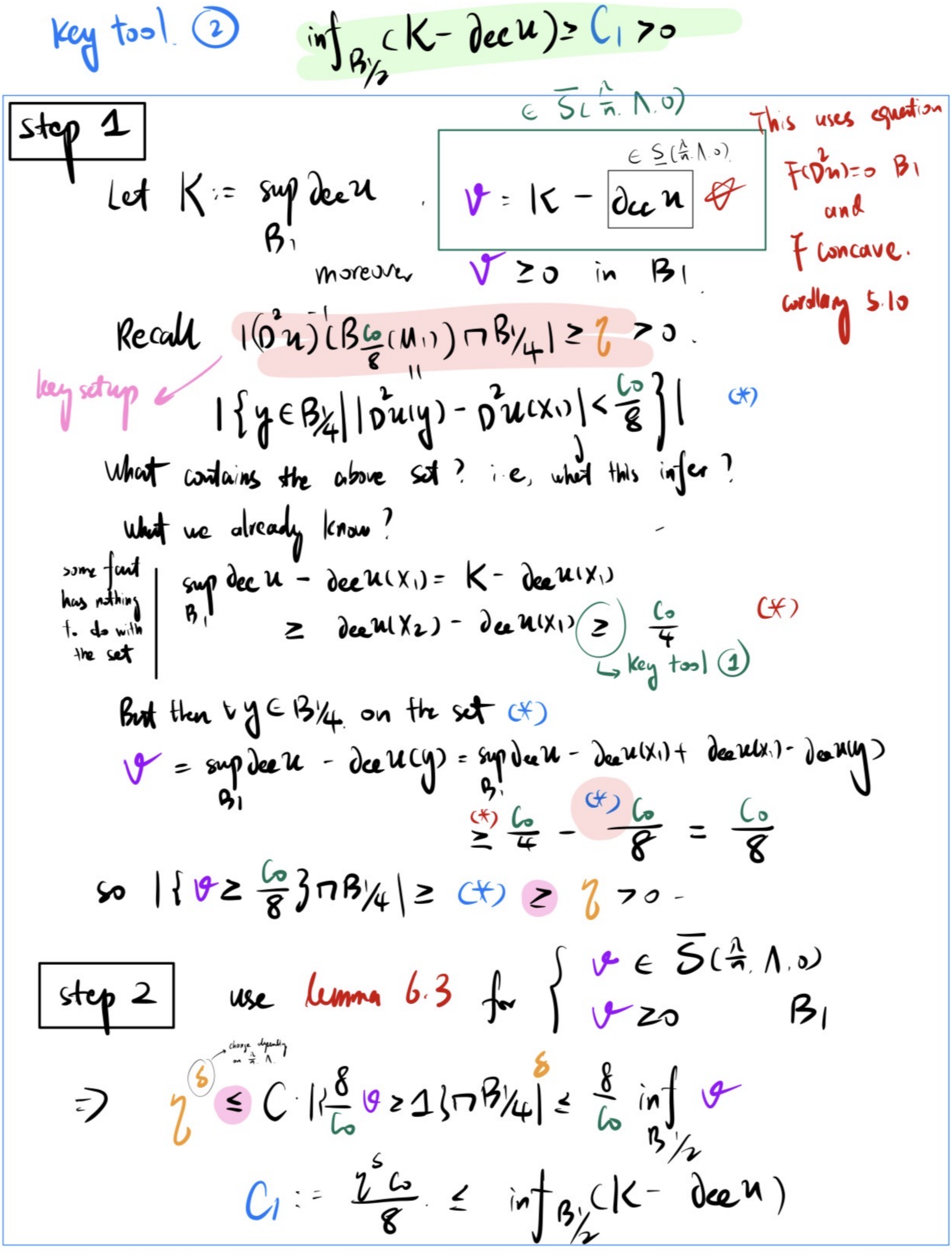


Figure 8.25: Lemma 8.5.5 Key Tool 2

Key Tool 3.
By definition of K as supremum, for our given $\varepsilon > 0$, there exists $x \in B_1$ s.t.

$$K \leq \partial_{ee} u(x) + \varepsilon$$

But for $D^2u(x) \in B_1$, there exists M_j s.t. $D^2u(x) \in B_{2\varepsilon}(M_j)$ so

$$\begin{aligned} K &< \partial_{ee}u(x) + \varepsilon = \partial_{ee}u(x) - \partial_{ee}u(x_j) + \partial_{ee}u(x_j) + \varepsilon \\ &\leq 3\varepsilon + \partial_{ee}u(x_j) \end{aligned}$$

so

$$K - \partial_{ee}u(x_j) < 3\varepsilon \tag{8.100}$$

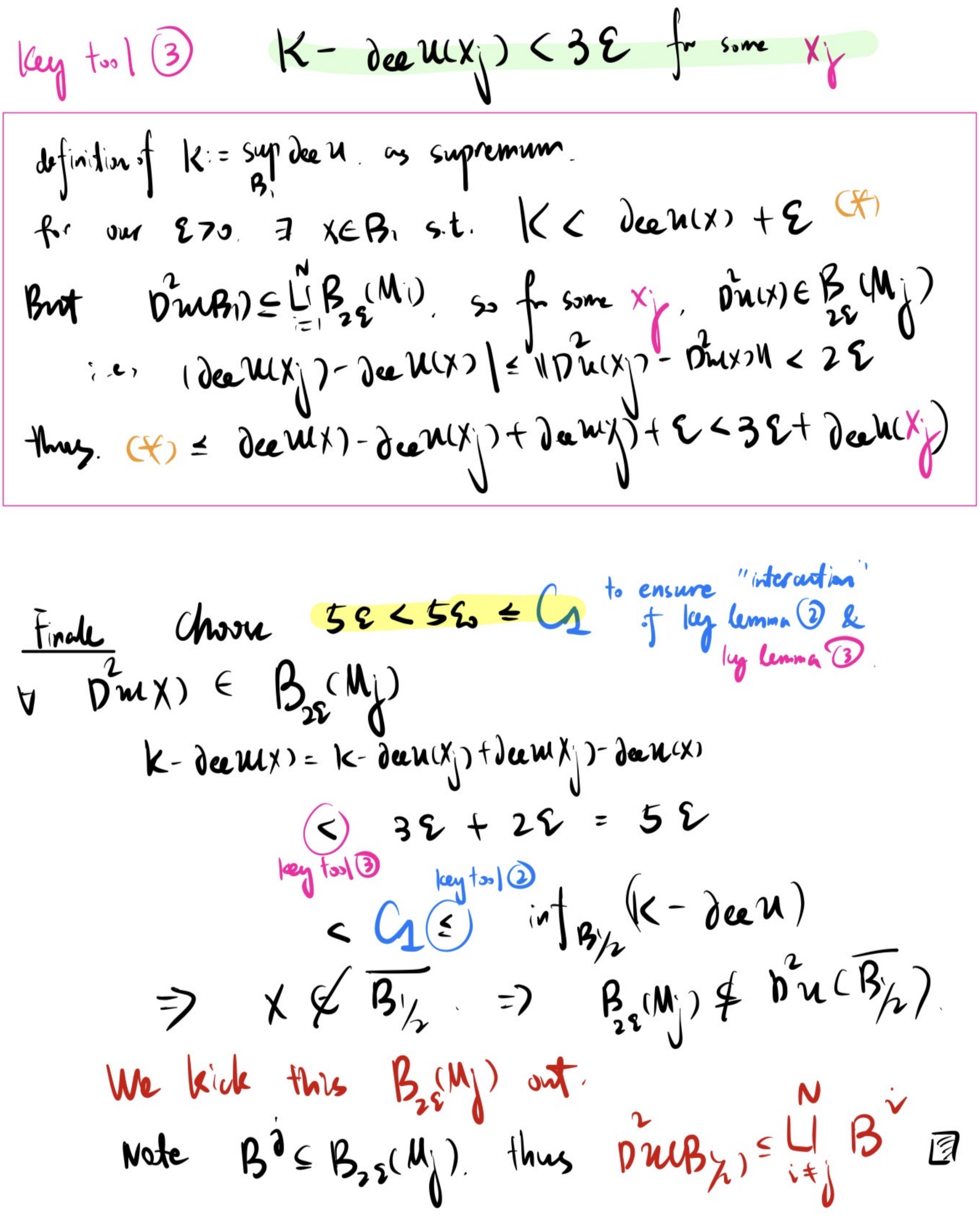


Figure 8.26: Lemma 8.5.5 Key Tool 3

Conclusion.
Now we choose at last

$$5\varepsilon < 5\varepsilon_0 < C_1$$

so that for any $D^2u(x) \in B_{2\varepsilon}(M_j)$, using Key Tool 2 and Key Tool 3

$$\begin{aligned} K - \partial_{ee}u(x) &= K - \partial_{ee}u(x_j) + \partial_{ee}u(x_j) - \partial_{ee}u(x) \\ &\overset{(8.100)}{\leq} 3\varepsilon + 2\varepsilon \qquad \text{and using } D^2u(x) \in B_{2\varepsilon}(M_j) \\ &< C_1 \overset{(8.99)}{\leq} \inf_{x \in B_{1/2}} K - \partial_{ee}u \end{aligned}$$

Consequently $x \notin B_{1/2}$. This is to say

$$B_{2\varepsilon}(M_j) \nsubseteq B_{1/2}$$

We throw away $B_{2\varepsilon}(M_j)$ as the member for the covering $D^2u(B_1) \subseteq \bigcup_{i=1}^{N} B_{2\varepsilon}(M_i)$. In particular since $B_{2\varepsilon}(M_j) \supseteq B^j$ by construction, we've thrown away B^j. This concludes our result.

□

8.5.3 Conclusion of Proof for Lemma 8.5.1

In the subsection we prove Lemma 8.5.1.

Since we start with $\text{diam}(D^2u(B_1)) = 2$, we cover $D^2u(B_1)$ with N balls where N is universal number, of radius ε_0 as in Lemma 8.5.5. Now apply Lemma 8.5.5, once we're in $B_{1/2}$, we drop one ball from the collection.

If $\text{diam}(D^2u(B_{1/2})) > 1$, one may rescale

$$w(y) = 4u(\frac{y}{2}) \qquad \forall\ y \in B_1$$

so that

$$\text{diam}(D^2w(B_1)) = \text{diam}(D^2u(B_{1/2})) \in (1, 2]$$

On the other hand, $D^2w(B_1)$ is covered by the same $N-1$ balls of radius ε_0, result of the first iteration. Now since w again solves the equation

$$\mathcal{F}(D^2w) = 0$$

with $\mathcal{F}$ concave, we apply Lemma 8.5.5 again so that

$$D^2w(B_{1/2}) = D^2u(B_{1/4})$$

is covered by $N-2$ balls of universal radius ε_0.

But one cannot run out of balls. Thus there exists $k \leq N$ s.t.

$$\text{diam}(D^2u(B_{2^{-k}})) \leq 1$$

One conclude (8.90) with

$$\delta_0 = \frac{1}{2^N}$$

8.6 Dirichlet Problem for Concave Equations

8.6.1 Higher Regularity

Standard Elliptic regularity theory applies to $C^{2,\alpha}$ solutions of fully nonlinear equations, saying once we have $C^{2,\alpha}$, we may bootstrap to C^∞.

Here we have no assumptions on concavity of $\mathcal{F}$.

Proposition 8.6.1 ([CC95] Proposition 9.1). *Let $0<\alpha<1$. Let $u \in C^{2,\alpha}$ be a solution to*

$$\mathcal{F}(D^2u, x) = f(x) \qquad \Omega$$

Assume $\mathcal{F} \in C^\infty(\mathrm{Sym}(n) \times \Omega)$, and $f \in C^\infty(\Omega)$.

Then $u \in C^\infty(\Omega)$.

Proof. **Base case**. Let $h \in \mathbb{R}$ small and $k \in \{1, \cdots, n\}$. Then define the first-order difference quotient

$$D_k^h u(x) := \frac{1}{h}\left(u(x+he_k) - u(x)\right)$$

We study what equation the difference quotient solves. First note $u(x+he_k)$ remains a solution, so

$$\begin{aligned} \mathcal{F}(D^2u(x+he_k), x+he_k) &= f(x+he_k) \\ \mathcal{F}(D^2u(x), x) &= f(x) \\ \frac{1}{h}\left(\mathcal{F}(D^2u(x+he_k), x+he_k) - \mathcal{F}(D^2u(x), x)\right) &= D_k^h f(x) \end{aligned}$$

We would be very happy to differentiate the equation. To do so consider the origin equation as integration of the linearized operator as follows

$$a_{ij}(x) := \int_0^1 \mathcal{F}_{ij}(tD^2u(x+he_k) + (1-t)D^2u(x), t(x+he_k) + (1-t)x)dt$$

which is a uniformly elliptic operator with elliptic constants λ, $n\Lambda$. Now one may instead differentiate in t ($\mathcal{F}_{ij}$ denotes the derivative w.r.t. (i,j)-entry of the matrix coefficient and $\mathcal{F}_k$ denotes the kth derivative in x variable)

$$\begin{aligned} \frac{d}{dt}\left(\mathcal{F}(tD^2u(x+he_k) + (1-t)D^2u(x), x+the_k)\right) &= \mathcal{F}_{ij}(\cdots)\left(\partial_{ij}u(x+he_k) - \partial_{ij}u(x)\right) + \mathcal{F}_k(\cdots)h \\ \frac{1}{h}\frac{d}{dt}\left(\mathcal{F}(tD^2u(x+he_k) + (1-t)D^2u(x), x+the_k)\right) &= \mathcal{F}_{ij}(\cdots)D_k^h\partial_{ij}u(x) + \mathcal{F}_k(\cdots) \\ \frac{1}{h}\left(\mathcal{F}(D^2u(x+he_k), x+he_k) - \mathcal{F}(D^2u(x), x)\right) &= a_{ij}(x)D_k^h\partial_{ij}u(x) + \int_0^1 \mathcal{F}_k(\cdots)dt \qquad \text{integrate from 0 to 1} \end{aligned}$$

If we define

$$G(x) := \int_0^1 \mathcal{F}_k(tD^2u(x+he_k) + (1-t)D^2u(x), x+the_k)dt$$

Then we observe that $D_k^h u$ solves the linearized equation

$$a_{ij}(x)\partial_{ij}\left(D_k^h u(x)\right) = D_k^h f(x) - G(x) \in C^{0,\alpha}$$

Now with the assumption $u \in C^{2,\alpha}$, $G \in C^{0,\alpha}$, $a_{ij} \in C^{0,\alpha}$, and since $f \in C^\infty$, the interior Schauder estimate applies so that for some $\overline{H} \subseteq \Omega$ where h are sufficiently small so $H + he_k \in \Omega$

$$\left\|D_k^h u\right\|_{C^{2,\alpha}(H)} \le C\left(\|f\|_{C^{1,\alpha}} + \|\mathcal{F}\|_{C^\infty}\right)$$

uniformly in h. Thus by Ascoli-Arzela, up to subsequence, $D_k^h u \to \partial_k u \in C^{2,\alpha}$. Do this for every k and gain $u \in C^{3,\alpha}$.

