Geometry

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Chapter 1

Smooth Manifolds

1.1 Topological Manifolds

We begin by introducing manifolds. First recall that a topological space $(M, \mathcal{T})$ is a set M with $\mathcal{T}$ a collection of subsets of M, which we call open sets, s.t.

1. $\varnothing$, $M \in \mathcal{T}$
2. arbitrary unions of elements in $\mathcal{T}$ belong to $\mathcal{T}$
3. finite intersection of elements in $\mathcal{T}$ belong to $\mathcal{T}$.

Definition 1.1.1 (Topological Manifold; [Lee12] Chapter 1, Liu 2024)**.** *A topological manifold M of dimension n is a non-empty topological space that is Hausdorff and second countable, s.t. for any $p \in M$, there exists open set $U \subseteq M$ containing p and a homeomorphism ϕ from U to open subset of $\mathbb{R}^n$ (continuous bijection with continuous inverse)*

$$\begin{aligned} \phi : U \subseteq M &\to \phi(U) \subseteq \mathbb{R}^n \\ q &\mapsto \phi(q) := (x_1(q), \cdots, x_n(q)) \end{aligned}$$

The pair (U, ϕ) is a coordinate chart around p. The continuous maps $x_i : U \to \mathbb{R}$ are local coordinates on U.

To check if M is a topological space, of course we need to specify its topology so that we can choose the open set U and homeomorphism ϕ. What are some immediate examples of Topological Manifolds?

Examples of Topological Manifolds

1. *Graphs of Continuous Functions* ([Lee12] Example 1.3). Let $U \subseteq \mathbb{R}^n$ be open and

$$f : U \subseteq \mathbb{R}^n \to \mathbb{R}^k \qquad \text{be continuous function}$$

The graph of f is defined as

$$\operatorname{Graph}(f) := \{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R}^k \mid x \in U\}$$

We equip $\operatorname{Graph}(f) \subseteq \mathbb{R}^n \times \mathbb{R}^k$ with subspace topology, where $\mathbb{R}^n \times \mathbb{R}^k$ is equipped with the product topology. Subspace topology composed with finite product topology preserves 2nd countability and T_2.

Consider the coordinate map

$$\begin{aligned} \pi : \mathbb{R}^n \times \mathbb{R}^k &\to \mathbb{R}^n \\ (x, y) &\mapsto x \end{aligned}$$

and its restriction

$$\begin{aligned} \phi := \pi|_{\operatorname{Graph}(f)} : \operatorname{Graph}(f) &\to U \subseteq \mathbb{R}^n \\ (x, f(x)) &\mapsto x \end{aligned}$$

(a) Since the product topology $\mathbb{R}^n \times \mathbb{R}^k$ makes π continuous, our $\operatorname{Graph}(f)$ with subspace topology preserves continuity of ϕ.

(b) That ϕ is a restriction of coordinate map ensures surjectivity of ϕ. That f is a function ensures injectivity of ϕ. Thus ϕ is bijection.

(c) One define the inverse

$$\phi^{-1} : U \subseteq \mathbb{R}^n \to \text{Graph}(f) \subseteq \mathbb{R}^n \times \mathbb{R}^k$$
$$x \mapsto (x, f(x))$$

ϕ^{-1} is continuous because its restriction to each coordinate is continuous, i.e., let π_i denote restriction to ith coordinate

$$\pi_1 \circ \phi^{-1} = \text{identity}, \qquad \pi_2 \circ \phi^{-1} = f$$

Thus $(\text{Graph}(f), \phi)$ is a global coordinate chart. Graph(f) is homeomorphic to U, and Graph(f) is a topological manifold of dimension n.

2. *Sphere* ([Lee12] Example 1.4). Let $n \geq 1$. Define

$$\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} \mid |x| = (\sum_{i=1}^{n+1} x_i^2)^{\frac{1}{2}} = 1\}$$

We equip $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ with subspace topology, hence inherits 2nd countability and T_2.

For $i = 1, \cdots, n+1$, consider the neighborhoods

$$U_i^+ := \{x \in \mathbb{S}^n \mid x_i > 0\} \qquad U_i^- := \{x \in \mathbb{S}^n \mid x_i < 0\}$$

For each i consider the maps

$$\varphi_i^{\pm} : \{x \in \mathbb{R}^{n+1} \mid |x| < 1,\ x_i = 0\} \to U_i^{\pm}$$
$$x = (x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_{n+1}) \mapsto (x_1, \cdots, x_{i-1}, \pm\sqrt{1 - |x|^2}, x_{i+1}, \cdots, x_{n+1})$$

(a) $\varphi_i^{\pm}$ are well-defined, and continuous because each coordinate is a continuous function.

(b) $\varphi_i^{\pm}$ are bijective. Injectivity is trivial due to matching coordinates. On the other hand, for any $y \in U_i^{\pm} \subseteq \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$, write

$$y = (y_1, \cdots, y_{i-1}, y_i, y_{i+1}, \cdots, y_{n+1})$$

Taking out

$$\tilde{y}_i := (y_1, \cdots, y_{i-1}, 0, y_{i+1}, \cdots, y_{n+1})$$

necessarily yields

$$|\tilde{y}_i|^2 = |y|^2 - y_i^2 = 1 - y_i^2 < 1 \iff y_i \neq 0$$

Thus $\tilde{y}_i \in \{x \in \mathbb{R}^{n+1} \mid |x| < 1,\ x_i = 0\}$. For $y_i > (<)0$, noticing

$$\pm\sqrt{1 - |\tilde{y}_i|^2} = y_i$$

yields $\varphi_i^{\pm}$ are surjective.

(c) As above, the inverse maps are well-defined

$$\phi_i^{\pm} := (\varphi_i^{\pm})^{-1} : U_i^{\pm} \to \{x \in \mathbb{R}^{n+1} \mid |x| < 1,\ x_i = 0\} \subseteq \mathbb{R}^{n+1} \times \{x_i = 0\} \cong \mathbb{R}^n$$
$$(y_1, \cdots, y_{n+1}) \mapsto \tilde{y}_i := (y_1, \cdots, y_{i-1}, 0, y_{i+1}, \cdots, y_{n+1})$$

$\phi_i^{\pm}$ are continuous because each coordinate is a continuous function (notice constant is continuous).

Thus $\mathbb{S}^n$ is equipped with coordinate charts $\{(U_i^{\pm}, \phi_i^{\pm})\}_{i=1}^{n+1}$. Since each $\{x \in \mathbb{R}^{n+1} \mid |x| < 1,\ x_i = 0\}$ is identified as subset of $\mathbb{R}^n$, $\mathbb{S}^n$ is a topological manifold of dimension n.

3. *Product Manifolds* ([Lee12] Example 1.8). Let $M_1, \cdots, M_k$ be topological manifolds of dimension $n_1, \cdots, n_k$. Then the product space is defined as

$$\prod_{i=1}^k M_i := \{(x_1, \cdots, x_k) \mid x_i \in M_i,\ i = 1, \cdots, k\}$$

We equip $\prod_{i=1}^k M_i$ with the product topology, hence inherits 2nd countability and T_2.

For each point $(x_1, \cdots, x_k) \in \prod_{i=1}^k M_i$, one may pick a coordinate chart (U_i, ϕ_i) around x_i for M_i. We define the product map

$$\phi_1 \times \cdots \times \phi_k : (\prod_{i=1}^k U_i, \mathcal{T}_{U\,\textbf{product}}) \to (\prod_{i=1}^k \phi_i(U_i), \mathcal{T}_{\phi(U)\,\textbf{product}})$$
$$(y_1, \cdots, y_k) \mapsto (\phi_1(y_1), \cdots, \phi_k(y_k))$$

(a) By definition of product topology, and characterisation using coordinate maps, the product map is continuous.

(b) Using each ϕ_i is bijection, the product map remains bijection.

(c) Define the inverse

$$(\phi_1 \times \cdots \times \phi_k)^{-1} : (\prod_{i=1}^{k} \phi_i(U_i), \mathcal{T}_{\phi(U)\,\textbf{product}}) \to (\prod_{i=1}^{k} U_i, \mathcal{T}_{U\,\textbf{product}})$$
$$(y_1, \cdots, y_k) \mapsto (\phi_1^{-1}(y_1), \cdots, \phi_k^{-1}(y_k))$$

Note

$$(\phi_1 \times \cdots \times \phi_k)^{-1} = \phi_1^{-1} \times \cdots \times \phi_k^{-1}$$

remains a product map of continuous functions, hence the inverse is continuous.

Thus $\prod_{i=1}^k M_i$ is equipped with coordinate charts $\{(U_1 \times \cdots \times U_k, \phi_1 \times \cdots \times \phi_k)\}$ around each $x = (x_1, \cdots, x_k)$. Since each

$$\prod_{i=1}^{k} \phi_i(U_i) \subseteq \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$$

the product manifold $\prod_{i=1}^k M_i$ is a topological manifold of dimension $n_1 + \cdots n_k$.

Consequently, the n-torus as the n-product of $\mathbb{S}^1$ ([Lee12] Example 1.9)

$$\mathbb{T}^n := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$$

is a topological manifold of dimension n.

4. *Real Projective Space* ([Lee12] Example 1.5). Let $n \geq 1$. Define

$$P_n(\mathbb{R}) \equiv \mathbb{RP}^n := \{\ell \subseteq \mathbb{R}^{n+1} \mid \ell \text{ is 1-dim vector subspace}\}$$

(a) One has a natural bijection between $\mathbb{RR}^n$ with the following collections of equivalence classes

$$(\mathbb{R}^{n+1} \setminus \{0\})/\sim$$

where

$$x \sim y \in \mathbb{R}^{n+1} \setminus \{0\} \iff \exists\, \lambda \in \mathbb{R} \setminus \{0\} \qquad \text{s.t.} \qquad y = \lambda x$$

To equip a topology on the quotient space $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim$, consider the canonical surjection

$$\pi : \mathbb{R}^{n+1} \setminus \{0\} \to (\mathbb{R}^{n+1} \setminus \{0\})/\sim$$
$$x = (x_1, \cdots, x_{n+1}) \mapsto \pi(x) = [x_1, \cdots, x_{n+1}]$$

and we declare the quotient topology on $\mathbb{RP}^n$ induced by the surjection π.

(b) Equivalently, consider another collection of equivalence classes

$$\mathbb{S}^n/\sim$$

where in the case the relation is defined as

$$x \sim y \in \mathbb{S}^n \iff y = -x$$

One may equivalently equip a topology on $\mathbb{RP}^n = \mathbb{S}^n/\sim$ by considering the canonical surjection

$$\tilde{\pi} : \mathbb{S}^n \to \mathbb{S}^n/\sim$$
$$x = (x_1, \cdots, x_{n+1}) \mapsto \tilde{\pi}(x) = [x_1, \cdots, x_{n+1}]$$

and declaring the quotient topology on $\mathbb{RP}^n$ as induced by π.

The two definitions induce the same topology on $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim = \mathbb{S}^n/\sim$. Now we check compactness and Hausdorff.

(a) Since $\mathbb{S}^n$ is compact and $\pi|_{\mathbb{S}^n}$ is continuous, $\mathbb{RP}^n = \mathbb{S}^n/\sim$ remains compact.

(b) Take any $[x] \neq [y] \in \mathbb{S}^n/\sim$, and consider the four points $\pm x$, $\pm y$. Using $\mathbb{S}^n$ is Hausdorff, one may take disjoint open sets U, V on $\mathbb{S}^n$ s.t.

$$(U \cup -U) \cap (V \cup -V) = \varnothing$$

Then after projection $\pi(U)$ and $\pi(V)$ are disjoint open sets covering x and y. Hence Hausdorff.

Finally let's see $\mathbb{RP}^n$ is a topological space. For $i = 1, \cdots, n+1$ define

$$U_i := \{[x_1, \cdots, x_{n+1}] \in \mathbb{RP}^n \mid (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\},\ x_i \neq 0\} \tag{1.1}$$

Via the quotient topology U_i are indeed open sets. Define for $i = 1, \cdots, n+1$

$$\begin{aligned} \phi_i : U_i \subseteq \mathbb{RP}^n &\to \mathbb{R}^n \\ [x_1, \cdots, x_{n+1}] &\mapsto \left(\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_{n+1}}{x_i}\right) \end{aligned} \tag{1.2}$$

(a) ϕ_i are well-defined using U_i, and ϕ_i are continuous because one can identify

$$\begin{aligned} \varphi_i := \phi_i \circ \pi : \pi^{-1}(U_i) \subseteq \mathbb{R}^{n+1} \setminus \{0\} &\to \mathbb{R}^n \\ (x_1, \cdots, x_{n+1}) &\mapsto \left(\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_{n+1}}{x_i}\right) \end{aligned}$$

Indeed φ_i are continuous, so using characterisation property of quotient map one conclude ϕ_i are continuous.

(b) ϕ_i are injective because assuming

$$\left(\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_{n+1}}{x_i}\right) = (\frac{y_1}{y_i}, \cdots, \frac{y_{i-1}}{y_i}, \frac{y_{i+1}}{y_i}, \cdots, \frac{y_{n+1}}{y_i})$$

yields

$$x_j = \frac{x_i}{y_i} y_j \qquad \forall\ j = 1, \cdots, n+1$$

which is to say

$$x \sim y$$

Thus ϕ_i is indeed bijection onto its range.

(c) Define its inverse as

$$\begin{aligned} \phi_i^{-1} : \phi_i(U_i) \subseteq \mathbb{R}^n &\to U_i \subseteq \mathbb{RP}^n \\ (y_1, \cdots, y_n) &\mapsto [y_1, \cdots, y_{i-1}, 1, y_i, \cdots, y_n] \end{aligned}$$

Consider function for $i = 1, \cdots, n+1$

$$\begin{aligned} s_i : \mathbb{R}^n &\to \mathbb{R}^{n+1} \setminus \{0\} \\ (y_1, \cdots, y_n) &\mapsto (y_1, \cdots, y_{i-1}, 1, y_i, \cdots, y_n) \end{aligned}$$

which is indeed continuous. Then identifying

$$\phi_i^{-1} = \pi \circ s_i$$

as composition of continuous functions yields continuity of ϕ_i^{-1}. In fact, one has the following diagram for each $i = 1, \cdots, n+1$

$$\begin{array}{ccccc} \mathbb{R}^{n+1} \setminus \{0\} & \supseteq & \pi^{-1}(U_i) & \overset{s_i}{\nwarrow} & \\ \downarrow \pi & & \downarrow \pi_i & & \\ \mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim & \supseteq & U_i & \underset{\phi_i^{-1}}{\overset{\phi_i}{\rightleftarrows}} & \phi_i(U_i) \subseteq \mathbb{R}^n \end{array}$$

Thus $\mathbb{RP}^n$ is equipped with coordinate charts $\{(U_i, \phi_i)\}_{i=1}^{n+1}$ which are local homeomorphisms to subsets of $\mathbb{R}^n$. $\mathbb{RP}^n$ is therefore a topological manifold of dimension n.

1.2 Smooth Structure

Now one need to put structure indicating 'smoothness' on the manifold.

Definition 1.2.1 (Smooth Structure; [Lee12] Chapter 1, Liu 2024)**.** *Let M be a topological manifold of dimension n. Let $k \in \mathbb{N}$ or ∞.*

1. *An atlas of a topological manifold M is a collection of charts that covers the whole manifold*

$$\Phi = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{A}} \qquad s.t. \qquad \bigcup_\alpha U_\alpha = M$$

and that the transition functions for any two charts (U_α, ϕ_α), (U_β, ϕ_β)

$$\begin{aligned} \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n &\to \phi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n \\ (x_1, \cdots, x_n) &\mapsto (y_1, \cdots, y_n) \end{aligned} \tag{1.3}$$

are homeomorphisms.

2. *A C^k atlas is an atlas whose transition functions are C^k diffeomorphisms, i.e., for any two charts (U_α, ϕ_α), (U_β, ϕ_β), the transition functions $\phi_\alpha \circ \phi_\beta^{-1}$ (1.3) is C^k, bijective, and admits a C^k inverse $\phi_\beta \circ \phi_\alpha^{-1}$.*

3. *Two C^k atlas $\Phi = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{A}}$ and $\Psi = \{(V_\beta, \psi_\beta)\}_{\beta \in \mathcal{B}}$ are equivalent if*

$$\Phi \cup \Psi = \{(U_\alpha, \phi_\alpha), (V_\beta, \psi_\beta)\}_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}}$$

is again a C^k atlas. An equivalence class of some C^k atlas is called a C^k differentiable structure.

4. *A topological manifold of dimension n equipped with a C^k differentiable structure is called a C^k manifold of dimension n.*

Remark 1.2.1. *To show whether two given charts admits a C^k transition function in between, it suffices to check*

1. *$\phi_\alpha \circ \phi_\beta^{-1}$ is C^k*

2. *$\phi_\alpha \circ \phi_\beta^{-1}$ is injective (hence taking the range yields bijection)*

3. *$\phi_\alpha \circ \phi_\beta^{-1}$ has non-singular Jacobian at each point, i.e.,*

$$\forall\, x \in \phi_\beta(U_\alpha \cap U_\beta), \qquad |\det(\frac{\partial(\phi_\alpha \circ \phi_\beta^{-1})_i}{\partial x_j})(x)| \neq 0$$

Hence by Inverse Function Theorem, the inverse defines C^k function.

Checking the above gives a C^k atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{A}}$. One may consider its equivalence class, which gives us a C^k differentiable structure. Alternatively, given our C^k atlas Φ, one may consider taking the unique maximal C^k atlas containing Φ (not properly contained in any larger C^k atlas). Such maximal C^k atlas is in fact the equivalence class of Φ ([Lee12] Proposition 1.17).

Examples of Smooth Manifolds

1. For M C^k-manifold of dimension m, $U \subset M$ open. $\Phi := \{(U_\alpha,\, \phi_\alpha) \mid \alpha \in I\}$ some C^k-atlas of M. Then $\Phi_U := \{(U_\alpha \cap U,\, \phi_\alpha|_{U_\alpha \cap U}) \mid \alpha \in I,\, U_\alpha \cap U \neq \varnothing\}$ is C^k-atlas for U. So U is a C^k-manifold of dimension m.

2. *Real Projective Space* ([Lee12] Example 1.33). Recall one has constructed open charts $\{(U_i, \phi_i)\}_{i=1}^{n+1}$ (1.1), (1.2) for $\mathbb{RP}^n$. Since ϕ_i are homeomorphism, the compositions over (assume $1 < i < j < n+1$)

$$\begin{aligned} \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \subseteq \mathbb{R}^n &\to \phi_i(U_i \cap U_j) \subseteq \mathbb{R}^n \\ (x_1, \cdots, x_n) &\mapsto \phi_i([x_1, \cdots, x_{j-1}, 1, x_j, \cdots, x_n]) \\ &= \left(\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_{j-1}}{x_i}, \frac{1}{x_i}, \frac{x_j}{x_i}, \cdots, \frac{x_n}{x_i}\right) \end{aligned} \tag{1.4}$$

is indeed homeomorphism. Alternatively from the explicit expression (1.4), $\phi_i \circ \phi_j^{-1}$ is indeed C^∞ over $\phi_j(U_i \cap U_j)$ since $x_i \neq 0$. Bijection follows from both as homeomorphisms, while smooth inverse follows merely by switching the roles of ϕ_i and ϕ_j. Thus $(\mathbb{RP}^n, \Phi)$ where $\Phi = \{(U_i, \phi_i)\}_{i=1}^{n+1}$ with (1.1), (1.2), defines a C^∞ manifold of dimension n.

3. *Finite Dimensional Vector Spaces* ([Lee12] Example 1.24). Let V be any finite n-dimensional vector space, any norm on V determines a topology (independent of choice of norm). Each ordered basis $\{e_1, \cdots, e_n\}$ for V defines a basis linear isomorphism

$$E : \mathbb{R}^n \to V$$
$$x = (x_1, \cdots, x_n) \mapsto \sum_{i=1}^{n} x_i e_i$$

For any topology on V, E is a homeomorphism. Thus (V, E^{-1}) is a chart. For any other basis linear isomorphism $\tilde{E}$ w.r.t. basis $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$, note there is an invertible matrix $A = a_{ij}$ s.t.

$$e_i = \sum_{j=1}^{n} a_{ij}\tilde{e}_j$$

Thus the transition map between two charts is given by

$$\tilde{E}^{-1} \circ E : \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto \tilde{x} = (\sum_{i=1}^{n} a_{ij} x_i)_{1 \le j \le n}$$

which is given by

$$\sum_{j=1}^{n} \tilde{x}_j \tilde{e}_j = \sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} a_{ij}\tilde{e}_j$$

Thus $\tilde{E}^{-1} \circ E$ is an invertible map and hence a diffeomorphism. The collection of all such charts defines a smooth structure on V.

1.3 Differentiable Maps

C^k Differentiable/ Smooth Map

Definition 1.3.1 (C^k maps)**.** *Let M be C^ℓ manifold of dimension m and N a C^ℓ manifold of dimension n.*

Let $1 \le k \le \ell \le \infty$. A continuous map $f : M \to N$ is called C^k-differentiable if for any $p \in M$, there exists a chart (U, ϕ) for M around p and (V, ψ) for N around $f(p)$ s.t. $f(U) \subset V$, and $g := \psi \circ f \circ \phi^{-1}$ is C^k.

$$\begin{array}{ccccccc} M & \overset{open}{\supseteq} & p \in U & \xrightarrow{f} & V & \overset{open}{\subseteq} & N \\ & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \\ \mathbb{R}^m & \overset{open}{\supseteq} & \phi(U) & \xrightarrow{g} & \psi(V) & \overset{open}{\subseteq} & \mathbb{R}^n \end{array}$$

When $k = \infty$, C^∞ maps are called smooth maps.

Well-definedness. Indeed if $\tilde{g} := \tilde{\psi} \circ f \circ \tilde{\phi}^{-1}$ is another composition for $(\tilde{U}, \tilde{\phi})$ chart of M around p and $(\tilde{V}, \tilde{\psi})$ chart of N around $f(p)$ then

$$\tilde{g} = (\tilde{\psi} \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \tilde{\phi}^{-1}) = (\tilde{\psi} \circ \psi^{-1}) \circ g \circ (\phi \circ \tilde{\phi}^{-1})$$

remains C^k. This is because transition functions are C^ℓ diffeomorphisms and g is C^k. Hence Definition 1.3.1 works for any charts, and f as C^k map is well-defined. □

Example 1.3.1 (Projection for $\mathbb{RP}^n$)**.** *Recall*

$$\pi : \mathbb{R}^{n+1} \setminus \{0\} \to P_n(\mathbb{R})$$
$$p = (p_1, \cdots, p_{n+1}) \mapsto [p_1, \cdots, p_{n+1}]$$

We prove π is C^∞ map.

Proof. For any $p \in \mathbb{R}^{n+1} \setminus \{0\}$, recall U_i and ϕ_i as in (1.1) and (1.2). $\pi(p) \in P_n(\mathbb{R})$, so there exists some i s.t. $\pi(p) \in U_i$. Hence $p \in \pi^{-1}(U_i)$.

$$\begin{array}{ccccccc} \mathbb{R}^{n+1} \setminus \{0\} & \overset{open}{\supseteq} & p \in \pi^{-1}(U_i) & \xrightarrow{\pi} & U_i & \overset{open}{\subseteq} & P_n(\mathbb{R}) \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \phi_i} & & \\ \mathbb{R}^{n+1} & \overset{open}{\supseteq} & \pi^{-1}(U_i) & \xrightarrow{g} & \mathbb{R}^n & \overset{open}{\subseteq} & \mathbb{R}^n \end{array}$$

$g := \phi_i \circ \pi \circ id^{-1} : \pi^{-1}(U_i) \subset \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^n$ s.t.

$$g(x_1, \cdots, x_{n+1}) := (\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_{n+1}}{x_i})$$

is a C^∞ map. □

Diffeomorphism

Definition 1.3.2 (Diffeomorphism). *M, N C^∞ manifold. $f : M \to N$ continuous. $\dim M = m$, $\dim N = n$.*

- *f is C^∞ diffeomorphism if f is a homeomorphism, and f, f^{-1} are C^∞ maps. In particular, $m = n$.*
- *For $p \in M$, f is a local diffeomorphism (C^∞) at p if there exist a open neighborhood U of p in M and V of $f(p)$ in N s.t. $f|_U : U \to V$ is a C^∞-diffeomorphism. In particular, $m = n$.*

1.4 Submersion/ Immersion/ Embedding

Submersion/Immersion

Definition 1.4.1 (Submersion/Immersion in $\mathbb{R}^m$). *Let $f = (f_1, \cdots, f_n) : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be C^k-map for $1 \le k \le \infty$, and $U \subseteq \mathbb{R}^m$ open.*

We call f a submersion (resp. immersion) at $x = (x_1, \cdots, x_m) \in U$ if the 'differential'

$$df_x := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_m}(x) \\ \frac{\partial f_n}{\partial x_1}(x) & \cdots & \frac{\partial f_n}{\partial x_m}(x) \end{pmatrix} : \mathbb{R}^m \to \mathbb{R}^n \qquad \textit{is surjective (resp. injective)}$$

We call f a submersion (resp. immersion) if f is a submersion (resp. immersion) at every $x \in U$.

Note if f is a submersion (resp. immersion) at some point $x \in U$, then $n \le m$ (resp. $n \ge m$)

Example 1.4.1 (Canonical Examples). *1. For $m \ge n$, the projection*

$$\begin{aligned} \pi : \mathbb{R}^m &\to \mathbb{R}^n \\ (x_1, \cdots, x_m) &\mapsto \pi(x_1, \cdots, x_m) := (x_1, \cdots, x_n) \end{aligned}$$

is the canonical submersion. Here $d\pi_x = \pi : \mathbb{R}^m \to \mathbb{R}^n$ for any $x \in \mathbb{R}^m$.

2. For $m \le n$,

$$\begin{aligned} i : \mathbb{R}^m &\to \mathbb{R}^n \\ (x_1, \cdots, x_m) &\mapsto i(x_1, \cdots, x_m) := (x_1, \cdots, x_m, 0, \cdots, 0) \end{aligned}$$

is the canonical immersion. Here $di_x = i : \mathbb{R}^m \to \mathbb{R}^n$ for any $x \in \mathbb{R}^m$.

Definition 1.4.2 (Submersion/Immersion). *Let M and N be C^∞-manifold of dimension m, n. Let $f : M \to N$ be a C^∞ map.*

We call $f : M \to N$ a submersion(immersion) at $p \in M$ if there exists (U, ϕ) chart for M around p and (V, ψ) chart for N around $f(p)$ s.t.

- *$f(U) \subseteq V$ and*
- *$g := \psi \circ f \circ \phi^{-1}$ the C^∞ map is a submersion (resp. immersion) at $\phi(p)$, which implies $n \le m$ (resp. $n \ge m$).*

$$\begin{array}{ccccccc} M & \overset{open}{\supseteq} & p \in U & \xrightarrow{f} & f(p) \in V & \overset{open}{\subseteq} & N \\ & & \downarrow \phi & & \downarrow \psi & & \\ \mathbb{R}^m & \overset{open}{\supseteq} & \phi(p) \in \phi(U) & \xrightarrow{g} & \psi(V) & \overset{open}{\subseteq} & \mathbb{R}^n \end{array}$$

We call f a submersion (resp. immersion) if f is a submersion (resp. immersion) at any point $p \in M$.

Well-definedness. $\tilde{g} = (\tilde{\psi} \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \tilde{\phi}^{-1}) = (\tilde{\psi} \circ \psi^{-1}) \circ g \circ (\phi \circ \tilde{\phi}^{-1})$ and so

$$d\tilde{g}_{\tilde{\phi}(p)} = d(\tilde{\psi} \circ \psi^{-1})_{g(\phi(p))} \circ (dg)_{\phi(p)} \circ d(\phi \circ \tilde{\phi}^{-1})_{\tilde{\phi}(p)} \qquad \text{is surjective (resp. injective)}$$

for $(\tilde{U}, \tilde{\phi})$ another chart of M around p and $(\tilde{V}, \tilde{\psi})$ another chart of N around $f(p)$ s.t. $f(\tilde{U}) \subset \tilde{V}$. □

Proposition 1.4.1 (Rank Theorem). *Let M be C^∞-manifold of dimension m and N C^∞-manifold of dimension n.*

- *If f is a submersion (resp. immersion) at $p \in M$ (so $m \geq n$ ($m \leq n$)), then there exists charts (U, ϕ) for M around p and (V, ψ) for N around $f(p)$ s.t.*

$$\phi(p) = 0 \in \mathbb{R}^m \qquad \psi(f(p)) = 0 \in \mathbb{R}^n$$

and

$$g = \psi \circ f \circ \phi^{-1} : \phi(U) \subseteq \mathbb{R}^m \to \psi(V) \subseteq \mathbb{R}^n \qquad \textit{is the canoncial submersion (resp. immersion)}$$

i.e.

$$g(x_1, \cdots, x_m) = (x_1, \cdots, x_n) \qquad (\textit{resp. } g(x_1, \cdots, x_m) = (x_1, \cdots, x_m, 0, \cdots, 0))$$

- *If f is both a submersion and an immersion at p, i.e., $dg_0 : \mathbb{R}^m \to \mathbb{R}^{n=m}$ is a linear isomorphism, then f is a local diffeomorphism at p.*

Smooth Embeddings

Definition 1.4.3 (C^∞ Embedding)**.** *Let $f : M \to N$ be C^∞ map between C^∞-manifolds, where dimension $M = m$, dimension $N = n$. We say f is a smooth embedding if*

- *f is a smooth immersion at any point $p \in M$ (implies $m \leq n$) and*
- *$f : M \to f(M) \subseteq N$ is a homeomorphism (bijection onto its range, and has continuous inverse) w.r.t. the subspace topology.*

Heuristically, we call $f(M)$ a C^∞ submanifold of N of dimension m.

Example 1.4.2. *$f : \mathbb{R} \to \mathbb{R}^2$ for $f(t) := (x(t), y(t))$, $f'(t) = (x'(t), y'(t))$, then*

$$df_t : \mathbb{R} \to \mathbb{R}^2 \qquad s.t. \qquad df_t(v) := \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} v$$

f is immersion at t iff $f'(t) \neq (0, 0)$. For example

- *$f(t) = (t, t^2)$, $f'(t) = (1, 2t)$ is a immersion, and in fact, C^∞-embedding since f is a homeomorphism (in particular, bijective) from $\mathbb{R}$ onto $f(\mathbb{R})$.*
- *$f(t) = (\cos t, \sin t)$ then $f'(t) = (-\sin t, \cos t)$ so $f(\mathbb{R}) = \mathbb{S}^1$. This is immersion but not embedding because f is not injective.*
- *$f(t) = (t^3 - 4t, t^2 - 4)$ then $f'(t) = (3t^2 - 4, 2t)$. f is a immersion but not an embedding because f is not injective at $(0, 0)$. Note both $t = -2$ and $t = 2$ correspond to $f(-2) = f(2) = (0, 0)$.*
- *$f(t) = (t^3, t^2)$, $f'(t) = (3t^2, 2t)$. This is not immersion at $t = 0$. But $f(\mathbb{R})$ is homeomorphic to $\mathbb{R}$.*

Example 1.4.3 (counter-example for injective immersion but not embedding)**.** *$f : (-3, 0) \to \mathbb{R}^2$ smooth*

$$f(t) = \begin{cases} (0, -t-2) & -3 < t < -1 \\ \cdots & -1 < t < \frac{-1}{\pi} \\ (-t, -\sin(\frac{1}{t})) & \frac{-1}{\pi} < t < 0 \end{cases}$$

This is not an embedding because $f(-3, 0) \subseteq \mathbb{R}^2$ is not a topological manifold. In particular, f^{-1} is not continuous at the point $(0, 0)$, hence that f needs to be homeomorphism fails.

1.5 Submanifolds

Submanifold In a more direct manner, one define submanifold by directly viewing M as subset of N.

Definition 1.5.1 (Submanifold)**.** *Let N be C^∞ manifold of dimension n, M subset of N.*
We call M a C^∞ submanifold of N of dimension $m \leq n$ if

- *for any $p \in M$, there exists chart (U, ϕ) for N around p s.t. $\phi(p) = 0 \in \mathbb{R}^n$ and*
- *$\phi(U \cap M) = \phi(U) \cap (\mathbb{R}^m \times \{0\})$.*

$$\begin{array}{ccccccc} M & \overset{open}{\supseteq} & p \in U \cap M & \xrightarrow{\quad id \quad} & p \in U & \overset{open}{\subseteq} & N \\ & & \Big\downarrow \phi|_{U \cap M} & & \Big\downarrow \phi & & \\ \mathbb{R}^m & \overset{open}{\supseteq} & \phi(U) \cap (\mathbb{R}^m \times \{0\}) & \longrightarrow & \phi(p) = 0 \in \phi(U) & \overset{open}{\subseteq} & \mathbb{R}^n \end{array}$$

Well-definedness. For any $p \in M$, there exists local charts (U_p, ϕ_p) for N around p s.t. $\phi_p(p) = 0 \in \mathbb{R}^n$. Moreover, $\phi_p(U_p \cap M) = \phi_p(U_p) \cap (\mathbb{R}^m \times \{0\})$. One wish to define an Atlas on M. Indeed, let $\Phi_M := \left\{(U_p \cap M,\ \phi_p|_{U_p \cap M}) \mid p \in M\right\}$. Since U_p are open in N, $M \subseteq N$, so w.r.t. the subspace topology, $U_p \cap M$ are open neighborhoods of p in M. Moreover,

$$\phi_p(U_p \cap M) = \phi_p(U_p) \cap (\mathbb{R}^m \times \{0\}) \subseteq (\mathbb{R}^m \times \{0\}) \cong \mathbb{R}^m$$

are open w.r.t. subspace topology. Hence $\phi_p|_{U_p \cap M}$ are local homeomorphisms to subsets of $\mathbb{R}^m$, equipping M with topological m-manifold structure. That $M = M \cap N = \bigcup_{p \in M} M \cap U_p$ and transition functions inherits C^∞ w.r.t. subspace topology make M a m-dim C^∞ manifold. □

Alternatively, one may define as follows.

Definition 1.5.2 (Embedded Submanifold). *Let N be C^∞ manifold. An embedded submanifold $M \subseteq N$ is a subset that is a manifold in the subspace topology, endowed with a smooth structure s.t. the inclusion map*

$$M \hookrightarrow N$$

is a smooth embedding.

Let's justify the equivalence between definitions for submanifolds.

Proposition 1.5.1 ([Lee12] Proposition 5.2). *Let M, N be C^∞ manifold, and $f : M \to N$ be smooth embedding.*

Then w.r.t. the subspace topology, $f(M)$ is a topological manifold, and there is a unique smooth structure making it into an embedded submanifold of N s.t. f is a diffeomorphism onto its image.

Proof. Let $\{(U_\alpha, \phi_\alpha)\}_\alpha$ be C^∞ atlas on M. Define an atlas on $f(M)$ via $\{f(U_\alpha), \phi_\alpha \circ f^{-1}\}_\alpha$. This is indeed a smooth structure on $f(M)$ since

$$\phi_\alpha \circ f^{-1} \circ (\phi_\beta \circ f^{-1})^{-1} = \phi_\alpha \circ \phi_\beta^{-1} \qquad \text{are diffeomorphisms}$$

Now we check $f(M)$ is embedded submanifold of N, i.e., $f(M) \hookrightarrow N$ is smooth embedding. Indeed, since the inclusion is a composition of diffeomorphism and a smooth embedding.

$$f(M) \xrightarrow{f^{-1}} M \xrightarrow{f} N$$

□

Preimage Theorem Now we discuss tool to construct a smooth submanifold using preimage of a regular value.

Remark 1.5.1. *An immediate observation says preimage of singletons are closed subsets.*

- *A topological manifold M may not be a Hausdorff (T_2) space (though we require so...). But this is always a T_1 space, i.e., for any p, $q \in M$ s.t. $p \neq q$, there exists U, V open subsets of M s.t. $p \in U$ but $p \notin U$ and $q \in V$ but $p \notin V$. This is equivalent to saying for any $p \in M$, $\{p\}$ the singleton is closed in M.*
- *Hence for any $f : M \to N$ continuous map between topological manifolds, for any $q \in N$, $f^{-1}(q) \subset M$ is in fact closed.*

Definition 1.5.3 (Critical Value & Regular Value). *Let M, N smooth manifolds, and $f : M \to N$ smooth map.*

- *We say $p \in M$ is a critical point of f if f is not a submersion at p.*
- *$q \in N$ is a critical value of f is there exists $p \in M$ critical point of f s.t. $p \in f^{-1}(q)$.*
- *$q \in N$ is a regular value of f if q is not a critical value of f. In other words, for any $p \in f^{-1}(q)$, f is a submersion at p.*

In particular, if $f^{-1}(q)$ is empty, then $q \in N$ is regular value of f.

The following Theorem constructs smooth submanifolds.

Theorem 1.5.1 (Preimage Theorem). *Let M, N be smooth manifolds, and $f : M \to N$ smooth map. Suppose $q \in N$ is a regular value of f, and suppose $f^{-1}(q)$ is not empty (hence $\dim(M) = m \geq \dim(N) = n$).*

Then $f^{-1}(q)$ is a closed smooth submanifold of M of dimension $m - n \geq 0$.

Let's see some examples.

Example 1.5.1 (Spheres as Submanifold). *Let* $f : \mathbb{R}^{n+1} \to \mathbb{R}$ *s.t.* $f(x_1, \cdots, x_{n+1}) = x_1^2 + \cdots x_{n+1}^2$. f *is* C^∞ *map, and* $df_x : \mathbb{R}^{n+1} \to \mathbb{R}$

$$df_x = (2x_1, \cdots, 2x_{n+1})$$

the only critical point is $0 \in \mathbb{R}^{n+1}$ *and the only critical value is* $0 \in \mathbb{R}$. *Regular values are* $\mathbb{R} \setminus \{0\}$. *By Preimage Theorem, for any* $a > 0$

$$f^{-1}(a) = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_i x_i^2 = a\} \subset \mathbb{R}^{n+1} =: \mathbb{S}^n(\sqrt{a})$$

is a C^∞*-submanifold of dimension* n. $\mathbb{S}^n(1) = \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ *is a* C^∞ *submanifold of dimension* n. *If* $a = 0$, $f^{-1}(0) = 0$ *is just single point. If* $a < 0$, $f^{-1}(0) = \varnothing$.

Example 1.5.2 (Orthogonal Group as Submanifold). $O(n) := \{A \in M_n(\mathbb{R}) \mid AA^T = I_n \ n \times n \text{ identity}\} \subseteq M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ *where the latter is linear isomorphism. The subset* $O(n) \subseteq M_n(\mathbb{R})$ *is a* C^∞ *submanifold of* $M_n(\mathbb{R})$ *of dimension* $\frac{n(n-1)}{2}$.

Proof. Define $f : M_n(A) \cong \mathbb{R}^{n^2} \to S_n(\mathbb{R}) \cong \mathbb{R}^{\frac{n(n+1)}{2}}$ where $S_n(\mathbb{R})$ are real $n \times n$ symmetric matrices. Define $f(A) = AA^T - I_n$ so $O(n) = f^{-1}(0)$. Now if $B = f(A)$, $b_{ij} = \sum_{k=1}^n a_{ik}a_{kj} - \delta_{ij}$. So f is C^∞ map. It remains to show that 0 is a regular value of the map f. For any $A \in M_n(\mathbb{R})$, $df_A : \mathbb{R}^{n^2} \to \mathbb{R}^{\frac{n(n+1)}{2}}$

$$df_A(B) = \lim_{h \to 0} \frac{f(A + hB) - f(A)}{h} = \lim_{h \to 0} \frac{(A + hB)(A^T + hB^T) - I_n - (AA^T - I_n)}{h} = BA^T + AB^T \tag{1.5}$$

Claim: for any $A \in f^{-1}(0) = O(n)$, df_A is surjective, i.e., for any $C \in S_n(\mathbb{R})$, there exists $B \in M_n(\mathbb{R})$ s.t. $C = df_A(B) = BA^T + AB^T$. Indeed

$$C = df_A(B) = BA^T + AB^T = BA^T + (BA^T)^T$$
$$\implies \text{Let } BA^T = \frac{1}{2}C \iff B = \frac{1}{2}CA$$

so $B = \frac{1}{2}CA \in M_n(\mathbb{R})$ gives $df_A(B) = \frac{1}{2}CAA^T + A\frac{1}{2}A^TC = C$. Thus we conclude that $O(n)$ is submanifold of $M_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$. □

Similarly, $O(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid A\overline{A}^T = I_n\} \subseteq M_n(\mathbb{C})$. $O(n, \mathbb{C})$ *is* C^∞ *submanifold of* $M_n(\mathbb{C})$ *of dimension* n^2. *(*$M_n(\mathbb{C}) \cong \mathbb{C}^n \cong \mathbb{R}^{2n^2}$*).*

1.6 Orientation

Definition 1.6.1 (Orientation). *Let* M *be* C^k *manifold of dimension* n.

We say M *is orientable if there exists a* C^k*-atlas* $\Phi = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ *on* M *s.t. for any* $U_\alpha \cap U_\beta \neq \varnothing$,

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n \to \phi_\beta(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n$$

is C^k *diffeomorphism, and for any* $x \in \phi_\alpha(U_\alpha \cap U_\beta)$,

$$\det(d(\phi_\beta \circ \phi_\alpha^{-1})_x) > 0 \tag{1.6}$$

Note for a diffeomorphism, indeed $d(\phi_\beta \circ \phi_\alpha^{-1})_x \in \mathrm{GL}(n, \mathbb{R}) := \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$. *The point is that orientability requires a sign on the determinant on any overlapping charts.*

Note we only require there exists one such Atlas.

- *If* M *is orientable, an orientation* Φ *on* M *is a choice of* C^k*-altas satisfying* (1.6).
- *if both* Φ *and* Ψ *on* M *satisfy* (1.6)*, we say they define the same orientation if* $\Phi \cup \Psi$ *still satisfies* (1.6).

Proposition 1.6.1. *If* M *is a complex manifold of complex dimension* n*, then* M *is an orientable* C^∞ *manifold of real dimension* $2n$.

Proof. One compute

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \subseteq \mathbb{C}^n \to \phi_j(U_i \cap U_j) \subseteq \mathbb{C}^n$$

its differential is

$$d(\phi_j \circ \phi_i^{-1})_{y_1, \cdots, y_n} : \mathbb{C}^n \to \mathbb{C}^n \qquad \mathbb{C}\text{-linear map}$$

In general, for L a $\mathbb{C}$-linear map, one has realization over the reals, i.e.

$$\begin{array}{ccc} x+iy \in \mathbb{C}^n & \xrightarrow{L} & L(x+iy) \in \mathbb{C}^n \\ \downarrow & & \downarrow \\ (x,y) \in \mathbb{R}^{2n} & \xrightarrow{L_{\mathbb{R}}} & L_{\mathbb{R}}(x,y) \in \mathbb{R}^{2n} \end{array}$$

This is done via the following: for the L, $\mathbb{C}$-linear map, there exists $C \in M_n(\mathbb{C})$ s.t.

$$x + iy \mapsto C(x+iy)$$

Now assume C of the form $C = A + iB$ where A, $B \in M_n(\mathbb{R})$. Then

$$C(x+iy) = (A+iB)(x+iy) = (Ax - By) + i(Bx + Ay)$$

Therefore $L_{\mathbb{R}}$ is realized via

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $\det(\begin{bmatrix} A & -B \\ B & A \end{bmatrix}) = |\det(C)|^2$. So L being linear isomorphism implies $\det(\begin{bmatrix} A & -B \\ B & A \end{bmatrix}) > 0$. Hence

$$\det(d(\phi_j \circ \phi_i^{-1})_{y_1,\cdots,y_n}) > 0$$

□

Example 1.6.1. *$P_n(\mathbb{C})$ is orientable. $P_n(\mathbb{R})$ is orientable iff n is odd. For example $P_1(\mathbb{R}) \cong \mathbb{S}^1$ so orientable, but $P_2(\mathbb{R})$ is not.*

1.7 Tangent Space and Differential

Tangent Space

Definition 1.7.1 (Tangent Space)**.** *Let M be C^k manifold of dimension n, $k \geq 1$. For any fixed $p \in M$, define the set*

$$T_pM := \{(U, \phi, u) \mid (U, \phi) \text{ is a } C^k \text{ chart around } p,\ u \in \mathbb{R}^n\} / \sim_p$$

where for p fixed, we define the equivalence relation

$$(U, \phi, u) \sim_p (V, \psi, v) \iff d(\psi \circ \phi^{-1})_{\phi(p)}(u) = v$$

1. *Fix (U, ϕ) chart around p, define the map*

$$\begin{aligned} \theta_{U,\phi,p} : \mathbb{R}^n &\to T_pM \\ u &\mapsto [U, \phi, u] \end{aligned} \tag{1.7}$$

   This is bijection.

2. *We use this $\theta_{U,\phi,p}$ to equip T_pM with the structure of a vector space over $\mathbb{R}$. This structure is well-defined because diagram commutes.*

$$\begin{array}{ccc} \mathbb{R}^n & & \\ {\scriptstyle d(\psi\circ\phi^{-1})|_{\phi(p)}}\downarrow & \searrow^{\theta_{U,\phi,p}} & \\ \mathbb{R}^n & \xrightarrow{\theta_{V,\psi,p}} & T_pM \end{array}$$

3. *The diagram is equivalent to saying*

$$d(\psi \circ \phi^{-1})\big|_{\phi(p)} = \theta^{-1}_{V,\psi,p} \circ \theta_{U,\phi,p} \tag{1.8}$$

We call T_pM tangent space to M at p. A tangent vector to M at p is an element in T_pM.

Differential

Definition 1.7.2 (Differential)**.** *Let M, N be C^k manifolds of dimension m, n. Let $f : M \to N$ be a C^k map. We define its differential at $p \in M$ as the linear map*

$$df_p : T_pM \to T_{f(p)}N$$

s.t. for any (U, ϕ) C^k chart around p and (V, ψ) C^k chart around $f(p)$, if one denote $g = \psi \circ f \circ \phi^{-1}$ as the local representation, df_p denotes the composition

$$\begin{aligned} df_p := \theta_{V,\psi,f(p)} \circ dg_{\phi(p)} \circ \theta^{-1}_{U,\phi,p} : T_pM &\to T_{f(p)}N \\ [U, \phi, u] &\mapsto [V, \psi, dg_{\phi(p)}(u)] \end{aligned}$$

The following diagram for the differential commutes

$$\begin{array}{ccccccc} M & \overset{open}{\supseteq} & p \in U & \xrightarrow{f} & V & \overset{open}{\subseteq} & N \\ & & \downarrow \phi & & \downarrow \psi & & \\ \mathbb{R}^m & \overset{open}{\supseteq} & \phi(p) \in \phi(U) & \xrightarrow{g} & \psi(V) & \overset{open}{\subseteq} & \mathbb{R}^n \end{array} \qquad \begin{array}{ccc} T_pM & \xrightarrow{df_p} & T_{f(p)}N \\ \theta_{U,\phi,p} \uparrow & & \theta_{V,\psi,f(p)} \uparrow \\ \mathbb{R}^m & \xrightarrow{dg_{\phi(p)}} & \mathbb{R}^n \end{array}$$

One has immediate observations

1. f is a submersion (resp. immersion) at p if $df_p : T_pM \to T_{f(p)}N$ is surjective (resp. injective).
2. *Chain Rule for manifolds.* If $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are C^k maps between C^k manifolds, where $k \geq 1$. Then
   - $g \circ f : M_1 \to M_3$ is C^k
   - For any $p \in M_1$, $df_p : T_pM_1 \to T_{f(p)}M_2$, $dg_{f(p)} : T_{f(p)}M_2 \to T_{g(f(p))}M_3$, one has

$$d(g \circ f)_p = dg_{f(p)} \circ df_p : T_pM_1 \to T_{g\circ f(p)}M_3$$

Tangent Space for Submanifolds constructed via Preimage Theorem Let $i : M \to N$ inclusion map be smooth embedding so M is a embedded submanifold, i.e., for any $p \in M$

$$di_p : T_pM \to T_pN \qquad \text{is injective}$$

Thus one regard T_pM as linear subspace of T_pN.

More generally, let M, N be C^∞ manifolds of dimension m and n. Let $f : M \to N$ be C^∞ map, and $q \in N$ be a regular value s.t. $S = f^{-1}(q)$ is non-empty. By our Preimage Theorem, we know that $S \subseteq M$ is a closed smooth submanifold of dimension $m - n$.

Now for any $p \in S$,

$$T_pS = \ker(df_p : T_pM \cong \mathbb{R}^m \to T_{f(p)}N \cong \mathbb{R}^n) \tag{1.9}$$

In other words, there is a short exact sequence of real vector spaces

$$0 \to T_pS \to T_pM \to T_{f(p)}N \to 0$$

Hence **if a manifold is constructed via Preimage Theorem, we can compute its tangent spaces**. Let's make use of (1.9) to compute explicitly tangent space of some submanifolds.

Example 1.7.1. *For any $p \in \mathbb{R}^n$, we have linear isomorphism $T_p\mathbb{R}^n \cong \mathbb{R}^n$ given by* (1.7)

$$\begin{aligned} \theta^{-1}_{\mathbb{R}^n,\mathrm{id},p} : T_p\mathbb{R}^n &\to \mathbb{R}^n \\ [\mathbb{R}^n, \mathrm{id}, u] &\mapsto u \end{aligned}$$

Example 1.7.2 ($T_x\mathbb{S}^n$)**.** *Consider*

$$\begin{aligned} f : \mathbb{R}^{1+n} &\to \mathbb{R} \\ (x_1, \cdots, x_{n+1}) &\mapsto \sum_{i=1}^{n+1} x_i^2 \end{aligned}$$

f is C^∞ map, 1 is regular value of f. Thus $\mathbb{S}^n := f^{-1}(1)$ is a C^∞ submanifold of $\mathbb{R}^{n+1}$ of dimension n.

For any $x \in \mathbb{R}^{1+n}$, $df_x(v) = 2x \cdot v$ for any $v \in T_x\mathbb{R}^{n+1}$. Thus for any $x \in \mathbb{S}^n$, using (1.9)

$$T_x\mathbb{S}^n := \{v \in T_x\mathbb{R}^{1+n} \mid df_x(v) = 0\} = \{v \in \mathbb{R}^{1+n} \mid x \cdot v = 0\} \subseteq T_x\mathbb{R}^{1+n} \cong \mathbb{R}^{1+n}$$

where the linear isomorphism is viewed via $\theta_{\mathbb{R}^{1+n},\mathrm{id},x}$ (1.7).

Example 1.7.3 ($T_AO(n)$). *Consider*

$$f : M_n(\mathbb{R}) \cong \mathbb{R}^{n^2} \to S_n(\mathbb{R}) \cong \mathbb{R}^{\frac{n(n+1)}{2}}$$
$$A \mapsto AA^T$$

f is C^∞ map, I_n is a regular value of f. Thus $O(n) = f^{-1}(I_n)$ is a C^∞ submanifold of $M_n(\mathbb{R})$ of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

Now for any $A \in O(n)$, using (1.9)

$$T_AO(n) = \{B \in M_n(\mathbb{R}) \mid df_A(B) = 0\} \subseteq T_AM_n(\mathbb{R}) \cong M_n(\mathbb{R})$$

where $\cong$ is done via $\theta_{M_n(\mathbb{R}),\mathrm{id},A}$ (1.7). Then recalling $df_A(B) = BA^T + AB^T$ (1.5)

$$T_AO(n) = \{B \in M_n(\mathbb{R}) \mid BA^T + AB^T = 0\}$$

In particular at identity, this is

$$T_{I_n}O(n) = \{B \in M_n(\mathbb{R}) \mid B + B^T = 0\} \qquad \textit{skew symmetric matrices}$$

1.8 Tangent Bundle

Construction of Tangent Bundle Given C^k manifold M of dimension n where $k \in \mathbb{N}$. We will construct the tangent bundle TM of M as a C^{k-1} manifold of dimension $2n$.

1. *Set.* The tangent bundle of M is

$$TM = \{(p, v) \mid p \in M, v \in T_pM\} = \bigsqcup_{p\in M} T_pM$$

Define the projection as

$$\pi : TM \to M$$
$$(p, v) \mapsto p$$

π is a surjective map.

2. *Topology.* For any (U, ϕ) C^k chart of M, define

$$\tilde{\phi} : \pi^{-1}(U) \subseteq TM \to \phi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$$
$$(p, v) \mapsto (\phi(p), \theta^{-1}_{U,\phi,p}([U, \phi, v]))$$

where we recall (1.7). Such $\tilde{\phi}$ is a bijection.

Now take any C^k atlas $\Phi = \{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$ on M. Define the surjective map

$$F : \bigcup_{\alpha\in I} \phi_\alpha(U_\alpha) \times \mathbb{R}^n \subseteq \mathbb{R}^n \times \mathbb{R}^n \to TM$$
$$(x, u) \mapsto (\phi^{-1}_\alpha(x), \theta_{U_\alpha,\phi_\alpha,\phi^{-1}_\alpha(x)}(u))$$

We equip TM with the topology induced by the surjective map F. Thus the quotient map F is continuous by definition.

3. *Topological Manifold.* TM is a topological $2n$-manifold with Atlas

$$\tilde{\Phi} = \{(\pi^{-1}(U_\alpha), \tilde{\phi}_\alpha) \mid \alpha \in I\} \tag{1.10}$$

where

$$\tilde{\phi}_\alpha : \pi^{-1}(U_\alpha) \subseteq TM \to \phi_\alpha(U_\alpha) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$$
$$(p, v) \mapsto (\phi(p), \theta^{-1}_{U_\alpha,\phi_\alpha,p}([U_\alpha, \phi_\alpha, v]))$$

and the following diagram commutes

$$\begin{array}{ccccccc} TM & \overset{\text{open}}{\supseteq} & (p,v) \in \pi^{-1}(U_\alpha) & \xrightarrow{\pi} & p \in U_\alpha & \overset{\text{open}}{\subseteq} & M \\ & & \downarrow \tilde{\phi}_\alpha & & \downarrow \phi_\alpha & & \\ \mathbb{R}^{2n} & \overset{\text{open}}{\supseteq} & \phi_\alpha(U_\alpha) \times \mathbb{R}^n & \xrightarrow{\pi_{\text{can}}} & \phi_\alpha(U_\alpha) & \overset{\text{open}}{\subseteq} & \mathbb{R}^n \end{array}$$

Here

$$\pi_{\text{can}} = \phi_\alpha \circ \pi \circ \tilde{\phi}^{-1}_\alpha : \phi_\alpha(U_\alpha) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n} \to \phi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$$
$$(\phi_\alpha(p), u) \mapsto \phi_\alpha(p)$$

is the canonical submersion onto the first n coordinates.

4. C^{k-1} *manifold*. We wish to compute transition functions.

   For any U open set of M, one may identify

$$\pi^{-1}(U) = TU = \bigsqcup_{p \in U} T_pU$$

   So given two charts (U_α, ϕ_α), (U_β, ϕ_β) for M, we have two corresponding charts $(TU_\alpha, \tilde{\phi}_\alpha)$, $(TU_\beta, \tilde{\phi}_\beta)$ for TM.

   Now notice

$$\tilde{\phi}_\alpha(\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)) = \tilde{\phi}_\alpha(\pi^{-1}(U_\alpha \cap U_\beta)) = \phi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

   Thus for any $U_\alpha \cap U_\beta \neq \varnothing$

$$\begin{aligned} \tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n} &\to \phi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^n \\ (x, u) &\mapsto (\phi_\beta \circ \phi_\alpha^{-1}(x),\, \theta^{-1}_{U_\beta, \phi_\beta, \phi_\beta^{-1}(x)} \circ \theta_{U_\alpha, \phi_\alpha, \phi_\alpha^{-1}(x)}(u)) \\ &= \big(\phi_\beta \circ \phi_\alpha^{-1}(x),\, d(\phi_\beta \circ \phi_\alpha^{-1})_x(u)\big) \end{aligned} \tag{1.11}$$

   where the last identification uses diagram (1.8).

   Since $\phi_\beta \circ \phi_\alpha^{-1}$ is C^k in $x \in \phi_\alpha(U_\alpha \cap U_\beta)$ while $d(\phi_\beta \circ \phi_\alpha^{-1})_x$ in C^{k-1} in $u \in \mathbb{R}^n$, our $\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1}$ are C^{k-1} maps in $(x, u) \in \phi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n$.

   Thus $\tilde{\Phi}$ is a C^{k-1} atlas on TM. $(TM, \tilde{\Phi})$ is a C^{k-1} manifold of dimension $2n$.

One has two immediate observations

1. Our surjective map

$$\begin{aligned} \pi : TM &\to M \\ (p, v) &\mapsto p \end{aligned}$$

   is C^{k-1} map due to $\pi = \phi_\alpha^{-1} \circ \pi_{\text{can}} \circ \tilde{\phi}_\alpha$ as composition with C^{k-1} charts. For $k \geq 2$, π is a submersion.

2. TM is an orientable C^{k-1} manifold of dimension $2n$, even though M might not be.

   Proof. Compute

$$\det(d(\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1})_x) = \det(d(\phi_\beta \circ \phi_\alpha^{-1})_x)^2 \geq 0$$

   □

Tangent Map

Definition 1.8.1. *Let M, N be C^k manifolds and $f : M \to N$ be C^k map. Then define its tangent map as*

$$\begin{aligned} df : TM &\to TN \\ (p, v) &\mapsto (f(p), df_p(v)) \end{aligned}$$

One has immediate observations. Let $f : M \to N$ is C^k map between C^k manifolds where $k \geq 1$.

1. $df : TM \to TN$ is a C^{k-1} map between C^{k-1} manifolds. For $k \geq 2$, $d(df) : T(TM) \to T(TN)$ is defined.

2. If f is a submersion (resp. immersion), then df is a submersion (resp. immersion). If f is submersion (resp. immersion) at some point $p \in M$, then df is a submersion (resp. immersion) at (p, v) for any $v \in T_pM$.

3. If N is smooth manifold of dimension n and $M \hookrightarrow N$ smooth submanifold of dimension $m \leq n$. Then

$$TM = \{(p, v) \mid p \in M,\, v \in T_pM\} \subseteq TN = \{(p, v) \mid p \in N,\, v \in T_pN\}$$

   TM is C^∞ submanifold for TN of dimension $2m$.

Examples of Tangent Bundles

Example 1.8.1 ($T\mathbb{S}^n$). *For chart*

$$\begin{aligned} \mathrm{id} : \mathbb{S}^n &\to \mathbb{R}^{n+1} \\ x &\mapsto x \end{aligned}$$

the corresponding chart writes

$$\begin{aligned} \tilde{\mathrm{id}} : T\mathbb{S}^n \subseteq T\mathbb{R}^{n+1} &\to T\mathbb{R}^{n+1} = \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \\ (x, v) &\mapsto (x, v) \end{aligned}$$

One has

$$\begin{aligned} T\mathbb{S}^n &= \{(x, v) \in \mathbb{R}^{1+n} \times \mathbb{R}^{1+n} \mid x \in \mathbb{S}^n \;\; v \in T_x\mathbb{S}^n\} \\ &= \{(x, v) \in \mathbb{R}^{1+n} \times \mathbb{R}^{1+n} \mid x \cdot x = 1, \;\; x \cdot v = 0\} \end{aligned}$$

$T\mathbb{S}^n$ is C^∞ submanifold of dimension $2n$.

Example 1.8.2 ($TO(n)$).

$$TO(n) = \{(A, B) \in M_n(\mathbb{R}) \times M_n(\mathbb{R}) \mid AA^T = I_n, \;\; BA^T + AB^T = 0\} \subseteq TM_n(\mathbb{R}) \cong M_n(\mathbb{R}) \times M_n(\mathbb{R})$$

$TO(n)$ is C^∞ submanifold of dimension $n(n-1)$.

1.9 Vector Bundle

Definition for Vector Bundle Given C^k manifold M of dimension n where $k \in \mathbb{N}$. We will define a C^ℓ ($\ell \leq k$) *real vector bundle of rank r over M* with total space E and base M

$$\pi : E \to M$$

as E a C^ℓ manifold of dimension $r + n$, and π a surjective C^ℓ map, s.t.

1. *Local Trivialization.* There exists an open cover $\{U_\alpha\}$ of M and a family of C^ℓ diffeomorphisms $\{h_\alpha\}$

$$h_\alpha : \pi^{-1}(U_\alpha) \subseteq E \to U_\alpha \times \mathbb{R}^r \subseteq M \times \mathbb{R}^r \tag{1.12}$$

s.t. the diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) \subseteq E & & \\ \downarrow h_\alpha & \searrow \pi & \\ U_\alpha \times \mathbb{R}^r \subseteq M \times \mathbb{R}^r & \xrightarrow{\mathrm{pr}_1} & U_\alpha \subseteq M \end{array}$$

where pr_1 is the projection onto the first coordinates

$$\begin{aligned} \mathrm{pr}_1 : U_\alpha \times \mathbb{R}^r \subseteq M \times \mathbb{R}^r &\to U_\alpha \subseteq M \\ (p, v) &\mapsto p \end{aligned}$$

2. *Transition Functions.* For any $U_\alpha \cap U_\beta \neq \varnothing$ open subsets of M, and the local trivializations

$$h_\alpha : \pi^{-1}(U_\alpha) \subseteq E \to U_\alpha \times \mathbb{R}^r, \qquad h_\beta : \pi^{-1}(U_\beta) \subseteq E \to U_\beta \times \mathbb{R}^r$$

The transition function takes the form

$$\begin{aligned} h_\beta \circ h_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^r &\to (U_\alpha \cap U_\beta) \times \mathbb{R}^r \\ (p, v) &\mapsto (p, g_{\beta\alpha}(p)(v)) \end{aligned}$$

where $g_{\beta\alpha}$ is a C^ℓ map s.t.

$$\begin{aligned} g_{\beta\alpha} : U_\alpha \cap U_\beta \subseteq M &\to \mathrm{GL}(r, \mathbb{R}) = \{A \in M_r(\mathbb{R}) \mid \det(A) \neq 0\} \\ p &\mapsto g_{\beta\alpha}(p) = (g_{\beta\alpha}(p))_{ij} \qquad \text{a linear isomorphism between } \mathbb{R}^r \end{aligned} \tag{1.13}$$

Following the definition, for any $x \in M$ one define the *fiber of E at x* as

$$E_x := \pi^{-1}(x)$$

Now for $x \in U_\alpha$, and h_α as the local trivialization (1.12),

$$h_\alpha|_{E_x} : E_x \to \{x\} \times \mathbb{R}^r \overset{\mathrm{pr}_2}{\cong} \mathbb{R}^r \qquad \forall\, x \in U_\alpha \tag{1.14}$$

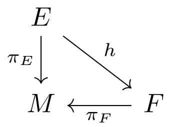
is a bijection. Here pr_2 denotes projection onto the second coordinate. One may thus define a vector space structure on E_x using h_α. Moreover, using the linear isomorphism (1.13), E_x is defined independent of local trivializations, and thus the linear structure is well-defined.

Therefore, set-wise, the vector bundle writes

$$E = \bigsqcup_{x \in M} E_x$$

Vector Bundle Isomorphism

Definition 1.9.1. *Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be two C^ℓ vector bundles over the same C^k manifold of dimension n. A C^ℓ vector bundle isomorphism from $\pi_E : E \to M$ to $\pi_F : F \to M$ is a C^ℓ diffeomorphism $h : E \to F$ s.t. the diagram commutes*



and for any $x \in M$

$$h|_{E_x} : E_x \to F_x \qquad \textit{is linear isomorphism of vector spaces}$$

Two C^ℓ vector bundles are isomorphic if there exists such a C^ℓ isomorphism.

Examples of Vector Bundles

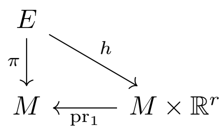
Example 1.9.1 (Product Vector Bundle)**.** *Let M be C^k manifold of dimension n. Then $E = M \times \mathbb{R}^r$ with*

$$\begin{aligned} \mathrm{pr}_1 : E = M \times \mathbb{R}^r &\to M \\ (p, v) &\mapsto p \end{aligned}$$

defines the product vector bundle.

Example 1.9.2 (Trivial Vector Bundle)**.** *We say a C^ℓ vector bundle $\pi : E \to M$ is trivial vector bundle of rank r if it is isomorphic to the product vector bundle $\mathrm{pr}_1 : M \times \mathbb{R}^r \to M$. In other words, there exists C^ℓ diffeomorphism $h : E \to M \times \mathbb{R}^r$ s.t.*

1. *the diagram commutes*



2. *for any $x \in E$, the restriction of h to each fiber E_x is a linear isomorphism*

$$h|_{E_x} : E_x \subseteq E \to \{x\} \times \mathbb{R}^r$$

In a word, $\pi : E \to M$ is trivial vector bundle if there exists only one global trivialization $h : E \to M \times \mathbb{R}^r$.