Bootstrap Step. With $u \in C^{3,\alpha}$ one is able to differentiate the equation in $k \in \{1, \cdots, n\}$ so that

$$\mathcal{F}_{ij}(D^2u, x)\partial_{ij}(\partial_k u) = -\mathcal{F}_k(D^2u, x) + \partial_k f(x)$$

Using Schauder theory with $C^{1,\alpha}$ coefficients one get $\partial_k u \in C^{3,\alpha}$ for any k, thus $u \in C^{4,\alpha}$. Since the regularity of the coefficients keeps improving because of improved regularity for D^2u, one bootstrap to C^∞. □

One has the global $C^{2,\alpha}$ to C^∞ result due to the Global Schauder Estimates and improvement of regularity.

Proposition 8.6.2 ([CC95] Remark 9.1 1). *Let $u \in C^{2,\alpha}(\overline{\Omega})$ be solution to*

$$\mathcal{F}(D^2u, x) = f(x) \qquad \Omega$$

Assume $\partial\Omega \in C^\infty$, $u|_{\partial\Omega} \in C^\infty$, $\mathcal{F} \in C^\infty(\mathrm{Sym}(n), \overline{\Omega})$, and $f \in C^\infty(\overline{\Omega})$. Then $u \in C^\infty(\overline{\Omega})$.

8.6.2 Berstein's Technique: $C^{1,1}$ A priori estimates

Concave Equation and its Linearization

Lemma 8.6.1 ([CC95] Lemma 9.2)**.** *Let $\mathcal{F}$ be uniformly elliptic,concave and smooth. Assume $u \in C^4(\Omega)$ solves*

$$\mathcal{F}(D^2u) = 0$$

and assume $F(0) = 0$. Consider the linearized equation

$$\mathcal{L}v = a_{ij}\partial_{ij}v := \mathcal{F}_{ij}(D^2u(x))\partial_{ij}v \tag{8.101}$$

Then for any $e \in \mathbb{S}^{n-1}$,

$$\mathcal{L}u \leq 0 \tag{8.102}$$
$$\mathcal{L}\partial_e u = 0 \tag{8.103}$$
$$\mathcal{L}\partial_{ee} u \geq 0 \tag{8.104}$$

Proof. Consider the interplay between 0 and D^2u under $\mathcal{F}$

$$f(t) := \mathcal{F}((1-t)D^2u)$$

Since u solves $\mathcal{F}(D^2u) = 0$ we know $f(0) = 0$, and since $\mathcal{F}(0) = 0$ we know $f(1) = 0$. Now that $\mathcal{F}$ is concave tells us that $f(t) \geq 0$ over $t \in [0,1]$ and

$$f''(t) < 0$$

Now in particular, at $t = 0$

$$f'(0) \geq 0$$

u **is supersolution**. Let's differentiate $f(t)$

$$f'(t) = -\mathcal{F}_{ij}((1-t)D^2u)\partial_{ij}u$$

Plugging in $t = 0$ gives

$$f'(0) = -\mathcal{F}_{ij}(D^2u)\partial_{ij}u \geq 0$$
$$\mathcal{L}u \leq 0$$

$\partial_e u$ **is solution**. We directly differentiate the equation

$$0 = \partial_e\left(\mathcal{F}(D^2u)\right) = \mathcal{F}_{ij}(D^2u)\partial_{ij}(\partial_e u)$$

which is doable since we've assumed $u \in C^4$.

$\partial_{ee}u$ **is subsolution**. We differentiate the equation one more time to obtain

$$0 = \partial_e\left(\mathcal{F}_{ij}(D^2u)\partial_{ij}(\partial_e u)\right) = \mathcal{F}_{ij,k\ell}(D^2u)\partial_{ij}(\partial_e u)\partial_{k\ell}(\partial_e u) + \mathcal{F}_{ij}(D^2u)\partial_{ij}(\partial_{ee}u)$$
$$\mathcal{L}\partial_{ee}u = -\mathcal{F}_{ij,k\ell}(D^2u)\partial_{ij}(\partial_e u)\partial_{k\ell}(\partial_e u) \geq 0$$

Using concavity of the operator $\mathcal{F}$, so its second derivatives are negative semi-definite. □

Berstein's Technique for $\|\nabla u\|_{L^\infty}$

Proposition 8.6.3 ([CC95] Proposition 9.3 (i))**.** *Let $\mathcal{F}$ be uniformly elliptic, concave, $F(0) = 0$ and smooth. Let $u \in C^3(\overline{B_1})$ satisfy $\mathcal{F}(D^2u) = 0$ in B_1. Then*

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)} \tag{8.105}$$

Proof. Take test function $\varphi \in C_0^\infty(B_1)$ with $0 \leq \varphi \leq 1$, $\varphi = 1$ on $B_{1/2}$. Let $\|\varphi\|_{C^2} \leq C(n)$.

For $\delta > 0$ large to be chosen, for $M := \sup_{B_1} u(x)$, consider the function

$$h(x) := \delta(M-u)^2 + \varphi^2|\nabla u(x)|^2$$

If we're able to show $\mathcal{L}h \geq 0$ where $\mathcal{L}$ is as in (8.101), then one may apply Weak Maximum Principle so that

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq \sup_{B_1} h \leq \sup_{\partial B_1} h \leq C(n,\delta)\|u\|_{L^\infty(B_1)}$$

which directly concludes the proof. To achieve this one need to compute $\mathcal{L}h$ and take $\delta > 0$ large as follows:

$$\begin{aligned}
\partial_i \delta(M-u)^2 &= -2\delta(M-u)\partial_i u \\
\partial_{ij}\delta(M-u)^2 &= 2\delta\partial_i u\partial_j u - 2\delta(M-u)\partial_{ij}u \\
\partial_i \varphi^2|\nabla u(x)|^2 &= 2\varphi\varphi_i|\nabla u|^2 + 2\varphi^2\partial_k u\partial_{ki}u \\
\partial_{ij}\varphi^2|\nabla u(x)|^2 &= 2\left(\varphi_i\varphi_j + \varphi\varphi_{ij}\right)|\nabla u|^2 + 8\varphi\varphi_i\partial_k u\partial_{kj}u + 2\varphi^2\partial_{ki}u\partial_{kj}u + 2\varphi^2\partial_k u\partial_{kij}u
\end{aligned}$$

Let's study how to bound them.

$$\begin{aligned}
a_{ij}\partial_{ij}\delta(M-u)^2 &= 2\delta a_{ij}\partial_i u\partial_j u - 2\delta(M-u)a_{ij}\partial_{ij}u \\
&\geq 2\lambda\delta|\nabla u|^2 \qquad \text{for the first we use ellipticity, for the second we use (8.102)} \\
a_{ij}\partial_{ij}\left(\varphi^2|\nabla u(x)|^2\right) &\geq -C\Lambda|\nabla u|^2 + 8a_{ij}\varphi\varphi_i\partial_k u\partial_{kj}u + 2\varphi^2 a_{ij}\partial_{ki}u\partial_{kj}u + \underbrace{\cancel{2\varphi^2\partial_k u a_{ij}\partial_{kij}u}}_{\text{use (8.103)}} \\
&\geq -C\Lambda|\nabla u|^2 - \underbrace{C\Lambda\left(C(\varepsilon)|\nabla u|^2 + \varepsilon\varphi^2|D^2u|^2\right)}_{\text{Young's to hide } \varphi^2|D^2u|^2} + \underbrace{2\varphi^2\lambda|D^2u|^2}_{\text{use ellipticity}} \\
&\geq -C\Lambda|\nabla u|^2 + c(\lambda,\Lambda)\varphi^2|D^2u|^2 \geq -C\Lambda|\nabla u|^2
\end{aligned}$$

Take δ sufficiently large to conclude. □

Berstein's Technique for $\left\|D^2u\right\|_{L^\infty}$

Proposition 8.6.4 ([CC95] Proposition 9.3 (ii)). *Given same assumption as in Proposition 8.6.3 except we assume $u \in C^4(\overline{B_1})$. Then*

$$\left\|D^2u\right\|_{L^\infty(B_{1/2})} \leq C\left\|\nabla u\right\|_{L^\infty(B_1)} \tag{8.106}$$

Proof. Take any $e \in \mathbb{S}^{n-1}$. Take test function $\varphi \in C_0^\infty(B_1)$ with $0 \leq \varphi \leq 1$, $\varphi = 1$ on $B_{1/2}$. Let $\|\varphi\|_{C^2} \leq C(n)$.

For $\delta > 0$ large to be chosen, consider the function

$$g := \delta(\partial_e u)^2 + \varphi^2(\partial_{ee}u)^2$$

on the set

$$\Omega = \{x \in B_1 \mid \partial_{ee}u(x) > 0\}$$

If we're able to show $\mathcal{L}g \geq 0$ in the positive set Ω for $\partial_{ee}u$ where $\mathcal{L}$ is (8.101), then using the Weak Maximum Principle and the fact that

$$\partial\Omega \subseteq \partial B_1 \cup \{x \in B_1 \mid \partial_{ee}u(x) = 0\}$$

we obtain

$$\begin{aligned}
\sup_{B_{1/2}}((\partial_{ee}u)^+)^2 = \sup_{B_{1/2}\cap\Omega}(\partial_{ee}u)^2 \leq \sup_{\Omega} g &\overset{\text{Maximum Principle}}{=} \sup_{\partial\Omega} g \\
&\leq C(n,\delta)\sup_{B_1}(\partial_e u)^2 = C\left\|\nabla u\right\|_{L^\infty(B_1)}
\end{aligned}$$

Now using Lemma 8.5.4 (this uses uniform ellipticity, and that $\mathcal{F}(D^2u) = \mathcal{F}(0)$!) we know the positive and negative parts are comparable. Thus one conclude the result.

It suffices to compute and show $\mathcal{L}g \geq 0$ on Ω.

$$\begin{aligned}
\partial_{ij}(\delta(\partial_e u)^2) &= 2\delta\partial_{ei}u\partial_{ej}u + 2\delta\partial_e u\partial_{eij}u \\
\partial_i(\varphi^2(\partial_{ee}u)^2) &= 2\varphi\varphi_i(\partial_{ee}u)^2 + 2\varphi^2\partial_{ee}u\partial_{eei}u \\
\partial_{ij}(\varphi^2(\partial_{ee}u)^2) &= 2(\varphi_i\varphi_j + \varphi\varphi_{ij})(\partial_{ee}u)^2 + 8\varphi\varphi_i\partial_{ee}u\partial_{eej}u + 2\varphi^2\partial_{eei}u\partial_{eej}u + 2\varphi^2\partial_{ee}u\partial_{eeij}u
\end{aligned}$$

Let's bound them

$$\begin{aligned}
a_{ij}\partial_{ij}(\delta(\partial_e u)^2) &\geq 2\lambda\delta|\nabla\partial_e u|^2 + \cancel{2\delta\partial_e u a_{ij}\partial_{ij}\partial_e u} \qquad \text{ellipticity and (8.103)} \\
a_{ij}\partial_{ij}(\varphi^2(\partial_{ee}u)^2) &\geq -C\Lambda|\partial_{ee}u|^2 - C\Lambda\left(C(\varepsilon)|\partial_{ee}u|^2 + \varepsilon\varphi^2|\nabla\partial_{ee}u|^2\right) \\
&\quad + 2\lambda\varphi^2|\nabla\partial_{ee}u|^2 + \underbrace{2\varphi^2\partial_{ee}u a_{ij}\partial_{ij}(\partial_{ee}u)}_{\geq 0 \text{ using (8.104)}} \\
&\geq -C\Lambda|\partial_{ee}u|^2 + c(\lambda,\Lambda)\varphi^2|\nabla\partial_{ee}u|^2 \geq -C\Lambda|\partial_{ee}u|^2
\end{aligned}$$

Take δ sufficiently large to conclude. □

$C^{2,\alpha}$ **Interior Estimates** Assuming $u \in C^2(B_1)$, one immediately obtain $C^{2,\alpha}$ estimate along with $u \in C^\infty$.