Example 1.9.3 (Tangent Bundle)**.** *Let M be a C^k manifold where $k \geq 1$ equipped with Atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$. Then $\pi : TM \to M$ is a C^{k-1} vector bundle over M of rank $n = \dim M$.*

Recall we've constructed the $2n$ dimensional C^{k-1} manifold

$$TM = \bigsqcup_{p \in M} T_p M$$

wit Atlas (1.10)

$$\tilde{\Phi} = \{(TU_\alpha, \tilde{\phi}_\alpha)\}_{\alpha \in I}$$

and transition functions (1.11)

$$\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1}(x, u) := (\phi_\beta \circ \phi_\alpha^{-1}(x), d(\phi_\beta \circ \phi_\alpha^{-1})_x(u))$$

To check $\pi : TM \to M$ is the vector bundle

1. *Local Trivialization. Define*

$$\begin{aligned} h_\alpha : \pi^{-1}(U_\alpha) \subseteq TM &\to U_\alpha \times \mathbb{R}^n \subseteq M \times \mathbb{R}^n \\ (p, [U_\alpha, \phi_\alpha, u]) &\mapsto (p, u = \theta^{-1}_{U_\alpha,\phi_\alpha,p}([U_\alpha, \phi_\alpha, u])) \end{aligned}$$

Indeed the diagram commutes since

$$\mathrm{pr}_1 \circ h_\alpha((p, [U_\alpha, \phi_\alpha, u])) = p = \pi(p, [U_\alpha, \phi_\alpha, u])$$

2. *Transition Functions. Compute*

$$\begin{aligned} h_\beta \circ h_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n &\to (U_\alpha \cap U_\beta) \times \mathbb{R}^n \\ (p, u) &\mapsto (p, \theta^{-1}_{U_\beta,\phi_\beta,p} \circ \theta_{U_\alpha,\phi_\alpha,p}(u)) \\ &= (p, d(\phi_\beta \circ \phi_\alpha^{-1})_{\phi_\alpha(p)}(u)) \end{aligned}$$

Then indeed

$$g_{\beta\alpha}(p) = d(\phi_\beta \circ \phi_\alpha^{-1})_{\phi_\alpha(p)} \in \mathrm{GL}(n, \mathbb{R})$$

1.10 Section

$C^\ell(M, E)$ **Section** Let M be a C^k manifold of dimension n. Let $\ell \le k$. We denote

$$C^\ell(M) := \{C^\ell \text{ functions } f : M \to \mathbb{R}\}$$

Definition 1.10.1 (Sections)**.** *Let M be a C^k manifold of dimension n. Let $\pi : E \to M$ be a C^ℓ vector bundle over M of rank r.*

We call $s : M \to E$ a C^ℓ section of π if it is a C^ℓ map s.t.

$$\pi \circ s = \mathrm{id} : M \to M$$

i.e.

$$s(x) \in E_x = \pi^{-1}(x) \qquad \forall\, x \in M$$

We denote

$$C^\ell(M, E) := \{C^\ell \textit{ sections } s : M \to E\}$$

Lemma 1.10.1. *$C^\ell(M, E)$ is a $C^\ell(M)$-module.*

Proof. For any $f \in C^\ell(M)$ and $s \in C^\ell(M, E)$, one define

$$\begin{aligned} fs : M &\to E \\ x &\mapsto f(x)s(x) \end{aligned}$$

Since $s(x) \in E_x$ where the fiber is equipped with linear structure over $\mathbb{R}$, $fs \in C^\infty(M, E)$. □

Basis construction for E_x via local trivialization Let $\pi : E \to M$ be C^ℓ vector bundle of rank r. Take open set $U_\alpha \subseteq M$ and

$$h_\alpha : \pi^{-1}(U_\alpha) \subseteq E \to U_\alpha \times \mathbb{R}^r \subseteq M \times \mathbb{R}^r$$

How do we use such local trivialization h_α to define a basis for the vector space E_x for each $x \in U_\alpha$? Take $\{e_1, \cdots, e_r\}$ as standard basis of $\mathbb{R}^r$. Define

$$\begin{aligned} s_{\alpha,i} : U_\alpha \subseteq M &\to \pi^{-1}(U_\alpha) \subseteq E \\ x &\mapsto h_\alpha^{-1}(x, e_i) \end{aligned} \tag{1.15}$$

We claim $\{s_{\alpha,i}(x)\}_{1 \le i \le r}$ is a basis for E_x for any $x \in U_\alpha$.

Proof. For any $x \in U_\alpha$, recall (1.14)

$$\mathrm{pr}_2 \circ h_\alpha|_{E_x} : E_x \to \mathbb{R}^r \qquad \text{defines a linear isomorphism}$$

We show $\{s_{\alpha,i}(x)\}_{1 \le i \le r}$ **spans** E_x. For any $v \in E_x$, there exists $w \in \mathbb{R}^r$ s.t.

$$\mathrm{pr}_2 \circ h_\alpha|_{E_x}(v) = \mathrm{pr}_2((x, w)) = w$$

Now write w in the standard basis $\sum_{i=1}^r w^i e_i$. Compute

$$h_\alpha|_{E_x}(\sum_{i=1}^r w^i s_{\alpha,i}(x)) = \sum_{i=1}^r w^i \; h_\alpha|_{E_x}(s_{\alpha,i}(x)) \overset{(1.15)}{=} \sum_{i=1}^r w^i(x, e_i)$$

$$\mathrm{pr}_2 \circ h_\alpha|_{E_x}(\sum_{i=1}^r w^i s_{\alpha,i}(x)) = \sum_{i=1}^r w^i e_i = w$$

Thus using $h_\alpha|_{E_x}$ is linear isomorphism,

$$v = \sum_{i=1}^r w^i s_{\alpha,i}(x)$$

We show $\{s_{\alpha,i}(x)\}_{1\leq i\leq r}$ **is linearly independent**. Assume $0 = \sum_{i=1}^r w^i s_{\alpha,i}(x)$, then using above computations and that linear isomorphism maps 0 to 0

$$(x, 0) = h_\alpha|_{E_x}(\sum_{i=1}^r w^i s_{\alpha,i}(x)) = \sum_{i=1}^r w^i(x, e_i)$$

$$0 = \sum_{i=1}^r w^i e_i$$

Using $\{e_i\}$ is standard basis, $w_i = 0$ for any i. □

Trivial Vector Bundle iff Sections as Global Basis

Theorem 1.10.1. *Let M be C^k manifold of dimension n. Let $\pi : E \to M$ be C^ℓ vector bundle of rank r. Then $\pi : E \to M$ is trivial iff there exists C^ℓ sections $\{s_i\}_{1\leq i\leq r}$ s.t.*

$$\{s_i(x)\}_{1\leq i\leq r} \qquad \textit{is basis for } E_x \qquad \forall\, x \in M$$

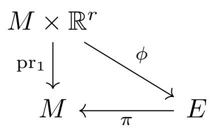
Proof. ($\Longrightarrow$) If π is trivial, then it has a global trivialization h. Take such h and define s_i via (1.15).

($\Longleftarrow$) If there are C^ℓ sections $\{s_i\}_{1\leq i\leq r}$ s.t. $\{s_i(x)\}$ are basis for E_x, define

$$\phi : M \times \mathbb{R}^r \to E$$
$$(x, v) \mapsto \sum_{i=1}^r v^i s_i(x) \in E_x$$

To check $\pi : E \to M$ is trivial, one claim such ϕ is a C^ℓ vector bundle isomorphism from the product bundle $\mathrm{pr}_1 : M \times \mathbb{R}^r \to M$ to $\pi : E \to M$. Using Definition 1.9.1, one need to check

1. The diagram commutes



   Indeed, for any $(x, v) \in M \times \mathbb{R}^r$, using $\sum_{i=1}^r v^i s_i(x) \in E_x = \pi^{-1}(x)$

$$\pi \circ \phi(x, v) = \pi(\sum_{i=1}^r v^i s_i(x)) = x = \mathrm{pr}_1(x, v)$$

2. For any $x \in M$

$$\phi|_{\{x\}\times\mathbb{R}^r} : \{x\} \times \mathbb{R}^r \to E_x$$
$$(x, v) \mapsto \sum_{i=1}^r v^i s_i(x)$$

   This indeed defines a linear isomorphism.

3. It suffices to check that ϕ is C^ℓ diffeomorphism. Since $\pi : E \to M$ is a C^ℓ vector bundle, there exists open cover $\{U_\alpha\}$ of M and local trivializations $\{h_\alpha\}$ s.t. $\pi = \mathrm{pr}_1 \circ h_\alpha$ diagram commutes. It suffices to check that

$$h_\alpha \circ \phi : U_\alpha \times \mathbb{R}^r \to U_\alpha \times \mathbb{R}^r$$

defines a C^ℓ diffeomorphism. But what is this map? For any $i \in \{1, \cdots, r\}$, since both h_α and s_i are C^ℓ maps, the composition remains a C^ℓ map, thus

$$\begin{aligned} h_\alpha \circ s_i : U_\alpha \subseteq M &\to U_\alpha \times \mathbb{R}^r \\ x &\mapsto (x, \begin{pmatrix} a_{1i}(x) \\ \vdots \\ a_{ri}(x) \end{pmatrix}) \end{aligned}$$

for certain $\{a_{ji}\}_{1\leq j\leq r} \subseteq C^\ell(U_\alpha)$. Now using $h_\alpha|_{E_x}$ defines linear isomorphism

$$\begin{aligned} h_\alpha \circ \phi(x,v) = h_\alpha(\sum_{i=1}^r v^i s_i(x)) &= \sum_{i=1}^r v^i h_\alpha \circ s_i(x) \\ &= \sum_{i=1}^r v^i (x, \begin{pmatrix} a_{1i}(x) \\ \vdots \\ a_{ri}(x) \end{pmatrix}) = (x, \begin{pmatrix} \sum_{i=1}^r v^i a_{1i}(x) \\ \vdots \\ \sum_{i=1}^r v^i a_{ri}(x) \end{pmatrix}) = (x, A(x)v) \end{aligned}$$

where $A(x)$ defines the matrix

$$A(x) = (a_{ij}(x))_{1\leq i,j\leq r}$$

Since $\{s_i(x)\}_{1\leq i\leq r}$ are basis of E_x, under linear isomorphism $\mathrm{pr}_2 \circ h_\alpha|_{E_x} : E_x \to \mathbb{R}^r$

$$\mathrm{pr}_2 \circ h_\alpha|_{E_x}(s_i(x)) = \begin{pmatrix} a_{1i}(x) \\ \vdots \\ a_{ri}(x) \end{pmatrix} \qquad \forall\ 1 \leq i \leq r \qquad \text{are basis for } \mathbb{R}^r$$

Since the matrix A consists of n columns of a set of basis, A is invertible ($A \in \mathrm{GL}(r, \mathbb{R})$). Consequently $h_\alpha \circ \phi$ defines C^ℓ diffeomorphism.

□

1.11 Derivation on $C_p^k(M)$

$C_p^k(M)$ Germs Let M be a C^k manifold of dimension n. Let $k \in \mathbb{N} \cup \{\infty\}$.

Definition 1.11.1 (Germs)**.** *For any $p \in M$, define*

$$C_p^k(M) := \{(f : U \to \mathbb{R}) \mid U \text{ open neighborhood of } p \text{ in } M,\ f \in C^k(U)\}/\sim_p$$

where the equivalence relation is defined via

$$(f : U \to \mathbb{R}) \sim_p (g : V \to \mathbb{R})$$

iff there exists an open neighborhood W around p s.t.

$$W \subseteq U \cap V, \qquad f|_W = g|_W$$

Elements of $C_p^k(M)$ are called germs of C^k functions at p. When the open neighborhood doesn't matter, one may denote the equivalence class as

$$[f]_p = [f : U \to \mathbb{R}]$$

Lemma 1.11.1. *$C_p^k(M)$ is equipped with an $\mathbb{R}$-algebra structure.*

Proof. Define ring addition and multiplication as

$$\begin{aligned} [f : U \to \mathbb{R}] + [g : V \to \mathbb{R}] &:= [f + g : U \cap V \to \mathbb{R}] \\ [f : U \to \mathbb{R}] \cdot [g : V \to \mathbb{R}] &:= [fg : U \cap V \to \mathbb{R}] \end{aligned}$$

Define scalar multiplication for $\mathbb{R}$-vector space structure as

$$c[f : U \to \mathbb{R}] := [cf : U \to \mathbb{R}]$$

□

Ring Homomorphism for $C_p^k(M)$ For $p \in M$ and (U, ϕ) a C^k chart for M around p s.t. $\phi(p) = 0$, the map

$$\Phi_1 : C_p^k(M) \to C_0^k(\mathbb{R}^n)$$
$$[f : V \to \mathbb{R}] = [f|_{U \cap V} : U \cap V \to \mathbb{R}] \mapsto [f \circ \phi^{-1} : \phi(U \cap V) \to \mathbb{R}]$$

defines a ring isomorphism.

The map

$$\Phi_2 : C^k(M) \to C_p^k(M)$$
$$(f : M \to \mathbb{R}) \mapsto [f : M \to \mathbb{R}]$$

is a surjective ring homomorphism, but not injective.

Proof of Surjectivity. Given $[f : V \to \mathbb{R}] \in C_p^k(M)$, there exists $\beta \in C^k(V)$ with $\mathrm{supp}(\beta) \subseteq V$ s.t.

$$(\beta : V \to \mathbb{R}) \overset{p}{\sim} (1 : M \to \mathbb{R})$$

Hence

$$[f : V \to \mathbb{R}] = [\beta f : V \to \mathbb{R}]$$

and βf can be extended to M due to Hausdorff topology on M. □

D_pM **Derivation on $C_p^k(M)$**

Definition 1.11.2 (Derivation)**.** *A Derivation on $C_p^k(M)$ is a $\mathbb{R}$-linear map*

$$\delta : C_p^k(M) \to \mathbb{R}$$

s.t. the Leibniz Rule is satisfied, i.e.,

$$\delta([f]_p \cdot [g]_p) = \delta([f]_p)g(p) + f(p)\delta([g]_p)$$

Lemma 1.11.2. *The set of derivations on $C_p^k(M)$ has vector space structure.*

Proof. For any c_1, $c_2 \in \mathbb{R}$ and $\delta_1, \delta_2 \in C_p^k(M)$, one may define

$$c_1\delta_1 + c_2\delta_2 : C_p^k(M) \to \mathbb{R}$$
$$[f]_p \mapsto c_1\delta([f]_p) + c_2\delta([f]_p)$$

To check this remains a derivation, for any $[f]_p$ and $[g]_p$

$$\begin{aligned}
&(c_1\delta_1 + c_2\delta_2)([f]_p \cdot [g]_p) \\
&= c_1\delta_1([f]_p \cdot [g]_p) + c_2\delta_2([f]_p \cdot [g]_p) \\
&= c_1\delta_1([f]_p)g(p) + c_1 f(p)\delta_1([g]_p) + c_2\delta_2([f]_p)g(p) + c_2 f(p)\delta_2([g]_p) \\
&= (c_1\delta_1 + c_2\delta_2)([f]_p)g(p) + f(p)(c_1\delta_1 + c_2\delta_2)([g]_p)
\end{aligned}$$

□

Example 1.11.1. $k \geq 1$.

$$\frac{\partial}{\partial x_i}(0) : C_0^k(\mathbb{R}^n) \to \mathbb{R}$$
$$[f : U \to \mathbb{R}] \mapsto \frac{\partial}{\partial x_i} f(0)$$

Then $\frac{\partial}{\partial x_i}(0)$ is a derivation for any $1 \leq i \leq n$.

Consequently, for any $a_i \in \mathbb{R}$,

$$\sum_i a_i \frac{\partial}{\partial x_i}(0) : C_0^k(\mathbb{R}^n) \to \mathbb{R}$$
$$[f : U \to \mathbb{R}] \mapsto \sum_i a_i \frac{\partial}{\partial x_i} f(0)$$

is a derivation.

Lemma 1.11.3. *Let $k \in \mathbb{N} \cup \{\infty\}$.*

- *If $\delta : C_0^k(\mathbb{R}) \to \mathbb{R}$ is a derivation and c is a constant, then $\delta([c]_p) = 0$.*

  Proof. Since δ is $\mathbb{R}$-linear

  $$\delta([c]_0) = \delta(c\,[1]_0) = c\delta([1]_0)$$

  But

  $$\delta([1]_0) = \delta([1]_0 \cdot [1]_0) = \delta([1]_0) \cdot 1 + 1 \cdot \delta([1]_0) \implies \delta([1]_0) = 0$$

  □

- *δ is a derivation on $C_0^0(\mathbb{R}) \iff \delta \equiv 0$.*

  Proof. Using δ as derivation is $\mathbb{R}$-linear, for $[f]_0 \in C_0^0(\mathbb{R})$, since

  $$\delta([f]_0) = \delta([f - f(0)]_0)$$

  It suffices to assume $f(0) = 0$. Using decomposition

  $$f = f^+ - f^-, \qquad f^\pm \geq 0$$

  it suffices to assume $f \geq 0$ and $f(0) = 0$. Now define $g := \sqrt{f}$, so

  $$\delta([f]_0) = \delta([g]_0 \cdot [g]_0) = \delta([g]_0)g(0) + g(0)\delta([g]_0) = 0$$

  □

Lemma 1.11.4. *If δ is a derivation on $C_0^1(\mathbb{R}^n)$, then*

$$\delta = \sum_{i=1}^n \delta([x_i]_0)\frac{\partial}{\partial x_i}(0) \tag{1.16}$$

Proof. Want to show for any $[f]_0 \in C_0^1(\mathbb{R}^n)$,

$$\delta([f]_0) = \sum_{i=1}^n \delta([x_i]_0)\frac{\partial f}{\partial x_i}(0)$$

To do so fix $x \in \mathbb{R}^n$, define $g(t) := f(tx)$ so that $g'(t) = \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(tx)$ Then

$$f(x) - f(0) = g(1) - g(0) = \int_0^1 g'(t)\,dt = \sum_i x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx)\,dt$$

Define $h_i(x) := \int_0^1 \frac{\partial f}{\partial x_i}(tx)\,dt$ so that $[h_i : U \to \mathbb{R}] \in C_0^1(\mathbb{R}^n)$ with

$$h_i(0) = \int_0^1 \frac{\partial f}{\partial x_i}(0)\,dt = \frac{\partial f}{\partial x_i}(0)$$

Consequently

$$\delta([f]_0) = \delta([f - f(0)]_0) = \sum_i \delta([x_i h_i]_0) = \sum_i \delta([x_i]_0)h_i(0) + \sum_i x_i(0)\delta([h_i]_0) = \sum_i \delta([x_i]_0)\frac{\partial f}{\partial x_i}(0)$$

□

$D_pM \cong T_pM$ Linear Isomorphism Let M be C^∞ manifold of dimension n. Let $p \in M$. We denote D_pM as the vector space of Derivations on $C_p^\infty(M)$.

Define (U, ϕ) a C^∞ chart for M around p, whose coordinates we write

$$\phi(p) = 0 \in \mathbb{R}^n, \qquad \phi = (x_1, \cdots, x_n) \in C^\infty(U; \mathbb{R}^n)$$

We define a derivation on $C_p^\infty(M)$ as

$$\begin{aligned} \frac{\partial}{\partial x_i}(p) &: C_p^\infty(M) \to \mathbb{R} \\ [f : V \to \mathbb{R}] &\mapsto \frac{\partial}{\partial x_i}(f \circ \phi^{-1})(\phi(p)) = \frac{\partial}{\partial x_i}(f \circ \phi^{-1})(0) \end{aligned} \tag{1.17}$$

Lemma 1.11.5. *$\{\frac{\partial}{\partial x_i}(p)\}_{1\le i\le n}$ forms a basis for D_pM. Thus*

$$D_pM = \bigoplus_{i=1}^{n} \mathbb{R}\frac{\partial}{\partial x_i}(p) \tag{1.18}$$

Proof. Using ring isomorphism and (1.16), one see that $\{\frac{\partial}{\partial x_i}(p)\}_{1\le i\le n}$ spans D_pM via

$$\delta = \sum_{i=1}^{n} \delta([x_i]_p)\frac{\partial}{\partial x_i}(p) \tag{1.19}$$

To see linear independence, assume

$$0 = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}(p) \qquad \text{in } D_pM$$

Then acting on the germs $[x_j]_p \in C_p^\infty(M)$ gives

$$0 = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}(x_j \circ \phi^{-1})(0) = \sum_{i=1}^{n} a_i\delta_{ij} = a_j \qquad \forall\ j$$

□

Now one may define a linear isomorphism between vector spaces via

$$\begin{aligned} \Phi : T_pM &\to D_pM \\ [U,\phi,u] &\mapsto \sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i}(p) \end{aligned}$$

1.12 Derivation on $C^\infty(M)$

$\mathfrak{X}(M)$ Smooth Vector Field

Definition 1.12.1 (Smooth Vector Field)**.** *A smooth vector field on a C^∞ manifold M is $X \in C^\infty(M, TM)$, i.e., a smooth section of the tangent bundle $\pi : TM \to M$. In particular,*

$$X(p) \in T_pM \qquad \forall\ p \in M$$

We denote

$$\mathfrak{X}(M) := C^\infty(M, TM)$$

$D(M)$ Derivation on $C^\infty(M)$

Definition 1.12.2 (Derivation on $C^\infty(M)$)**.** *Let M be C^∞ manifold. A derivation on M is an $\mathbb{R}$-linear map*

$$\delta : C^\infty(M) \to C^\infty(M)$$

s.t. the Leibniz Rule holds

$$\delta(fg) = \delta(f)g + f\delta(g)$$

Let $D(M)$ be set of all derivations $C^\infty(M) \to C^\infty(M)$.

Lemma 1.12.1. *The set of derivations $D(M)$ has $C^\infty(M)$-module structure.*

Proof. If δ_1, $\delta_2 \in D(M)$, $c_1\, c_2 \in C^\infty(M)$, then

$$\begin{aligned} c_1\delta_1 + c_2\delta_2 : C^\infty(M) &\to C^\infty(M) \\ f &\mapsto c_1\delta(f) + c_2\delta(f) \end{aligned}$$

is also a derivation. □

For any $p \in M$, there is a localizing $\mathbb{R}$-linear map as follows.

$$\begin{aligned} D(M) &\to D_pM \\ \delta &\mapsto \delta(p) \end{aligned}$$

where

$$\begin{aligned} \delta(p) : C_p^\infty(M) &\to \mathbb{R} \\ [f : M \to \mathbb{R}] &\mapsto (\delta f)(p) \end{aligned}$$

One also define

$$\begin{aligned} \delta_p : C_p^\infty(M) &\to C_p^\infty(M) \\ [f : M \to \mathbb{R}] &\mapsto [\delta f : M \to \mathbb{R}] \end{aligned}$$

$D(M) \cong \mathfrak{X}(M)$ **Isomorphism as $C^\infty(M)$-module** Let M be C^∞ manifold of dim n.

Let (U, ϕ) be C^∞ chart with $\phi = (x_1, \cdots, x_n) \in \mathbb{R}^n$. We define smooth vector field on U as

$$\begin{aligned} \frac{\partial}{\partial x_i} &: U \subseteq M \to TU = \pi^{-1}(U) \subseteq TM \\ p &\mapsto \frac{\partial}{\partial x_i}(p) \in D_pM \cong T_pM \end{aligned} \tag{1.20}$$

where $\frac{\partial}{\partial x_i}(p)$ is defined via (1.17). $\frac{\partial}{\partial x_i}$ as C^∞ vector fields on U implies by definition that $\frac{\partial}{\partial x_i} \in C^\infty(U, TU)$.

In view of (1.18), one has isomorphism as free $C^\infty(M)$-module

$$\mathfrak{X}(U) = \bigoplus_{i=1}^n C^\infty(U) \frac{\partial}{\partial x_i}$$

In view of (1.19), for any $X : U \to TU$ continuous section, there are $a_i \in C(U)$ continuous functions s.t. for any $p \in U$ with local chart $(U, \phi = (x_1, \cdots, x_n))$

$$X(p) = \sum_{i=1}^n a_i(p) \frac{\partial}{\partial x_i}(p)$$

$X \in C^k(U, TU)$ is C^k vector field iff $a_i \in C^k(U)$.

L_X **Lie Derivative on** $C^\infty(M)$

Definition 1.12.3 (Lie Derivative)**.** *Let M be $C^\infty(M)$ manifold of dimension n. Define an assignment*

$$\begin{aligned} \mathfrak{X}(M) &\to D(M) \\ X &\mapsto L_X \end{aligned} \tag{1.21}$$

where L_X known as the Lie Derivative of X, is a derivation

$$\begin{aligned} L_X : C^\infty(M) &\to C^\infty(M) \\ f &\mapsto Xf \end{aligned} \tag{1.22}$$

s.t.

$$\begin{aligned} Xf : M &\to \mathbb{R} \\ p &\mapsto X(p)([f]_p) \end{aligned}$$

Usually one denote $X(p)([f]_p) = X(p)f$.

Proof that $Xf \in C^\infty(M)$. One use local coordinates to check this is C^∞ function. Let $(U, \phi = (x^1, \ldots, x^n))$ be a chart. On U we can write

$$X = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i} \qquad \text{with } a^i \in C^\infty(U).$$

Then for $p \in U$,

$$(Xf)(p) = \sum_{i=1}^n a^i(p) \, \frac{\partial(f \circ \phi^{-1})}{\partial x^i}\big(\phi(p)\big),$$

which is smooth in p since a^i and the partial derivatives of $f \circ \phi^{-1}$ are smooth. Hence $Xf \in C^\infty(M)$. □

In fact, the assignment via Lie Derivative (1.21) is the $C^\infty(M)$-module isomorphism between smooth vector fields and derivations on $C^\infty(M)$.

Lemma 1.12.2. (1.21) *$\mathfrak{X}(M) \cong D(M)$ is an isomorphism as $C^\infty(M)$-module*

Proof. We have surjectivity. Given any $\delta \in D(M)$, define

$$\begin{aligned} X : M &\to TM \\ p &\mapsto \delta(p) \in D_pM = T_pM \end{aligned}$$

One use local coordinates to check that X is C^∞.

For injectivity, if $X \neq 0$, there exists $p \in M$ s.t. $X(p) \neq 0$. Then there exists $[f]_p \in C_p^\infty(M)$ s.t. $X(p)([f]_p) \neq 0$ implying $L_X f \neq 0$. □

1.13 Lie Bracket

Lie Bracket

Definition 1.13.1 (Lie Bracket)**.** *For X, $Y \in \mathfrak{X}(M) = D(M)$, define*

$$\begin{aligned} [X,Y] : C^\infty(M) &\to C^\infty(M) \\ f &\mapsto XYf - YXf \end{aligned} \tag{1.23}$$

Notice

1. $[X,Y]$ is a $\mathbb{R}$-linear map.
2. $[X,Y]$ satisfies the Leibniz rule

$$[X,Y](fg) = ([X,Y]f)g + f([X,Y]g)$$

so $[X,Y] \in D(M) = \mathfrak{X}(M)$ defines a derivation.

More explicitly, for $(U, \phi = (x_1, \cdots, x_n))$ C^∞ chart on M, one may write on U

$$X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}, \qquad Y = \sum_{j=1}^n b_j \frac{\partial}{\partial x_j} \qquad \text{for } a_i,\ b_j \in C^\infty(U)$$

So

$$[X,Y] = \sum_j^n \left(\sum_i^n a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}$$

Lie Algebra over $\mathbb{R}$ Define assignment

$$\begin{aligned} [\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ (X,Y) &\mapsto [X,Y] \end{aligned}$$

Notice

1. $[\cdot,\cdot]$ is $\mathbb{R}$-linear in both X, Y, but not C^∞-linear

$$[c_1 X_1 + c_2 X_2, Y] = c_1[X_1, Y] + c_2[X_2, Y]$$

2. $[X,Y] = -[Y,X]$
3. Jacobi Identity.

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0 \tag{1.24}$$

with these above, $(\mathfrak{X}(M), [\cdot,\cdot])$ is a Lie algebra over $\mathbb{R}$.

1.14 Differential

Pullback of $C^\ell(N)$

Definition 1.14.1 (Pullback of $C^\ell(N)$)**.** *Let $F : M \to N$ be C^k-map between C^k manifolds, and let $\ell \le k$ be a positive integer. Then the map F induces the pullback*

$$\begin{aligned} F^* : C^\ell(N) &\to C^\ell(M) \\ f &\mapsto f \circ F \end{aligned} \tag{1.25}$$

Definition 1.14.2 (Local pull back of $C^\ell_{F(p)}(N)$)**.** *For a point $p \in M$, we get a map F_p^* local pullback s.t.*

$$\begin{aligned} F_p^* : C^\ell_{F(p)}(N) &\to C^\ell_p(M) \\ [f : V \to \mathbb{R}] &\mapsto [f \circ F : F^{-1}(V) \to \mathbb{R}] \end{aligned}$$

Differential as Map Between Derivations

Lemma 1.14.1. *Let $F : M \to N$ be a smooth map between smooth manifolds. For each $p \in M$, the differential*

$$dF_p : T_pM = D_pM \to T_{F(p)}N = D_{F(p)}N$$
$$X \mapsto dF_p(X)$$

is given by the map

$$\begin{aligned} dF_p(X) &: C^\infty_{F(p)}(N) \to \mathbb{R} \\ [f : V \to \mathbb{R}] &\mapsto X([F^*f : F^{-1}(V) \to \mathbb{R}]) = X([f \circ F : F^{-1}(V) \to \mathbb{R}]) \end{aligned} \tag{1.26}$$

Note here we denote $X \in T_pM = D_pM$.

Proof. Pass to local coordinates. Assume $M \subseteq \mathbb{R}^m$ open subset and $N \subseteq \mathbb{R}^n$ open subset. $p = 0 \in \mathbb{R}^m$ and $F(p) = 0 \in \mathbb{R}^n$. Then one write

$$F(x) = (y_1(x), \cdots, y_n(x)) \qquad \forall\, x \in \mathbb{R}^m$$

For any tangent vector $X \in T_0\mathbb{R}^m$, $X = \sum_{i=1}^m a_i \frac{\partial}{\partial x_i}(0)$

$$dF_p(X) = \sum_{j=1}^n \left(\sum_{i=1}^m \frac{\partial y_j}{\partial x_i}(0) a_i \right) \frac{\partial}{\partial y_j}(0) \in T_0(N) \tag{1.27}$$

To compute explicitly

$$\text{LHS} = dF_p(X)([f]_{F(p)}) = \sum_{i=1}^m \sum_{j=1}^n a_i \frac{\partial y_j}{\partial x_i}(0) \frac{\partial f}{\partial y_j}(0)$$
$$\text{RHS} = X([f \circ F]_p) = \sum_{i=1}^m a_i \frac{\partial (f \circ F)}{\partial x_i}(0)$$

which is equal by chain rule. □

Differential as map between curve velocity

Definition 1.14.3 (smooth curve)**.** *Let M be smooth manifold. A smooth curve in M is a smooth map*

$$\gamma : (a, b) \to M$$

for $-\infty \le a < b \le \infty$.

For any $t \in (a, b)$, let $\gamma'(t)$ or $\frac{d\gamma}{dt}(t)$ denote the tangent vector

$$d\gamma_t(\frac{\partial}{\partial t}(t)) \in T_{\gamma(t)}M$$

Example 1.14.1. *If $M = \mathbb{R}^n$ then the smooth map*

$$\gamma : (a, b) \to M$$
$$t \mapsto (x_1(t), \cdots, x_n(t))$$

where $x_i : (a, b) \to \mathbb{R}$ are C^∞ functions on (a, b). Then

$$\gamma'(t) = (x_1'(t), \cdots, x_n'(t)) = \sum_{i=1}^n x_i'(t) \frac{\partial}{\partial x_i}(\gamma(t))$$

Lemma 1.14.2. *Let M be a smooth manifold and $\gamma : (-\varepsilon, \varepsilon) \to M$ be a smooth curve. Let $\gamma(0) = p$. Then $\gamma'(0)$ is a derivation at p s.t.*

$$\gamma'(0)([f]_p) = \frac{\partial}{\partial t}(0)([f \circ \gamma]_0)$$

Usually we denote this as

$$\gamma'(0)f = \left.\frac{d}{dt}\right|_{t=0} (f \circ \gamma)$$

Proof. This is special case of $dF_p(X)([f]_{F(p)}) = X([F^*f]_p)$. □

Notice the above only depends on how f changes along $\gamma \subseteq M$. In particular, this means the tangent derivative is an intrinsic object of the manifold.

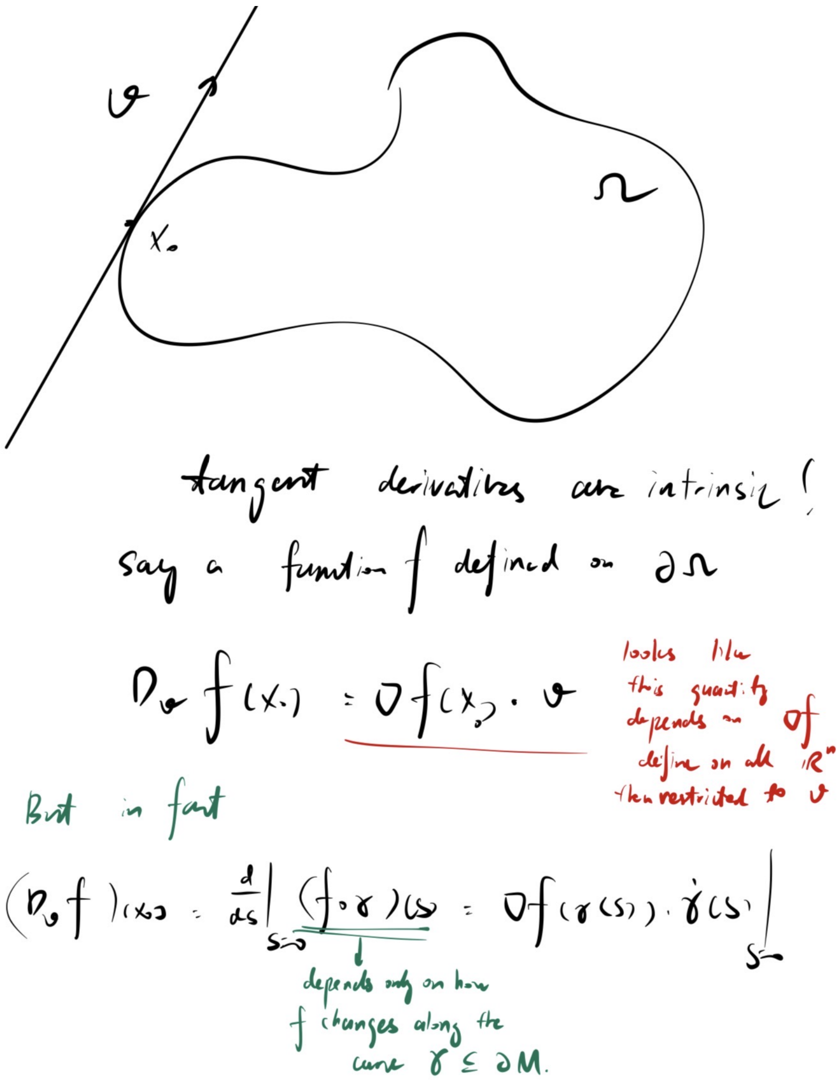


Figure 1.1: Tangent is Intrinsic

Remark 1.14.1. *One may alternatively define the tangent space T_pM as collection of all such*

$$\gamma'(0) : C_p^\infty(M) \to \mathbb{R}$$

Under this definition, $dF_p : T_pM \to T_{F(p)}N$ of a smooth map $F : M \to N$ at $p \in M$ is defined by

$$\begin{aligned} dF_p : T_pM &\to T_{F(p)}N \\ \gamma'(0) &\mapsto (F \circ \gamma)'(0) \end{aligned} \tag{1.28}$$

1.15 Integral Curves

Integral Curve

Definition 1.15.1 (Integral Curves)**.** *Let $X \in \mathfrak{X}(M)$ be a smooth vector field on a smooth manifold M. Let $\gamma : I \to M$ be a smooth curve. We say that γ is a integral curve of X if*

$$\gamma'(t) = X(\gamma(t)) \qquad \forall\ t \in I$$

Example 1.15.1. *$M = \mathbb{R}^n$ and $\gamma(t) = (x_1(t), \cdots, x_n(t))$ for $x_i : I \to \mathbb{R}$ smooth functions on I. A smooth vector field on $\mathbb{R}^n$ is of the form*

$$X(x) = (a_1(x), \cdots, a_n(x)) = \sum_i a_i(x) \frac{\partial}{\partial x_i}$$

where a_i are smooth functions s.t. $a_i : \mathbb{R}^n \to \mathbb{R}$.

Therefore X can be viewed as a smooth map from $\mathbb{R}^n \to \mathbb{R}^n$. γ is an integral curve of X is equivalent to the solution to the system of ODEs

$$\frac{dx_i}{dt}(t) = a_i(x_1(t), \cdots, x_n(t)) \qquad \forall\ i = 1, \cdots, n$$

Local Existence and Uniqueness of Integral Curves

Theorem 1.15.1 (Local Existence and Uniqueness of Integral Curves)**.** *Let M be a smooth manifold and $X \in \mathfrak{X}(M)$.*

(i) For any $p \in M$ there is an open interval $I_p \subseteq \mathbb{R}$ containing 0 and an integral curve

$$\phi_p : I_p \to M$$

of X s.t.

$$\phi_p(0) = p$$

and I_p is a maximal interval for such ϕ_p.

(ii) Moreover, this integral curve is unique in the following sense. If $\gamma : I' \to M$ is integral curve of the vector field X on I' s.t. $\gamma(0) = p$, then the interval $I' \subseteq I_p$ and the curve γ is the restriction

$$\gamma = \phi_p|_{I'}$$

(iii) Existence of Local Flow. For any $p \in M$, there is

- *an open neighborhood U of p in M*
- *an open interval I of 0 in $\mathbb{R}$*
- *a smooth map, known as local flow*

$$\begin{aligned} \phi : I \times U &\to M \\ (t, q) &\mapsto \phi(t, q) \end{aligned}$$

s.t.

$$\begin{cases} \frac{\partial}{\partial t}\phi(t, q) = X(\phi(t, q)) \\ \phi(0, q) = q \end{cases} \qquad \forall\ (t, q) \in I \times U \tag{1.29}$$

Proof. Assume $M = \mathbb{R}^n$ and $p = 0$ then the proof is a theorem in ODE. □

Example 1.15.2. *$M = \mathbb{R}^n$ and $p = (a_1, \cdots, a_n) \in \mathbb{R}^n$. Suppose X is the identity vector field, i.e*

$$X(x) = x \qquad \forall\ x = (x_1, \cdots, x_n) \in \mathbb{R}^n$$

Then

$$\begin{cases} \frac{d}{dt}x_i(t) = x_i(t) \\ x_i(0) = a_i \end{cases} \qquad \forall\ \ i = 1, \cdots n$$

hence $x_i = a_i e^t$. We conclude that the integral curves are straight lines emanating the origin. We also calculate the local flow

$$\begin{aligned} \phi : \mathbb{R} \times \mathbb{R}^n &\to \mathbb{R}^n \\ (t, x_1, \cdots, x_n) &\mapsto (x_1 e^t, \cdots, x_n e^t) \end{aligned}$$

or in short, $\phi(t, x) = e^t x$.

Example 1.15.3. *$M = \{x \in \mathbb{R}^n \mid |x| < 1\}$, and $X(x) = x$ is identity vector field. If $p = a = (a_1, \cdots, a_n)$ then*

$$\begin{aligned} \phi_p : I_p &\to \mathbb{R}^n \\ t &\mapsto e^t a \end{aligned}$$

where

$$I_p = (-\infty, -\log|a|)$$

Example 1.15.4. *Given flow*

$$\begin{aligned}\phi : \mathbb{R}\times\mathbb{R}^2 &\to \mathbb{R}^2\\ (t,(x,y)) &\mapsto \begin{pmatrix}\cos(t) & -\sin(t)\\ \sin(t) & \cos(t)\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\end{aligned}$$

To find the corresponding vector field, use

$$\frac{\partial}{\partial t}\phi(0,q) = X(\phi(0,q)) = X(q)$$

So

$$X((x,y)) = \frac{\partial}{\partial t}\phi(0,(x,y)) = \begin{pmatrix}-\sin(t) & -\cos(t)\\ \cos(t) & -\sin(t)\end{pmatrix}\Bigg|_{t=0}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-y\\ x\end{pmatrix}$$

Hence $X(x,y) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$.

Lemma 1.15.1. *For* M C^∞ *manifold,* $X, Y \in \mathfrak{X}(M)$ *with* $[X,Y]=0$*, let* $p\in M$*, and suppose* $\phi^X_s\circ\phi^Y_t(p)$ *and* $\phi^Y_t\circ\phi^X_s(p)$ *are defined for* $(s,t)\in I\times J$ *with* I, J *open intervals containing* 0*, then one has*

$$\phi^X_s\circ\phi^Y_t(p) = \phi^Y_t\circ\phi^X_s(p) \qquad \forall\ (s,t)\in I\times J$$

Global Existence Let M be C^∞ manifold. Denote

$$\begin{aligned}\phi_t : U &\to M\\ q &\mapsto \phi(t,q)\end{aligned}$$

This tells us where the point in M gets mapped after flowing a certain time t.

Lemma 1.15.2. *Let* X *be smooth vector field on a smooth manifold* M *s.t. the support of* X *is compact, where*

$$\operatorname{supp}(X) := \overline{\{p\in M \mid X(p)\neq 0\}}$$

Then there exists a unique smooth map $\phi:\mathbb{R}\times M\to M$ *s.t.*

$$\begin{cases}\frac{\partial\phi}{\partial t}(t,q) = X(\phi(t,q))\\ \phi(0,q) = q\end{cases}$$

In other words, we have a global flow

$$\phi_t : M\to M$$

which exists for all times $t\in\mathbb{R}$.

Proof. It suffices to prove existence. Let $K=\operatorname{supp}(X)$. First step, look at $V = M\setminus K$ open, $X(q)=0$ for any $q\in V$. Then define

$$\begin{aligned}\phi:\mathbb{R}\times V &\to M\\ (t,q) &\mapsto q\end{aligned}$$

Then ϕ is smooth and

$$\begin{cases}\frac{\partial\phi}{\partial t}(t,q) = 0 = X(q) = X(\phi(t,q))\\ \phi(0,q) = q\end{cases}$$

Step 2, given $p\in K$, there exists open neighborhood U_p of p in M and $\varepsilon_p>0$ s.t. there is a C^∞ map

$$\psi_p : (-\varepsilon_p,\varepsilon_p)\times U_p \to M$$

a local flow which satisfies

$$\begin{cases}\frac{\partial\psi_p}{\partial t}(t,q) = X(\psi_p(t,q))\\ \psi_p(0,q) = q\end{cases}$$

Moreover, if p_1, $p_2\in K$ and $U_{p_1}\cap U_{p_2}\neq\varnothing$, then

$$\psi_{p1}|_{(-\varepsilon,\varepsilon)\times(U_{p_1}\cap U_{p_2})} = \psi_{p2}|_{(-\varepsilon,\varepsilon)\times(U_{p_1}\cap U_{p_2})}$$

where $\varepsilon := \min\{\varepsilon_{p_1},\varepsilon_{p_2}\} > 0$. So we obtain a smooth map $\psi(t,q)$ defined on $(-\varepsilon,\varepsilon)\times(U_{p_1}\cup U_{p_2})$ Since K is compact, $K\subset\bigcup_{p\in K}U_p$ hence there are finitely many $p_1,\cdots,p_N\in K$ s.t. $K\subset\bigcup_{i=1}^N U_{p_i}$. Let $\varepsilon := \min\{\varepsilon_{p_1},\cdots,\varepsilon_{p_N}\}>0$ and $U := \bigcup_{i=1}^N U_{p_i}$ we obtain a smooth map

$$\psi : (-\varepsilon,\varepsilon)\times U\to M$$

s.t.

$$\begin{cases} \frac{\partial \psi}{\partial t}(t,q) = X(\psi(t,q)) \\ \psi(0,q) = q \end{cases}$$

Step 3, again by uniqueness

$$\phi|_{(-\varepsilon,\varepsilon)\times(U\cap V)} = \phi : \mathbb{R}\times V \to M \qquad \text{and} \qquad \psi : (-\varepsilon,\varepsilon)\times U \to M$$

We also have $U \cup V = M$ so we obtain

$$\phi : (-\varepsilon,\varepsilon)\times M \to M$$

satisfying assumptions. Step 4, for any $t \in \mathbb{R}$, there exists $n \in \mathbb{N}$ with $|t| < n\varepsilon$, we define $\phi(t,q) = \phi(\frac{t}{n}, \phi(\frac{t}{n}, \cdots, \phi(\frac{t}{n}, q))\cdots)$ Then $\phi : \mathbb{R}\times M \to M$ satisfy the assumptions. □

1.16 Lie Derivative

L_X **Lie Derivative on** $\mathfrak{X}(M)$ Recall $\mathfrak{X}(M)$ is C^∞-module. Also recall (1.22).

Definition 1.16.1 (Lie Derivative on smooth vector fields)**.** *Let M be $C^\infty(M)$ manifold. Define*

$$\begin{aligned} L_X : \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ Y &\mapsto [X,Y] \end{aligned} \tag{1.30}$$

where $[X,Y]$ denotes the Lie Bracket (1.23).

Notice

1. Leibniz Rule holds

$$L_X(fY) = (L_X f)Y + fL_XY \qquad \forall\ f \in C^\infty(M) \qquad \forall\ Y \in \mathfrak{X}(M) \tag{1.31}$$

Proof. Recall $L_XY = [X,Y]$ and $L_Xf = X(f)$. To show two vector fields are equal, it suffices to check they act the same on every $g \in C^\infty(M)$. Fix $g \in C^\infty(M)$. Then

$$\begin{aligned} L_X(fY)(g) &= [X, fY](g) \\ &= X\big((fY)(g)\big) - (fY)\big(X(g)\big) \\ &= X\big(f\,Y(g)\big) - f\,Y\big(X(g)\big) \\ &= X(f)\,Y(g) + f\,X\big(Y(g)\big) - f\,Y\big(X(g)\big) \\ &= X(f)\,Y(g) + f\,[X,Y](g). \end{aligned}$$

□

2. Recall Lie Derivative on $C^\infty(M)$ is $C^\infty(M)$-linear in

$$L_{fX}(g) = fL_X(g) \qquad \forall\ f,\,g \in C^\infty(M), \qquad \forall\ X \in \mathfrak{X}(M)$$

But in general $L_{fX}(Y) \neq fL_XY$ since

$$L_{fX}(Y) = fL_XY - Y(f)X$$

Proof. For any $g \in C^\infty(M)$

$$\begin{aligned} L_{fX}(Y)(g) &= [fX,Y](g) = fX(Y(g)) - Y(fX(g)) \\ &= fX(Y(g)) - Y(f)X(g) - fY(X(g)) \\ &= f[X,Y](g) - Y(f)X(g) \end{aligned}$$

□

Pullback and Pushforward of $\mathfrak{X}(M)$ Let M, N be C^∞ manifolds.

Definition 1.16.2. *Let $F : M \to N$ be C^∞ **diffeomorphism**. Define the pushforward*

$$\begin{aligned} F_* : \mathfrak{X}(M) &\to \mathfrak{X}(N) \\ X &\mapsto F_* X \end{aligned}$$

where

$$\begin{aligned} F_*X : N &\to TN \\ p &\mapsto dF_{F^{-1}(p)}(X(F^{-1}(p))) \in T_{F(p)}N \end{aligned} \tag{1.32}$$

Define pullback

$$\begin{aligned} F^* := (F^{-1})_* : \mathfrak{X}(N) &\to \mathfrak{X}(M) \\ Y &\mapsto F^*Y \end{aligned}$$

so that

$$\begin{aligned} F^*Y : M &\to TM \\ p &\mapsto dF^{-1}_{F(p)}(Y(F(p))) \in T_pM \end{aligned} \tag{1.33}$$

Remark that in this definition, F being diffeomorphism is essential.

L_X Lie Derivative as the Derivative under Pullback of Local Flow Let M be smooth manifold, $X \in \mathfrak{X}(M)$. Take $p \in M$ and U open neighborhood of p in M.

Recall definitions for pullback of $C^\infty(M)$ function (1.25) and pullback of $\mathfrak{X}(M)$ smooth vector field (1.33).

Proposition 1.16.1 (Lie Derivative using Flow)**.** *Let $\phi_t : U \to M$ smooth be flow of X at p for $t \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$.*

Then

1. *For $[f : M \to \mathbb{R}] = [f]_p \in C^\infty_p(M)$, pick a representative $f \in C^\infty(M)$, then*

$$(L_X f)(p) := X(p)([f]_p) = \left.\frac{d}{dt}\right|_{t=0} (\phi_t^* f)(p) \tag{1.34}$$

2. *For V open neighborhood of p, for any $Y \in \mathfrak{X}(V)$*

$$\begin{aligned} (L_X Y)(p) := [X, Y](p) &= \left.\frac{d}{dt}\right|_{t=0} (\phi_t^* Y)(p) \\ &= -\left.\frac{d}{dt}\right|_{t=0} (\phi_{t*} Y)(p) = \lim_{t\to 0} \frac{Y(p) - (d\phi_t)_{\phi_{-t}(p)}(Y(\phi_{-t}(p)))}{t} \end{aligned} \tag{1.35}$$

Note the equivalence in (1.35) is given by the fact

$$\phi_{t*} Y = -(\phi_{-t})_* Y = -\phi_t^* Y$$

Before the proof one need a Lemma.

Lemma 1.16.1. *If*

$$\begin{aligned} h : (-\delta, \delta) \times U &\to \mathbb{R} \\ (t, q) &\mapsto h(t, q) \end{aligned}$$

is C^∞ map for $U \subset M$ open, $\delta > 0$, and suppose that $h(0, q) = 0$. Then there exists C^∞ map $g : (-\delta, \delta) \times U \to \mathbb{R}$ s.t.

$$h(t, q) = tg(t, q)$$

Proof. Fix t, q. Let $u(s) := h(st, q)$. Then $\frac{d}{ds}u(s) = t\frac{\partial}{\partial t}h(st, q)$ with

$$h(t, q) = h(t, q) - h(0, q) = u(1) - u(0) = \int_0^1 \frac{d}{ds}u(s)\, ds = t\int_0^1 \frac{\partial}{\partial t}h(st, q)\, ds = tg(t, q)$$

where $g(t, q) = \int_0^1 \frac{\partial}{\partial t}h(st, q)\, ds$. Here g is C^∞ map. Notice $g(0, q) = \int_0^1 \frac{\partial}{\partial t}h(0, q)\, ds = \frac{\partial}{\partial t}h(0, q)$. □

Proof of Proposition 1.16.1. For $f \in C_p^\infty(M)$,

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} (\phi_t^* f)(p) &= \left.\frac{d}{dt}\right|_{t=0} f(\phi_t(p)) \\ &= \left.\frac{d}{dt}\right|_{t=0} (f \circ \phi_p)(t) \\ &= \phi_p'(0) f = X(p) f\end{aligned}$$

since $\phi_p(t) = \phi_t(p)$ for $\phi_p : (-\varepsilon, \varepsilon) \to M$ integral curves of X s.t. $\phi_p(0) = p$ and $\phi_p'(t) = X(\phi_p(t))$. Now for the second item, claim that

$$\left.\frac{d}{dt}\right|_{t=0} (\phi_{t*} Y)(p)(f) = -[X, Y](p) f \quad \forall\, f \in C_p^\infty(M)$$

To see this, let

$$h(t, q) = f \circ \phi_t(q) - f(q)$$

Here $h : (-\delta, \delta) \times V \to \mathbb{R}$ is C^∞ with $h(0, q) = 0$. By lemma 1.16.1, there exists C^∞ $g : (-\delta, \delta) \times V \to \mathbb{R}$ s.t. $h(t, q) = t g(t, q)$. For fixed $t \in (-\delta, \delta)$, $g_t : V \to \mathbb{R}$ smooth with $g_t(q) := g(t, q)$. So

$$f \circ \phi_t(q) = f(q) + h(t, q) = (f + t g_t)(q)$$

Also note

$$g_0(q) = \frac{\partial}{\partial t} h(0, q) = \left.\frac{d}{dt}\right|_{t=0} f \circ \phi_t(q) = X(q) f$$

from first item. Hence using Lemma 1.14.1

$$\begin{aligned}(\phi_{t*} Y)(p)(f) &= (d\phi_t)_{\phi_{-t}(p)}(Y(\phi_{-t}(p))) f = Y(\phi_{-t}(p))(f \circ \phi_t) \\ &= Y(\phi_{-t}(p))(f + t g_t) = Y(\phi_{-t}(p)) f + Y(\phi_{-t}(p))(t g_t)\end{aligned}$$

$$\left.\frac{d}{dt}\right|_{t=0} Y(\phi_{-t}(p))(f \circ \phi_t) = \left.\frac{d}{dt}\right|_{t=0} (Y f)(\phi_{-t}(p)) + Y(p) g_0 = -X(p) Y f + Y(p) X f = -[X, Y](p) f$$

□

1.17 Frobenius Theorem

Subbundle

Definition 1.17.1 (subbundle)**.** *Let $\pi : E \to M$ be C^∞ vector bundle of rank r over a C^∞ manifold M.*
Let $F \hookrightarrow E$ be embedded smooth submanifold. We say

$$\pi|_F : F \to M$$

is a C^∞ subbundle of rank $k \leq r$ if for any $p \in M$, there exists open neighborhood U of p in M and a local trivialization

$$h : \pi^{-1}(U) \to U \times \mathbb{R}^r \qquad C^\infty \text{ diffeomorphism}$$

s.t. diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U) \subseteq E & & \\ \big\downarrow h & \searrow^{\pi} & \\ U \times \mathbb{R}^r \subseteq M \times \mathbb{R}^r & \xrightarrow{\text{pr}_1} & U \subseteq M \end{array}$$

and

$$h(F \cap \pi^{-1}(U)) = U \times (\mathbb{R}^k \times \{0\}) \qquad \text{for } \mathbb{R}^k \times \{0\} \subset \mathbb{R}^r \tag{1.36}$$

Recall for any $x \in U$, $E_x \cong \mathbb{R}^r$ via (1.14)

$$h|_U : E_x = \pi^{-1}(x) \to \{x\} \times \mathbb{R}^r \overset{\text{pr}_2}{\cong} \mathbb{R}^r$$

For F a subbundle,

$$F_x := F \cap E_x$$

are vector subspaces of E_x of dimension $k \leq r$, with the inherited linear structure.

Given $\pi : E \to M$ smooth vector bundle of rank r over C^∞ manifold M, for any $x \in M$, with $F_x \subseteq E_x$ linear subspaces of dimension $k \leq r$, one may define C^∞ *subbundle of rank k* as the disjoint union

$$F := \bigsqcup_{x \in M} F_x \subseteq E = \bigsqcup_{x \in M} E_x$$

Proposition 1.17.1. *F is a C^∞ subbundle of E of rank k iff for any $p \in M$, there exists open neighborhood U of p in M and C^∞ sections*

$$\{s_1, \cdots, s_k\} \subseteq C^\infty(U; \pi^{-1}(U) = E|_U)$$

s.t. for any $q \in U$

$$\{s_1(q), \cdots s_k(q)\} \qquad \text{forms a basis for } F_q \qquad \forall\, q \in U$$

Example 1.17.1. *Let*

$$E = \{(\ell, v) \mid \ell \in P_n(\mathbb{R}),\, v \in \ell\} \subseteq P_n(\mathbb{R}) \times \mathbb{R}^{n+1}$$

Now

$$\begin{aligned} \mathrm{pr}_1 : E \subseteq P_n(\mathbb{R}) \times \mathbb{R}^{n+1} &\to P_n(\mathbb{R}) \\ (\ell, v) &\mapsto \ell \end{aligned}$$

is a smooth subbundle of rank 1 of the product vector bundle.

Distribution Let M be C^∞ manifold of dimension n.

Definition 1.17.2 (distribution)**.** *A C^∞ distribution of dimension $k \leq n$ on M is a collection*

$$\{F_p \subseteq T_pM \mid p \in M\}$$

where F_p are k-dimensional subspaces of T_pM s.t.

$$F = \bigsqcup_{p \in M} F_p \subseteq TM = \bigsqcup_{p \in M} T_pM$$

is a C^∞ subbundle of TM of rank k.

One has an equivalent definition for smooth distribution using Prop 1.17.1.

Lemma 1.17.1. *The collection*

$$\{F_p \subseteq T_pM \mid p \in M\}$$

of k-dimensional subspaces of T_pM is a smooth distribution iff for any $p \in M$, there exists open neighborhood U of p in M and $X_1, \cdots, X_k \in \mathfrak{X}(U)$ s.t. for any $q \in U$

$$F_q = \bigoplus_{i=1}^{k} \mathbb{R} X_i(q)$$

Given a smooth subbundle $F \to M$ of $\pi : TM \to M$, and denoting $C^\infty(M, F)$ as space of smooth sections of the subbundle $F \to M$. Notice

$$C^\infty(M, F) \subseteq C^\infty(M, TM) = \mathfrak{X}(M)$$

is $C^\infty(M)$-submodule.

Involutive and Completely Integrable Let F be C^∞ distribution of dimension k on a C^∞ manifold M of dimension n.

Definition 1.17.3 (Involutive)**.** *We say F is involutive if $C^\infty(M, F)$ is a Lie subalgebra of $(\mathfrak{X}(M), [\cdot, \cdot])$, i.e.*

$$[X, Y] \in C^\infty(M, F), \qquad \forall\, X,\, Y \in C^\infty(M, F)$$

Definition 1.17.4 (Completely Integrable)**.** *F is completely integrable if for any $p \in M$, there exists (U, ϕ) C^∞-chart for M around p with coordinates $\phi = (x_1, \cdots, x_n)$ s.t.*

$$F_q = \bigoplus_{i=1}^{k} \mathbb{R} \frac{\partial}{\partial x_i}(q) \qquad \forall\, q \in U$$

Complete Integrability is equivalent to saying for any $p \in M$, there is a k-dimensional submanifold $S \subset M$ s.t. $p \in S$ and for any $q \in S$, the subspace $T_qS = F_q$.

Example 1.17.2.

- *For $\dim F = \dim M$, then $F_p = T_pM$ for any $p \in M$, here F is involutive and completely integrable.*
- *For $\dim F = 1$, F is involutive and completely integrable.*
- *For $U \subset \mathbb{R}^3$ open, there exists $\dim = 2$ distributions not involutive and not completely integrable.*

Frobenius Theorem

Theorem 1.17.1 (Frobenius Theorem)**.** *A C^∞ distribution F on a C^∞ manifold is completely integrable if and only if it is involutive.*

Proof. Let $k := \operatorname{rank} F \le n = \dim M = \operatorname{rank} TM$.

($\Longrightarrow$) If F completely integrable, for any $X, Y \in C^\infty(M,F)$, for any $p \in M$, there exists (U,ϕ) C^∞ chart for M around p s.t. for any $q \in U$

$$F_q = \bigoplus_{i=1}^k \mathbb{R}\frac{\partial}{\partial x_i}(q)$$

On U, $X = \sum_{i=1}^k a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_{j=1}^k b_j \frac{\partial}{\partial x_j}$ so

$$[X,Y] = \sum_j^k \left(\sum_i^k a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i}\right)\frac{\partial}{\partial x_j} \implies [X,Y] \in C^\infty(M,F)$$

($\Longleftarrow$) Let F involutive. As a distribution, since F is smooth subbundle of TM, for any $p \in M$, there exists open neighborhood U of p in M and $X_1, \cdots, X_k \in \mathfrak{X}(U)$ s.t.

$$F_q = \bigoplus_{i=1}^k \mathbb{R}X_i(q) \qquad \forall\, q \in U$$

For any $p \in M$, there exists (U,ϕ) $\phi = (x_1, \cdots, x_n)$ so $X_i = \sum_{j=1}^n a_{ij}\frac{\partial}{\partial x_j}$ for $a_{ij} \in C^\infty(U)$, $i = 1, \cdots, k$. For any $p \in U$, consider

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix}(q) \qquad \text{of rank } k$$

By permuting $x_1, \cdots, x_n$ if necessary, we may assume the minor matrix

$$\det \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \cdots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}(p) \neq 0$$

Due to smoothness of a_{ij}, by shrinking U if necessary, we may assume

$$\det \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \cdots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}(q) \neq 0 \qquad \forall\, q \in U$$

Let $A := \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \cdots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}$ so $A = (a_{ij})_{i,j=1}^k : U \to \mathrm{GL}(r,\mathbb{R})$ and $A^{-1} =: (a^{ij})_{i,j=1}^k : U \to \mathrm{GL}(r,\mathbb{R})$ are smooth. Using $A^{-1}A = I_k$ we write

$$\sum_{\ell=1}^k a^{i\ell}a_{\ell j} = \delta_{ij}$$

For $i = 1, \cdots, k$, define

$$E^i := \sum_{j=1}^k a^{ij}X_j \in \mathfrak{X}(U) \qquad \textit{for any } q \in U$$

Hence for any $q \in U$, $F_q = \bigoplus_{i=1}^k \mathbb{R}E^i(q)$. Using $X_j = \sum_{\ell=1}^n a_{i\ell}\frac{\partial}{\partial x_\ell}$

$$\begin{aligned} E^i &:= \sum_{j=1}^k a^{ij}\left(\sum_{\ell=1}^n a_{j\ell}\frac{\partial}{\partial x_\ell}\right) = \sum_{\ell=1}^k \delta_{i\ell}\frac{\partial}{\partial x_\ell} + \sum_{\ell=k+1}^n \gamma_\ell^i \frac{\partial}{\partial x_\ell} \\ &= \frac{\partial}{\partial x_i} + \sum_{\ell=k+1}^n \gamma_\ell^i \frac{\partial}{\partial x_\ell} \\ \implies [E^i, E^j] &= [\frac{\partial}{\partial x_i} + \sum_{\ell=k+1}^n \gamma_\ell^i \frac{\partial}{\partial x_\ell}, \frac{\partial}{\partial x_j} + \sum_{\ell=k+1}^n \gamma_\ell^j \frac{\partial}{\partial x_\ell}] \\ &= \sum_{m=k+1}^n c_m^{ij}\frac{\partial}{\partial x_m} \end{aligned}$$

For any $q \in U$

$$[E^i, E^j](q) \in \bigoplus_{m=k+1}^{n} \mathbb{R}\frac{\partial}{\partial x_m}(q) =: G_q$$

where $\dim G_q = n - k$. Now G is completely integrable distribution of dimension $n - k$ on U. Since F is involutive with $E^i \in C^\infty(U, F|_U)$, for any $q \in U$

$$[E^i, E^j](q) \in F_q = \bigoplus_{i=1}^{k} \mathbb{R}E^i(q)$$

But as vector spaces $F_q \cap G_q = \{0\}$, so

$$[E^i, E^j](q) = 0$$

Conclusion: If F is an involutive C^∞ distribution of dimension k on M, then for any $p \in M$, there exists smooth chart (U, ϕ) for $\phi = (x_1, \cdots, x_n)$ of p in M and $E^1, \cdots, E^k \in \mathfrak{X}(U)$ s.t. $E^i = \frac{\partial}{\partial x_i} + \sum_{\ell=k+1}^n \gamma_\ell^i \frac{\partial}{\partial x_\ell}$

$$[E^i, E^j] = 0 \qquad \text{and} \qquad \forall\ q \in U \quad F_q = \bigoplus_{i=1}^{k} \mathbb{R}E^i(q)$$

The strategy is to construct new coordinates $(t_1, \cdots, t_n)$ on $U' \subseteq U$ s.t. $E^i = \frac{\partial}{\partial t_i}$ for $i = 1, \cdots, k$ on U'. One want to apply Lemma 1.15.1. To do so, we may assume $\phi(p) = 0 \in \mathbb{R}^n$. Define for V open neighborhood of $0 \in \mathbb{R}^n$

$$\begin{aligned} &\psi : V \subseteq \mathbb{R}^n \to M \\ &(t_1, \cdots, t_n) \mapsto \phi_{t_1}^{E^1} \circ \phi_{t_2}^{E^2} \circ \cdots \circ \phi_{t_k}^{E^k} \circ \phi^{-1}(0, \cdots, 0, t_{k+1}, \cdots, t_n) \end{aligned}$$

Then ψ is a C^∞ map. But for each $i \in \{1, \cdots, k\}$ one in fact has

$$\psi(t_1, \cdots, t_k) = \phi_{t_i}^{E^i}\left(\psi(t_1, \cdots, t_{i-1}, 0, t_{i+1}, \cdots, t_k)\right)$$

For fixed $t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_k$. Integral curve of E^i are

$$\gamma(s) := \psi(t_1, \cdots, t_{i-1}, s, t_{i+1}, \cdots, t_n) \qquad \text{with} \qquad \gamma(0) = \psi(t_1, \cdots, t_{i-1}, 0, t_{i+1}, \cdots, t_n)$$

Thus for $\psi : V \subseteq \mathbb{R}^n \to M$

$$d\psi_t(\frac{\partial}{\partial t_i}) = \frac{\partial \psi}{\partial t_i}(t_1, \cdots, t_n) = E^i(\psi(t_1, \cdots, t_n)) \qquad \forall t = (t_1, \cdots, t_n) \in V$$

At $t = 0$,

$$d\psi_0(\frac{\partial}{\partial t_i}) = \begin{cases} E^i(p)\ 1 \leq i \leq k \\ \frac{\partial}{\partial x_i}(p)\ k+1 \leq i \leq n \end{cases}$$

Hence $d\psi_0 : T_0V \cong \mathbb{R}^n \to T_pM$ is a linear isomorphism. There exists open neighborhood V' of 0 in $V \subset \mathbb{R}^n$, U' of p in M $U' \subseteq U$ s.t.

$$\psi|_{V'} : V' \to U'$$

is a C^∞ diffeomorphism. Then define $\phi' := (\psi|_{V'})^{-1} : U' \to V' \subseteq \mathbb{R}^n$ with $E^i = \frac{\partial}{\partial t_i}$ on $U' \subseteq U$, where $\phi' = (t_1, \cdots, t_n)$. □

1.18 Operations on Vector Bundles

Recall operations on vector spaces. V, W finite dimensional vector spaces of dimension r, s. Then

- V^* dual vector space is of dimension r
- $V \oplus W$ direct sum dimension $r + s$
- $V \otimes W$ tensor product dimension of rs
- $V^{\otimes k} = V \otimes \cdots \otimes V$ k-tensor product of V, dimension of r^k.
- $\Lambda^k V$ Wedge product, dimension $\binom{r}{k}$.

Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be C^∞ vector bundles of rank r, s over a C^∞ manifold M. Let the fibers be denoted as $E_p := \pi_E^{-1}(p) \cong \mathbb{R}^r$ and $F_p := \pi_F^{-1}(p) \cong \mathbb{R}^s$ for any $p \in M$, i.e.,

$$\pi_E : E = \bigsqcup_{p\in M} E_p \to M \qquad \text{and} \qquad \pi_F : F = \bigsqcup_{p\in M} F_p \to M$$

Since each E_p, F_p has structure of a vector space, one may perform the above vector space operations to fibers and define the following bundles at the set level.