Proposition 8.6.5 ([CC95] Proposition 9.4)**.** *Let $\mathcal{F}$ be uniformly elliptic, concave, $\mathcal{F} \in C^\infty$. If $u \in C^2(B_1)$ and solve*

$$\mathcal{F}(D^2u) = 0 \qquad B_1$$

Then $u \in C^\infty(B_1)$ and for some $0 < \alpha < 1$

$$\|u\|_{C^{2,\alpha}(\overline{B_{1/2}})} \le C \|u\|_{L^\infty(B_1)} \tag{8.107}$$

Proof. Since we're working with $u \in C^2$, we first use Evans-Krylov 8.5.1 to obtain $u \in C^{2,\alpha}$ with RHS bounded by $C^{1,1}$. Now running $C^{2,\alpha}$ to C^∞ bootstrap argument, Proposition 8.6.1, one obtain $u \in C^\infty$.

Now indeed $u \in C^4$ is valid, hence Berstein's Technique (8.105) and (8.106) applies to bound $\|u\|_{C^{1,1}}$ with $\|u\|_{L^\infty(B_1)}$ provided $F(0) = 0$. If $F(0) \ne 0$, one conduct simplification, and one may bound $F(0)$ via $\|u\|_{L^\infty(B_1)}$ as in (8.110). □

8.6.3 $C^{2,\alpha}$ A priori Estimate up to the Boundary

In this section we prove the $C^{2,\alpha}$ estimates up to the boundary for solutions of concave equations

$$\mathcal{F}(D^2u) = 0$$

Theorem 8.6.1 ([CC95] Theorem 9.5)**.** *Let $\mathcal{F}$ be concave and smooth. Let $g \in C^\infty(\overline{B_1})$.*
Then there exists $\alpha \in (0,1)$ s.t. if $u \in C^{2,\alpha}(\overline{B_1})$ solves

$$\begin{cases} \mathcal{F}(D^2u) = 0 & B_1 \\ u = g & \partial B_1 \end{cases} \tag{8.108}$$

Then for $C > 0$ universal constant

$$\|u\|_{C^{2,\alpha}(\overline{B_1})} \leq C\left(\|g\|_{C^3(\overline{B_1})} + |\mathcal{F}(0)|\right) \tag{8.109}$$

Simplification 1: $\mathcal{F}(0) = 0$ One may assume $\mathcal{F}(0) = 0$.

Lemma 8.6.2 ([CC95] Remark 6.2 1)**.** *Let $\mathcal{F}$ be uniformly elliptic, and assume u is viscosity solution of*

$$\mathcal{F}(D^2u) = 0 \qquad B_1$$

Then for $C = C(\Lambda) > 0$

$$|\mathcal{F}(0)| \leq C\,\|u\|_{L^\infty(B_1)} \tag{8.110}$$

Proof. **Step 1**. We claim there exists $t \in \mathbb{R}$ s.t.

$$\mathcal{F}(tI) = 0, \qquad \frac{|\mathcal{F}(0)|}{\Lambda} \leq |t| \leq \frac{|\mathcal{F}(0)|}{\lambda} \tag{8.111}$$

Using definition for uniform ellipticity (8.9), we know that $f(t) := F(tI)$ is an increasing, Lipschitz function with Lipschitz bounds, for any $t_1 > t_2$

$$\lambda(t_1 - t_2) \leq f(t_1) - f(t_2) \leq \Lambda(t_1 - t_2)$$

Thus there necessarily exists $t \in \mathbb{R}$ s.t. $f(t) = 0$. At such t, using (8.10)

$$\begin{aligned} \lambda\left\|(tI)^+\right\| - \Lambda\left\|(tI)^-\right\| \leq \mathcal{F}(tI) - \mathcal{F}(0) \leq \Lambda\left\|(tI)^+\right\| - \lambda\left\|(tI)^-\right\| \\ \lambda\left\|(tI)^-\right\| - \Lambda\left\|(tI)^+\right\| \leq \mathcal{F}(0) \leq \Lambda\left\|(tI)^-\right\| - \lambda\left\|(tI)^+\right\| \end{aligned}$$

If $t = 0$ there's nothing to show. If $t > 0$ then

$$-\Lambda t \leq F(0) \leq -\lambda t \implies \frac{|\mathcal{F}(0)|}{\Lambda} \leq t \leq \frac{|\mathcal{F}(0)|}{\lambda}$$

If $t < 0$ then

$$-\lambda t \leq F(0) \leq -\Lambda t \implies \frac{|\mathcal{F}(0)|}{\Lambda} \leq -t \leq \frac{|\mathcal{F}(0)|}{\lambda}$$

Thus we conclude

$$\frac{|\mathcal{F}(0)|}{\Lambda} \leq |t| \leq \frac{|\mathcal{F}(0)|}{\lambda}$$

Step 2. Assume $\|u\|_{L^\infty(B_1)} = M$. For $t > 0$, define

$$P(x) = M - \frac{t}{2}(1 - |x|^2)$$

Then

$$P = M \geq u \qquad \partial B_1$$

Also

$$D^2P = tI$$

so that

$$\mathcal{F}(D^2P) = \mathcal{F}(tI) = 0$$

Now using comparison principle, we know

$$u(x) \leq M - \frac{t}{2}(1 - |x|^2) \qquad \forall\, x \in B_1$$

In particular at 0

$$-M \le u(0) \le M - \frac{t}{2} \implies t \le 4M$$

For $t < 0$, define

$$P(x) = -M - \frac{t}{2}(1 - |x|^2)$$

Then

$$\begin{aligned} P = -M \le u \qquad & \partial B_1 \\ \mathcal{F}(D^2P) = \mathcal{F}(tI) = \mathcal{F}(D^2u) \qquad & B_1 \end{aligned}$$

Comparison principle again yields in particular at 0

$$P(0) = -M - \frac{t}{2} \le u(0) \le M \implies -4M \le t$$

Thus

$$\frac{|\mathcal{F}(0)|}{\Lambda} \le |t| \le 4M$$

and one conclude

$$|\mathcal{F}(0)| \le 4\Lambda \|u\|_{L^\infty(B_1)}$$

□

On the other hand, for t as in (8.111) one may define for any $x_0 \in \mathbb{R}^n$

$$P(x) := \frac{t}{2}|x - x_0|^2$$

so that $D^2P = tI$. Now

$$\mathcal{F}(D^2u) = \mathcal{F}(D^2(u - P) + tI) =: G(D^2(u - P))$$

The operator G remains uniformly elliptic and $G(0) = 0$. And whenever u solves $\mathcal{F}(D^2u) = 0$, $u - P$ solves $G(D^2(u - P)) = 0$.

Simplification 2: $\|g\|_{C^3(\overline{B_1})} \le 1$ Let $v = \frac{u}{\|g\|_{C^3(\overline{B_1})}} := \frac{1}{K}u$. Then v solves

$$\tilde{\mathcal{F}}(D^2v) = \frac{1}{K}\mathcal{F}(KD^2v) = 0$$

where $\tilde{\mathcal{F}}$ has the same ellipticity constants as $\mathcal{F}$. Thus it's safe to assume that

$$\|g\|_{C^3(\overline{B_1})} \le 1$$

Goal Therefore the goal reduces to bounding

$$\|u\|_{C^{2,\alpha}(\overline{B_1})} \le C$$

First thing, in view of our assumption $u \in C^{2,\alpha}(\overline{B_1})$, using Proposition 8.6.2, we know $u \in C^\infty(\overline{B_1})$ so we can differentiate infinitely-many often up to the boundary.

8.6.3.1 Bound for $\|u\|_{L^\infty(B_1)}$

This is simply done by using u solves the Dirichlet problem (8.108) with 0 right hand side is hence belonging to $u \in \mathcal{S}(\frac{\lambda}{n}, \Lambda, 0)$ via Proposition 8.1.5. Now the Maximum Principle

$$\|u\|_{L^\infty(B_1)} \le \|u\|_{L^\infty(\partial B_1)} = \|g\|_{L^\infty(\partial B_1)} \le 1$$

where the last step uses our simplification.

8.6.3.2 Bound for $\|\nabla u\|_{L^\infty(B_1)}$

Consider $x_0 = 0 \in \partial B_1$ where we take $B_1 = B_1(0, \cdots, 0, 1)$.

The first goal is to bound $\|\nabla u\|_{L^\infty(\partial B_1)}$. We want to translate u down via

$$\tilde{u} := u - g(0) - \sum_{i=1}^{n-1} \partial_i g(0) x_i$$

so that

$$\tilde{u}(0) = 0$$
$$\partial_i \tilde{u}(0) = \partial_i u(0) - \partial_i g(0) = 0 \qquad \forall\, 1 \le i \le n-1$$

Now for any $x \in \partial B_1$, notice $|x|^2 = 2x_n$. Using $g = u$ is C^2 on the boundary, and all first order derivatives along ∂B_1 for $\tilde{u}$ are zero, for any $x \in \partial B_1$ one may expand so

$$|\tilde{u}(x)| \le \frac{C}{2}|x|^2 = Cx_n$$

Now since $\tilde{u} \pm Cx_n$ solves

$$\tilde{u} \pm Cx_n = u - g(0) - \sum_{i=1}^{n-1} \partial_i g(0) x_i \pm Cx_n \in \mathcal{S}(\frac{\lambda}{n}, \Lambda, 0)$$

one may apply the Maximum Principle on both sides to obtain (**This is trapping the solution with two planes** $\pm Cx_n$)

$$-Cx_n \le \tilde{u}(x) \le Cx_n \qquad \forall\, x \in \overline{B_1}$$

Since $\tilde{u}(0) = 0$, we take the normal derivative ∂_n at 0 so that

$$|\partial_n u(0)| = |\partial_n \tilde{u}(0)| \le C$$

Recall we also have $|\partial_i u(0)| = |\partial_i g(0)| \le 1$. Therefore translating every point $x \in \partial B_1$ to 0, one obtain

$$\|\nabla u\|_{L^\infty(\partial B_1)} \le C$$

The next goal is to apply Maximum Principle. Since u solves $\mathcal{F}(D^2 u) = 0$ implies (8.78) that for any $e \in \mathbb{S}^{n-1}$, $\partial_e u \in \mathcal{S}(\frac{\lambda}{n}, \Lambda, 0)$. Therefore applying Maximum Principle

$$|\partial_i u| \le C \qquad \forall\, x \in B_1$$

8.6.3.3 Bound for $\|D^2 u\|_{L^\infty(B_1)}$

Consider $x_0 = 0 \in \partial B_1$ where we take $B_1 = B_1(0, \cdots, 0, 1)$. One would like to bound $\|D^2 u\|_{L^\infty(\partial B_1)}$ first, and then apply the Maximum Principle.

Computations for Tangent Derivatives. Consider tangential derivatives w.r.t. ∂B_1. The tangent vectors are computed by taking $(x_1, \cdots, x_{n-1}, x_n - 1)$ as unit normal on ∂B_1 and ensuring orthogonality w.r.t. the unit normal, i.e. $(a_1, \cdots, a_n) \in T_x \mathbb{S}^{n-1}$ iff

$$\sum_{i=1}^{n-1} a_i x_i + a_n(x_n - 1) = 0$$

For example one may choose

$$\partial_{T_k} u := (1 - x_n)\partial_k u + x_k \partial_n u \qquad \forall\, 1 \le k \le n-1 \tag{8.112}$$

The vectors $\{(0, \cdots, 0, 1 - x_n, 0, \cdots, x_k)\}_{1 \le k \le n-1}$ are linearly independent thus they span the tangent space. Moreover, since u by assumption are smooth up to the boundary, the tangent derivatives match with the boundary data

$$\partial_{T_k} u = \partial_{T_k} g \qquad \partial B_1$$

The reason we do this is because the tangential derivatives of our solution exactly matches the tangent derivatives for g.

Bound on $|\partial_{k\ell}u(0)|$ for $1 \le k,\, \ell \le n-1$. Let's compute the mixed tangent derivatives for any $1 \le k, \ell \le n-1$

$$\begin{aligned}\partial_{T_k}\partial_{T_\ell}u &= (1-x_n)\partial_k\left((1-x_n)\partial_\ell u + x_\ell\partial_n u\right) + x_k\partial_n\left((1-x_n)\partial_\ell u + x_\ell\partial_n u\right)\\ &= (1-x_n)^2\partial_{k\ell}u + (1-x_n)\delta_{k\ell}\partial_n u + (1-x_n)x_\ell\partial_{nk}u - x_k\partial_\ell u + x_k(1-x_n)\partial_{\ell n}u + x_kx_\ell\partial_{nn}u\end{aligned}$$

Now evaluating at 0 yields

$$\begin{aligned}\partial_{T_k}u(0) &= \partial_k u(0) = \partial_k g(0)\\ \partial_{T_k}\partial_{T_\ell}u(0) &= \partial_{k\ell}u(0) + \delta_{k\ell}\partial_n u(0)\end{aligned}$$

Thus to bound the tangential derivatives of u, one simply obtain

$$|\partial_{k\ell}u(0)| = |\partial_{T_k}\partial_{T_\ell}u(0) - \delta_{k\ell}\partial_n u(0)| \le |\partial_{T_k}\partial_{T_\ell}g(0)| + |\nabla u(0)| \le 1 + C$$

for any $1 \le k, \ell \le n-1$.

Bound on mixed derivative $|\partial_{kn}u(0)|$. Let $\mu > 0$ universal to be chosen later. Fix $1 \le k \le n-1$. We first compute

$$\begin{aligned}\partial_n\partial_{T_k}u &= \partial_n\left((1-x_n)\partial_k u + x_k\partial_n u\right) = -\partial_k u + (1-x_n)\partial_{kn}u + x_k\partial_{nn}u\\ \partial_n\partial_{T_k}u(0) &= -\partial_k u(0) + \partial_{kn}u(0)\end{aligned} \tag{8.113}$$

We define $h_\pm$ and extend in quadratic fashion in n-direction to B_1

$$h_\pm(x) := \partial_{T_k}g(x) - \partial_{T_k}g(0) \underbrace{-\sum_{\ell=1}^{n-1}\partial_{T_\ell}\partial_{T_k}g(0)x_\ell + \partial_{T_k}g(0)x_n}_{\text{subtract the first order expansion of } \partial_{T_k}g \text{ at } 0} \pm\mu\sum_{\ell=1}^{n-1}(\partial_{T_\ell}g(x) - \partial_{T_\ell}g(0))^2 \pm \frac{\mu}{2}x_n^2$$

We define the same with u in place of g

$$\begin{aligned}v_\pm(x) &:= \partial_{T_k}u(x) - \partial_{T_k}u(0) - \sum_{\ell=1}^{n-1}\partial_{T_\ell}\partial_{T_k}u(0)x_\ell + \partial_{T_k}u(0)x_n \pm \mu\sum_{\ell=1}^{n-1}(\partial_{T_\ell}u(x) - \partial_{T_\ell}u(0))^2 \pm \frac{\mu}{2}x_n^2\\ &= (1-x_n)\partial_k u(x) + x_k\partial_n u(x) - \partial_k g(0) - \sum_{\ell=1}^{n-1}(\partial_{T_\ell}\partial_{T_k}g)(0)x_\ell + \partial_{T_k}g(0)x_n\\ &\pm \mu\sum_{\ell=1}^{n-1}((1-x_n)\partial_\ell u(x) + x_\ell\partial_n u(x) - \partial_{T_\ell}g(0))^2 \pm \frac{\mu}{2}x_n^2\end{aligned}$$

Finally we define the barrier functions (**We want our $w_\pm$ to be sub/supersolutions to $\mathcal{L}$**)

$$w_\pm(x) = \partial_{T_k}u(x) - \partial_k u(0) - \sum_{\ell=1}^{n-1}\partial_{T_\ell}\partial_{T_k}u(0)x_\ell \underbrace{+\partial_k u(0)x_n}_{\text{because (8.113)}} \pm \underbrace{\frac{\mu}{2}\sum_{\ell=1}^{n-1}(\partial_\ell u(x) - \partial_\ell u(0))^2}_{\text{modify by this, which is subsolution}} \pm\frac{\mu}{2}x_n^2$$

The key is that the Cartesian derivatives $\partial_\ell u$ satisfy a clean equation under the linearized operator. $\partial_\ell u$ is the only difference between $w_\pm$ and $v_\pm$.

In particular notice

$$h_\pm(0) = v_\pm(0) = w_\pm(0) = 0$$

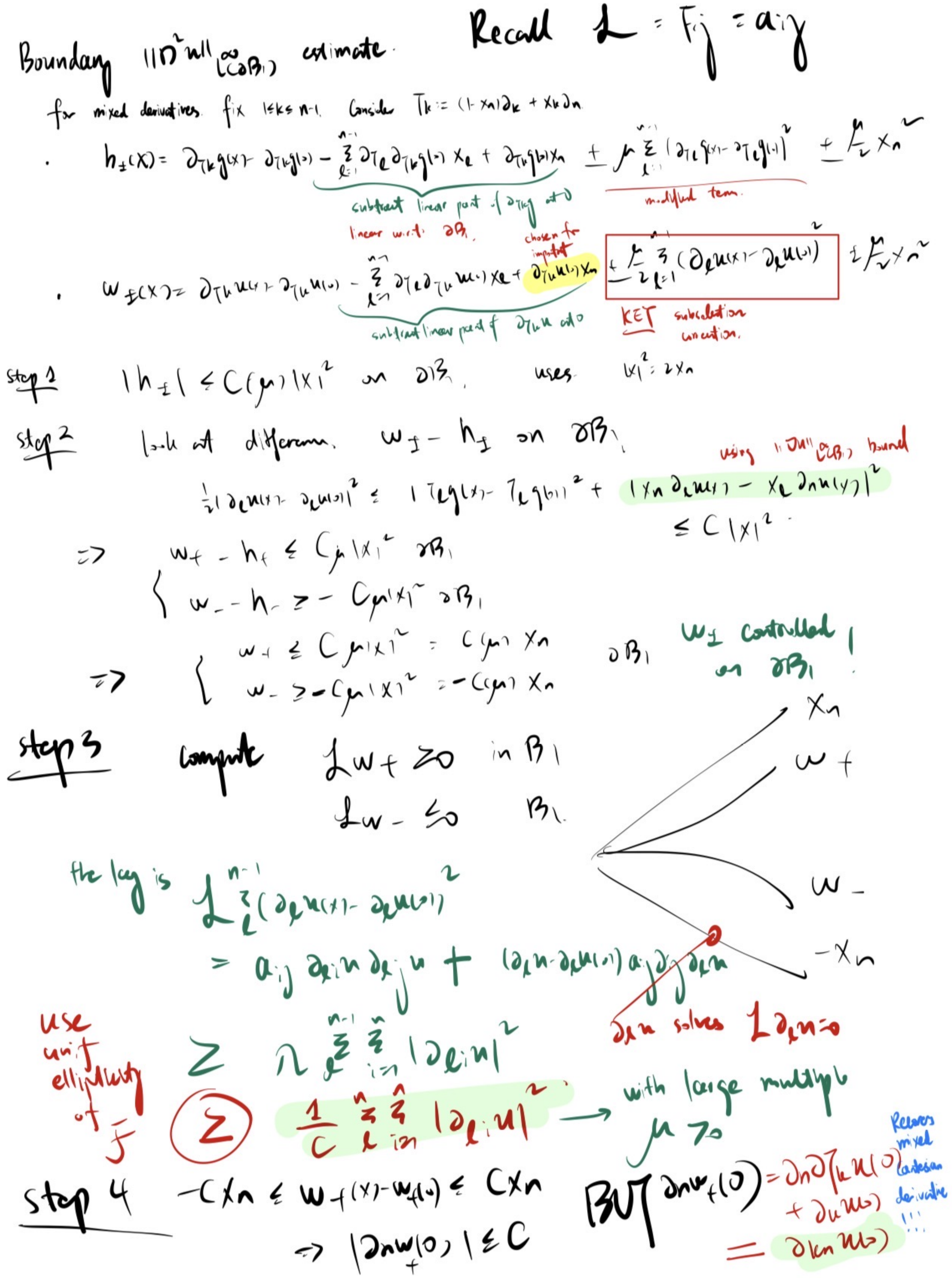


Figure 8.27: Bound on $|\partial_{kn}u(0)|$

We bound $|v_\pm(x)| = |h_\pm(x)| \le C|x^2|$ **on** ∂B_1. Immediately notice $v_\pm = h_\pm$ on ∂B_1. Let's see why we can bound $|h_\pm(x)|$ on ∂B_1. We compute for any $1 \le i \le n-1$

$$\partial_i(h_\pm)(x) = \partial_i\partial_{T_k}g(x) - \partial_{T_i}\partial_{T_k}g(0) \pm 2\mu\sum_{\ell=1}^{n-1}(\partial_{T_\ell}g(x) - \partial_{T_\ell}g(0))\partial_i\partial_{T_\ell}g(x)$$
$$\partial_i(h_\pm)(0) = 0$$

and note $h_\pm(0) = 0$. Let's compute the rest as well

$$\partial_n(h_\pm)(x) = \partial_n\partial_{T_k}g(x) + \partial_{T_k}g(0) \pm 2\mu\sum_{\ell=1}^{n-1}(\partial_{T_\ell}g(x) - \partial_{T_\ell}g(0))\partial_n\partial_{T_\ell}g(x) \pm \mu x_n$$
$$\partial_n(h_\pm)(0) = \partial_n\partial_{T_k}g(0) + \partial_{T_k}g(0)$$

and for $1 \le j \le n$, using $\|g\|_{C^3(\overline{B_1})} \le 1$

$$\partial_{ij}(h_\pm)(x) = \partial_{ij}\partial_{T_k}g(x) \pm 2\mu\sum_{\ell=1}^{n-1}\partial_i\partial_{T_\ell}g(x)\partial_i\partial_{T_\ell}g(x) \pm 2\mu\sum_{\ell=1}^{n-1}(\partial_{T_\ell}g(x) - \partial_{T_\ell}g(0))\partial_{ij}\partial_{T_\ell}g(x)$$
$$\|\partial_{ij}(h_\pm)\|_{L^\infty(B_1)} \le C(1+\mu)$$
$$\partial_{nn}(h_\pm)(x) \le \partial_{nn}\partial_{T_k}g(x) \pm 2\mu\sum_{\ell=1}^{n-1}\partial_n\partial_{T_\ell}g(x)\partial_n\partial_{T_\ell}g(x) \pm 2\mu\sum_{\ell=1}^{n-1}(\partial_{T_\ell}g(x) - \partial_{T_\ell}g(0))\partial_{nn}\partial_{T_\ell}g(x) \pm \mu$$
$$\|\partial_{nn}h\|_{L^\infty(B_1)} \le C(1+\mu)$$

Thus one may Taylor Expand and control $x \in \partial B_1$

$$\begin{aligned} h_\pm(x) &= h_\pm(0) + \sum_{i=1}^{n}\partial_i h_\pm(0)x_i + \frac{1}{2}x^T D^2 h_\pm(0)x + o(|x|^2) \\ &= \underbrace{\partial_n h_\pm(0)x_n}_{\text{quadratic on } \partial B_1} + \frac{1}{2}x^T D^2 h_\pm(0)x + o(|x|^2) \\ |h_\pm(x)| &\le C(1+\mu)|x|^2 \qquad \text{using } |x|^2 = 2x_n, \text{ i.e., } x \in \partial B_1 \end{aligned}$$

Then

$$|v_\pm(x)| = |h_\pm(x)| \le C(1+\mu)|x|^2 \qquad x \in \partial B_1 \tag{8.114}$$

We bound $|w_\pm(x)| \le Cx_n$ **on** ∂B_1. We add and subtract

$$\begin{aligned} \frac{1}{2}|\partial_\ell u(x) - \partial_\ell g(0)|^2 &= \frac{1}{2}|(1-x_n)\partial_\ell u(x) + x_\ell\partial_n u(x) - \partial_\ell g(0) + x_n\partial_\ell u(x) - x_\ell\partial_n u(x)|^2 \\ &\le |(1-x_n)\partial_\ell u(x) + x_\ell\partial_n u(x) - \partial_\ell g(0)|^2 + |x_n\partial_\ell u(x) - x_\ell\partial_n u(x)|^2 \end{aligned}$$

and using $|\nabla u| \le C$

$$|x_n\partial_\ell u(x) - x_\ell\partial_n u(x)|^2 \le C|x|^2$$

Thus notice upon subtraction

$$\begin{aligned} w_+(x) - v_+(x) &\le \mu|x_n\partial_\ell u(x) - x_\ell\partial_n u(x)|^2 \le C\mu|x|^2 \qquad \forall\, x \in \overline{B_1} \\ w_-(x) - v_-(x) &\ge -\mu|x_n\partial_\ell u(x) - x_\ell\partial_n u(x)|^2 \ge -C\mu|x|^2 \qquad \forall\, x \in \overline{B_1} \end{aligned}$$

Now in combination with (8.114) one obtain

$$-C(\mu)|x|^2 \le w_-(x) \le w_+(x) \le C(\mu)|x|^2 \qquad x \in \partial B_1$$

Since on the boundary ∂B_1 one has $|x|^2 = 2x_n$, we conclude that

$$\begin{aligned} w_+(x) &\le C(\mu)x_n \qquad x \in \partial B_1 \\ w_-(x) &\ge -C(\mu)x_n \qquad x \in \partial B_1 \end{aligned}$$

To conclude $|\partial_{kn}u(0)| \le C$. Now we claim there is a universal constant $\mu > 0$ large s.t. for $\mathcal{L}$ defined as in (8.101)

$$\mathcal{L}w_+ \ge 0 \qquad B_1, \qquad \mathcal{L}w_- \le 0 \qquad B_1 \tag{8.115}$$

Now by the Comparison Principle applied to subsolution w_+ and supersolution w_-, one get

$$\begin{aligned} w_+(x) &\le C(\mu)x_n \qquad x\in B_1\\ w_-(x) &\ge -C(\mu)x_n \qquad x\in B_1 \end{aligned}$$

Let's take the normal derivative ∂_n at 0

$$\begin{aligned} -Cx_n &\le w_+(x)-w_+(0)\le Cx_n\\ |\partial_n(w_+)(0)| &\le C \end{aligned}$$

But what is $|\partial_n(w_+)(0)|$?

$$\begin{aligned} \partial_n(w_+)(x) &= \partial_n\partial_{T_k}u(x)+\partial_k u(0)+\mu\sum_{\ell=1}^{n-1}(\partial_\ell u(x)-\partial_\ell u(0))\partial_{n\ell}u(x)+\mu x_n\\ |\partial_n(w_+)(0)| &= |\partial_n\partial_{T_k}u(0)+\partial_k u(0)| \overset{(8.113)}{=} |\partial_{nk}u(0)| \end{aligned}$$

Thus if one is able to show for (8.115), one may conclude the bound $|\partial_{nk}u(0)|\le C$ for $1\le k\le n-1$.