- $E^* := \bigsqcup_{p\in M} E_p^*$ where $E_p^* := (E_p)^*$.
- $E \oplus F := \bigsqcup_{p\in M}(E \oplus F)_p$ where $(E \oplus F)_p := E_p \oplus F_p$.
- $E \otimes F := \bigsqcup_{p\in M}(E \otimes F)_p$ where $(E \otimes F)_p := E_p \otimes F_p$.
- $E^{\otimes k} := \bigsqcup_{p\in M}(E^{\otimes k})_p$ where $(E^{\otimes k})_p := E_p^{\otimes k}$.
- $\Lambda^k E := \bigsqcup_{p\in M}(\Lambda^k E)_p$ where $(\Lambda^k E)_p := \Lambda^k E_p$.

1.18.1 Dual Bundle

Let $\pi_E : E \to M$ be C^∞ vector bundles of rank r over a C^∞ manifold M.

- As a set, let $E^* := \bigsqcup_{p\in M} E_p^*$.
- As a map, let
$$\begin{aligned} \pi_{E^*} : E^* &\to M \\ E_p^* &\mapsto \{p\} \end{aligned}$$

One wish to construct $\pi_{E^*} : E^* \to M$ a smooth vector bundle of rank r.

First recall the local trivializations and smooth frame on E. Since $\pi_E : E \to M$ is vector bundle of rank r, there exists $\{U_\alpha \mid \alpha \in I\}$ open cover of M and local trivializations

$$h_\alpha^E : \pi_E^{-1}(U_\alpha) \subseteq E \to U_\alpha \times \mathbb{R}^r$$

C^∞ diffeomorphisms s.t. $\pi_E = \mathrm{pr}_1 \circ h_\alpha^E$. For any $x \in U_\alpha$, $h_\alpha^E\big|_{E_x} : E_x = \pi_E^{-1}(x) \to \{x\} \times \mathbb{R}^r$ are linear isomorphisms. One shall notice that

- h_α^E are local trivialization iff
- h_α^E are isomorphisms from $\pi_E^{-1}(U_\alpha)$ to the product vector bundle of rank r over U_α iff
- There exists C^∞ frame $e_{\alpha_1}, \cdots, e_{\alpha_r}$ where $e_{\alpha_i} \in C^\infty(U_\alpha, \pi_E^{-1}(U_\alpha))$. In particular, for any $x \in U_\alpha$, $\{e_{\alpha_i}(x)\}_{i=1}^r$ are defined as
$$\begin{aligned} e_{\alpha_i} : U_\alpha \subseteq M &\to \pi_E^{-1}(U_\alpha) \subseteq E \\ x &\mapsto (h_\alpha^E)^{-1}(x, e_i) \end{aligned} \tag{1.37}$$
where $e_i = (0, \cdots, 1, \cdots 0)$ are standard basis in $\mathbb{R}^r$. Notice the inverse of local trivializations can be recovered by the smooth frame
$$\begin{aligned} (h_\alpha^E)^{-1} : U_\alpha \times \mathbb{R}^r &\to \pi_E^{-1}(U_\alpha) \\ (x, v) &\mapsto (x, \sum_{i=1}^r v_i e_{\alpha_i}(x)) \end{aligned}$$

Then recall the smooth transition functions for E. On $U_\alpha \cap U_\beta$, one has smooth frames $\{e_{\alpha_i}(x)\}_{i=1}^r$ defined by h_α^E and $\{e_{\beta_i}(x)\}_{i=1}^r$ defined by h_β^E. Due to definition of vector bundle, one has the linear isomorphisms in $\mathbb{R}^r$

$$(g_{\beta\alpha}^E(x))_{i,j=1}^r \in C^\infty(U_\alpha \cap U_\beta; \mathrm{GL}(r, \mathbb{R}))$$

s.t.

$$e_{\alpha j}(x) = \sum_{i=1}^r e_{\beta i}(x) g_{\beta\alpha}^E(x)_{ij} \tag{1.38}$$

or in short

$$e_\alpha = e_\beta g_{\beta\alpha}^E$$

with notation $e_\alpha = [e_{\alpha_1}, \cdots, e_{\alpha_r}]$ and $e_\beta = [e_{\beta_1}, \cdots, e_{\beta_r}]$. The $g^E_{\beta\alpha}$ corresponds to the transition functions

$$h^E_\beta \circ (h^E_\alpha)^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^r \to (U_\alpha \cap U_\beta) \times \mathbb{R}^r$$

via the following

$$\begin{aligned} h^E_\beta \circ (h^E_\alpha)^{-1}(x, v) &= h^E_\beta(x, \sum_{j=1}^r v_j e_{\alpha_j}(x)) \\ &= h^E_\beta(x, \sum_{j=1}^r v_j \sum_{i=1}^r e_{\beta_i}(x) g^E_{\beta\alpha}(x)_{ij}) \\ &= h^E_\beta(x, \sum_{i=1}^r (\sum_{j=1}^r v_j g^E_{\beta\alpha}(x)_{ij}) e_{\beta_i}(x)) \\ &= (x, g^E_{\beta\alpha}(x) v) \end{aligned}$$

So the transition functions $h^E_\beta \circ (h^E_\alpha)^{-1}$ are given by

$$h^E_\beta \circ (h^E_\alpha)^{-1}(x, v) = (x, g^E_{\beta\alpha}(x) v)$$

Now one wish to define the smooth structure on the set E^*.

(i) For Smooth Frame, define

$$e^*_{\alpha_i} : U_\alpha \to \pi^{-1}_{E^*}(U_\alpha) = \bigsqcup_{x \in U_\alpha} E^*_x \subseteq E^*$$

s.t. for any $x \in U_\alpha$ with $e_{\alpha_j}(x) \in E_x$, $e^*_{\alpha_i}(x) \in (E^*)_x := (E_x)^*$, we have

$$\langle e^*_{\alpha_i}(x), e_{\alpha_j}(x) \rangle = \delta_{ij} \tag{1.39}$$

i.e., $\{e^*_{\alpha_i}(x)\}_{i=1}^r$ is a dual basis for the dual space E^*_x w.r.t. $\{e_{\alpha_i}(x)\}_{i=1}^r$ as basis of E_x.

(ii) For Local trivializations, define

$$\begin{aligned} h^{E^*}_\alpha : \pi^{-1}_{E^*}(U_\alpha) \subseteq E^* &\to U_\alpha \times \mathbb{R}^r \\ (x, \sum_{i=1}^r v_i e^*_{\alpha_i}(x)) &\mapsto (x, v = \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix}) \end{aligned}$$

bijection. We use this bijection to equip $\pi^{-1}_{E^*}(U_\alpha)$ with topology and a smooth structure s.t. the map $h^{E^*}_\alpha$ is C^∞ diffeomorphism. Then $\pi^{-1}_{E^*}(U_\alpha)$ is a C^∞ manifold of dimension $n + r$ where $n = \dim M$. Indeed $\pi_{E^*} = \mathrm{pr}_1 \circ h^{E^*}_\alpha$ for any $x \in U_\alpha$ and $E^*_x \cong \mathbb{R}^r$.

(iii) Smooth Transition Functions. On $U_\alpha \cap U_\beta \neq \varnothing$, recall

$$e_{\alpha_j}(x) = \sum_{i=1}^r e_{\beta_i}(x) g^E_{\beta\alpha}(x)_{ij} \in E_x$$

Then by our definition of $e^*_{\beta_k}$ (1.39)

$$\begin{aligned} \langle e^*_{\beta_k}(x), e_{\alpha_j}(x) \rangle &= \sum_{i=1}^r \delta_{ik} g^E_{\beta\alpha}(x)_{kj} = g^E_{\beta\alpha}(x)_{kj} \\ \implies e^*_{\beta_k}(x) &= \sum_{i=1}^r g^E_{\beta\alpha}(x)_{ki} e^*_{\alpha_i}(x) \\ &= \sum_{i=1}^r e^*_{\alpha_i}(x) \left(g^E_{\beta\alpha}(x)\right)^T_{ik} \\ &:= \sum_{i=1}^r e^*_{\alpha_i}(x) g^{E^*}_{\alpha\beta}(x)_{ik} \\ \implies (g^{E^*}_{\beta\alpha})^{-1} = g^{E^*}_{\alpha\beta} &= (g^E_{\beta\alpha})^T \end{aligned}$$

Now

$$g_{\beta\alpha}^{E^*} = ((g_{\beta\alpha}^E)^T)^{-1} : U_\alpha \cap U_\beta \to \mathrm{GL(r,\mathbb{R})} \qquad \text{is } C^\infty \text{ map}$$

The transition map

$$h_\alpha^{E^*} \circ \left(h_\beta^{E^*}\right)^{-1} : U_\alpha \cap U_\beta \times \mathbb{R}^r \to U_\alpha \cap U_\beta \times \mathbb{R}^r$$

is given by

$$h_\alpha^{E^*} \circ \left(h_\beta^{E^*}\right)^{-1}(x,v) = \left(x, g_{\alpha\beta}^{E^*}(x)v\right) = \left(x, (g_{\beta\alpha}^E)^T(x)v\right)$$

while its inverse is given by

$$h_\beta^{E^*} \circ \left(h_\alpha^{E^*}\right)^{-1}(x,v) = \left(x, g_{\beta\alpha}^{E^*}(x)v\right) = \left(x, ((g_{\beta\alpha}^E)^T)^{-1}(x)v\right)$$

The above smooth structure gives

$$\pi_{E^*} : E^* \to M$$

a C^∞ vector bundle of rank r.

Other Operations Similarly, for $\{e_{\alpha_i}\}_{i=1}^r$ C^∞ frame of $E|_{U_\alpha} := \pi_E^{-1}(U_\alpha)$ and $\{f_{\alpha_j}\}_{j=1}^s$ C^∞ frame of $F|_{U_\alpha} := \pi_F^{-1}(U_\alpha)$

- $\{e_{\alpha_i}\}_{i=1}^r \cup \{f_{\alpha_j}\}_{j=1}^s$ is C^∞ frame of $(E\oplus F)|_{U_\alpha}$.
- $\{e_{\alpha_i} \otimes f_{\alpha_j} \mid 1 \le i \le r,\, 1 \le j \le s\}$ is C^∞ frame of $(E\otimes F)|_{U_\alpha}$.
- $\{e_{\alpha_{i_1}} \wedge \cdots \wedge e_{\alpha_{i_k}} \mid 1 \le i_1 \le \cdots \le i_k \le r\}$ is C^∞ frame of $(\Lambda^k E)\big|_{U_\alpha}$ for $k \le r$.

1.19 Tensor Bundles

1.19.1 Basics on Tensors

Let $V_1, \cdots, V_k$ be vector spaces of dimension n. A map

$$\begin{aligned} T : V_1 \times \cdots \times V_k &\to \mathbb{R} \\ (v_1, \cdots, v_k) &\mapsto T(v_1, \cdots, v_k) \end{aligned}$$

is *multi-linear* if it is linear as a function of each variable separately while holding the other fixed, i.e.

$$T(v_1, \cdots, av_i + a'v_i', \cdots, v_k) = aT(v_1, \cdots, v_i, \cdots, v_k) + a'T(v_1, \cdots, v_i', \cdots, v_k)$$

Definition 1.19.1. *Given a finite-dimensional vector space V. A covariant k-tensor on V is a multi-linear map $T \in T_k^0(V)$*

$$\begin{aligned} T : \underbrace{V \times \cdots \times V}_{k \text{ times}} &\to \mathbb{R} \\ (v_1, \cdots, v_k) &\mapsto T(v_1, \cdots, v_k) \end{aligned}$$

A contravariant k-tensor on V is a multi-linear map $T \in T_0^k(V)$

$$\begin{aligned} T : \underbrace{V^* \times \cdots \times V^*}_{k \text{ times}} &\to \mathbb{R} \\ (\alpha_1, \cdots, \alpha_k) &\mapsto T(\alpha_1, \cdots, \alpha_k) \end{aligned}$$

A mixed tensor $T \in T_s^r(V)$ of type (r,s), also called a r-contravariant, s-covariant tensor, is a multi-linear map

$$\begin{aligned} T : \underbrace{V^* \times \cdots \times V^*}_{r \text{ times}} \times \underbrace{V \times \cdots \times V}_{s \text{ times}} &\to \mathbb{R} \\ (\alpha_1, \cdots, \alpha_r, v_1, \cdots, v_s) &\mapsto T(\alpha_1, \cdots, \alpha_k, v_1, \cdots, v_s) \end{aligned}$$

The rank of a tensor is the total number of arguments it takes.

Tensor Product There is a natural product, called the tensor product, linking various tensor spaces over V.

Definition 1.19.2. *Let $S \in T^r_s(V)$ and $T \in T^k_\ell(V)$, then the tensor product $S \otimes T \in T^{r+k}_{s+\ell}(V)$ is defined as*

$$S \otimes T : \underbrace{V^* \times \cdots \times V^*}_{r+k \text{ times}} \times \underbrace{V \times \cdots \times V}_{s+\ell \text{ times}} \to \mathbb{R}$$
$$(\alpha_1, \cdots, \alpha_{r+k}, v_1, \cdots, v_{s+\ell}) \mapsto S(\alpha_1, \cdots, \alpha_r, v_1, \cdots, v_s)\, T(\alpha_{r+1}, \cdots, \alpha_{r+k}, v_{s+1}, \cdots, v_{s+\ell})$$

Given $\{e_i\}$ a basis for V, and $\{e^i\}$ its dual basis in V^*, we may define a 'basis' tensor that acts on basis elements via

$$e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s} \left(e^{k_1}, \cdots, e^{k_r}, e_{\ell_1}, \cdots, e_{\ell_s}\right) := \delta^{k_1}_{i_1} \cdots \delta^{k_r}_{i_r} \delta^{j_1}_{\ell_1} \cdots \delta^{j_s}_{\ell_s}$$

Thus the vector space $T^r_s(V)$ has dimension n^{r+s} with the basis

$$\left\{e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}\right\}_{\substack{1 \le i_1, \cdots, i_r \le n \\ 1 \le j_1, \cdots, j_s \le n}}$$

We can write any tensor $T \in T^r_s(V)$ in components

$$T = \sum_{\substack{1 \le i_1, \cdots, i_r \le n \\ 1 \le j_1, \cdots, j_s \le n}} T^{i_1, \cdots, i_r}_{j_1, \cdots, j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}$$

where the coefficients are

$$T^{i_1, \cdots, i_r}_{j_1, \cdots, j_s} := T(e^{i_1}, \cdots, e^{i_r}, e_{j_1}, \cdots, e_{j_s})$$

Tensor Contraction Let V be a vector space of dimension n. Recall $T \in T^r_s(V) = V^{\otimes r} \otimes V^{*\otimes s}$ is a (r, s)-tensor, in particular

$$T : \underbrace{V^* \times \cdots \times V^*}_{r \text{ times}} \times \underbrace{V \times \cdots \times V}_{s \text{ times}} \to \mathbb{R}$$
$$(\alpha_1, \cdots, \alpha_r, v_1, \cdots, v_s) \mapsto T(\alpha_1, \cdots, \alpha_r, v_1, \cdots, v_s)$$

For any index $a \in \{1, \cdots, r\}$, $b \in \{1, \cdots, s\}$ we define Tensor Contraction (trace) as the unique linear map s.t.

$$\begin{aligned} \mathrm{tr}^b_a : T^r_s(V) &\to T^{r-1}_{s-1}(V) \\ T &\mapsto \mathrm{tr}^b_a T \end{aligned}$$

where

$$\mathrm{tr}^b_a T : \underbrace{V^* \times \cdots \times V^*}_{r-1 \text{ times}} \times \underbrace{V \times \cdots \times V}_{s-1 \text{ times}} \to \mathbb{R}$$
$$(\alpha_1, \cdots, \alpha_{r-1}, v_1, \cdots, v_{s-1}) \mapsto \sum_{i=1}^n T(\alpha_1, \cdots, \alpha_{a-1}, e^i, \alpha_a, \cdots, \alpha_{r-1}, v_1, \cdots, v_{b-1}, e_i, v_b, \cdots, v_{s-1}) \tag{1.40}$$

with $\{e_i\}$ a basis of V and $\{e^i\}$ its dual basis in V^*. This definition is independent of the choice of basis, and hence is well-defined.

Taking the trace lowers the rank of the tensor by 2

In particular, if $T \in T^r_s(V)$ has coefficients

$$T = \sum_{\substack{1 \le i_1, \cdots, i_r \le n \\ 1 \le j_1, \cdots, j_s \le n}} T^{i_1, \cdots, i_r}_{j_1, \cdots, j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}$$

w.r.t. $\{e_i\}$ basis for V and $\{e^j\}$ dual basis in V^*, where

$$T^{i_1, \cdots, i_r}_{j_1, \cdots, j_s} := T(e^{i_1}, \cdots, e^{i_r}, e_{j_1}, \cdots, e_{j_s})$$

Then its contraction $\mathrm{tr}^b_a T$ has coefficients

$$(\mathrm{tr}^b_a T)^{i_1, \cdots, i_{a-1}, \hat{i}_a, \cdots, i_r}_{j_1, \cdots, j_{b-1}, \hat{j}_b, \cdots, j_s} = \sum_{m=1}^n T^{i_1, \cdots, i_{a-1}, m, \cdots, i_r}_{j_1, \cdots, j_{b-1}, m, \cdots, j_s} \tag{1.41}$$

Notice the base example for contraction identifies

$$V \times V^* = T^1_1(V) \cong \mathrm{End}(V)$$

where

$$\begin{aligned} \mathrm{tr}^1_1 : V \otimes V^* = T^1_1(V) &\to \mathbb{R} \\ Y \otimes \alpha &\mapsto \alpha(Y) \end{aligned} \tag{1.42}$$

1.19.2 Definitions

T^*M Cotangent Bundle

Definition 1.19.3 (Cotangent Bundle). *Let M be C^∞ manifold with dimension n. Let $p \in M$*

- *A cotangent vector at $p \in M$ is a vector in $T_p^*M := (T_pM)^*$.*
- *T_p^*M is the cotangent vector space at p.*
- *$T^*M := (TM)^* = \bigsqcup_{p\in M} T_p^*M$ a C^∞ vector bundle of rank n is the cotangent bundle.*

$C^\infty(M, \mathcal{T}_s^r(M))$ Smooth (r,s)-Tensors

Definition 1.19.4. *Let M be C^∞ manifold with dimension n. Consider the C^∞ vector bundle over M of rank n^{r+s}*

$$T_s^r(M) := (TM)^{\otimes r} \otimes (T^*M)^{\otimes s}$$

A C^∞ (r,s)-tensor is a smooth section of $T_s^r(M)$, i.e.

$$\{\textit{Space of smooth } (r,s)\textit{-tensors on } M\} := C^\infty(M, \mathcal{T}_s^r(M))$$

One has the useful Tensor Characterisation Lemma.

Lemma 1.19.1 ([Lee12] Lemma 12.24). *A map*

$$T : (T^*M)^{\otimes r} \times TM^{\otimes s} \to C^\infty(M)$$
$$(\alpha_1, \cdots, \alpha_r, X_1, \cdots, X_s) \mapsto T(\alpha_1, \cdots, \alpha_r, X_1, \cdots, X_s)$$

is induced by a smooth (r,s)-tensor iff it is multi-linear over $C^\infty(M)$.

$\Omega^s(M)$ Smooth $(0,s)$-Forms

Definition 1.19.5. *Let M be C^∞ manifold with dimension n. Consider the C^∞ vector bundle over M of rank $\binom{n}{k}$, Λ^sT^*M.*

A C^∞ s-form on M is a C^∞ section of Λ^sT^*M.

$$\{\textit{Space of smooth } s\textit{-forms on } M\} \equiv \Omega^s(M) := C^\infty(M, \Lambda^sT^*M)$$

Notice $\Lambda^sT^*M \subseteq T_s^0(M) = (T^*M)^{\otimes s}$.

One has some first remarks.

Remark 1.19.1. *Given smooth manifold M.*

- *$f \in C^\infty(M)$ is $(0,0)$-tensor.*

$$C^\infty(M) = C^\infty(M, \mathbb{R}) = C^\infty(M, T_0^0M) = \Omega^0(M)$$

- *$X \in \mathfrak{X}(M)$ is $(1,0)$-tensor.*

$$\mathfrak{X}(M) = C^\infty(M, TM) = C^\infty(M, T_0^1M)$$

- *1-form are exactly $(0,1)$-tensors.*

$$\Omega^1(M) = C^\infty(M, T^*M) = C^\infty(M, T_1^0M)$$

- *s-forms are examples of $(0,s)$-tensors.*

$$\Omega^s(M) \subseteq C^\infty(M, T_s^0M)$$

Example 1.19.1 (Differential of a smooth function df as a 1-form). *Let M be a smooth manifold and (U,ϕ) a smooth chart with coordinates $\phi = (x_1, \ldots, x_n)$.*

For $f \in C^\infty(U)$, the differential defines a map df

$$\begin{aligned} df : U &\to T^*M|_U \\ q &\mapsto df_q \end{aligned} \tag{1.43}$$

where each $df_q \in T_q^*M$ is the covector

$$df_q : T_qM \to T_{f(q)}\mathbb{R} = \mathbb{R}$$
$$v \mapsto v(f)$$

Thus $df \in \Omega^1(U)$ is a smooth $(0,1)$-tensor field (a 1-form).

Moreover, the pairing $\langle df, \frac{\partial}{\partial x_i}\rangle$ defines a smooth mapping for each $i = 1, \cdots, n$ via

$$\langle df, \frac{\partial}{\partial x_i}\rangle = \frac{\partial f}{\partial x_i} : U \to \mathbb{R}$$
$$q \mapsto df_q\left(\frac{\partial}{\partial x_i}\Big|_q\right) = \frac{\partial}{\partial x_i}\Big|_q([f]_q) \overset{(1.17)}{:=} \frac{\partial(f \circ \phi^{-1})}{\partial x_i}(\phi(q))$$

Tensors in Local Coordinates We pass to local coordinates. Let (U, ϕ) be C^∞ chart for M with $\phi = (x_1, \cdots, x_n)$ for $x_i \in C^\infty(U)$.

Differentials of coordinate functions $\{dx_i\}$. $dx_i \in \Omega^1(U)$

$$dx_i : U \subseteq M \to T^*M|_U$$
$$p \mapsto (dx_i)_p$$

where

$$(dx_i)_p : T_pM \to T_{\phi(p)}\mathbb{R} \cong \mathbb{R}$$

is linear map defined s.t.

$$(dx_i)_p(\frac{\partial}{\partial x_j}(p)) := \delta_{ij} = \frac{\partial x_i}{\partial x_j}$$

where $\{\frac{\partial}{\partial x_j}\}$ is C^∞ frame of $TM|_U = TU$. Hence $\{dx_i\}$ is the C^∞ dual frame of $T^*M|_U = T^*U$.

Differentials of $f \in C^\infty(U)$. For any $f \in C^\infty(U)$ one writes

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \in \Omega^1(U) \tag{1.44}$$

More generally, on U, C^∞ vector fields as $(1,0)$-tensors are

$$\sum_i^n a_i \frac{\partial}{\partial x_i}$$

where $a^i \in C^\infty(U)$, and C^∞ 1 -forms as $(0,1)$-tensors are

$$\sum_i a_i dx_i$$

where $a^i \in C^\infty(U)$.

C^∞ (r,s)**-tensors**.

$$\sum_{\substack{1\le i_1,\cdots,i_r\le n \\ 1\le j_1,\cdots,j_s\le n}} a^{i_1,\cdots,i_r}_{j_1,\cdots,j_s} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s} \tag{1.45}$$

for $a^{i_1,\cdots,i_r}_{j_1,\cdots,j_s} \in C^\infty(U)$. And C^∞ s-form is

$$\sum_{1\le j_1,\cdots,j_s\le n} a_{j_1,\cdots,j_s} dx_{j_1} \wedge \cdots \wedge dx_{j_s}$$

with convection $dx_1 \wedge dx_2 = dx_1 \otimes dx_2 - dx_2 \otimes dx_1$.

1.19.3 Pullback and Pushforward of Tensor Bundles

Pullback Dual Map Let M, N be smooth manifolds.

Definition 1.19.6. *Let $\phi : M \to N$ be C^∞ map. Recall its differential (1.26) writes*

$$d\phi_p : T_pM \to T_{\phi(p)}N$$

Define its Pullback Dual Map

$$d\phi_p^* : T^*_{\phi(p)}N \to T_p^*M$$
$$Y \mapsto d\phi_p^*(Y)$$

where

$$d\phi_p^*(Y) : T_pM \to \mathbb{R}$$
$$X \mapsto Y \circ d\phi_p(X) = Y(d\phi_p(X)) \tag{1.46}$$

Pullback of $(0,s)$-tensor Let M, N be smooth manifolds.

Definition 1.19.7. *Let $\phi : M \to N$ be C^∞ map. Define pull back of $(0,s)$-tensors under ϕ as*

$$\begin{aligned}\phi^* : C^\infty(N, T^0_sN) &\to C^\infty(M, T^0_sM)\\ T &\mapsto \phi^*T\end{aligned}$$

where

$$\begin{aligned}\phi^*T : M &\to T^0_sM\\ p &\mapsto (\phi^*T)_p \in (T^0_sM)_p = (T^*M)_p^{\otimes s}\end{aligned}$$

s.t.

$$\begin{aligned}(\phi^*T)_p : T_pM^{\otimes s} &\to \mathbb{R}\\ (v_1, \cdots, v_s) &\mapsto T_{\phi(p)}(d\phi_p(v_1), \cdots, d\phi_p(v_s))\end{aligned}$$

Or equivalently, one simply define in view of (1.46)

$$(\phi^*T)(p) := (d\phi_p^*)^{\otimes s}(T(\phi(p))) \tag{1.47}$$

Pullback of $(0,s)$-forms Let M, N be smooth manifolds. Let $\phi : M \to N$ be C^∞ map. Consider the pullback of $(0,s)$-forms

$$\begin{aligned}\phi^* : \Omega^s(N) \subseteq C^\infty(N, T^0_sN) &\to \Omega^s(M) \subseteq C^\infty(M, T^0_sM)\\ \alpha &\mapsto \phi^*\alpha\end{aligned}$$

where

$$\begin{aligned}\phi^*\alpha : M &\to \Lambda^sT^*M\\ p &\mapsto (\phi^*\alpha)_p\end{aligned}$$

s.t.

$$\begin{aligned}(\phi^*\alpha)_p : \Lambda^sT_pM &\to \mathbb{R}\\ (v_1, \cdots, v_s) &\mapsto \alpha_{\phi(p)}(d\phi_p(v_1), \cdots, d\phi_p(v_s))\end{aligned} \tag{1.48}$$

In particular, the **pullback of $(0,1)$-tensor (equivalently 1-forms)** are, for any $\alpha \in \Omega^1(N)$, $Y \in \mathfrak{X}(N)$, for any $p \in M$

$$\begin{aligned}(\phi^*\alpha)_p : T_pM &\to \mathbb{R}\\ v &\mapsto \alpha_{\phi(p)}(d\phi_p(v))\end{aligned} \tag{1.49}$$

If $f \in C^\infty(N) = \Omega^0(N)$, $df \in \Omega^1(N)$ as in Example 1.19.1. The question is, how does df behave under pullbacks?

One has commutative lemma for on $\Omega^1(N)$.

Lemma 1.19.2.

$$\phi^*(df) = d(\phi^*f) \in \Omega^1(M) \tag{1.50}$$

Proof. For any $p \in M$

$$(\phi^*df)(p) \overset{(1.47)}{=} d\phi_p^*(df_{\phi(p)}) \overset{(1.46)}{=} df_{\phi(p)} \circ d\phi_p = d(f \circ \phi)_p \overset{(1.25)}{=} d(\phi^*f)_p$$

□

Let's see the formula for coordinate changes.

Lemma 1.19.3. *Let $\phi : M \to N$ be smooth, with $(x_1, \cdots, x_n)$ coordinates for $U \subseteq M$ and $(y_1, \cdots, y_n)$ coordinates for $\phi(U) \subseteq V$. Then*

$$\phi^*(dy_i) = \sum_{j=1}^n (\frac{\partial y_i}{\partial x_j} \circ \phi) dx_j$$

Now for open set $V \subseteq N$ with coordinates $(y_1, \cdots, y_n)$ in $\mathbb{R}^n$, one has local coordinates expression

$$df = \sum_{i=1}^n \frac{\partial f}{\partial y_i} dy_i \qquad V$$

Denoting $(x_1, \cdots, x_n)$ as local coordinates on U one has pullback

$$\begin{aligned}d(\phi^*f) &= \sum_{i=1}^n \frac{\partial(f \circ \phi)}{\partial x_i} dx_i\\ \phi^*(df) &= \sum_{i=1}^n (\frac{\partial f}{\partial y_i} \circ \phi)\phi^*(dy_i) = \sum_{i=1}^n (\frac{\partial f}{\partial y_i} \circ \phi) \left(\sum_{j=1}^n \frac{\partial y_i}{\partial x_j} dx_j \right) = \sum_{j=1}^n \left(\sum_{i=1}^n (\frac{\partial f}{\partial y_i} \circ \phi) \frac{\partial y_i}{\partial x_j} \right) dx_j\end{aligned}$$

If more generally take any 1-form over N with smooth frame $\{dy_i\}_{i=1}^n$ in local coordinates, one has

$$\phi^*(dy_i) = \sum_{j=1}^n \frac{\partial y_i}{\partial x_j} dx_j \ \in \Omega^1(M)$$

So for the local coordinate representation,

$$\phi^*(\sum_{i=1}^n a_i dy_i) = \sum_{i=1}^n (a_i \circ \phi)\ \phi^* dy_i \ \in \Omega^1(M)$$

for $a_i \in C^\infty(N)$.

Lemma 1.19.4. *For* $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$

$$(g \circ f)^* = f^* g^* : C^\infty(M_3, T_s^0(M_3)) \to C^\infty(M_1, T_s^0(M_1))$$

One also has commutative lemma for wedge product.

Lemma 1.19.5. *Let* α*,* β *be a 1-forms on a manifold* N*, and let* $\phi : M \to N$ *be smooth. Then*

$$\phi^*(\alpha \wedge \beta) = (\phi^*\alpha) \wedge (\phi^*\beta).$$

Proof. Fix $p \in M$ and $v_1, v_2 \in T_pM$. By definition of pullback of a 2-form,

$$\big(\phi^*(\alpha \wedge \beta)\big)_p(v_1, v_2) = (\alpha \wedge \beta)_{\phi(p)}\big(d\phi_p(v_1),\, d\phi_p(v_2)\big).$$

Since α, β are 1-forms, the wedge product is given by

$$(\alpha \wedge \beta)_q(w_1, w_2) = \alpha_q(w_1)\beta_q(w_2) - \alpha_q(w_2)\beta_q(w_1) \qquad (q \in N,\ w_1, w_2 \in T_qN).$$

Applying this with $q = \phi(p)$ and $w_i = d\phi_p(v_i)$, we obtain

$$\begin{aligned}(\alpha \wedge \beta)_{\phi(p)}\big(d\phi_p(v_1),\, d\phi_p(v_2)\big) &= \alpha_{\phi(p)}(d\phi_p(v_1))\, \beta_{\phi(p)}(d\phi_p(v_2)) - \alpha_{\phi(p)}(d\phi_p(v_2))\, \beta_{\phi(p)}(d\phi_p(v_1)) \\ &= (\phi^*\alpha)_p(v_1)\, (\phi^*\beta)_p(v_2) - (\phi^*\alpha)_p(v_2)\, (\phi^*\beta)_p(v_1) \\ &= \big((\phi^*\alpha) \wedge (\phi^*\beta)\big)_p(v_1, v_2)\end{aligned}$$

□

Example 1.19.2. *Let*

$$\begin{aligned}\phi : (0, \infty) \times \mathbb{R} &\to \mathbb{R}^2 \\ (r, \theta) &\mapsto (r\cos(\theta), r\sin(\theta))\end{aligned}$$

We'd like to compute ϕ^*dx*,* ϕ^*dy *and* $\phi^*(dx \wedge dy)$*. First of all, what is* $\phi^*(x)$ *and* $\phi^*(y)$*? Here*

$$\begin{aligned}x : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto x \\ y : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto y\end{aligned}$$

Thus

$$\begin{aligned}\phi^*(x) = x \circ \phi : (0, \infty) \times \mathbb{R} &\to \mathbb{R} \\ (r, \theta) &\to r\cos(\theta) \\ \phi^*(y) = y \circ \phi : (0, \infty) \times \mathbb{R} &\to \mathbb{R} \\ (r, \theta) &\to r\sin(\theta)\end{aligned}$$

Furthermore, in view of (1.44)

1. $\phi^*(dx) = d(\phi^*x) = d(r\cos(\theta)) = \cos(\theta)dr - r\sin(\theta)d\theta.$
2. $\phi^*(dy) = d(\phi^*y) = d(r\sin(\theta)) = \sin(\theta)dr + r\cos(\theta)d\theta.$
3. $\phi^*(dx \wedge dy) = d(\phi^*x) \wedge d(\phi^*y) = r\cos^2(\theta)dr \wedge d\theta + r\sin^2(\theta)dr \wedge d\theta = rdr \wedge d\theta.$

We may also compute

$$\begin{aligned}\phi^*(-ydx + xdy) &= -r\sin(\theta)(\cos(\theta)dr - r\sin(\theta)d\theta) + r\cos(\theta)(\sin(\theta)dr + r\cos(\theta)d\theta) \\ &= r^2 d\theta\end{aligned}$$

Pullback of (r,s)-tensor under smooth diffeomorphism Let M, N be smooth manifolds with the same dimension. Let $F: M \to N$ be C^∞ diffeomorphism with inverse $F^{-1}: N \to M$.