We prove (8.115) **by lifting** μ **large**. Since both $\mathcal{F}(0)=\mathcal{F}(D^2u)=0$, by ellipticity

$$\underbrace{|\partial_{nn}u(x)|^2\le C\sum_{\ell=1}^{n-1}\sum_{i=1}^{n}|\partial_{\ell i}u(x)|^2}_{\text{The key step that we use our equation}} \qquad \forall\ x\in\overline{B_1} \tag{8.116}$$

Recall $\mathcal{L}\partial_\ell u=0$ for any ℓ (8.103). Now for any $1\le i,j\le n-1$

$$\partial_i w_+ = (1-x_n)\partial_{ki}u+\delta_{ki}\partial_n u+x_k\partial_{ni}u-\partial_{T_i}\partial_{T_k}u(0)+\mu\sum_{\ell=1}^{n-1}(\partial_\ell u(x)-\partial_\ell u(0))\partial_{\ell i}u(x)$$

$$\partial_n w_+ = -\partial_k u+(1-x_n)\partial_{kn}u+x_k\partial_{nn}u+\partial_k u(0)+\mu\sum_{\ell=1}^{n-1}(\partial_\ell u(x)-\partial_\ell u(0))\partial_{\ell n}u(x)+\mu x_n$$

$$\partial_{ij}w_+ = (1-x_n)\partial_{kij}u+\delta_{ki}\partial_{nj}u+\delta_{kj}\partial_{ni}u+x_k\partial_{nij}u+\mu\sum_{\ell=1}^{n-1}\partial_{\ell i}u\partial_{\ell j}u+\mu\sum_{\ell=1}^{n-1}(\partial_\ell u(x)-\partial_\ell u(0))\partial_{\ell ij}u$$

$$\partial_{in}w_+ = -\partial_{ik}u+(1-x_n)\partial_{kin}u+\delta_{ik}\partial_{nn}u+x_k\partial_{nni}u+\mu\sum_{\ell=1}^{n-1}\partial_{\ell i}u(x)\partial_{\ell n}u(x)+\mu\sum_{\ell=1}^{n-1}(\partial_\ell u(x)-\partial_\ell u(0))\partial_{\ell in}u(x)$$

$$\partial_{nn}w_+ = -2\partial_{kn}u+(1-x_n)\partial_{knn}u+x_k\partial_{nnn}u+\mu\sum_{\ell=1}^{n-1}\partial_{\ell n}u(x)\partial_{\ell n}u(x)+\mu\sum_{\ell=1}^{n-1}(\partial_\ell u(x)-\partial_\ell u(0))\partial_{\ell nn}u(x)+\mu$$

Now we calculate

$$\begin{aligned}
\mathcal{L}w_+ &= a_{ij}\partial_{ij}w_+ = \sum_{1\le i,\,j\le n-1}a_{ij}\partial_{ij}w_+ + 2\sum_{1\le i\le n-1}a_{in}\partial_{in}w_+ + a_{nn}\partial_{nn}w_+\\
&= \sum_{1\le i,\,j\le n-1}a_{ij}\left((1-x_n)\partial_{kij}u+\delta_{ki}\partial_{nj}u+\delta_{kj}\partial_{ni}u+x_k\partial_{nij}u+\mu\sum_{\ell=1}^{n-1}\partial_{\ell i}u\partial_{\ell j}u+\mu\sum_{\ell=1}^{n-1}(\partial_\ell u(x)-\partial_\ell u(0))\partial_{\ell ij}u\right)\\
&+2\sum_{1\le i\le n-1}a_{in}\left(-\partial_{ik}u+(1-x_n)\partial_{kin}u+\delta_{ik}\partial_{nn}u+x_k\partial_{nni}u+\mu\sum_{\ell=1}^{n-1}\partial_{\ell i}u(x)\partial_{\ell n}u(x)+\mu\sum_{\ell=1}^{n-1}(\partial_\ell u(x)-\partial_\ell u(0))\partial_{\ell in}u(x)\right)\\
&+a_{nn}\left(-2\partial_{kn}u+(1-x_n)\partial_{knn}u+x_k\partial_{nnn}u+\mu\sum_{\ell=1}^{n-1}\partial_{\ell n}u(x)\partial_{\ell n}u(x)+\mu\sum_{\ell=1}^{n-1}(\partial_\ell u(x)-\partial_\ell u(0))\partial_{\ell nn}u(x)+\mu\right)\\
&= \underbrace{\cancel{(1-x_n)\sum_{1\le i,j\le n}a_{ij}\partial_{ij}\partial_k u}}_{(8.103)} + \underbrace{2\sum_{1\le j\le n}\left(\underbrace{a_{kj}\partial_{nj}u}_{\text{includes }\partial_{nn}u}-a_{nj}\partial_{kj}u\right)}_{\text{call this }A} + \underbrace{\cancel{x_k\sum_{1\le i,j\le n}a_{ij}\partial_{ij}\partial_n u}}_{(8.103)}\\
&+\underbrace{\mu\sum_{\ell=1}^{n-1}\sum_{1\le i,j\le n}a_{ij}\partial_{\ell i}u\partial_{\ell j}u}_{\text{call this }B} + \underbrace{\cancel{\mu\sum_{\ell=1}^{n-1}(\partial_\ell u(x)-\partial_\ell u(0))\left(\sum_{1\le i,j\le n}a_{ij}\partial_{ij}\right)\partial_\ell u}}_{(8.103)} + \mu a_{nn}
\end{aligned}$$

Notice the term B that comes from the modification of $w_\pm$ by large multiple of the subsolution

$$\frac{\mu}{2}\sum_{\ell=1}^{n-1}(\partial_\ell u(x)-\partial_\ell u(0))^2$$

is precisely the term that kills the leftover, which is the term A.

Now we further estimate

$$\begin{aligned}\mathcal{L}w_+ &= A+B+\mu a_{nn}\\ &\geq -C\Lambda\left(\sum_{\ell,j=1}^{n}|\partial_{\ell j}u|^2\right)^{\frac{1}{2}} + \underbrace{\mu\lambda\sum_{\ell=1}^{n-1}\sum_{i=1}^{n}|\partial_{\ell i}u|^2}_{\text{using ellipticity}} + \mu\lambda\\ &\geq -C\Lambda\left(\sum_{\ell,j=1}^{n}|\partial_{\ell j}u|^2\right)^{\frac{1}{2}} + \underbrace{\frac{\mu}{C}\sum_{\ell,i=1}^{n}|\partial_{\ell i}u|^2}_{\text{use (8.116)}} + \mu\lambda\end{aligned}$$

Now for μ sufficiently large (8.115) holds.

Bound on Pure derivative $|\partial_{nn}u(0)|$. This is direct consequence of bounds on mixed derivative and (8.116).

Finally we apply Maximum Principle. The above deals with the bound

$$\left\|D^2u\right\|_{L^\infty(\partial B_1)} \leq C$$

To get bound on all of $\overline{B_1}$, take any $e\in\mathbb{S}^{n-1}$, and consider $\partial_{ee}u$ which is bounded on ∂B_1. Using essentially Concavity Corollary 8.4.3 we know

$$\partial_{ee}u \in \underline{S}(\frac{\lambda}{n},\Lambda,0)$$

so it is subsolution. Then Maximum Principle applies so that

$$\sup_{B_1}\partial_{ee}u = \sup_{\partial B_1}\partial_{ee}u \leq C$$

Now Lemma 8.5.4 applied to D^2u and 0 yields comparability

$$\left\|D^2u(x)\right\| \leq C\sup_{|e|=1}(\partial_{ee}u)^+ \leq C \qquad \forall\ x\in B_1$$

8.6.3.4 Bound for $\left\|D^2u\right\|_{C^{0,\alpha}(\overline{B_1})}$

We claim it suffices to bound for $0<\beta<1$ universal that

$$\left\|D^2u(x_1)-D^2u(x_0)\right\| \leq C|x_1-x_0|^\beta \qquad \forall\ x_0,\ x_1\in\partial B_1 \tag{8.117}$$

From (8.117) to global $C^{2,\alpha}(\overline{B_1})$ Bound. For any $e\in\mathbb{S}^{n-1}$, using $\mathcal{F}$ is concave, we get from Corollary 8.4.3 (or via Berstein's $\mathcal{L}\partial_{ee}u\geq 0$) that

$$\partial_{ee}u \in \underline{S}(\frac{\lambda}{n},\Lambda,0)$$

Now combining with the Main Claim yet to prove (8.117), which tells us that $\partial_{ee}u|_{\partial B_1}\in C^{0,\beta}(\partial B_1)$, the conditions for Proposition 8.3.3 are satisfied. The result (8.66) reads

$$|\partial_{ee}u(x)-\partial_{ee}u(x_0)| \leq C|x-x_0|^{\frac{\beta}{2}} \qquad \forall\ x\in\overline{B_1},\ x_0\in\partial B_1$$

Now using both D^2u and 0 are solutions and Lemma 8.5.4 that they are comparable, we get

$$\left\|D^2u(x)-D^2u(x_0)\right\| \leq C|x-x_0|^{\frac{\beta}{2}} \qquad \forall\ x\in\overline{B_1},\ x_0\in\partial B_1$$

On the other hand, Evans-Krylov Theorem 8.5.1 gives interior estimate for $\|u\|_{C^{2,\gamma}(\overline{B_{1/2}})}$ via $\|u\|_{C^2(\overline{B_1})}$ for universal $0<\gamma<1$. Since we already have the bound on Hessian $\left\|D^2u\right\|_{L^\infty(B_1)} \le C$, one combine the Boundary and Interior estimates via a Global Argument Proposition 8.3.4 that

$$\|u\|_{C^{2,\alpha}(\overline{B_1})} \le C \qquad \alpha = \min(\gamma, \frac{\beta}{2})$$

Thus the goal is to show (8.117).

Flatten out the boundary and reduce to (8.119). For $x_0 \in \partial B_1$, let A be a neighborhood of x_0 that contains a ball centered at x_0 of universal radius. Then consider smooth diffeomorphism

$$\begin{aligned}\varphi : B_1 &\to \Omega = \varphi(B_1)\\ x &\mapsto \varphi(x) = (\varphi^1(x), \cdots, \varphi^n(x)) = y\end{aligned}$$

s.t. $\varphi(x_0)=0$, and

$$\begin{aligned}\varphi(A\cap B_1) &= B_4^+ := \{y\in\mathbb{R}^n \mid |y|<4,\ y_n>0\}\\ \varphi(A\cap\partial B_1) &= \Gamma_4 := \{y=(y',y_n)\in\mathbb{R}^{n-1}\times\mathbb{R} \mid |y'|<4,\ y_n=0\}\end{aligned}$$

We denote $\psi(y) := x$. Let $\|\varphi\|_{C^3(\overline{B_1})} + \|\psi\|_{C^3(\overline{\Omega})} \le C$.

Define the stretched function as

$$v(y) := u(x) - g(x) = u(\psi(y)) - g(\psi(y)) \qquad \forall\ y\in\overline{\Omega}$$

which vanishes on $\partial\Omega$.

We compute for $u(x) = v(\varphi(x)) + g(x)$

$$\begin{aligned}\partial_i u(x) &= \partial_k v(y)\frac{\partial\varphi^k}{\partial x_i} + \partial_i g(x)\\ \partial_{ij}u(x) &= \partial_{k\ell}v(y)\frac{\partial\varphi^k}{\partial x_i}\frac{\partial\varphi^\ell}{\partial x_j} + \partial_k v(y)\frac{\partial^2\varphi^k}{\partial x_i\partial x_j} + \partial_{ij}g(x)\end{aligned}$$

Since $\mathcal{F}(D^2u)=0$, if we alternatively define

$$\mathcal{G}(M,p,y) := \mathcal{F}\left(\left(M_{k\ell}\frac{\partial\varphi^k}{\partial x_i}(\psi(y))\frac{\partial\varphi^\ell}{\partial x_j}(\psi(y)) + p_k\frac{\partial^2\varphi^k}{\partial x_i\partial x_j}(\psi(y)) + \partial_{ij}g((\psi(y)))\right)_{ij}\right)$$

Then we see that v solves

$$\mathcal{G}(D^2v,\nabla v,y) = 0 \qquad \Omega \tag{8.118}$$

It remains to prove

$$\left\|D^2v(y) - D^2v(y_0)\right\| \le C|y|^\beta \qquad \forall\ y\in\Gamma_1 \tag{8.119}$$

where

$$\Gamma_1 := \{y=(y',y_n)\in\mathbb{R}^{n-1}\times\mathbb{R} \mid |y'|<1,\ y_n=0\}$$

Reduce to bound on Mixed Derivatives (8.121). We show that one can bound $|\partial_{nn}v(0)|$ by mixed derivatives.