Definition 1.19.8 (Pullback of (r,s)-tensor under C^∞ diffeomorphism)**.** *We define the Pullback of (r,s)-tensor, that takes (r,s)-tensor T on N to F^*T, a (r,s)-tensor on M. Let*

$$F^*: C^\infty(N, T^r_s N) \to C^\infty(M, T^r_s M)$$
$$T \mapsto F^*T$$

where

$$F^*T: M \to T^r_s M$$
$$p \to (F^*T)(p) \in (T^r_s M)_p = T_pM^{\otimes r} \otimes T^*_pM^{\otimes s}$$

is defined via

$$(F^*T)(p): T^*_pM^{\otimes r} \otimes T_pM^{\otimes s} \to \mathbb{R}$$
$$(\alpha_1, \cdots, \alpha_r, v_1, \cdots, v_s) \mapsto T(F(p))((dF_p^{-1})^*(\alpha_1), \cdots, (dF_p^{-1})^*(\alpha_r), dF_p(v_1), \cdots, dF_p(v_s))$$

Here $(dF_p^{-1})^*$ are the pullback dual map for F^{-1} as in (1.46).

Or equivalently

$$(F^*T)(p) = \left(dF_p^{-1}\right)^{\otimes r} \otimes ((dF_p)^*)^{\otimes s} (T(F(p)))$$

Note $T(F(p)) \in (T^r_s N)_{F(p)} = (T_{F(p)}N)^{\otimes r} \otimes (T^*_{F(p)}N)^{\otimes s}$.

One can check $F^*T: M \to T^r_s M$ is a C^∞ section using local coordinates.

Definition 1.19.9 (Pushforward of (r,s)-tensor under C^∞ diffeomorphism)**.** *Define pushforward*

$$F_* := (F^{-1})^*: C^\infty(M, T^r_s M) \to C^\infty(N, T^r_s N)$$
$$T \mapsto F_*T$$

where

$$F_*T: N \to T^r_s N$$
$$p \mapsto (F_*T)(p) \in (T^r_s N)_p = T_pN^{\otimes r} \otimes T^*_pN^{\otimes s}$$

is defined via

$$(F_*T)(p): T^*_pN^{\otimes r} \otimes T_pN^{\otimes s} \to \mathbb{R}$$
$$(\alpha_1, \cdots, \alpha_r, v_1, \cdots, v_s) \mapsto T(F^{-1}(p))((dF_p)^*(\alpha_1), \cdots, (dF_p)^*(\alpha_r), dF_p^{-1}(v_1), \cdots, dF_p^{-1}(v_s))$$

Lemma 1.19.6. *For $M_1 \xrightarrow{F} M_2 \xrightarrow{G} M_3$ C^∞ diffeomorphism.*

$$(G \circ F)^* = G^* \circ F^*$$

Example 1.19.3. *Let $M = \{(r,\theta) \mid r > 0, |\theta| < \frac{\pi}{2}\}$ and*

$$F: M \to \mathbb{R}^2$$
$$(r,\theta) \mapsto (r\cos(\theta), r\sin(\theta))$$

Consider the pullback of tensor field $A = \frac{1}{x^2} dy \otimes dy$ by F

$$\begin{aligned} F^*A &= \frac{1}{r^2\cos^2(\theta)} d(r\sin(\theta)) \otimes d(r\sin(\theta)) \\ &= \frac{1}{r^2\cos^2(\theta)} (\sin(\theta)dr + r\cos(\theta)d\theta) \otimes (\sin(\theta)dr + r\cos(\theta)d\theta) \\ &= \frac{\tan^2(\theta)}{r^2} dr \otimes dr + \frac{\tan(\theta)}{r}(dr \otimes d\theta + d\theta \otimes dr) + d\theta \otimes d\theta \end{aligned}$$

1.19.4 Lie Derivative of Tensors

We discuss Lie Derivative L_X on (r,s)-tensors for $X \in \mathfrak{X}(M)$ where M is a C^∞ manifold.

We want to define

$$L_X: C^\infty(M, T^r_s M) \to C^\infty(M, T^r_s M)$$
$$T \mapsto L_X T$$

extending $(0,0)$-tensors (1.22)

$$\begin{aligned} L_X : C^\infty(M) &\to C^\infty(M) \\ f &\mapsto Xf \\ Xf : M &\to \mathbb{R} \\ p &\mapsto X(p)([f]_p) \end{aligned}$$

and extending $(1,0)$-tensors (1.30)

$$\begin{aligned} L_X : \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ Y &\mapsto [X,Y] \\ [X,Y] : C^\infty(M) &\to C^\infty(M) \\ f &\mapsto XYf - YXf \end{aligned}$$

L_X Lie Derivative on $\Omega^1(M)$ We want to define $L_X : \Omega^1(M) \to \Omega^1(M)$ $(0,1)$-tensors by requiring that it is $\mathbb{R}$-linear and satisfies the following Leibnitz rule: For any

$$\alpha \in \Omega^1(M) \in C^\infty(M, T_1^0(M)) \qquad \text{and} \qquad Y \in \mathfrak{X}(M) = C^\infty(M, T_0^1(M))$$

Note $\alpha(Y) \in C^\infty(M)$ takes the form

$$\begin{aligned} \alpha(Y) : M &\to \mathbb{R} \\ p &\mapsto \alpha(p)(Y(p)) \end{aligned}$$

To satisfy the Leibniz Rule, one needs

$$\begin{aligned} L_X(\alpha(Y)) &= (L_X\alpha)(Y) + \alpha(L_X Y) \\ (L_X\alpha)(Y) &= L_X(\alpha(Y)) - \alpha(L_X Y) \\ &= X(\alpha(Y)) - \alpha([X,Y]) \end{aligned}$$

Definition 1.19.10 (Lie Derivative on $\Omega^1(M)$)**.** *The only way to define L_X is as following*

$$\begin{aligned} L_X : \Omega^1(M) &\to \Omega^1(M) \\ \alpha &\mapsto L_X\alpha \end{aligned} \tag{1.51}$$

s.t.

$$\begin{aligned} L_X\alpha : \mathfrak{X}(M) &\to C^\infty(M) \\ Y &\mapsto X(\alpha(Y)) - \alpha([X,Y]) \end{aligned}$$

Using tensor product

$$L_X(S \otimes T) = (L_X S) \otimes T + S \otimes (L_X T)$$

this extends to tensors of any type.

Lie Derivative as the Derivative under Pullback of Local Flow Given $X \in \mathfrak{X}(M)$ we want to define $L_X T$ where T is (r,s)-tensor on M, using the local flow of X.

For any $p \in M$, there exists open neighborhood U of p in M, $\varepsilon > 0$ and a local flow defined for any $t \in (-\varepsilon, \varepsilon)$

$$\begin{aligned} \phi_t : U &\overset{C^\infty}{\to} M \\ q &\mapsto \phi(t,q) \end{aligned}$$

where the flow ϕ is the smooth map defined via (1.29)

$$\begin{cases} \frac{\partial}{\partial t}\phi(t,q) = X(\phi(t,q)) & (t,q) \in (-\varepsilon,\varepsilon) \times U \\ \phi(0,q) = q & (0,q) \end{cases}$$

Define

$$\left(\tilde{L}_X T\right)(p) := \left.\frac{d}{dt}\right|_{t=0} (\phi_t^* T)(p) \tag{1.52}$$

where the flow gives smooth map

$$\begin{aligned} (-\varepsilon,\varepsilon) &\to (T_s^r M)_p = (T_p M)^{\otimes r} \otimes (T_p^* M)^{\otimes s} \\ t &\mapsto (\phi_t^* T)(p) \end{aligned}$$

We've already checked the equivalence for $(0,0)$ and $(1,0)$ tensors as in (1.34) and (1.35).

Claim: $\tilde{L}_X T = L_X T$ for any T tensor on M of any type (r,s). It suffices to check that

(a) For any $\alpha \in \Omega^1(M)$ and $Y \in \mathfrak{X}(M)$

$$(\tilde{L}_X\alpha)(Y) = X(\alpha(Y)) - \alpha([X,Y])$$

(b) Leibniz Rule applies to Tensor product structure

$$\tilde{L}_X(S \otimes T) = (\tilde{L}_X S) \otimes T + S \otimes (\tilde{L}_X T)$$

To do so, one use local flow

$$\begin{cases} \phi_t^*(\alpha(Y)) = \phi_t^*(\alpha)\phi_t^*(Y) \\ \phi_t^*(\alpha(S \otimes T)) = \phi_t^*(S) \otimes \phi_t^*(T) \end{cases}$$

and take derivative $\frac{d}{dt}\Big|_{t=0}$ to determine uniquely.

Lemma 1.19.7. *For $\omega \in \Omega^k(M)$, $\tau \in \Omega^\ell(M)$ and $X \in \mathfrak{X}(M)$*

$$L_X(\omega \wedge \tau) = (L_X\omega) \wedge \tau + \omega \wedge (L_X\tau)$$

Lemma 1.19.8. *For $\omega \in \Omega^k(M)$, $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$*

$$L_X(f\omega) = L_X(f)\omega + f(L_X\omega) = (Xf)\omega + fL_X\omega$$

Lemma 1.19.9 (Leibnitz Rule for Lie Derivative)**.** *For any $\omega \in \Omega^s(M)$, $X \in \mathfrak{X}(M)$ and $Y_1, \cdots, Y_s \in \mathfrak{X}(M)$*

$$L_X(\omega(Y_1, \cdots, Y_s)) = (L_X\omega)(Y_1, \cdots, Y_s) + \sum_{i=1}^{s} \omega(Y_1,\ \cdots,\ Y_{i-1},\ [X, Y_i],\ Y_{i+1},\ \cdots,\ Y_s)$$

1.19.5 d Exterior Derivative on $\Omega^s(M)$

Let $\mathcal{L}_X : \Omega^s(M) \to \Omega^s(M)$ be the Lie Derivative on s-forms.

Definition 1.19.11 (Exterior Derivative on forms)**.** *The exterior derivative is $\mathbb{R}$-linear*

$$\begin{aligned} d : \Omega^s(M) &\to \Omega^{s+1}(M) \\ \alpha &\mapsto d\alpha \end{aligned} \tag{1.53}$$

that satisfies

(a) On $\Omega^0(M)$

$$\begin{aligned} d : C^\infty(M) = \Omega^0(M) &\to \Omega^1(M) \\ f &\mapsto df \end{aligned}$$

d sends f to its differential df (1.43), i.e,

$$\begin{aligned} df : M &\to T^*M \\ p &\mapsto df_p \end{aligned}$$

s.t.

$$\begin{aligned} df_p : T_pM &\to \mathbb{R} \\ v &\to v([f]_p) \end{aligned}$$

(b) For any $f \in \Omega^0(M)$ we have $df \in \Omega^1(M)$ and moreover

$$d(df) = 0$$

(c) For $\alpha \in \Omega^r(M)$ and $\beta \in \Omega^s(M)$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta \tag{1.54}$$

Exterior Derivative in Local Coordinates In local coordinates, let (U, ϕ) be C^∞ chart on M. For $\alpha \in \Omega^s(M)$, on U

$$\alpha = \sum_{1 \le j_1, \cdots, j_s \le n} a_{j_1,\cdots,j_s} dx_{j_1} \wedge \cdots \wedge dx_{j_s}$$

for $a_{j_1,\cdots,j_s} \in C^\infty(U)$. Then we compute

$$\begin{aligned} d\alpha &= d\left(\sum_{1 \le j_1, \cdots, j_s \le n} a_{j_1,\cdots,j_s} dx_{j_1} \wedge \cdots \wedge dx_{j_s} \right) \\ &= \sum_{1 \le j_1, \cdots, j_s \le n} da_{j_1,\cdots,j_s} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_s} \\ &= \sum_{1 \le j_1, \cdots, j_s \le n} \sum_{k=1}^{n} \frac{\partial a_{j_1,\cdots,j_s}}{\partial x_k} dx_k \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_s} \end{aligned}$$

Properties for Exterior Derivative

Proposition 1.19.1. *Let d be the exterior derivative.*

(i) $dd\omega = 0$ *for any* $\omega \in \Omega^s(M)$.

(ii) *For* $F : M \to N$ C^∞ *map, for any* $\omega \in \Omega^s(N)$

$$d(F^*\omega) = F^*(d\omega) \in \Omega^{s+1}(M)$$

It is the nature of d that it commutes with pullbacks $d \circ F^* = F^* \circ d$

(iii) *For* $X \in \mathfrak{X}(M)$ *and* $\omega \in \Omega(M)$

$$d(L_X\omega) = L_X(d\omega) \in \Omega^{s+1}(M)$$

so d commutes with Lie derivatives $d \circ L_X = L_X \circ d$

(iv) *For* $\alpha \in \Omega^s(M)$ *and* $X_0 \cdots X_s \in \mathfrak{X}(M)$

$$(d\alpha)(X_0 \cdots X_s) = \sum_{i=0}^{s}(-1)^i X_i\left(\alpha(X_0, \cdots, \hat{X}_i, \cdots, X_s)\right) + \sum_{0\le i<j\le s}(-1)^{i+j}\alpha\left([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_s\right)$$

or in short, for $\alpha \in \Omega^1(M)$, X, $Y \in \mathfrak{X}(M)$

$$(d\alpha)(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]) \tag{1.55}$$

Proof for Prop 1.19.1 (iv) $\Omega^1(M)$ case. By linearity in $\mathbb{R}$, it suffices to assume $\alpha = fdg$ where f, $g \in C^\infty(U)$ for U open set on M.

$$\begin{aligned}(d\alpha)(X, Y) &= (df \wedge dg)(X, Y) = df(X)dg(Y) - dg(X)df(Y) = (Xf)Yg - (Xg)Yf\\ X\alpha(Y) &= X((fdg)(Y)) = X(f)dg(Y) + fX(dg(Y)) = (Xf)Yg + fX(Yg)\\ Y\alpha(X) &= Y(fdg(X)) = YfXg + fY(Xg)\\ \alpha([X, Y]) &= fdg(XY - YX) = fXYg - fYXg\end{aligned}$$

□

Example 1.19.4.

- *Let* $f \in C^\infty(\mathbb{R}^3)$, *then*

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

- *Let* $\alpha = Adx + Bdy + Cdz$ *for* A, B, $C \in C^\infty(\mathbb{R}^3)$. *Then*

$$\begin{aligned}d\alpha &= dA \wedge dx + dB \wedge dy + dC \wedge dz\\ &= \left(\frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial z}dz\right) \wedge dx + \left(\frac{\partial B}{\partial x}dx + \frac{\partial B}{\partial y}dy + \frac{\partial B}{\partial z}dz\right) \wedge dy + \left(\frac{\partial C}{\partial x}dx + \frac{\partial C}{\partial y}dy + \frac{\partial C}{\partial z}dz\right) \wedge dz\\ &= -\frac{\partial A}{\partial y}dx \wedge dy + \frac{\partial A}{\partial z}dz \wedge dx + \frac{\partial B}{\partial x}dx \wedge dy - \frac{\partial B}{\partial z}dy \wedge dz - \frac{\partial C}{\partial x}dz \wedge dx + \frac{\partial C}{\partial y}dy \wedge dz\\ &= \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy + \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) dy \wedge dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) dz \wedge dx\end{aligned}$$

- *Let* $\alpha = Cdx \wedge dy + Ady \wedge dz + Bdz \wedge dx$ *for* A, B, $C \in C^\infty(\mathbb{R}^3)$

$$\begin{aligned}d\alpha &= dC \wedge dx \wedge dy + dA \wedge dy \wedge dz + dB \wedge dz \wedge dx\\ &= \frac{\partial C}{\partial z}dz \wedge dx \wedge dy + \frac{\partial A}{\partial x}dx \wedge dy \wedge dz + \frac{\partial B}{\partial y}dy \wedge dz \wedge dx\\ &= \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) dx \wedge dy \wedge dz\end{aligned}$$

Since $d^2 = 0$, *this is to say for any* $f \in C^\infty(M)$, $\text{curl}(\nabla f) = 0$, *and for any* $X \in \mathfrak{X}(\mathbb{R}^3)$, $\text{div}(\text{curl}(X)) = 0$.

1.19.6 i_X Interior Derivative on $\Omega^s(M)$

Let $X \in \mathfrak{X}(M)$.

Definition 1.19.12 (Interior Derivative on forms). *Define interior derivative*

$$\begin{aligned} i_X : \Omega^s(M) &\to \Omega^{s-1}(M) \\ \alpha &\mapsto i_X\alpha \end{aligned}$$

by satisfying the following

(a) $i_X f = 0$ *for any* $f \in C^\infty(M) = \Omega^0(M)$.

(b) For $Y_1, \cdots, Y_{s-1} \in \mathfrak{X}(M)$

$$(i_X\alpha)(Y_1, \cdots, Y_{s-1}) = \alpha(X, Y_1, \cdots, Y_{s-1})$$

Proposition 1.19.2. *Let* i_X *denote interior derivative*

(i) $i_X \circ i_X\omega = 0$ *for any* $\omega \in \Omega^s(M)$

(ii) $\alpha \in \Omega^r(M)$, $\beta \in \Omega^s(M)$

$$i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^r \alpha \wedge i_X\beta$$

(iii) Cartan's formula.

$$d \circ i_X + i_X \circ d = L_X \tag{1.56}$$

Lemma 1.19.10. *For any* $\omega \in \Omega^s(M)$, X, $Y \in \mathfrak{X}(M)$

$$L_X(i_Y\omega) - i_Y(L_X\omega) = i_{[X,Y]}\omega$$

1.20 F-related

Definition 1.20.1 (F-related smooth vector fields). *Let* $F : M \overset{C^\infty}{\to} N$ *between smooth manifolds* M *and* N. $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$.

We say X *and* Y *are* F*-related if for any* $p \in M$

$$dF_p(X(p)) = Y(F(p))$$

Lemma 1.20.1 (Characterisation for F-related). *Given* $F : M \overset{C^\infty}{\to} N$, *and* $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$

- X *and* Y *are* F*-related iff*

$$X(F^*f) = F^*(Y(f)) \qquad \forall\ f \in C^\infty(N)$$

Recall definition of pullback of smooth functions (1.25).

- *If* F *is diffeomorphism, then* X *and* Y *are* F*-related iff*

$$Y = F_*X$$

Recall pushforward of smooth vector fields (1.32).

Lemma 1.20.2 (F-related preserves Lie-Bracket). *For* $F : M \overset{C^\infty}{\to} N$ *where* X_1, $X_2 \in \mathfrak{X}(M)$ *and* Y_1, $Y_2 \in \mathfrak{X}(N)$ *and* X_i, Y_i *are* F*-related. Then* $[X_1, X_2]$ *and* $[Y_1, Y_2]$ *are* F*-related.*

Proof. Let $f \in C^\infty(N)$

$$\begin{aligned} [X_1, X_2](F^*f) &= X_1(X_2(F^*f)) - X_2(X_1(F^*f)) \\ &= X_1(F^*(Y_2(f))) - X_2(F^*(Y_1(f))) \\ &= F^*(Y_1(Y_2(f))) - F^*(Y_2(Y_1(f))) = F^*[Y_1, Y_2](f) \end{aligned}$$

□

Corollary 1.20.1. $F : M \overset{C^\infty}{\to} N$ *is smooth diffeomorphism, hence pushforward under* F

$$\begin{aligned} F_* : \mathfrak{X}(M) &\to \mathfrak{X}(N) \\ X &\mapsto F_*X \end{aligned}$$

defines X *and* F_*X *as* F*-related vector fields. Thus*

$$F_*[X_1, X_2] = [F_*X_1, F_*X_2]$$

1.21 Connections on Vector Bundles

Connections on C^∞ Vector Bundles Let M be a C^∞ manifold. Let $\pi : E \to M$ be a C^∞ vector bundle of rank r over M.

Definition 1.21.1 (Connection on E)**.** *A connection on E is a $\mathbb{R}$-linear map*

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times C^\infty(M,E) &\to C^\infty(M,E) \\ (X,s) &\mapsto \nabla_X s \end{aligned} \tag{1.57}$$

s.t.

1. *$\nabla_X s$ is $C^\infty(M)$-linear in X, i.e. for any $f, g \in C^\infty(M)$, for any $X, Y \in \mathfrak{X}(M)$*

$$\nabla_{fX+gY} s = f\nabla_X s + g\nabla_Y s \tag{1.58}$$

2. *For any fixed $X \in \mathfrak{X}(M)$, the map*

$$\begin{aligned} \nabla_X : C^\infty(M,E) &\to C^\infty(M,E) \\ s &\mapsto \nabla_X s \end{aligned}$$

satisfies the Leibniz Rule, i.e.

$$\nabla_X(fs) = X(f)s + f\nabla_X s \qquad \forall\ f \in C^\infty(M),\ \ s \in C^\infty(M,E) \tag{1.59}$$

Notice the above requirements make sense, since both $\mathfrak{X}(M)$ and $C^\infty(M,E)$ are $C^\infty(M)$-modules.

Affine Connection on C^∞ manifold Let M be a C^∞ manifold.

Definition 1.21.2 (Affine Connection)**.** *An affine connection is a connection on the tangent bundle $\pi : TM \to M$.*

In other words, it is a $\mathbb{R}$-linear map

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ (X,Y) &\mapsto \nabla_X Y \end{aligned} \tag{1.60}$$

s.t.

1. *$\nabla_X Y$ is $C^\infty(M)$-linear in X, i.e., for any $f, g \in C^\infty(M)$ and $X, Y \in \mathfrak{X}(M)$*

$$\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$$

2. *For any fixed $X \in \mathfrak{X}(M)$, the map*

$$\begin{aligned} \nabla_X : \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ Y &\mapsto \nabla_X Y \end{aligned}$$

satisfies the Leibniz rule, i.e.

$$\nabla_X(fY) = X(f)Y + f\nabla_X Y \qquad \forall\ f \in C^\infty(M),\ \ Y \in \mathfrak{X}(M) \tag{1.61}$$

Affine Connection in Local Coordinates Immediately one can write down in local coordinates. Let

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial x_i}, \qquad Y = \sum_{j=1}^n Y^j \frac{\partial}{\partial x_j}$$

then

$$\begin{aligned} \nabla_X Y = \nabla_{X^i \frac{\partial}{\partial x_i}} (Y^j \frac{\partial}{\partial x_j}) = X^i \nabla_{\frac{\partial}{\partial x_i}} (Y^j \frac{\partial}{\partial x_j}) &\overset{(1.61)}{=} X^i \frac{\partial Y^j}{\partial x_i}\frac{\partial}{\partial x_j} + X^i Y^j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \\ &= X^i \frac{\partial Y^j}{\partial x_i}\frac{\partial}{\partial x_j} + X^i Y^j \Gamma_{ij}^k \frac{\partial}{\partial x_k} \\ &= \left(X^i \frac{\partial Y^k}{\partial x_i} + X^i Y^j \Gamma_{ij}^k \right) \frac{\partial}{\partial x_k} \end{aligned}$$

where the Christoffel symbols are to be defined in (1.66).

$\Omega^p(M,E)$ **E-valued p-form**

Lemma 1.21.1. *Let E and F be C^∞ vector bundles on a C^∞ manifold M, and let*

$$\phi : C^\infty(M,E) \to C^\infty(M,F)$$

be $C^\infty(M)$-linear, i.e. for $f \in C^\infty(M)$ and $s \in C^\infty(M,E)$

$$\phi(fs) = f\phi(s)$$

Then $\phi \in C^\infty(M, E^* \otimes F)$.

Proof. On $U \subseteq M$ open, let $\{e_1, \cdots, e_r\}$, $\{f_1, \cdots, f_s\}$ be C^∞ frame of $E|_U$ and $F|_U$ respectively. Then in local coordinates

$$\phi(e_i) = \sum_{j=1}^{s} a_{ij} f_j \qquad \text{for} \qquad a_{ij} \in C^\infty(U)$$

we can thus rewrite

$$\phi = \sum_{i=1}^{r}\sum_{j=1}^{s} a_{ij} e_i^* \otimes f_j$$

for $\{e_1^*, \cdots, e_r^*\}$ C^∞ frame of $E^*|_U$ dual to $(e_1, \cdots, e_r)$. □

We introduce the following notation. Let M be C^∞ manifold of dimension n, and let $\pi : E \to M$ be a vector bundle of rank r.

Definition 1.21.3 (E-valued p-forms)**.** *We define the space of E-valued p-forms*

$$\Omega^p(M,E) := C^\infty(M, \Lambda^p T^*M \otimes E) \tag{1.62}$$

In particular we check the standard cases

1. $\Omega^0(M,E) = C^\infty(M,E)$
2. $\Omega^1(M,E) = C^\infty(M, T^*M \otimes E)$
3. *For $E = TM$ this recovers $\Omega^0(M,TM) = C^\infty(M,TM) = \mathfrak{X}(M)$ and $\Omega^1(M,TM) = C^\infty(M, T^*M \otimes TM)$.*

Interpretation of $\nabla s \in \Omega^1(M,E)$ Now one may interpret ∇s for any $s \in C^\infty(M,E) = \Omega^0(M,E)$.

$$\begin{aligned} \nabla s : \mathfrak{X}(M) = C^\infty(M,TM) &\to C^\infty(M,E) \\ X &\mapsto \nabla_X s \end{aligned} \tag{1.63}$$

Thanks to (1.58), ∇s is $C^\infty(M)$-linear in X. Immediately notice Lemma 1.21.1 is applicable, so we view

$$\nabla s \in C^\infty(M, T^*M \otimes E) = \Omega^1(M,E)$$

Interpretation of $\nabla \in \Omega^1(M, \mathrm{End}(E))$ Let M be C^∞ manifold and $\pi : E \to M$ be C^∞ vector bundle of rank r over M. One may give an alternative definition of ∇.

Definition 1.21.4 (Connection on E)**.** *A connection on E is a $\mathbb{R}$-linear map*

$$\begin{aligned} \nabla : \Omega^0(M,E) &\to \Omega^1(M,E) \\ s &\mapsto \nabla s \end{aligned} \tag{1.64}$$

where ∇s is given in (1.63), s.t. Leibniz rule holds

$$\nabla(fs) = df \otimes s + f\nabla s \qquad \forall\ f \in C^\infty(M),\ s \in C^\infty(M,E) \tag{1.65}$$

Well-definedness. Recall in general, for any $\alpha \in \Omega^p(M) = C^\infty(M, \Lambda^p T^*M)$ and $s \in C^\infty(M,E)$

$$\alpha \otimes s \in \Omega^p(M,E) = C^\infty(M, \Lambda^p T^*M \otimes E)$$

Hence for $f \in C^\infty(M)$, $df \in \Omega^1(M) = C^\infty(M, T^*M)$, and so

$$df \otimes s \in C^\infty(M, T^*M \otimes E) = \Omega^1(M,E)$$

□

We make the following remark.

Remark 1.21.1 ($\Omega^1(M, \text{End}(E))$)**.** *Given E as C^∞ vector bundle over M. Let $F = T^*M \otimes E$. Then any $C^\infty(M)$-linear map*

$$\phi : C^\infty(M, E) = \Omega^0(M, E) \to C^\infty(M, T^*M \otimes E) = \Omega^1(M, E)$$

can be viewed as

$$\phi \in C^\infty(M, E^* \otimes (T^*M \otimes E)) = C^\infty(M, T^*M \otimes \text{End}(E)) = \Omega^1(M, \text{End}(E))$$

via Lemma 1.21.1.

In particular, the connection on E is such an object because of definition (1.58),

$$\nabla \in \Omega^1(M, \text{End}(E)) = C^\infty(M, E^* \otimes T^*M \otimes E)$$

$A(E)$ **Space of connections on** E In fact, the difference of any two connections remains such an element.

Lemma 1.21.2 ($\nabla_1 - \nabla_0 \in \Omega^1(M, \text{End}(E))$)**.** *If ∇_0 and ∇_1 are two connections on the same vector bundle $\pi : E \to M$, then*

$$\nabla_1 - \nabla_0 : \Omega^0(M, E) = C^\infty(M, E) \to \Omega^1(M, E) = C^\infty(M, T^*M \otimes E)$$
$$s \mapsto \nabla_1 s - \nabla_0 s$$

is $C^\infty(M)$-linear.

Therefore, this corresponds to a section of

$$E^* \otimes T^*M \otimes E = T^*M \otimes End(E)$$

according to Lemma 1.21.1, i.e.,

$$\nabla_1 - \nabla_0 \in C^\infty(M, T^*M \otimes \text{End}(E)) = \Omega^1(M, \text{End}(E))$$

Proof. It suffices to check $C^\infty(M)$-linearity. For any $f \in C^\infty(M)$ and $s \in C^\infty(M, E)$

$$\begin{aligned}(\nabla_1 - \nabla_0)(fs) &= \nabla_1(fs) - \nabla_0(fs)\\ &= (df \otimes s + f\nabla_1 s) - (df \otimes s + f\nabla_0 s)\\ &= f(\nabla_1 s - \nabla_0 s) = f(\nabla_1 - \nabla_0)s\end{aligned}$$

□

Definition 1.21.5 ($A(E)$ Space of Connections on Vector Bundle)**.** *Denote $A(E)$ as the set of connections on E.*

Then $A(E)$ is an affine space associated to the vector space $\Omega^1(M, \text{End}(E))$.

Indeed, for any $\nabla_0 \in A(E)$, $\phi \in \Omega^1(M, \text{End}(E))$

$$(\nabla_0 + \phi) : \Omega^0(M, E) \to \Omega^1(M, E)$$

so $\nabla_0 + \phi \in A(E)$. Note $\Omega^1(M, \text{End}(E))$ is ∞-dimensional if $\dim M > 0$ and rank$E > 0$.

1.21.1 Connection on C^∞ Vector Bundle in Local Coordinates

Let $\pi : E \to M$ be C^∞ vector bundle of rank r over C^∞ manifold M of dimension n.

We write our connection on E (1.64)

$$\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$$
$$s \mapsto \nabla s$$

in local coordinates.

We setup our local coordinate. Suppose (U, ϕ) with coordinates $\phi = (x_1, \cdots, x_n)$ is a C^∞ chart for M such that $E|_U := \pi^{-1}(U)$ is trivial. Then

$$h : \pi^{-1}(U) = E|_U \subseteq E \to U \times \mathbb{R}^r \subseteq M \times \mathbb{R}^r$$

is local trivialization. We have $\{e_1, \cdots, e_r\} \subseteq C^\infty(U, E|_U)$ as a C^∞ frame of $\pi|_U : E|_U \to U$. Recall the frames e_j are constructed via (1.37)

$$e_j : U \subseteq M \to \pi^{-1}(U) \subseteq E$$
$$x \mapsto h^{-1}(x, \hat{e}_j)$$

where $\hat{e}_j = (0, \cdots, 1, \cdots, 0)$ are the standard basis in $\mathbb{R}^r$.

For any $s \in C^\infty(U, E|_U)$, one may write the smooth section

$$s = \sum_{k=1}^{r} a^k e_k \in C^\infty(U, E|_U)$$

in local coordinates for $a^k \in C^\infty(U)$.

Christoffel Symbol and Connection 1-form We have $\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\}$ as C^∞ frame of $TM|_U = TU$. To let ∇ act on s, we first discuss what ∇ is acting on e_j. In fact, on U we define the *Christoffel Symbols*

$$\Gamma_{ij}^k \in C^\infty(U)$$

via

$$\nabla_{\frac{\partial}{\partial x_i}} e_j := \sum_{k=1}^{r} \Gamma_{ij}^k e_k \in C^\infty(U, E|_U) \tag{1.66}$$

In other words, the Christoffel symbols are coordinate components for the section $\nabla_{\frac{\partial}{\partial x_i}} e_j$ w.r.t. the smooth frame $\{e_k\}$ itself.