Notice v solves (8.118). What inputs does $\mathcal{G}$ have? Consider differentiating w.r.t. M_{nn} so

$$\begin{aligned}\frac{\partial\mathcal{G}}{\partial M_{nn}} &= \mathcal{F}_{ij}\frac{\partial}{\partial M_{nn}}\left(M_{k\ell}\frac{\partial\varphi^k}{\partial x_i}(\psi(y))\frac{\partial\varphi^\ell}{\partial x_j}(\psi(y))\right)\\ &= \mathcal{F}_{ij}\frac{\partial\varphi^n}{\partial x_i}(\psi(y))\frac{\partial\varphi^n}{\partial x_j}(\psi(y)) \ge C\lambda\end{aligned}$$

using uniform ellipticity of the operator $\mathcal{F}_{ij} = a_{ij}$. There one may apply the Implicit Function Theorem so that (8.118) can be written as a solution to

$$\partial_{nn}v = H\left((\partial_{k\ell}v)_{\substack{1\le k\le n-1,\\ 1\le\ell\le n}},\ \nabla v,\ y\right) \tag{8.120}$$

But the uniform ellipticity implies the equation (8.118) is in fact equivalent to (8.120) globally. Now using again Implicit Function Theorem (hence the bound on the differential of H via that of $\mathcal{G}$ divided by $\partial_{M_{nn}}\mathcal{G}$)

$$\begin{aligned}|DH(\tilde{M},p,y)| &\le \frac{1}{|\partial_{M_{nn}}\mathcal{G}|}|D\mathcal{G}| \le \frac{1}{C\lambda}|D\mathcal{G}|\\ &\le C\left(|\tilde{M}| + |p| + 1\right)\end{aligned}$$

where $\tilde{M} = (M_{k\ell})_{\substack{1\le k\le n-1\\ 1\le \ell\le n}}$.

In previous steps we've achieved $\|\nabla v\|_{L^\infty(B_1)} + \left\|D^2 v\right\|_{L^\infty(B_1)} \le C$. Now

$$\begin{aligned}
|\partial_{nn} v(y) - \partial_{nn} v(0)| &= |H\left((\partial_{k\ell} v)_{\substack{1\le k\le n-1\\ 1\le \ell\le n}},\ \nabla v,\ y\right) - H\left((\partial_{k\ell} v)_{\substack{1\le k\le n-1\\ 1\le \ell\le n}}(0),\ \nabla v(0),\ 0\right)| \\
&\le |DH| \left(\sup_{\substack{1\le k\le n-1\\ 1\le \ell\le n}} |\partial_{k\ell} v(y) - \partial_{k\ell} v(0)| + \left\|D^2 v\right\|_{L^\infty(B_1)} |y| + |y|\right) \\
&\le C \left(\sup_{\substack{1\le k\le n-1\\ 1\le \ell\le n}} |\partial_{k\ell} v(y) - \partial_{k\ell} v(0)| + |y|\right)
\end{aligned}$$

Now if we restrict ourselves to $y \in \Gamma_1$, we know that all tangential derivatives vanish

$$\partial_{k\ell} v|_{\Gamma_1} = 0, \qquad 1 \le k, \ell \le n-1$$

due to construction of flattening out boundary $v|_{\Gamma_1} = 0$. Therefore, showing for (8.119) reduces to showing for the mixed derivatives only

$$|\partial_{kn} v(y) - \partial_{kn} v(0)| \le C|y|^\beta \qquad \forall\ y \in \Gamma_1, \qquad \forall\ 1 \le k \le n-1 \tag{8.121}$$

Reduce to the Theorem of Krylov. We show that $\partial_m v$ for $1 \le m \le k-1$ themselves satisfies a nice linearized equation so that one can direct apply a Theorem of Krylov's Theorem 8.6.2.

Since we wish to deal with (8.121), we differentiate the equation

$$\mathcal{F}\left(\left(\partial_{k\ell} v \frac{\partial \varphi^k}{\partial x_i}(\psi(y)) \frac{\partial \varphi^\ell}{\partial x_j}(\psi(y)) + \partial_k v \frac{\partial^2 \varphi^k}{\partial x_i \partial x_j}(\psi(y)) + \partial_{ij} g((\psi(y)))\right)_{ij}\right) = 0$$

w.r.t. y_m for $1 \le m \le n-1$.

$$0 = \mathcal{F}_{ij}\left(\partial_{k\ell}\partial_m v \frac{\partial \varphi^k}{\partial x_i}(\psi(y)) \frac{\partial \varphi^\ell}{\partial x_j}(\psi(y))\right) + \mathcal{F}_{ij}\left(\partial_{k\ell} v \partial_m \left(\frac{\partial \varphi^k}{\partial x_i}(\psi(y)) \frac{\partial \varphi^\ell}{\partial x_j}(\psi(y))\right) + \partial_m \left(\partial_k v \frac{\partial^2 \varphi^k}{\partial x_i \partial x_j}(\psi(y)) + \partial_{ij} g((\psi(y)))\right)_{ij}\right)$$

so that $\partial_m v$ solves

$$\tilde{\mathcal{L}} \partial_m v = f(y) \qquad \Omega$$

where

$$\tilde{\mathcal{L}} = \left(\sum_{k,\ell=1}^{n} \mathcal{F}_{ij} \frac{\partial \varphi^k}{\partial x_i}(\psi(y)) \frac{\partial \varphi^\ell}{\partial x_j}(\psi(y))\right) \partial_{k\ell}$$

is uniformly elliptic with elliptic constants $\frac{\lambda}{C}$ and $C\Lambda$ for $C \ge 1$ universal. On the other hand using

$$\left\|D^2 v\right\|_\infty + \|\nabla v\|_\infty + \|\varphi\|_{C^3(\overline{B_1})} + \|\psi\|_{C^3(\overline{\Omega})} + \|g\|_{C^3(\overline{B_1})} \le C$$

we get the force as remaining term

$$\|f\|_{L^\infty(\Omega)} \le C$$

Therefore

$$\begin{aligned}
\tilde{\mathcal{L}} \partial_m v = f(y) &\qquad B_4^+ \\
\partial_m v = 0 &\qquad \Gamma_4
\end{aligned}$$

$\tilde{L}$ is uniformly elliptic and one has bound

$$\|\partial_m v\|_{L^\infty(B_4^+)} + \|\nabla \partial_m v\|_{L^\infty(B_4^+)} + \|f\|_{L^\infty(B_4^+)} \le C$$

From the conditions above, one wish to estimate the normal derivative $\partial_n \partial_k v$ along the boundary Γ_1.

For this we refer to the Theorem of Krylov [CC95] Theorem 9.6 so that

$$\|\partial_n \partial_k v\|_{C^{0,\beta}(\overline{\Gamma_1})} \le C$$

Upon application this concludes the proof.

8.6.3.5 Krylov's Hölder Bound on Normal Derivative along Flat Boundary

In this section we prove the Krylov Theorem used above.

Theorem 8.6.2 ([CC95] Theorem 9.6; [Kaz85] Theorem 4.28)**.** *Consider half ball $B_4^+ \subseteq \mathbb{R}^n$, and a solution $u \in C^2(\overline{B_4^+})$ of*

$$\begin{cases} \mathcal{L}u := a_{ij}\partial_{ij}u = f & B_4^+ \\ u(x',0) = 0 & \partial B_4^+ \cap \{x_n = 0\} \end{cases} \tag{8.122}$$

where (a_{ij}) uniformly elliptic with constants λ, Λ, and both (a_{ij}), f bounded measurable. Also assume

$$\|u\|_{L^\infty(B_4^+)} + \|\nabla u\|_{L^\infty(B_4^+)} + \|f\|_{L^\infty(B_4^+)} \le K \tag{8.123}$$

Then there exists constant $0 < \alpha < 1$, and $c = c(n, K, \lambda, \Lambda)$ s.t.

$$\left\| \frac{\partial u}{\partial x_n} \right\|_{C^{0,\alpha}(\overline{\Gamma_1})} \le c \tag{8.124}$$

where $\Gamma_1 = \{(x', x_n) \in \mathbb{R}^n \mid |x'| \le 1,\ x_n = 0\}$

Since $u = 0$ on Γ_1

$$\frac{\partial u}{\partial x_n}(x',0) = \lim_{x_n \to 0} \frac{u(x',x_n) - 0}{x_n - 0} = \lim_{x_n \to 0} \frac{u(x)}{x_n}$$

so we estimate

$$v(x) = \frac{u(x)}{x_n}$$

In proving the theorem, we set up the domain we're working on. Let $R \le 1$ and for $\delta > 0$ small universal to be chosen, consider

$$Q(R) := \{(x', x_n) \in \mathbb{R}^n \mid |x'| \le R,\ 0 \le x_n \le \delta R\}$$
$$Q^+(R) := \{(x', x_n) \in \mathbb{R}^n \mid |x'| \le R,\ \frac{\delta}{2}R \le x_n \le \delta R\}$$

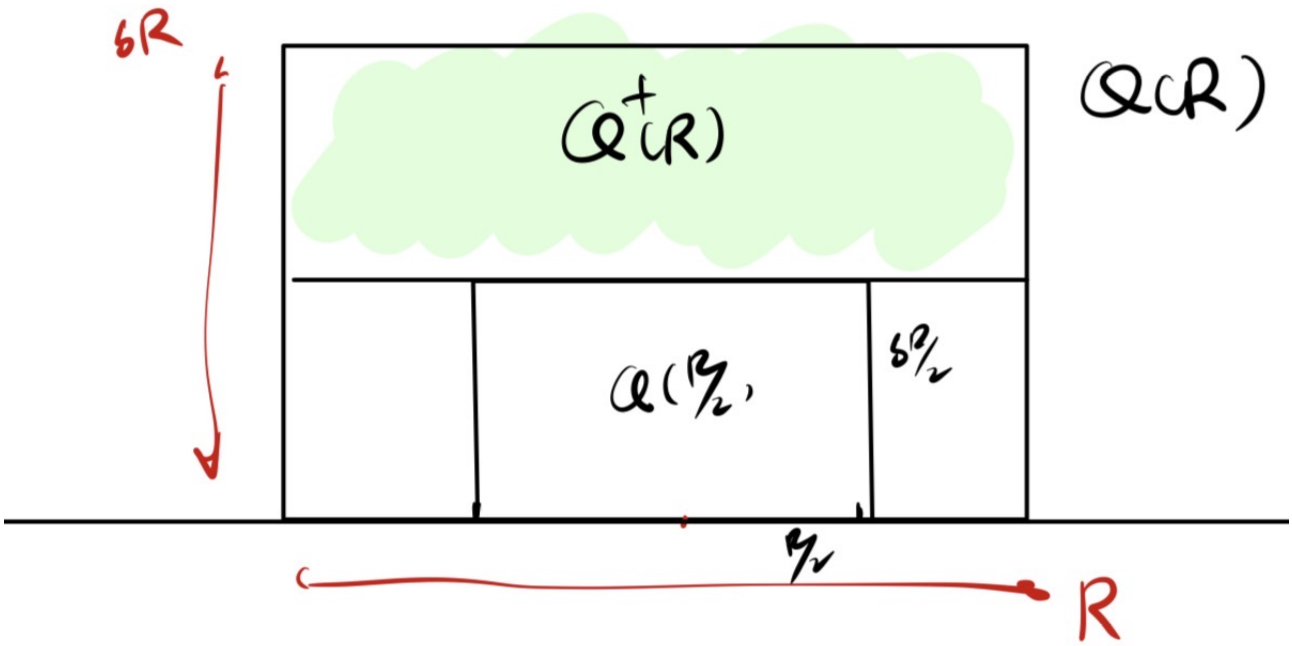


Figure 8.28: Cubes

Caffarelli's Simplification

Lemma 8.6.3 ([Kaz85] Lemma 4.31). *Let $\mathcal{L}u \le f$ in $Q(R)$, and $u \ge 0$ with $u(x',0)=0$. Then*

$$\inf_{Q^+(R)} v \le \frac{2}{\delta} \inf_{Q(R/2)} v + \frac{R}{\lambda} \sup_{B_4^+} |f| \tag{8.125}$$

Proof. Denote γ as the infimum of v on the top

$$\gamma := \inf_{x_n=\delta R,\ |x|'\le R} v(x)$$

Consider barrier

$$z(x) := \gamma x_n \left(\delta - 2\delta \frac{|x'|^2}{R^2} + \frac{x_n}{R}\right) - \frac{x_n(\delta R - x_n)}{2\lambda} \sup_{B_4^+}|f|$$

Let's see what z solves. For $1 \le i, j \le n-1$

$$\begin{aligned}
\partial_i z &= -4\gamma x_n \delta \frac{1}{R^2} x_i \\
\partial_n z &= \gamma \left(\delta - 2\delta \frac{|x'|^2}{R^2} + \frac{x_n}{R}\right) + \frac{\gamma}{R} x_n - \frac{(\delta R - x_n)}{2\lambda} \sup_{B_4^+}|f| + \frac{x_n}{2\lambda} \sup_{B_4^+}|f| \\
\partial_{ij} z &= -4\gamma x_n \delta \frac{1}{R^2} \delta_{ij} \\
\partial_{in} z &= -4\gamma\delta \frac{1}{R^2} x_i \\
\partial_{nn} z &= 2\frac{\gamma}{R} + \frac{1}{\lambda} \sup_{B_4^+}|f|
\end{aligned}$$

so

$$\begin{aligned}
a_{ij}\partial_{ij} z &= -4\gamma x_n \delta \frac{1}{R^2} a_{ii} - 4\gamma\delta \frac{1}{R^2} x_i a_{in} + \left(2\frac{\gamma}{R} + \frac{1}{\lambda}\sup_{B_4^+}|f|\right) a_{nn} \\
&\ge \sup_{B_4^+}|f| \ge f \qquad \text{for } \delta \text{ sufficiently small}
\end{aligned}$$

On the other hand, we consider the boundary values for z and u on $\partial Q(R)$. Note $u \ge 0$ in $Q(R)$ by assumption.

1. On $\{x_n = 0\}$, $u = 0 = z$
2. On the sides $\{|x|' = R,\ 0 < x_n < \delta R\}$

$$z(x) = \gamma x_n \left(-\delta + \frac{x_n}{R}\right) - \frac{x_n(\delta R - x_n)}{2\lambda} \sup_{B_4^+}|f| \le \gamma x_n \left(-\delta + \frac{x_n}{R}\right) < 0 \le u$$

3. On the top $\{|x|' < R,\ x_n = \delta R\}$

$$\begin{aligned}
z(x) &= \gamma\delta R \left(\delta - 2\delta \frac{|x'|^2}{R^2} + \delta\right) \le 2\gamma\delta^2 R \\
&\overset{\text{choose } \delta \text{ small}}{\le} \gamma\delta R = \inf_{x_n=\delta R,\ |x|'\le R} \frac{u(x)}{x_n} \cdot \delta R \le u
\end{aligned}$$

Therefore Comparison Principle applies so that

$$u \ge z \qquad Q(R)$$

Therefore

$$v(x) \ge \frac{z(x)}{x_n} \qquad \forall\, x \in Q(R),\ x_n > 0$$

In particular, restricting on $x \in Q(R/2)$, the RHS is

$$\begin{aligned}
\frac{z(x)}{x_n} &= \gamma\left(\delta - 2\delta\frac{|x'|^2}{R^2} + \frac{x_n}{R}\right) - \frac{(\delta R - x_n)}{2\lambda}\sup_{B_4^+}|f| \\
&\geq \gamma\frac{\delta}{2} - \frac{(\delta R - x_n)}{2\lambda}\sup_{B_4^+}|f| \qquad \text{using } |x'| \leq \frac{R}{2} \\
&\geq \gamma\frac{\delta}{2} - \frac{\delta R}{2\lambda}\sup_{B_4^+}|f| = \frac{\delta}{2}\left(\gamma - \frac{R}{\lambda}\sup_{B_4^+}|f|\right) \\
&\geq \frac{\delta}{2}\left(\inf_{Q^+(R)} v - \frac{R}{\lambda}\sup_{B_4^+}|f|\right)
\end{aligned}$$

Taking infimum in $x \in Q(R/2)$ on LHS yields the result. □

Harnack Inequality

Lemma 8.6.4 ([Kaz85] Lemma 4.35). *Let $u \in C^2(\overline{B_4^+})$ solve $\mathcal{L}u = f$ in B_4^+ with $u \geq 0$ in $B_{4r}(x) \subseteq B_4^+$. Then there is $c = c(n, \lambda, \Omega, K) > 0$ s.t.*

$$\sup_{B_r(x)} u \leq c\left(\inf_{B_r(x)} u + r^2\sup_{B_4^+}|f|\right) \tag{8.126}$$

Consequently, if $\mathcal{L}u = f$ in B_4^+ with $u \geq 0$ in $Q(2R)$, $R \leq 1$, then there is possibly different universal constant $c > 0$ s.t. for $v = \frac{u(x)}{x_n}$

$$\sup_{Q^+(R)} v \leq c\left(\inf_{Q^+(R)} v + R\sup_{B_4^+}|f|\right) \tag{8.127}$$

Proof. The first result (8.126) is a direct consequence of Harnack Inequality for [GT01] Theorem 9.20 and [GT01] Theorem 9.22. For (8.127), take any point $x \in Q^+(R)$ and consider applying (8.126) to ball $B_{4r}(x)$ where $4r = \frac{\delta R}{2}$. Up to different constant we get

$$\sup_{Q^+(R)} u \leq c\left(\inf_{Q^+(R)} u + R^2\sup_{B_4^+}|f|\right)$$

But in $Q^+(R)$ we have (this is why we stay a distance away from $x_n = 0$!)

$$\frac{\delta R}{2} \leq x_n \leq \delta R$$

substituting for $u = x_n v$ yields the result. □

Proof of Theorem 8.6.2 We denote

$$m(R) := \inf_{Q(R)} v, \qquad M(R) := \sup_{Q(R)} v$$

Oscillation Decay. We may apply (8.127) with $u - m(2R)x_n \geq 0$ in $Q(2R)$ so that

$$\begin{aligned}
\sup_{Q^+(R)}(v - m(2R)) &\overset{(8.127)}{\leq} c\left(\inf_{Q^+(R)}(v - m(2R)) + R\sup_{B_4^+}|f|\right) \\
&\overset{(8.125)}{\leq} c_1\left(\inf_{Q(R/2)} v - m(2R) + R\sup_{B_4^+}|f|\right) \\
&\leq c_1\left(m(R/2) - m(2R) + R\sup_{B_4^+}|f|\right)
\end{aligned} \tag{8.128}$$

Applying (8.127) with $M(2R)x_n - u \geq 0$ in $Q(2R)$ so that

$$\begin{aligned}\sup_{Q^+(R)}(M(2R)-v) &\overset{(8.127)}{\leq} c\left(\inf_{Q^+(R)}(M(2R)-u)+R\sup_{B_4^+}|f|\right)\\ &\overset{(8.125)}{\leq} c_1\left(M(2R)-\sup_{Q(R/2)}u+R\sup_{B_4^+}|f|\right)\\ &\leq c_1\left(M(2R)-M(R/2)+R\sup_{B_4^+}|f|\right)\end{aligned}\tag{8.129}$$

Adding (8.128) and (8.129) up yields

$$M(2R)-m(2R)\leq c_2\left(M(2R)-m(2R)-(M(R/2)-m(R/2))+R\sup_{B_4^+}|f|\right)$$

Denote $w(R) := M(R) - m(R)$ as the oscillation of v in $Q(R)$, one get

$$w(R/2)\leq \theta w(2R)+R\sup_{B_4^+}|f|\tag{8.130}$$

where

$$0<\theta=\frac{c_2-1}{c_2}=1-\frac{1}{c_2}<1$$

One may make c_2 larger so one may assume $\frac{1}{4}<\theta<1$.

Induction on Oscillation Decay. We show via induction that

$$w(\frac{1}{4^m})\leq\theta^m\left(w(1)+\frac{2\sup_{B_4^+}|f|}{4\theta-1}\left(1-\frac{1}{(4\theta)^m}\right)\right)\qquad\forall\ m=1,2,\cdots\tag{8.131}$$

$m=1$ is by (8.130). Now assume for $m-1$. To show for m

$$\begin{aligned}w(\frac{1}{4^m}) &\overset{(8.130)}{\leq} \theta w(\frac{1}{4^{m-1}})+2\frac{1}{4^m}\sup_{B_4^+}|f|\\ &\leq\theta\cdot\theta^{m-1}\left(w(1)+\frac{2\sup_{B_4^+}|f|}{4\theta-1}\left(1-\frac{1}{(4\theta)^{m-1}}\right)\right)+2\frac{1}{4^m}\sup_{B_4^+}|f| \qquad \text{inductive hypothesis}\\ &=\theta^m\left(w(1)+\frac{2\sup_{B_4^+}|f|}{4\theta-1}\left(\frac{(4\theta)^m-1+1-4\theta}{(4\theta)^m}\right)\right)+2\frac{1}{4^m}\sup_{B_4^+}|f|\\ &=\theta^m\left(w(1)+\frac{2\sup_{B_4^+}|f|}{4\theta-1}\left(1-\frac{1}{(4\theta)^m}\right)\right)-2\theta^m\sup_{B_4^+}|f|+2\frac{1}{4^m}\sup_{B_4^+}|f|\\ &\leq\theta^m\left(w(1)+\frac{2\sup_{B_4^+}|f|}{4\theta-1}\left(1-\frac{1}{(4\theta)^m}\right)\right)\end{aligned}$$

Thus induction (8.131) holds. In particular

$$w(\frac{1}{4^m})\leq c_3\theta^m\tag{8.132}$$

where

$$c_3=w(1)+\frac{2\sup_{B_4^+}|f|}{4\theta-1}\leq CK$$

Hölder Regularity Now for any $R \leq 1$, there is an integer m s.t.

$$\frac{1}{4^m}<R\leq\frac{1}{4^{m-1}}=4\cdot\frac{1}{4^m}$$

this is to say

$$m\log(\frac{1}{4}) < \log(R) \le (m-1)\log(\frac{1}{4})$$

Further simplify using the identity

$$a^{\log_c b} = a^{\frac{\log b}{\log c}} = b^{\frac{\log a}{\log c}} = b^{\log_c a}$$

gives

$$\begin{aligned} m-1 \le \frac{\log(R)}{\log(1/4)} < m \\ \theta^m < \theta^{\frac{\log(R)}{\log(1/4)}} = R^{\frac{\log(\theta)}{\log(1/4)}} \le \theta^{m-1} \end{aligned}$$

Now since $\frac{1}{4} < \theta < 1$ we may define

$$0 < \alpha := \frac{\log(\theta)}{\log(1/4)} < 1$$

so that

$$\theta^m < R^\alpha \le \theta^{m-1}$$

Now since the oscillation is monotone, and that $R \le \frac{1}{4^{m-1}}$, one obtain

$$\begin{aligned} w(R) \le w(\frac{1}{4^{m-1}}) &\overset{(8.132)}{\le} c_3\theta^{m-1} = \frac{c_3}{\theta}\theta^m \\ &< \frac{c_3}{\theta}R^\alpha \end{aligned}$$

Send $x_n \to 0$ to conclude.

8.6.4 The Dirichlet Problem

8.6.4.1 Method of Continuity for Smooth Concave Equation

Implicit Function Theorem We record the Implicit Function Theorem for Banach Space that we'll use.

Lemma 8.6.5 (Phong Spring 2025; Implicit Function Theorem on Banach Spaces)**.** *Let B_1 and B_2 be Banach Spaces. We consider a map*

$$\begin{aligned} \mathcal{F} : \mathbb{R} \times B_1 &\to B_2 \\ (t,x) &\mapsto \mathcal{F}(t,x) \end{aligned}$$

and we assume

$$\mathcal{F}(t_0, x_0) = 0$$

We assume also that $\mathcal{F} \in C^1$, and the linear mapping

$$\begin{aligned} D_x\mathcal{F}(t_0,x_0) : B_1 &\to B_2 \\ y &\mapsto D_x\mathcal{F}(t_0,x_0)(y) \end{aligned} \tag{8.133}$$

is invertible with a bounded inverse.

Then there exists an interval $(t_0-\varepsilon, t_0+\varepsilon)$ with the property that there exists a neighborhood $U \subseteq B_1$ of x_0, and a unique C^1 map

$$\begin{aligned} g : (t_0-\varepsilon, t_0+\varepsilon) &\to U \\ t &\mapsto g(t) \end{aligned}$$

s.t.

1. *$g(t_0) = x_0$*
2. *One obtain the equation*
$$\mathcal{F}(t, g(t)) = 0 \qquad \forall\ t \in (t_0-\varepsilon, t_0+\varepsilon)$$
3. *Moreover g is uniquely determined in the sense that, for any $(t,x) \in (t_0-\varepsilon, t_0+\varepsilon) \times U$ s.t. $\mathcal{F}(t,x) = 0$, necessarily*
$$x = g(t)$$

Existence Theory for Smooth Concave $\mathcal{F}$ In this section we use

1. $C^{2,\alpha}$ a priori estimate up to the boundary
2. Method of continuity

to prove Existence and Uniqueness result for the Dirichlet Problem

$$\mathcal{F}(D^2u) = 0$$

Assume $\Omega = B_1$.

Theorem 8.6.3 ([CC95] Theorem 9.7)**.** *Let $\mathcal{F}$ be uniformly elliptic, concave and smooth. Let $g \in C^\infty(\overline{B_1})$. Then there exists a unique viscosity solution $u \in C(\overline{B_1})$ to the Dirichlet Problem*

$$\begin{cases} \mathcal{F}(D^2u) = 0 & B_1 \\ u = g & \partial B_1 \end{cases} \tag{8.134}$$

Moreover $u \in C^\infty(\overline{B_1})$ along with the estimates

$$\|u\|_{C^{2,\alpha}(\overline{B_1})} \le C\left(\|g\|_{C^3(\overline{B_1})} + |F(0)|\right) \tag{8.135}$$

$$\|u\|'_{C^{2,\alpha}(\overline{B_{r/2}})} \le C\left(\|u\|_{L^\infty(B_r)} + r^2|F(0)|\right) \qquad \forall\ B_r(x_0) \Subset B_1 \tag{8.136}$$

where $0 < \alpha < 1$, $C > 0$ universal. $\|\cdot\|'_{C^{2,\alpha}}$ denotes the non-dimensional norm.

Proof. **We claim it suffices to show the existence of $u \in C^{2,\alpha}(\overline{B_1})$ for the $0 < \alpha < 1$ as in Theorem 8.6.1**. It follows that

1. From Corollary 8.4.2 we know u must be unique.
2. Using Proposition 8.6.2 we know such $u \in C^{2,\alpha}(\overline{B_1})$ is in fact $u \in C^\infty(\overline{\Omega})$ via bootstrapping.
3. Using Berstein, Evans-Krylov, one obtain interior $C^{2,\alpha}$ estimate once $u \in C^2$, which gives (8.136) upon rescaling.
4. Using $C^{2,\alpha}$ a priori estimate up to the boundary Theorem 8.6.1, we obtain the estimate (8.135).