We further define *connection* 1*-forms*

$$\omega_j^k \in \Omega^1(U)$$

s.t.

$$\nabla e_j = \sum_{k=1}^{r} \omega_j^k \otimes e_k \tag{1.67}$$

holds. Notice this uses only trivialization of $E|_U$ (but not trivialization of $T^*M|_U$). This also used the observation that the element ∇e_j is an E-valued one-form on U, i.e.

$$\nabla e_j \in \Omega^1(U, E|_U) = C^\infty(U, T^*U \otimes E|_U)$$

Now plugging (1.66) into above (1.67) we may identify

$$\begin{aligned}\sum_{k=1}^{r} \Gamma_{ij}^k e_k = \nabla_{\frac{\partial}{\partial x_i}} e_j &= \sum_{k=1}^{r} \omega_j^k \left(\frac{\partial}{\partial x_i}\right) e_k \\ \omega_j^k \left(\frac{\partial}{\partial x_i}\right) &= \Gamma_{ij}^k\end{aligned}$$

Thus we obtain the connection 1-forms in local coordinates

$$\omega_j^k = \sum_{i=1}^{n} \Gamma_{ij}^k dx_i \in \Omega^1(U) = C^\infty(U, T^*U) \tag{1.68}$$

Plugging back into (1.67) we have explicit form in both Christoffel Symbols and connection 1-forms.

$$\nabla e_j = \sum_{k=1}^{r} \omega_j^k \otimes e_k = \sum_{k=1}^{r} \sum_{i=1}^{n} \Gamma_{ij}^k dx_i \otimes e_k$$

∇s in Local Coordinates In this paragraph, we write smooth connection $s = e_\alpha s_\alpha$ in coordinates, and study how ∇ acts on s. In particular, we wish to compute the explicit formula for $(\nabla s)_\alpha$ where $\nabla s_\alpha = e_\alpha (\nabla s)_\alpha$.

We setup our transition functions. Take open cover $\{U_\alpha \mid \alpha \in I\}$ of the base M and

$$h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$$

local trivializations.

Define the frames for $j = 1, \cdots, r$

$$\begin{aligned} e_{\alpha_j} : U_\alpha \subseteq M &\to \pi^{-1}(U_\alpha) \subseteq E \\ x &\mapsto h_\alpha^{-1}(x, \hat{e}_j) \end{aligned}$$

where $\hat{e}_j$ are standard basis for $\mathbb{R}^r$. Now $e_{\alpha_1}, \cdots, e_{\alpha_r}$ are C^∞ frames of $E|_{U_\alpha}$.

For any $U_\alpha \cap U_\beta \neq \varnothing$, recall one has (1.13)

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \overset{C^\infty}{\to} \mathrm{GL}(r, \mathbb{R})$$
$$x \mapsto g_{\beta\alpha}(x)$$

s.t. the following holds

$$e_{\alpha_j}(x) = e_{\beta_i}(x) g_{\beta\alpha}(x)_{ij}$$

Using $g_{\beta\alpha}$ we defined transition functions

$$h_\beta \circ h_\alpha^{-1} : (U_\alpha \cap U_\beta) \cap \mathbb{R}^r \to (U_\alpha \cap U_\beta) \times \mathbb{R}^r$$
$$(x, v) \mapsto (x, g_{\beta\alpha}(x) v)$$

We setup $s = e_\alpha s_\alpha$. Since $s \in C^\infty(M, E)$ is a section, on U_α we have local coordinates representation

$$s = \sum_{j=1}^{r} s_\alpha^j e_{\alpha_j} = e_\alpha s_\alpha \qquad \text{for} \qquad s_\alpha^j \in C^\infty(U_\alpha) \tag{1.69}$$

where we denote by

$$e_\alpha = [e_{\alpha_1}, \cdots, e_{\alpha_r}], \qquad s_\alpha := \begin{pmatrix} s_\alpha^1 \\ \vdots \\ s_\alpha^r \end{pmatrix} \in C^\infty(U_\alpha, \mathbb{R}^r)$$

Now $s \in C^\infty(M, E)$ is a C^∞ section iff $s_\alpha \in C^\infty(U_\alpha, \mathbb{R}^r)$ and the sections transit via $g_{\beta\alpha}$

$$s_\beta = g_{\beta\alpha} s_\alpha$$

on $U_\alpha \cap U_\beta$. Note on $U_\alpha \cap U_\beta$

$$s = e_\alpha s_\alpha = e_\beta g_{\beta\alpha} s_\alpha = e_\beta s_\beta$$

We setup $\nabla e_\alpha = e_\alpha w_\alpha$. Now suppose that we're given a connection ∇ on E. On U_α we define connection 1-form $(\omega_\alpha)_j^k \in \Omega^1(U_\alpha)$ for $j, k = 1, \cdots, r$ as in (1.67) by

$$\nabla e_{\alpha_j} = \sum_{k=1}^{r} (\omega_\alpha)_j^k \otimes e_{\alpha_k} \qquad (\omega_\alpha)_j^k \in \Omega^1(U_\alpha) \tag{1.70}$$

So

$$\nabla e_\alpha = [\nabla e_{\alpha_1}, \cdots, \nabla e_{\alpha_r}] = e_\alpha \omega_\alpha \qquad \text{where} \qquad \omega_\alpha := \begin{pmatrix} (\omega_\alpha)_1^1 & \cdots & (\omega_\alpha)_r^1 \\ \vdots & \cdots & \vdots \\ (\omega_\alpha)_1^r & \cdots & (\omega_\alpha)_r^r \end{pmatrix} \in \Omega^1(U_\alpha, \mathfrak{gl}(r, \mathbb{R}) = M_r(\mathbb{R}))$$

where $\mathfrak{gl}(r, \mathbb{R})$ is the Lie algebra of $\mathrm{GL}(r, \mathbb{R})$.

We setup $\nabla s_\alpha = e_\alpha (\nabla s)_\alpha$. On U_α, we similarly define

$$(\nabla s)_\alpha := \begin{pmatrix} (\nabla s)_\alpha^1 \\ \vdots \\ (\nabla s)_\alpha^r \end{pmatrix} \in \Omega^1(U_\alpha, \mathbb{R}^r)$$

by

$$\nabla s = \sum_{j=1}^{r} (\nabla s)_\alpha^j \otimes e_{\alpha_j} \in \Omega^1(U_\alpha, E|_{U_\alpha}) = C^\infty(U_\alpha, T^*U_\alpha \otimes E|_{U_\alpha}) \tag{1.71}$$

for $(\nabla s)_\alpha^j \in \Omega^1(U_\alpha) = C^\infty(U_\alpha, T^*U_\alpha)$. So

$$\nabla s = e_\alpha (\nabla s)_\alpha$$

We compute $(\nabla s)_\alpha = ds_\alpha + \omega_\alpha s_\alpha$. By Leibniz Rule, we may unpack the definition

$$\begin{aligned}
\nabla s = \nabla \left(\sum_{j=1}^{r} s_\alpha^j e_{\alpha_j} \right) &\overset{(1.65)}{=} \sum_{j=1}^{r} ds_\alpha^j \otimes e_{\alpha_j} + \sum_{j=1}^{r} s_\alpha^j \nabla e_{\alpha_j} \\
&\overset{(1.70)}{=} \sum_{j=1}^{r} ds_\alpha^j \otimes e_{\alpha_j} + \sum_{j=1}^{r} \sum_{k=1}^{r} s_\alpha^j (\omega_\alpha)_j^k \otimes e_{\alpha_k} \\
&= \sum_{j=1}^{r} \left(ds_\alpha^j + \sum_{k=1}^{r} (\omega_\alpha)_k^j s_\alpha^k \right) \otimes e_{\alpha_j} \\
&\overset{(1.71)}{=} \sum_{j=1}^{r} (\nabla s)_\alpha^j \otimes e_{\alpha_j} \qquad \text{equating with the original definition}
\end{aligned}$$

Hence

$$(\nabla s)_\alpha = \begin{pmatrix} (\nabla s)^1_\alpha \\ \vdots \\ (\nabla s)^r_\alpha \end{pmatrix} = d\begin{pmatrix} s^1_\alpha \\ \vdots \\ s^r_\alpha \end{pmatrix} + \begin{pmatrix} (\omega_\alpha)^1_1 & \cdots & (\omega_\alpha)^1_r \\ \vdots & \cdots & \vdots \\ (\omega_\alpha)^r_1 & \cdots & (\omega_\alpha)^r_r \end{pmatrix}\begin{pmatrix} s^1_\alpha \\ \vdots \\ s^r_\alpha \end{pmatrix} = ds_\alpha + \omega_\alpha s_\alpha$$

Or in short hand notation

$$\nabla s \overset{(1.69)}{=} \nabla(e_\alpha s_\alpha) \overset{(1.65)}{=} \nabla e_\alpha s_\alpha + e_\alpha ds_\alpha \overset{(1.70)}{=} e_\alpha \omega_\alpha s_\alpha + e_\alpha ds_\alpha = e_\alpha\left(ds_\alpha + \omega_\alpha s_\alpha\right)$$

Combining with $\nabla s = e_\alpha (\nabla s)_\alpha$ we obtain

$$(\nabla s)_\alpha = ds_\alpha + \omega_\alpha s_\alpha \tag{1.72}$$

Transitions Now we discuss how ∇ transits between two intersecting coordinate charts.

One may ask: On $U_\alpha \cap U_\beta$, how are ω_α and ω_β related? On $U_\alpha \cap U_\beta$, we align both representations, and using (1.69)

$$\nabla e_\beta = e_\beta \omega_\beta = e_\alpha g_{\alpha\beta}\omega_\beta$$
$$\nabla e_\beta = \nabla\left(e_\alpha g_{\alpha\beta}\right) = \nabla e_\alpha g_{\alpha\beta} + e_\alpha dg_{\alpha\beta} = e_\alpha \omega_\alpha g_{\alpha\beta} + e_\alpha dg_{\alpha\beta}$$

for $g_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta, \mathfrak{gl}(r))$, $dg_{\alpha\beta} \in \Omega^1(U_\alpha \cap U_\beta, \mathfrak{gl}(r))$ and $\omega_\beta \in \Omega^1(U_\beta, \mathfrak{gl}(r))$.

Hence

$$g_{\alpha\beta}\omega_\beta = \omega_\alpha g_{\alpha\beta} + dg_{\alpha\beta} \in \Omega^1(U_\alpha, \mathfrak{gl}(r))$$

Rewriting yields

$$\omega_\beta = g^{-1}_{\alpha\beta}\omega_\alpha g_{\alpha\beta} + g^{-1}_{\alpha\beta} dg_{\alpha\beta} \tag{1.73}$$

Hence that

$$\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$$

is connection on E iff for any $\omega \in \Omega^1(U_\alpha, \mathfrak{gl}(r))$ it satisfies (1.73)

$$\omega_\beta = g^{-1}_{\alpha\beta}\omega_\alpha g_{\alpha\beta} + g^{-1}_{\alpha\beta} dg_{\alpha\beta} \qquad \text{on} \qquad U_\alpha \cap U_\beta$$

1.21.2 Pullback Section and Pullback Vector Bundle

1.21.2.1 F^*E Pullback Vector Bundle

Let $F : M \to N$ be a C^∞ map between C^∞ manifolds M and N. Let

$$\pi : E \to N$$

be a C^∞ vector bundle of rank r over N.

One wish to define the *pullback vector bundle*

$$\tilde{\pi} : F^*E \to M \tag{1.74}$$

as a C^∞ vector bundle of rank r over M.

Set. As a set let

$$F^*E := \bigsqcup_{p \in M} E_{F(p)}$$

where $E_{F(p)} \cong \mathbb{R}^r$ are fibers of the original bundle $\pi : E \to N$ at the point $F(p) \in N$.

$$\begin{array}{ccc} F^*E & \longrightarrow & E \\ \big\downarrow{\scriptstyle \tilde{\pi}} & & \big\downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\ F\ } & N \end{array}$$

In other words, in view of the set structure of disjoint union

$$F^*E := \{(x, (y, v)) \in M \times E \mid F(x) = y = \pi(y, v)\} \subseteq M \times E$$

for any $x \in M$, $y \in N$ and $v \in E_y$. In particular

$$(F^*E)_x = E_{F(x)} = \{(x, e) \in M \times E \mid F(x) = \pi(e)\} \cong \mathbb{R}^r$$

We define our surjective map as

$$\tilde{\pi} : F^*E \to M$$
$$(x, e) \mapsto x$$

F^*E is a C^∞ submanifold of $M \times E$.

Pullback Sections Let $\pi : E \to N$ be C^∞ vector bundle of rank r over a C^∞ manifold N. Let $F : M \to N$ be smooth map.

Definition 1.21.6 (Pullback Sections)**.** *Let $s \in C^\infty(N, E)$ be smooth section of E. We define $F^*s \in C^\infty(M, F^*E)$ via*

$$\begin{aligned} F^*s : M &\to F^*E \\ p &\mapsto s(F(p)) \in E_{F(p)} = (F^*E)_p \end{aligned} \tag{1.75}$$

One hence view

$$\begin{aligned} F^* : C^\infty(N, E) = \Omega^0(N, E) &\to C^\infty(M, F^*E) = \Omega^0(M, F^*E) \\ s &\mapsto F^*s \end{aligned} \tag{1.76}$$

One would like to equip F^*E with vector bundle structure s.t. the diagram commutes

$$\begin{array}{ccc} F^*E & \longrightarrow & E \\ {\scriptstyle F^*s}\uparrow & & \uparrow{\scriptstyle s} \\ M & \xrightarrow{F} & N \end{array}$$

In principle $F^*s \in C^\infty(M, F^*E)$ are smooth sections of F^*E.

Local Trivializations . To define local trivialization, we first defined smooth functions (1.75) and then recover the trivializations via acting on the frames.

Let $\{U_\alpha \mid \alpha \in I\}$ be open cover of N with

$$h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$$

as local trivializations of $\pi : E|_{U_\alpha} \to U_\alpha$. Let $\{e_{\alpha_1}, \cdots e_{\alpha_r}\}$ be C^∞ frame of $E|_{U_\alpha}$, defined via (1.37)

$$\begin{aligned} e_{\alpha_j} : U_\alpha &\to \pi^{-1}(U_\alpha) = E|_{U_\alpha} \\ y &\mapsto h_\alpha^{-1}(y, \hat{e}_j) \end{aligned}$$

where $\hat{e}_j$ are the standard basis for $\mathbb{R}^r$.

Then using $F : M \to N$ is C^∞ map,

$$\{F^{-1}(U_\alpha) \mid \alpha \in I\}$$

is open cover of M. We want to define

$$\tilde{h}_\alpha : \tilde{\pi}^{-1}(F^{-1}(U_\alpha)) \to F^{-1}(U_\alpha) \times \mathbb{R}^r$$

as local trivialization of the vector bundle $\tilde{\pi} : F^*E \to M$.

We wish to have $\{F^*e_{\alpha_1}, \cdots, F^*e_{\alpha_r}\}$ defined as pullback sections, i.e., C^∞ frame for $F^*E|_{F^{-1}(U_\alpha)}$. Recall from (1.75) they're defined via

$$\begin{aligned} F^*e_{\alpha_j} : F^{-1}(U_\alpha) \subseteq M &\to F^*E \\ x &\mapsto e_{\alpha_j}(F(x)) \end{aligned} \tag{1.77}$$

To do so, we want to match (1.37)

$$\tilde{h}_\alpha^{-1}(x, \hat{e}_j) = (F^*e_{\alpha_j})(x) = e_{\alpha_j}(F(x)) \tag{1.78}$$

Hence we instead define the inverse of local trivializations

$$\begin{aligned} \tilde{h}_\alpha^{-1} : F^{-1}(U_\alpha) \times \mathbb{R}^r &\to \tilde{\pi}^{-1}(F^{-1}(U_\alpha)) \subseteq M \times E \\ (x, v) &\mapsto \left(x, \sum_{j=1}^r v_j (F^*e_{\alpha_j})(x) \right) \end{aligned} \tag{1.79}$$

Requiring $\tilde{h}_\alpha^{-1}$ to be diffeomorphisms onto its range gives a valid definition for local trivializations.

Transition Functions On $U_\alpha \cap U_\beta$, for $e_\alpha = e_\beta g^E_{\beta\alpha}$ where $e_\alpha = [e_{\alpha_1}, \cdots, e_{\alpha_r}]$ are smooth frames $C^\infty(U_\alpha, E|_{U_\alpha})$, consider (1.38)

$$\begin{aligned} g^E_{\beta\alpha} : U_\alpha \cap U_\beta &\overset{C^\infty}{\to} \mathrm{GL}(r, \mathbb{R}) \\ x &\mapsto g^E_{\beta\alpha}(x) \end{aligned}$$

Note for $F^{-1}(U_\alpha) \cap F^{-1}(U_\beta) = F^{-1}(U_\alpha \cap U_\beta)$, one would like to define transition functions for the diagram to commute

$$\begin{array}{cccc} M & \overset{open}{\supseteq} & F^{-1}(U_\alpha \cap U_\beta) & \\ & & \downarrow F & \searrow^{F^* g^E_{\beta\alpha} = g^E_{\beta\alpha} \circ F} \\ N & \overset{open}{\supseteq} & U_\alpha \cap U_\beta & \xrightarrow{g^E_{\beta\alpha}} \mathrm{GL}(r, \mathbb{R}) \end{array}$$

and

$$F^* e_\alpha = [F^* e_{\alpha_1}, \cdots, F^* e_{\alpha_r}] = F^* e_\beta F^* g^E_{\beta\alpha}$$

Thus one take

$$\begin{aligned} g^{F^*E}_{\beta\alpha} := F^* g^E_{\beta\alpha} : F^{-1}(U_\alpha) \cap F^{-1}(U_\beta) \subseteq M &\to \mathrm{GL}(r, \mathbb{R}) \\ x &\mapsto (g^E_{\beta\alpha} \circ F)(x) \end{aligned} \tag{1.80}$$

Notice $s \in C^\infty(N, E)$ iff upon writing $s = e_\alpha s_\alpha$ for

$$s_\alpha = \begin{pmatrix} s^1_\alpha \\ \vdots \\ s^r_\alpha \end{pmatrix} \in C^\infty(U_\alpha, \mathbb{R}^r)$$

they transit via $s_\beta = g^E_{\beta\alpha} s_\alpha$ on $U_\alpha \cap U_\beta$. Hence we make sense of $F^* s \in C^\infty(M, F^* E)$ (1.76) via

$$(F^* s)_\alpha = F^* s_\alpha = \begin{pmatrix} F^* s^1_\alpha \\ \vdots \\ F^* s^r_\alpha \end{pmatrix} \in C^\infty(F^{-1}(U_\alpha), \mathbb{R}^r)$$

1.21.2.2 Pushforward and Pullback as $C^\infty(M, F^*TN)$

Now we consider the special case $E = TN$. Then the pullback tangent bundle writes

$$\tilde{\pi} : F^* TN \to M$$

We consider the space of connections on the C^∞ vector bundle F^*TN, i.e. $C^\infty(M, F^*TN)$.

Pushforward and Pullback as section of Pullback Tangent Bundle Let $F : M \to N$ smooth map. Note a key difference here is that we do not require F to be a diffeomorphism. Compare with (1.32) and (1.33).

Definition 1.21.7 (Pushforward and Pullback of Vector Field into Section of Pullback Tangent Bundle)**.** *Let $F : M \to N$ smooth map. Define the pushforward*

$$\begin{aligned} F_* : \mathfrak{X}(M) &\to C^\infty(M, F^*TN) \\ X &\mapsto F_* X \end{aligned}$$

as smooth section of pullback tangent bundle where

$$\begin{aligned} F_* X : M &\to F^*TN \\ p &\mapsto dF_p(X(p)) \in T_{F(p)}N = (F^*TN)_p \end{aligned} \tag{1.81}$$

Also, we have pull-back as particular example of Definition 1.21.6

$$\begin{aligned} F^* : \mathfrak{X}(N) &\to C^\infty(M, F^*TN) \\ Y &\mapsto F^* Y \end{aligned}$$

where

$$\begin{aligned} F^* Y : M &\to F^*TN \\ p &\mapsto Y(F(p)) \in T_{F(p)}N = (F^*TN)_p \end{aligned} \tag{1.82}$$

If moreover $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are F-related as in Definition 1.20.1 then

$$F_* X = F^* Y \in C^\infty(M, F^*TN)$$

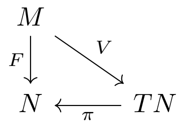
1.21.2.3 $C^\infty(M, F^*E)$ Smooth Sections along F

$C^\infty(M, F^*TN)$ Smooth Vector Field along F In particular, we study elements in $C^\infty(M, F^*TN)$, i.e., sections of pullback Tangent Bundle. This is extremely important concept as it captures how vector fields behave under smooth map F.

Definition 1.21.8 (C^∞ vector field along F)**.** *Let $F : M \to N$ be smooth map between C^∞ manifold.*
A C^∞ vector field along F is a C^∞ map

$$\begin{aligned} V : M &\to TN \\ p &\mapsto V(p) \in TN_{F(p)} = (F^*TN)_p \end{aligned} \tag{1.83}$$

We may view V as a C^∞ section of F^*TN, i.e., $V \in C^\infty(M, F^*TN)$.



$C^\infty(M, F^*E)$ Smooth Section along F More generally, for smooth vector bundle $\pi : E \to N$, we study elements in $C^\infty(M, F^*E)$.

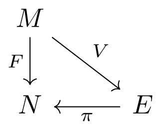
Definition 1.21.9 (C^∞ section along F)**.** *Let $F : M \to N$ be smooth map between C^∞ manifold. Let*

$$\pi : E \to N$$

be C^∞ vector bundle of rank r on N.
A C^∞ section of $\pi : E \to N$ along F is a C^∞ map

$$\begin{aligned} V : M &\to E \\ p &\mapsto V(p) \in E_{F(p)} = (F^*E)_p \end{aligned}$$

We may view V as a C^∞ section of $F^*E \to M$, i.e., $V \in C^\infty(M, F^*E)$.



1.21.3 Pullback Connection

$F^*\nabla$ Pullback Connection Let $F : M \to N$ be C^∞ map between C^∞ manifolds. Let

$$\pi : E \to N$$

be C^∞ vector bundle, and on it a connection (1.64)

$$\begin{aligned} \nabla : \Omega^0(N, E) &\to \Omega^1(N, E) \\ s &\mapsto \nabla s \end{aligned}$$

Definition 1.21.10 (Pullback Connection)**.** *There exists a unique connection on $\tilde{\pi} : F^*E \mapsto M$ called the pullback connection s.t. for any $s \in C^\infty(N, E)$*

$$\begin{aligned} F^*\nabla : \Omega^0(M, F^*E) &\to \Omega^1(M, F^*E) = C^\infty(M, T^*M \otimes F^*E) \\ F^*s &\mapsto (F^*\nabla)F^*s \end{aligned}$$

where

$$\begin{aligned} (F^*\nabla)F^*s : \mathfrak{X}(M) &\to C^\infty(M, F^*E) = \Omega^0(M, F^*E) \\ X &\mapsto (F^*\nabla)_X F^*s \end{aligned}$$

the last item is defined as

$$\begin{aligned} (F^*\nabla)_X F^*s : M &\to F^*E \\ p &\mapsto \left(\nabla_{dF_p(X(p))} s\right)(F(p)) \in (F^*E)_p = E_{F(p)} \end{aligned}$$

Thus

$$((F^*\nabla)_X (F^*s))(p) := \left(\nabla_{dF_p(X(p))} s\right)(F(p)) \qquad \forall\ p \in M,\ \forall\ X \in \mathfrak{X}(M),\ \forall\ s \in C^\infty(N, E) \tag{1.84}$$

F^* Pullback E-valued p-forms Recall we've defined pullback as in Definition 1.21.6

$$F^* : \Omega^0(N, E) = C^\infty(N, E) \to \Omega^0(M, F^*E) = C^\infty(M, F^*E)$$
$$s \mapsto F^*s$$

via (1.75), (1.76).

We may extend the definition

$$F^* : \Omega^p(N, E) \to \Omega^p(M, F^*E)$$

as $\mathbb{R}$-linear map s.t.

$$F^* : \Omega^p(N, E) = C^\infty(N, \Lambda^p T^*N \otimes E) \to \Omega^p(M, F^*E) = C^\infty(M, \Lambda^p T^*M \otimes F^*E)$$
$$\alpha \otimes s \mapsto (F^*\alpha) \otimes (F^*s) \tag{1.85}$$

Well-definedness. This is well-defined. Recall pullback of $(0, p)$-forms (1.48) s.t.

$$F^* : \Omega^p(N) \to \Omega^p(M)$$
$$\alpha \mapsto F^*\alpha$$

and pullback of E-valued sections (1.76). □

$F^*(\nabla s)$ Pullback Connection Thus for any $s \in \Omega^0(N, E)$, so that $\nabla s \in \Omega^1(N, E)$, the object $F^*(\nabla s) \in \Omega^1(M, F^*E)$ is understood via

$$F^*(\nabla s) : \mathfrak{X}(M) = C^\infty(M, TM) \to C^\infty(M, F^*E)$$
$$X \mapsto F^*(\nabla s)_X$$

where

$$F^*(\nabla s)_X : M \to F^*E$$
$$p \mapsto \left(\nabla_{dF_p(X(p))}s\right)(F(p)) \in (F^*E)_p = E_{F(p)}$$

In particular, (1.84) can thus be viewed as the

$$F^*(\nabla s) = (F^*\nabla)(F^*s) \in \Omega^1(M, F^*E) \tag{1.86}$$

Pullback Connection in Local Coordinates Given M a C^∞ manifold of dimension n. Let $\pi : E \to M$ be a C^∞ vector bundle of rank r over M.

For $\{U_\alpha \mid \alpha \in I\}$ as open cover of N, the local trivializations

$$h^E_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$$

corresponds to $\{e_{\alpha_1}, \cdots, e_{\alpha_r}\}$ a C^∞ frame of $E|_{U_\alpha}$.

On U_α

$$\nabla e_{\alpha_j} = \sum_{k=1}^r (\omega^{E,\nabla}_\alpha)^k_j \otimes e_{\alpha_k} \qquad \forall\ (\omega^{E,\nabla}_\alpha)^k_j \in \Omega^1(U_\alpha) \qquad U_\alpha \subseteq N \text{ open}$$

where $\omega^{E,\nabla}_\alpha \in \Omega^1(U_\alpha, \mathfrak{gl}(r, \mathbb{R}))$ are connection 1-forms (1.70) associated with ∇ on U_α.

On $U_\alpha \cap U_\beta$, recall (1.73)

$$\omega^{E,\nabla}_\beta = (g^E_{\alpha\beta})^{-1}\omega^{E,\nabla}_\alpha g^E_{\alpha\beta} + (g^E_{\alpha\beta})^{-1} dg^E_{\alpha\beta} \tag{1.87}$$

for transition functions $g^E_{\alpha\beta}$ on $\pi : E \to N$

$$g^E_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$$

Now, for $\{F^{-1}(U_\alpha) \mid \alpha \in I\}$ open cover of M, we have $F^*e_{\alpha_1}, \cdots, F^*e_{\alpha_r}$ C^∞ frame of $F^*E|_{F^{-1}(U_\alpha)}$. Using (1.85)

$$\begin{aligned}
(F^*\nabla)(F^*e_{\alpha_j}) &\overset{(1.86)}{=} F^*(\nabla e_{\alpha_j}) \\
&\overset{(1.70)}{=} F^*\Big(\sum_{k=1}^r (\omega^{E,\nabla}_\alpha)^k_j \otimes e_{\alpha_k}\Big) \\
&\overset{(1.85)}{=} \sum_{k=1}^r (F^*\omega^{E,\nabla}_\alpha)^k_j \otimes F^*e_{\alpha_k}
\end{aligned}$$

Now we denote

$$\omega_\alpha^{F^*E,F^*\nabla} := F^*\omega_\alpha^{E,\nabla} \in \Omega^1(F^{-1}(U_\alpha), \mathfrak{gl}(r,\mathbb{R}))$$

On $F^{-1}(U_\alpha)\cap F^{-1}(U_\beta)$, F^* acting on (1.87) yields

$$\omega_\beta^{F^*E,F^*\nabla} = (g_{\alpha\beta}^{F^*E})^{-1}\omega_\alpha^{F^*E,F^*\nabla}g_{\alpha\beta}^{F^*E} + (g_{\alpha\beta}^{F^*E})^{-1}dg_{\alpha\beta}^{F^*E}$$

Hence

$$\{\omega_\alpha^{F^*E,F^*\nabla}\} \subseteq \Omega^1(F^{-1}(U_\alpha), \mathfrak{gl}(r,\mathbb{R}))$$

defines a connection $F^*\nabla$ on $\tilde{\pi} : F^*E \to M$.

1.22 Covariant Derivative

$\frac{D}{dt}$ **Covariant Derivative** Let $\pi : E \to M$ be a C^∞ vector bundle over a C^∞ manifold M. We equip on E a connection (1.64)

$$\begin{aligned}\nabla : \Omega^0(M,E) &\to \Omega^1(M,E)\\ s &\mapsto \nabla s\end{aligned}$$

or equivalently (1.57)

$$\begin{aligned}\nabla : \mathfrak{X}(M)\times C^\infty(M,E) &\to C^\infty(M,E)\\ (X,s) &\mapsto \nabla_X s\end{aligned}$$

Definition 1.22.1 (Covariant Derivative)**.** *For any C^∞ curve*

$$\begin{aligned}c : I\subseteq\mathbb{R} &\to M\\ t &\mapsto c(t)\end{aligned}$$

recall the pullback section (1.75)

$$C^\infty(I,c^*E) = \{C^\infty \textit{sections of } E \textit{ along } c : I\to M\}$$

Define the covariant derivative along c as the pullback connection (1.84) *under c evaluated at $\frac{\partial}{\partial t}\in\mathfrak{X}(I)$*

$$\begin{aligned}\frac{D}{dt} : C^\infty(I,c^*E) &\to C^\infty(I,c^*E)\\ s &\mapsto (c^*\nabla)_{\frac{\partial}{\partial t}} s\end{aligned} \tag{1.88}$$

Definition 1.22.2 (Covariant Derivative for Affine Connection)**.** *In particular if pick $E = TM$ tangent bundle so that $C^\infty(M,E) = C^\infty(M,TM) = \mathfrak{X}(M)$, and*

$$\begin{aligned}\nabla : \mathfrak{X}(M)\times\mathfrak{X}(M) &\to \mathfrak{X}(M)\\ (X,Y) &\mapsto \nabla_X Y\end{aligned}$$

is an affine connection as in Definition 1.21.2.