Method of Continuity.
Consider for $t \in [0,1]$ the family of problems

$$\begin{cases} t\mathcal{F}(D^2u) + (1-t)\Delta u = 0 & B_1 \\ u = g & \partial B_1 \end{cases} \tag{8.137}$$

or letting $v = u - g$ one has an equivalent problem

$$\begin{cases} t\mathcal{F}(D^2(v+g)) + (1-t)\Delta(v+g) = 0 & B_1 \\ v = 0 & \partial B_1 \end{cases} \tag{8.138}$$

Now we denote the space

$$C_0^{2,\alpha}(\overline{B_1}) := \{v \in C^{2,\alpha}(\overline{B_1}) \mid v = 0\ \partial B_1\}$$

and denote the operator as

$$\phi(v,t) := t\mathcal{F}(D^2(v+g)) + (1-t)\Delta(v+g)$$

Notice for any $t \in [0,1]$

$$\begin{aligned} \phi(\cdot,t) : C_0^{2,\alpha}(\overline{B_1}) &\to C^{0,\alpha}(\overline{B_1}) \\ v &\mapsto \phi(v,t) \end{aligned}$$

Consider the set

$$A := \{t \in [0,1] \mid \text{there exists } v \in C_0^{2,\alpha}(\overline{B_1}) \text{ s.t. } \phi(v,t) = 0 \text{ in } \overline{B_1}\}$$

The goal is to show that $1 \in A$, in view of (8.137). To do so, we adopt the topological fact: Since $[0,1]$ is connected, if we're able to show A is nonempty, closed, and open, then $A = [0,1]$. In particular, $1 \in A$.
Let's check one by one.

A **is non-empty** because $0 \in A$ using Schauder Theory for Laplacian.

A **is open** essentially because of Implicit Function Theorem lemma 8.6.5. Recall what we want: That A is open is to say for any $t_0 \in A$, we know there exists a small neighborhood $B_\varepsilon(t_0) = (t_0 - \varepsilon, t_0 + \varepsilon) \cap [0,1]$ around s.t. $B_\varepsilon(t_0) \subseteq A$. The last condition means

$$B_\varepsilon(t_0) \subseteq A \iff \forall\ t \in B_\varepsilon(t_0),\ \text{there exists } v_t \in C_0^{2,\alpha}(\overline{B_1}) \text{ s.t. } \quad \phi(v_t, t) = 0$$

View ϕ via

$$\begin{aligned} \phi : C_0^{2,\alpha}(\overline{B_1}) \times \mathbb{R} &\to C^{0,\alpha}(\overline{B_1}) \\ (v,t) &\mapsto \phi(v,t) \end{aligned}$$

We're tempted to determine locally an implicit function

$$t \mapsto v_t, \qquad \phi(v_t, t) = 0, \qquad \overline{B_1}$$

with the information given

$$\phi(v_{t_0}, t_0) = 0$$

Can we do so? Yes as long as we ensure at the point (v_{t_0}, t_0), the linear mapping (8.133)

$$\begin{aligned} D_v\phi(v_{t_0}, t_0) : C_0^{2,\alpha}(\overline{B_1}) &\to C^{0,\alpha}(\overline{B_1}) \\ w &\mapsto D_v\phi(v_{t_0}, t_0)w \end{aligned}$$

is invertible with a bounded inverse. Let's compute what this is

$$D_v\phi(v,t) = \partial_v \left(t\mathcal{F}(D^2(v+g)) + (1-t)\Delta(v+g) \right)$$

We do so by computing the Fréchet Derivative in Banach Space, i.e., perturb $v + \varepsilon w$, expand, and then keep the $O(\varepsilon)$ term. So let $v + \varepsilon w$ for $v,\ w \in C_0^{2,\alpha}(\overline{B_1})$

$$\begin{aligned} \phi(v+\varepsilon w, t) &= t\mathcal{F}(D^2 v + \varepsilon D^2 w + D^2 g) + (1-t)\Delta(v+g) + (1-t)\varepsilon\Delta w \\ \frac{1}{\varepsilon}\left(\phi(v+\varepsilon w, t) - \phi(v,t)\right) &= t\frac{\mathcal{F}(D^2 v + D^2 g + \varepsilon D^2 w) - \mathcal{F}(D^2 v + D^2 g)}{\varepsilon} + (1-t)\Delta w \\ D_v\phi(v,t)(w) &= t\mathcal{F}_{ij}(D^2 v + D^2 g)\partial_{ij} w + (1-t)\Delta w \\ &= \left(t\mathcal{F}_{ij}(D^2 v + D^2 g) + (1-t)\delta_{ij} \right) \partial_{ij} w \end{aligned}$$

Since $\mathcal{F}$ is a uniformly elliptic operator with elliptic constants λ, Λ, the operator

$$a_{ij} := \left(t\mathcal{F}_{ij}(D^2 v + D^2 g) + (1-t)\delta_{ij} \right)$$

is uniformly elliptic with constants $t\lambda + (1-t)$ and $tn\Lambda + 1 - t$.

Now in particular at the point $t_0 \in A$, one may apply global Schauder Estimate for non-divergence form uniformly elliptic operator a_{ij}, and conclude that

$$\|w\|_{C^{2,\alpha}(\overline{B_1})} \le C(t_0, \lambda, \Lambda, n, g)\, \|a_{ij}\partial_{ij} w\|_{C^{0,\alpha}(\overline{B_1})}$$

Thus

$$D_v\phi(v_{t_0}, t_0)(w) = \left(t_0\mathcal{F}_{ij}(D^2 v_{t_0} + D^2 g) + (1-t_0)\delta_{ij} \right) \partial_{ij} w$$

is invertible with bounded inverse. Implicit Function Theorem Lemma 8.6.5 applies so that there exists a neighborhood $B_\varepsilon(t_0)$ s.t. for any $t \in B_\varepsilon(t_0)$, we know

$$\phi(v_t, t) = 0$$

The existence of $v_t \in C_0^{2,\alpha}(\overline{B_1})$ for each $t \in B_\varepsilon(t_0)$ gives $B_\varepsilon(t_0) \subseteq A$, i.e., openness of A.

A **is closed** essentially because of a priori $C^{2,\alpha}$ estimates. Recall what we want: That A is closed is to say for a sequence $\{t_n\} \subseteq A$ s.t. $t_n \to t_0 \in [0,1]$, in fact $t_0 \in A$. This translates to for a family of operators

$$\phi(\cdot, t_n) = t_n\mathcal{F}(D^2(\cdot + g)) + (1-t_n)\Delta(\cdot + g)$$

with a family of solution $\{v_{t_n}\} \subseteq C_0^{2,\alpha}(\overline{B_1})$

$$\phi(v_{t_n}, t_n) = 0 \qquad \overline{B_1}, \qquad \forall\ n$$

s.t. $t_n \to t_0 \in [0,1]$, i.e. the operators converge

$$\phi(\cdot, t_n) \to \phi(\cdot, t_0) \tag{8.139}$$

We want to show the existence of $v_{t_0} \in C_0^{2,\alpha}(\overline{B_1})$ that solves

$$\phi(v_{t_0}, t_0) = 0 \qquad \overline{B_1}$$

How to obtain the limit v_{t_0} that solves the equation with the desired regularity? We use Ascoli-Arzela! To do so we need to ensure two ingredients

1. Our sequence is uniformly bounded
2. Our sequence is equi-continuous.

We claim the key ingredient is that there exists C independent of t_n s.t.

$$\|v_{t_n}\|_{C^{2,\alpha}(\overline{B_1})} \le C \tag{8.140}$$

For if so, $\{v_{t_n}\}$ is in fact uniformly equi-continuous in C^2 norm. Thus applying Ascoli-Arzela([FRRO22] Theorem 1.7) one obtain $v_{t_0} \in C^{2,\alpha}(\overline{B_1})$ s.t. up to a subsequence

$$\|v_{t_n} - v_{t_0}\|_{C^2(\overline{B_1})} \to 0 \qquad n \to \infty$$

Combining the uniform convergence and convergence of operators (8.139), we know that, up to diagonalization,

$$0 = \lim_{n\to\infty} \phi(v_{t_n}, t_n) = \phi(v_{t_0}, t_0) \qquad \text{uniformly in } x \in \overline{B_1}$$

This is exactly what we want!

Now the final issue, why is (8.140) true? Let's use our a priori $C^{2,\alpha}$ global estimates Theorem 8.6.1. Since $\phi(\cdot, t_n)$ as the family of uniformly elliptic operators has elliptic constants $t_n\lambda + (1-t_n)$, and $t_n\Lambda + (1-t_n)$, one may in fact ensure a uniform bound on elliptic constants that depends on λ, Λ, t_0 by considering n sufficiently large. Thus our estimate reads (let's denote d as our dimension in case of confusion)

$$\begin{aligned}
\|v_{t_n}\|_{C^{2,\alpha}(\overline{B_1})} &\overset{(8.109)}{\le} C(d,\lambda,\Lambda,t_0,\alpha)|\phi(0,t_n)| \\
&\le C(d,\lambda,\Lambda,t_0,\alpha)|t_n\mathcal{F}(D^2 g) + (1-t_n)\Delta g| \\
&\le C(d,\lambda,\Lambda,t_0,\alpha,g) \qquad \text{for } n \text{ sufficiently large}
\end{aligned}$$

This gives our desired estimate (8.140) and we conclude the proof. □

8.6.4.2 Concave but Non-smooth Equations

As a consequence of Theorem 8.6.3, one get existence, uniqueness and regularity result for not necessarily smooth equations.

Proposition 8.6.6 ([CC95] Proposition 9.8)**.** *Let $\mathcal{F}$ be uniformly elliptic, **concave (no smoothness assumption!)**. Let $g \in C(\partial B_1)$.*

Then there exists a unique $u \in C(\overline{B_1})$ to the Dirichlet Problem (8.134)

$$\begin{cases} \mathcal{F}(D^2 u) = 0 & B_1 \\ u = g & \partial B_1 \end{cases}$$

Moreover $u \in C^{2,\alpha}(B_1)$ (not up to the boundary) for universal $0 < \alpha < 1$, along with estimate (8.136)

$$\|u\|'_{C^{2,\alpha}(\overline{B_{r/2}})} \le C\left(\|u\|_{L^\infty(B_r)} + r^2|F(0)|\right) \qquad \forall\ B_r(x_0) \Subset B_1$$

Furthermore, if $g \in C^3(\overline{B_1})$, then $u \in C^{2,\alpha}(\overline{B_1})$ and (8.135) holds

$$\|u\|_{C^{2,\alpha}(\overline{B_1})} \le C\left(\|g\|_{C^3(\overline{B_1})} + |F(0)|\right)$$

Proof. From (8.110) we may assume $\mathcal{F}(0) = 0$.

Convoluting with a mollifier in the space of $\mathrm{Sym}(n)$ symmetric matrices, one can approximate $\mathcal{F}$ with a sequence $\{\mathcal{F}_k\}$ of concave, smooth, uniformly elliptic operators with elliptic constants λ, Λ, that converges uniformly to $\mathcal{F}$ in compact subsets of $\mathrm{Sym}(n)$. Subtracting $\mathcal{F}_k(0)$ from $\mathcal{F}_k$ we may assume $\mathcal{F}_k(0) = 0$.

Also approximate boundary data $g \in C(\partial B_1)$ with a sequence of smooth functions $g_k \in C^\infty(\overline{B_1})$.

The upshot is: **We only constructed convergence in boundary data g_k and operators $\mathcal{F}_k$. What can we say about the convergence in solutions u_k?** There's two steps

1. Convergence in boundary data gives us uniform convergence in u_k.
2. Combining with a prior estimates

$$\|u_k\|_{C^{2,\alpha}_{\text{loc}}} \le C\left(\|u_k\|_{L^\infty(B_1)} + \|g_k\|\right)$$

one has uniformly bounded sequence in $C^{2,\alpha}$. Ascoli-Arzela extracts us a $C^{2,\alpha}$ in the interior solution u.

Solve the sequence of Dirichlet Boundary Value problems

$$\begin{cases} \mathcal{F}_k(D^2u_k) = 0 & B_1 \\ u_k = g_k & \partial B_1 \end{cases}$$

and obtain the solutions $\{u_k\} \subseteq C^\infty(\overline{B_1})$ with estimates (8.135), (8.136). BUT! If we want to use the estimates, one cannot pass information from g_k to g because we have very less regularity of g!

The only thing we can do for g is, by Maximum Principle

$$\|u_k\|_{L^\infty(B_1)} \leq \|g_k\|_{L^\infty(\partial B_1)} \leq \|g\|_{L^\infty(\partial B_1)} + 1$$

for k sufficiently large. Hence the sequence $\{u_k\}$ is uniformly bounded in L^∞.

On the other hand, $\{u_k\}$ are equi-continuous family of functions in $C(\overline{B_1})$ as inherited from the modulus of continuity of boundary data.

Thus via Ascoli-Arzela, up to subsequence, u_k converges uniformly to $u \in C(\overline{B_1})$ in $\overline{B_1}$.

Hence by Proposition 8.1.9, using uniform convergence of $\mathcal{F}_k$ and u_k, we know u is a viscosity solution to

$$\begin{cases} \mathcal{F}(D^2u) = 0 & B_1 \\ u = g & \partial B_1 \end{cases}$$

By Uniqueness Theorem 8.4.2 such viscosity solution $u \in C(\overline{B_1})$ is unique.

The remaining estimates follow from the result of Theorem 8.6.3, and compactness in C^2 of bounded sets in $C^{2,\alpha}$.

$\square$

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