Then one obtain Covariant Derivative for Affine Connection

$$\begin{aligned}\frac{D}{dt} : C^\infty(I,c^*TM) &\to C^\infty(I,c^*TM)\\ V &\mapsto \frac{DV}{dt} \equiv (c^*\nabla)_{\frac{\partial}{\partial t}} V\end{aligned}$$

Lemma 1.22.1. *Leibniz rule holds*

$$\frac{D}{dt}(fs) = \frac{df}{dt}s + f\frac{Ds}{dt} \qquad \forall\ f\in C^\infty(I), \qquad s\in C^\infty(I,c^*E) \tag{1.89}$$

Proof. WLOG assume $s = c^*e_j(t) = e_j(c(t))$ where $e_j\in C^\infty(N,E)$. Now extend f along c to $\tilde{f}$ defined on N s.t. locally

$$f(t) = \tilde{f}(c(t))$$

Then

$$\begin{aligned}\frac{D}{dt}(fc^*e_j) &= c^*\nabla_{\frac{\partial}{\partial t}}(\tilde{f}(c(t))e_j(c(t)) = \nabla_{c_*(\frac{\partial}{\partial t})}(\tilde{f}e_j)(c(t))\\ &= \frac{\partial}{\partial t}(\tilde{f}\circ c)e_j(c(t)) + f(t)(\nabla_{c_*(\frac{\partial}{\partial t})}e_j)(c(t))\\ &= \frac{\partial f}{\partial t}c^*e_j(t) + f(t)(c^*\nabla)_{\frac{\partial}{\partial t}}c^*e_j(t)\\ &= \frac{df}{dt}s(t) + f(t)\frac{D}{dt}s\end{aligned}$$

□

1.22.1 Covariant Derivative in Local Coordinates

Let M be C^∞ manifold of dimension n and $\pi : E \to M$ be C^∞ vector bundle of rank r. Let (U, ϕ) be a C^∞ chart with coordinates $\phi = (x_1, \cdots, x_n)$. We have $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$ smooth frame of $TM|_U = TU$ and $e_1, \cdots e_r$ a C^∞ frame of $E|_U$. Then the connection 1-form (1.67) and Christoffel symbol (1.66) writes

$$\nabla e_j = \sum_{k=1}^{r} \omega_j^k \otimes e_k = \sum_{i=1}^{n} \sum_{k=1}^{r} \Gamma_{ij}^k dx_i \otimes e_k$$

$$\nabla_{\frac{\partial}{\partial x_i}} e_j = \sum_{k=1}^{r} \Gamma_{ij}^k e_k \qquad \text{for} \qquad \Gamma_{ij}^k \in C^\infty(U)$$

Curve Velocity in Local Coordinates. If $E = TM$ and $r = n$, then $e_j = \frac{\partial}{\partial x_j}$. We have the curve c in local coordinates as

$$\phi \circ c(t) = (x_1(t), \cdots, x_n(t))$$

and the diagram commutes

$$\begin{array}{ccc} I & \xrightarrow{\;c\;} & M \\ \cup\, open & & \cup\, open \\ I' & \xrightarrow{\;c\;} & U \\ & \overset{\phi \circ c}{\searrow} & \downarrow \phi \\ & & \mathbb{R}^n \end{array}$$

Now what about the curve velocity in local coordinates? Via chain rule, this is

$$c'(t) = \sum_{i=1}^{n} \frac{dx_i}{dt}(t) \frac{\partial}{\partial x_i}(c(t)) \in C^\infty(I', c^*TM) \tag{1.90}$$

and for each fixed $t \in I$, this is tangent vector

$$(dc_t)(\frac{\partial}{\partial t}) = \frac{dc}{dt}(t) = \sum_{i=1}^{n} \frac{dx_i}{dt}(t) \frac{\partial}{\partial x_i}(c(t)) \in T_{c(t)}M$$

Pullback section in Local Coordinates. For $s \in C^\infty(I, c^*E)$, we have

$$s(t) = \sum_{j=1}^{r} s^j(t) e_j(c(t)) = \sum_{j=1}^{r} s^j(t)(c^* e_j)(t) \tag{1.91}$$

Covariant Derivative in Local Coordinates. Now we write, using Leibniz Rule (1.89)

$$\begin{aligned} \frac{Ds}{dt}(t) &\overset{(1.88)}{=} (c^*\nabla)_{\frac{\partial}{\partial t}} s \overset{(1.91)}{=} (c^*\nabla)_{\frac{\partial}{\partial t}} \left(\sum_{j=1}^{r} s^j c^* e_j \right) \\ &\overset{(1.89)}{=} \sum_{j=1}^{r} \frac{ds^j}{dt}(t) e_j(c(t)) + \sum_{j=1}^{r} s^j (c^*\nabla)_{\frac{\partial}{\partial t}} (c^* e_j) \end{aligned}$$

We wish to understand the last term on the RHS.

$$\begin{aligned} (c^*\nabla)_{\frac{\partial}{\partial t}} (c^* e_j) &\overset{(1.84)}{=} \nabla_{(dc_t)(\frac{\partial}{\partial t})} e_j(c(t)) = \nabla_{c'(t)} e_j(c(t)) \\ &\overset{(1.90)}{=} \nabla_{\sum_{i=1}^n \frac{dx_i}{dt} \frac{\partial}{\partial x_i}(c(t))} e_j(c(t)) \overset{(1.58)}{=} \sum_{i=1}^{n} \frac{dx_i}{dt}(t) \left(\nabla_{\frac{\partial}{\partial x_i}(c(t))} e_j(c(t)) \right) \\ &\overset{(1.66)}{=} \sum_{i=1}^{n} \sum_{k=1}^{r} \frac{dx_i}{dt}(t) \Gamma_{ij}^k(c(t)) e_k(c(t)) \end{aligned}$$

Hence for $s = \sum_{j=1}^r s^j(t) e_j(c(t))$ (1.91) we have **formula for covariant derivative for Connection in local coordinates**

$$\frac{Ds}{dt}(t) = \sum_{k=1}^{r} \left(\frac{ds^k}{dt}(t) + \sum_{i=1}^{n} \sum_{j=1}^{r} \Gamma_{ij}^k(c(t)) \frac{dx_i}{dt}(t) s^j(t) \right) e_k(c(t)) \tag{1.92}$$

Covariant Derivative for Affine Connection in Local Coordinates. In particular, if we have affine connection ∇, then

$$V(t) = \sum_{j=1}^{n} V^j(t) \frac{\partial}{\partial x_j}(c(t))$$

is a C^∞ vector field along $c : I \to M$, and we have expression

$$\frac{DV}{dt}(t) = \sum_{k=1}^{n} \left(\frac{dV^k}{dt}(t) + \sum_{i,j=1}^{n} (\Gamma_{ij}^k \circ c)(t) \frac{dx_i}{dt}(t) V^j(t) \right) \frac{\partial}{\partial x_k}(c(t)) \tag{1.93}$$

1.22.2 Parallel Transport

Parallel Section Let M be C^∞ manifold of dimension n, $\pi : E \to M$ a C^∞ vector bundle of rank r, and ∇ a connection on E.

Definition 1.22.3. *Let $V \in C^\infty(I, c^*E)$, i.e. a C^∞ section of E along c. We say V is parallel w.r.t. ∇ if*

$$\frac{DV}{dt} = 0 \qquad \forall\, t \in I$$

Existence and Uniqueness of Parallel Section

Proposition 1.22.1 ([dC92] Proposition 2.6)**.** *Let $c : I \overset{C^\infty}{\to} M$ be C^∞ curve. Given any $t_0 \in I$ and any $v \in E_{c(t_0)} \cong \mathbb{R}^r$ fiber of E over $c(t_0)$.*

Then there exists a unique parallel section V of E along c s.t. $V(t_0) = v$.

Proof. WLOG assume $c : I \to U \subset M$ open with $\phi = (x_1, \cdots, x_n)$ and $\phi(U) \subset \mathbb{R}^n$ open, i.e., (U, ϕ) is C^∞ chart for M. Recall $E|_U$ is trivialized iff there exists $e_1, \cdots, e_r$ C^∞ frame of $E|_U$. We thus have on U

$$\nabla_{\frac{\partial}{\partial x_i}} e_j = \sum_{k=1}^{r} \Gamma_{ij}^k e_k$$

For $(\phi \circ c)(t) = (x_1(t), \cdots, x_n(t))$ and $c'(t) = \sum_{i=1}^n \frac{dx_i}{dt}(t) \frac{\partial}{\partial x_i}(c(t))$ and hence

$$V(t) = \sum_{j=1}^{r} V^j(t) e_j(c(t))$$

Using (1.92), the condition $\frac{DV}{dt} = 0$ holds iff

$$\frac{dV^k}{dt} + \sum_{i=1}^{n} \sum_{j=1}^{r} (\Gamma_{ij}^k \circ c) \frac{dx_i}{dt} V^j = 0 \qquad k = 1, \cdots, r$$

For $v = \sum_{j=1}^r v^j e_j(c(t_0)) \in E_{c(t_0)}$ we have initial conditions $V(t_0) = v$ iff

$$V^k(t_0) = v^k \qquad k = 1, \cdots, r$$

Thus we have 1st order ODE. Directly Apply Existence and Uniqueness theorem. □

Parallel Transport

Definition 1.22.4 (Parallel Transport)**.** *Define for any $t \in I$*

$$\begin{aligned} P_{c,t_0,t} : E_{c(t_0)} &\to E_{c(t)} \\ v = V(t_0) &\mapsto V(t) \end{aligned} \tag{1.94}$$

where $V \in C^\infty(I, c^*E)$ is the unique C^∞ section of E along c s.t.

$$\frac{DV}{dt} = 0$$

and $V(t_0) = v$.

$P_{c,t_0,t}$ is the parallel transport along c (defined by (E, v)).

Example 1.22.1. *In particular, let $E = TM$, and ∇ be affine connection on M. Then we define parallel transport along $c : I \to M$ C^∞ curve, for any $t_0, t_1 \in I$,*

$$P_{c,t_0,t_1} : T_{c(t_0)}M \to T_{c(t_1)}M$$

This is a linear isomorphism.

1.23 Curvature on Smooth Vector Bundles

Let $\pi : E \to M$ be C^∞ vector bundle of rank r over a C^∞ manifold M of dimension n.

Let (1.57)

$$\begin{aligned}\nabla : \mathfrak{X}(M) \times C^\infty(M,E) &\to C^\infty(M,E)\\ (X,s) &\mapsto \nabla_X s\end{aligned}$$

be smooth connection on E.

F_∇ Curvature

Definition 1.23.1 (Curvature)**.** *A curvature on E is a $\mathbb{R}$-linear map*

$$\begin{aligned}F_\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \times C^\infty(M,E) &\to C^\infty(M,E)\\ (X,Y,s) &\mapsto F_\nabla(X,Y)s := \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s\end{aligned} \tag{1.95}$$

One may check that

1. F_∇ is anti-symmetric in $\mathfrak{X}(M) \times \mathfrak{X}(M)$, i.e.

$$F_\nabla(X,Y) = -F_\nabla(Y,X) \qquad \forall\ X,\ Y \in \mathfrak{X}(M) \tag{1.96}$$

Proof. Using ∇_X is $C^\infty(M)$-linear in X, and $[X,Y] = -[Y,X]$

$$\begin{aligned}F_\nabla(X,Y)(s) &= \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s = -\left(\nabla_Y\nabla_X s - \nabla_X\nabla_Y s\right) + \nabla_{[Y,X]}s\\ &= -F_\nabla(Y,X)s\end{aligned}$$

□

2. F_∇ is $C^\infty(M)$-linear in all three arguments $\mathfrak{X}(M) \times \mathfrak{X}(M) \times C^\infty(M,E)$, i.e.

$$F_\nabla(fX,Y)(s) = fF_\nabla(X,Y)(s) \tag{1.97}$$

$$F_\nabla(X,Y)(fs) = fF_\nabla(X,Y)(s) \tag{1.98}$$

for any $X,\ Y \in \mathfrak{X}(M)$, for any $s \in C^\infty(M,E)$ and for any $f \in C^\infty(M)$.

Proof. Since $F_\nabla(X,Y) = -F_\nabla(Y,X)$ it suffices to show (1.97) and (1.98). We compute

$$\begin{aligned}F_\nabla(fX,Y)(s) &= \nabla_{fX}\nabla_Y s - \nabla_Y\nabla_{fX}s - \nabla_{[fX,Y]}s\\ &= f\nabla_X\nabla_Y s - \nabla_Y(f\nabla_X s) - \nabla_{fXY-Y(fX)}s\\ &= f\nabla_X\nabla_Y s - Y(f)\nabla_X s - f\nabla_Y\nabla_X s - \nabla_{fXY-Y(f)X-fYX}s\\ &= f\nabla_X\nabla_Y s - f\nabla_Y\nabla_X s - Y(f)\nabla_X s - \nabla_{f[X,Y]-Y(f)X}s\\ &= fF_\nabla(X,Y)(s)\cancel{-Y(f)\nabla_X s + Y(f)\nabla_X s}\end{aligned}$$

and

$$\begin{aligned}F_\nabla(X,Y)(fs) &= \nabla_X\nabla_Y(fs) - \nabla_Y\nabla_X(fs) - \nabla_{[X,Y]}(fs)\\ &\overset{(1.59)}{=} \nabla_X(Y(f)s + f\nabla_Y s) - \nabla_Y(X(f)s + f\nabla_X s) - (XY(f) - YX(f))s - f\nabla_{[X,Y]}s\\ &= X(Y(f))s + \cancel{Y(f)\nabla_X s} + \cancel{X(f)\nabla_Y s} + f\nabla_X\nabla_Y s - Y(X(f))s \cancel{-X(f)\nabla_Y s} \cancel{-Y(f)\nabla_X s} - f\nabla_Y\nabla_X s\\ &\quad - (XY(f) - YX(f))s - f\nabla_{[X,Y]}s\\ &= \cancel{([X,Y](f))s} + f\nabla_X\nabla_Y s - f\nabla_Y\nabla_X s \cancel{-([X,Y](f))s} - f\nabla_{[X,Y]}s\\ &= fF_\nabla(X,Y)(s)\end{aligned}$$

□

Interpretation of $F_\nabla \in \Omega^2(M, \mathrm{End}(E))$ Since for any fixed $X,\ Y \in \mathfrak{X}(M)$

$$F_\nabla(X,Y) : C^\infty(M,E) \to C^\infty(M,E)$$
$$s \mapsto F_\nabla(X,Y)(s)$$

and is $C^\infty(M)$-linear in $C^\infty(M,E)$ (1.98). From Lemma 1.21.1 we understand the above via

$$F_\nabla(X,Y) \in C^\infty(M, E^* \otimes E) = C^\infty(M, \mathrm{End}(E))$$

Thus, using twice (1.97) and again Lemma 1.21.1, we know

$$F_\nabla \in C^\infty(M, T^*M \otimes T^*M \otimes \mathrm{End}(E))$$

Furthermore, using that F_∇ is anti-symmetric (1.96) we in fact have

$$F_\nabla \in C^\infty(M, (\Lambda^2 T^*M) \otimes \mathrm{End}(E)) \overset{(1.62)}{=} \Omega^2(M, \mathrm{End}(E)) \tag{1.99}$$

1.24 Metric on Smooth Vector Bundles

Let M be C^∞ manifold of dimension n. Let $\pi : E \to M$ be a C^∞ vector field of rank r.

Definition 1.24.1 (Metric on E). *A metric on E is a C^∞ section*

$$h \in C^\infty(M, \mathrm{Sym}^2 E^*)$$

such that for any $p \in M$

$$h(p) : E_p \times E_p \to \mathbb{R}$$
$$(u,v) \mapsto h_p(u,v)$$

defines an inner product on E_p.

Connection Compatible with Metric

Definition 1.24.2. *We say a connection ∇ on E*

$$\nabla : \mathfrak{X}(M) \times C^\infty(M,E) \to C^\infty(M,E)$$
$$(X,s) \mapsto \nabla_X s$$

is compatible with the metric h if

$$X(h(s,t)) = h(\nabla_X s, t) + h(s, \nabla_X t) \qquad \forall\ X \in \mathfrak{X}(M),\ \ s,\ t \in C^\infty(M,E) \tag{1.100}$$

where $h(s,t) \in C^\infty(M)$.

Compatibility implies Anti-Self adjointness of Curvature

Proposition 1.24.1 (Anti-Self adjoint). *Let ∇ be a connection on $E \to M$ compatible with a metric h. Then for any $X,\ Y \in \mathfrak{X}(M)$, the curvature $F_\nabla(X,Y) \in C^\infty(M, \mathrm{End}(E))$ is anti-self adjoint.*

$$h(F_\nabla(X,Y)s, t) = -h(F_\nabla(X,Y)t, s) \qquad \forall\ s,\ t \in C^\infty(M,E) \tag{1.101}$$

Proof. We compute using metric h admits inner product structure

$$h(F_\nabla(X,Y)s,t) + h(F_\nabla(X,Y)t,s) = h(F_\nabla(X,Y)(s+t),(s+t)) - h(F_\nabla(X,Y)s,s) - h(F_\nabla(X,Y)t,t)$$

It suffices to show that

$$h(F_\nabla(X,Y)s,s) = 0 \qquad \forall\ X,\ Y \in \mathfrak{X}(M) \qquad \forall\ s \in C^\infty(M,E)$$

so the RHS vanishes. Indeed

$$\begin{aligned} h(F_\nabla(X,Y)s,s) &= h(\nabla_X \nabla_Y s, s) - h(\nabla_Y \nabla_X s, s) - h(\nabla_{[X,Y]} s, s) \\ &\overset{(1.100)}{=} Xh(\nabla_Y s, s) - h(\nabla_Y s, \nabla_X s) - Yh(\nabla_X s, s) + h(\nabla_X s, \nabla_Y s) - \frac{1}{2}[X,Y]h(s,s) \\ &\overset{(1.100)}{=} \frac{1}{2}XYh(s,s) - \frac{1}{2}YXh(s,s) - \frac{1}{2}[X,Y]h(s,s) = 0 \end{aligned}$$

□

1.25 Covariant Derivative on Tensors

1.25.1 Connection and Covariant Derivative on Tensors

Consider the Affine Connection ∇ on a C^∞ manifold M.

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ (X, Y) &\mapsto \nabla_X Y \end{aligned}$$

By fixing $X \in \mathfrak{X}(M)$, this defines a $\mathbb{R}$-linear operator between $\mathfrak{X}(M) = C^\infty(M, T^1_0 M)$ to itself, that satisfies the Leibniz rule via (1.61)

$$\begin{aligned} \nabla_X : C^\infty(M, T^1_0 M) &\to C^\infty(M, T^1_0 M) \\ Y &\mapsto \nabla_X Y \end{aligned}$$

The above generalizes to tensor bundles.

∇_X Connection on $C^\infty(M, T^r_s M)$

Proposition 1.25.1. *∇_X admits a unique extension to any (r, s)-tensors*

$$\begin{aligned} \nabla_X : C^\infty(M, T^r_s M) &\to C^\infty(M, T^r_s M) \\ T &\mapsto \nabla_X T \end{aligned} \tag{1.102}$$

s.t.

1. *∇_X is $\mathbb{R}$-linear*
2. *∇_X commutes with any c contraction as in (1.40)*
$$\nabla_X(c(S)) = c(\nabla_X S) \qquad \forall\ S \in C^\infty(M, T^r_s M) \tag{1.103}$$
3. *∇_X satisfies the Leibniz Rule*
$$\nabla_X(S \otimes T) = \nabla_X S \otimes T + S \otimes \nabla_X T \qquad \forall\ S, T \in C^\infty(M, T^r_s M) \tag{1.104}$$

Proof. Because of Leibniz Rule extension (1.104), it suffices to define ∇_X on $C^\infty(M)$ the $(0, 0)$-tensors and ∇_X on $\Omega^1(M)$ the $(0, 1)$-tensors.

Let $X \in \mathfrak{X}(M)$ be fixed. We define ∇_X acting on $C^\infty(M)$ as the Lie Derivative (1.22)

$$\begin{aligned} \nabla_X : C^\infty(M) &\to C^\infty(M) \\ f &\mapsto X(f) \end{aligned} \tag{1.105}$$

To see ∇_X satisfies the desired properties, we check Leibniz Rule with $Y \in \mathfrak{X}(M)$. But this is the requirement for Leibniz Rule of Affine Connections (1.61)

$$\begin{aligned} \nabla_X(fY) &\overset{(1.61)}{=} X(f)Y + f\nabla_X Y \\ &= \nabla_X(f)Y + f\nabla_X Y \end{aligned}$$

We define ∇_X acting on $\Omega^1(M)$

$$\begin{aligned} \nabla_X : \Omega^1(M) &\to \Omega^1(M) \\ \alpha &\mapsto \nabla_X \alpha \end{aligned}$$

where

$$\begin{aligned} \nabla_X \alpha : \mathfrak{X}(M) &\to C^\infty(M) \\ Y &\mapsto X(\alpha(Y)) - \alpha(\nabla_X Y) \end{aligned} \tag{1.106}$$

To see ∇_X satisfies the desired properties, we check that its commutes with contraction

$$\begin{aligned} \nabla_X(c(Y \otimes \alpha)) &\overset{(1.42)}{=} \nabla_X(\alpha(Y)) \\ &\overset{(1.105)}{=} X(\alpha(Y)) \overset{(1.106)}{=} (\nabla_X \alpha)(Y) + \alpha(\nabla_X Y) \\ &\overset{(1.42)}{=} c(Y \otimes \nabla_X \alpha) + c(\nabla_X Y \otimes \alpha) \\ &= c\left(Y \otimes \nabla_X \alpha + \nabla_X Y \otimes \alpha\right) \qquad \text{contraction is linear} \\ &\overset{(1.104)}{=} c(\nabla_X(Y \otimes \alpha)) \end{aligned}$$

□

It is good to compare with Lie Derivative

$$(L_X\alpha)(Y) = X(\alpha(Y)) - \alpha(L_XY)$$

One has Leibniz Rule formula for any T as $(0,s)$-tensor, for any $Y_1,\cdots,Y_r \in \mathfrak{X}(M)$

$$(\nabla_X T)(Y_1,\cdots,Y_s) = \nabla_X(T(Y_1,\cdots,Y_s)) - \sum_{i=1}^{s} T(Y_1,Y_2,\cdots,Y_{i-1},\nabla_X Y_i,Y_{i+1},\cdots,Y_s) \tag{1.107}$$

Again, compare with Lie Derivative as in Lemma 1.19.9

$$(L_X T)(Y_1,\cdots,Y_s) = L_X(T(Y_1,\cdots,Y_s)) - \sum_{i=1}^{s} T(Y_1,Y_2,\cdots,Y_{i-1},L_X Y_i,Y_{i+1},\cdots,Y_s)$$

∇ Covariant Derivative on $C^\infty(M,T^r_sM)$ Let M be C^∞ manifold. Let ∇ be an affine connection on M viewed via (1.64)

$$\begin{aligned}\nabla : \Omega^0(M,TM) = C^\infty(M,T^1_0M) = \mathfrak{X}(M) &\to \Omega^1(M,E) = C^\infty(M,T^1_1M)\\ Y &\mapsto \nabla Y\end{aligned}$$

One may generalize this as well to general (r,s)-tensors.

Definition 1.25.1 (Covariant Derivative of (r,s)-tensor)**.** *Define*

$$\begin{aligned}\nabla : C^\infty(M,T^r_sM) &\to C^\infty(M,T^r_{s+1}M)\\ T &\mapsto \nabla T\end{aligned}$$

where

$$\begin{aligned}\nabla T : \underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{s+1\ times} &\to C^\infty(M,T^r_0M)\\ (X_1,\cdots,X_{s+1}) &\mapsto (\nabla_{X_{s+1}}T)(X_1,\cdots,X_s)\end{aligned} \tag{1.108}$$

where $\nabla_{X_{s+1}}T$ is defined as in (1.102) that satisfies (i) - (iii) in Proposition 1.25.1.

1.25.2 ∇T in Local Coordinates

Consider an affine ∇ connection on a C^∞ manifold M with a C^∞ chart (U,ϕ), and coordinates $\phi = (x_1,\cdots,x_n)$. From (1.66)

$$\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} = \sum_k \Gamma^k_{ij}\frac{\partial}{\partial x_k}$$

for $\Gamma^k_{ij} \in C^\infty(U)$.

$(0,1)$-tensor $\nabla_{\frac{\partial}{\partial x_i}}dx_j$ in Local Coordinates In general $(0,1)$-tensor in cotangent bundle $\alpha \in \Omega^1(M)$ takes the form

$$\alpha = \sum_{i=1}^n a_i dx_i, \qquad a_i = \alpha(\frac{\partial}{\partial x_i})$$

Recall from definition 1.102 that

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x_i}} : \Omega^1(M) &\to \Omega^1(M)\\ dx_j &\mapsto \nabla_{\frac{\partial}{\partial x_i}}dx_j\end{aligned}$$

Then $\nabla_{\frac{\partial}{\partial x_i}}dx_j$ in local coordinates take the form

$$\nabla_{\frac{\partial}{\partial x_i}}dx_j = \sum_k \left(\left(\nabla_{\frac{\partial}{\partial x_i}}dx_j\right)(\frac{\partial}{\partial x_k})\right)dx_k$$

Let's compute what the coefficients are. One has

$$\begin{aligned}\left(\nabla_{\frac{\partial}{\partial x_i}}dx_j\right)(\frac{\partial}{\partial x_k}) &\overset{(1.106)}{=} \frac{\partial}{\partial x_i}\left(dx_j(\frac{\partial}{\partial x_k})\right) - dx_j\left(\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_k}\right)\\ &= \frac{\partial}{\partial x_i}(\delta_{jk}) - dx_j\left(\sum_\ell \Gamma^\ell_{ik}\frac{\partial}{\partial x_\ell}\right)\\ &= -\Gamma^j_{ik} \qquad \text{using 1-form is } C^\infty\text{-linear}\end{aligned}$$

so

$$\nabla_{\frac{\partial}{\partial x_i}} dx_j = -\sum_{k=1}^{n} \Gamma^j_{ik} dx_k \tag{1.109}$$

We write in short for $e_i = \frac{\partial}{\partial x_i}$, $e^j = dx_j$ that

$$\nabla_{e_i} e_j = \Gamma^k_{ij} e_k \qquad \nabla_{e_i} e^j = -\Gamma^j_{ik} e^k \tag{1.110}$$

(r,s)-tensor $\nabla_{e_k} T$ in Local Coordinates For general (r,s)-tensors T we write in local coordinates

$$T = T^{i_1,\cdots,i_r}_{j_1,\cdots,j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}$$

where

$$T^{i_1,\cdots,i_r}_{j_1,\cdots,j_s} := T(e^{i_1}, \cdots, e^{i_r}, e_{j_1}, \cdots, e_{j_s}) \in C^\infty(U)$$

Now $\nabla T \in C^\infty(M, T^r_{s+1}M)$ is $(r, s+1)$-tensor, so we write in local coordinates

$$\nabla T = (\nabla T)^{i_1,\cdots,i_r}_{j_1,\cdots,j_{s+1}} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s} \otimes e^{j_{s+1}}$$

Denote

$$T^{i_1,\cdots,i_r}_{j_1,\cdots,j_s;k} := (\nabla T)^{i_1,\cdots,i_r}_{j_1,\cdots,j_s,k} \overset{(1.108)}{=} (\nabla_{e_k} T)^{i_1,\cdots,i_r}_{j_1,\cdots,j_s} \tag{1.111}$$

We want to express

$$T^{i_1,\cdots,i_r}_{j_1,\cdots,j_s;k}$$

in terms of $T^{i_1,\cdots,i_r}_{j_1,\cdots,j_s}$ and Γ^k_{ij} using Leibniz rule for Covariant Derivative (1.104).

We compute

$$\begin{aligned}
\nabla_{e_k} T &= \nabla_{e_k} \left(T^{i_1,\cdots,i_r}_{j_1,\cdots,j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s} \right) \\
&\overset{(1.104)}{=} e_k \left(T^{i_1,\cdots,i_r}_{j_1,\cdots,j_s} \right) e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s} \\
&+ \sum_{\alpha=1}^{r} T^{i_1,\cdots,i_r}_{j_1,\cdots,j_s} e_{i_1} \otimes \cdots \otimes e_{i_{\alpha-1}} \otimes \nabla_k e_{i_\alpha} \otimes e_{i_{\alpha+1}} \otimes \cdots \otimes \left(e^{j_1} \otimes \cdots \otimes e^{j_s} \right) \\
&+ \sum_{\beta=1}^{s} T^{i_1,\cdots,i_r}_{j_1,\cdots,j_s} (e_{i_1} \otimes \cdots \otimes e_{i_r}) \otimes e^{j_1} \otimes \cdots \otimes e^{j_{\beta-1}} \otimes \nabla_k e^{j_\beta} \otimes e^{j_{\beta+1}} \otimes \cdots \otimes e^{j_s}
\end{aligned}$$

Then we switch in $\nabla_k e_{i_\alpha} = \Gamma^\ell_{k i_\alpha} e_\ell$ and $\nabla_k e^{j_\beta} = -\Gamma^{j_\beta}_{k\ell} e^\ell$ as in (1.110) so

$$\nabla_{e_k} T = \left(e_k(T^{i_1,\cdots,i_r}_{j_1,\cdots,j_s}) + \Gamma^{i_\alpha}_{k\ell} T^{i_1,\cdots,i_{\alpha-1},\ell,i_{\alpha+1},\cdots i_r}_{j_1,\cdots,j_s} - \Gamma^\ell_{k,j_\beta} T^{i_1,\cdots,i_r}_{j_1,\cdots,j_{\beta-1},\ell,j_{\beta+1},\cdots,j_s} \right) e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}$$

Hence we have formula

$$T^{i_1,\cdots,i_r}_{j_1,\cdots,j_s;k} = e_k(T^{i_1,\cdots,i_r}_{j_1,\cdots,j_s}) + \sum_{\ell,\alpha}^{r} \Gamma^{i_\alpha}_{k\ell} T^{i_1,\cdots,i_{\alpha-1},\ell,i_{\alpha+1},\cdots i_r}_{j_1,\cdots,j_s} - \sum_{\ell,\beta}^{s} \Gamma^\ell_{k,j_\beta} T^{i_1,\cdots,i_r}_{j_1,\cdots,j_{\beta-1},\ell,j_{\beta+1},\cdots,j_s} \tag{1.112}$$

where $e_k = \frac{\partial}{\partial x_k}$.

1.26 Pullback of a Vector Bundle and Connection Revisited

In this section we discuss the very confusing facts about pullbacks of vector bundle and connection. This follows the short yet enlightening notes of Deane Yang.

Pullback Vector Bundle Consider M a C^∞ manifold of dimension m, N a C^∞ manifold of dimension n. Consider a smooth map

$$F : M \to N$$

Let $\pi : E \to N$ be a C^∞ vector bundle of rank r over N. We define the pullback bundle F^*E (1.74) as

1. Fiber at each $x \in M$ is $(F^*E)_x = E_{F(x)}$.
2. Given any local frame $\{e_i\}_{1 \le i \le r}$ of N,

   $$\{e_i \circ F\}_{1 \le i \le r}$$

   gives a local frame of F^*E (1.77). In particular, any section $s \in C^\infty(M, F^*E)$ can be written locally as

   $$s(x) = a^k(x) e_k(F(x))$$

   where a^k are smooth functions defined on M.

In other words, the space of smooth sections of F^*E, $C^\infty(M, F^*E)$, is the $C^\infty(M)$-module generated by the space of smooth sections of E pulled-back by F.

Pushforward of Vector Field Let $\phi = \{x_1, \cdots, x_m\}$ be coordinates on M, and the corresponding coordinate vector fields as $\{\frac{\partial}{\partial x_i}\}_{1\le i\le m}$. Let $\psi = \{y_1, \cdots, y_n\}$ be coordinates on N, and the corresponding coordinate vector fields as $\{\frac{\partial}{\partial y_i}\}_{1\le i\le n}$.

Now in local coordinates

$$(\psi \circ F \circ \phi^{-1})(x_1, \cdots, x_m) = (y_1(x_1, \cdots, x_m), \cdots, y_n(x_1, \cdots, x_m))$$

We ask, what is $F_*(\frac{\partial}{\partial x_i})$ for $1 \le i \le m$? First of all, in view of (1.81), we know $F_*(\frac{\partial}{\partial x_i}) \in C^\infty(M, F^*TN)$, thus

$$F_*(\frac{\partial}{\partial x_i}) = \sum_{j=1}^n F_*(\frac{\partial}{\partial x_i})^j \frac{\partial}{\partial y_j} \circ F$$

Now we make sense as in (1.90). By definition, and as in (1.27)

$$F_*(\frac{\partial}{\partial x_i}) = dF(\frac{\partial}{\partial x_i}) = \sum_{j=1}^n \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j} \circ F \tag{1.113}$$

More generally, for $X = \sum_{i=1}^m a_i \frac{\partial}{\partial x_i}$

$$F_*(X) = F_*(\sum_i a_i \frac{\partial}{\partial x_i}) = \sum_i dF(a_i \frac{\partial}{\partial x_i}) = \sum_{i=1}^m a_i \sum_{j=1}^n \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j} \circ F$$

Notice in particular, for any $1 \le i,\, k \le m$ and $1 \le j \le n$

$$\frac{\partial}{\partial x_k} F_*(\frac{\partial}{\partial x_i})^j = \frac{\partial}{\partial x_k}\frac{\partial y_j}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{\partial y_j}{\partial x_k} = \frac{\partial}{\partial x_i} F_*(\frac{\partial}{\partial x_k})^j$$

Pullback of Connection Let ∇ be connection on $\pi : E \to N$. Let $\{e_i\}_{1\le i\le n}$ be local frame of N. Then given any vector field $Y \in \mathfrak{X}(N)$, $\nabla_Y e_i \in C^\infty(N, E)$.

We define a pullback connection $D = F^*\nabla$ on F^*E as follows (1.84). For any $X \in \mathfrak{X}(M)$

$$D_X(e_j \circ F) = (F^*\nabla)_X(e_j \circ F) := \left(\nabla_{F_*(X)} e_j\right) \circ F \in C^\infty(M, F^*E)$$

Now given a smooth section of F^*E

$$f = \sum_{j=1}^n a_j e_j \circ F$$

Let's see how D acts on f. Via Leibniz Rule, for any $X \in \mathfrak{X}(M)$

$$\begin{aligned} D_X(\sum_{j=1}^n a_j e_j \circ F) &= \sum_{j=1}^n (X(a_j) e_j \circ F + a_j D_X(e_j \circ F)) \\ &= \sum_{j=1}^n \left(X(a_j) e_j \circ F + a_j (\nabla_{F_*(X)} e_j) \circ F\right) \end{aligned} \tag{1.114}$$

Pullback of Curvature Given a pullback section

$$f = \sum_{j=1}^n a_j e_j \circ F$$

Let $1 \le p,\, q \le m$ and consider

$$\begin{aligned} & D_{\frac{\partial}{\partial x_p}}(D_{\frac{\partial}{\partial x_q}} f) \\ &= D_{\frac{\partial}{\partial x_p}}\left(\sum_{j=1}^n \frac{\partial a_j}{\partial x_q} e_j \circ F + a_j (\nabla_{F_*(\frac{\partial}{\partial x_q})} e_j) \circ F\right) \\ &\overset{(1.113)}{=} D_{\frac{\partial}{\partial x_p}}\left(\sum_{j=1}^n \frac{\partial a_j}{\partial x_q} e_j \circ F + a_j (\nabla_{\sum_{k=1}^n \frac{\partial y_k}{\partial x_q}\frac{\partial}{\partial y_k}\circ F} e_j) \circ F\right) \\ &= D_{\frac{\partial}{\partial x_p}}\left(\sum_{j=1}^n \frac{\partial a_j}{\partial x_q} e_j \circ F + a_j \sum_{k=1}^n \frac{\partial y_k}{\partial x_q} (\nabla_{\frac{\partial}{\partial y_k}} e_j) \circ F\right) \\ &= \sum_{j=1}^n \left(\frac{\partial^2 a_j}{\partial x_p \partial x_q} e_j \circ F + \frac{\partial a_j}{\partial x_q} D_{\frac{\partial}{\partial x_p}}(e_j \circ F) + \sum_{k=1}^n \left(\left(\frac{\partial a_j}{\partial x_p}\frac{\partial y_k}{\partial x_q} + a_j \frac{\partial^2 y_k}{\partial x_p \partial x_q}\right)(\nabla_{\frac{\partial}{\partial y_k}} e_j) \circ F + a_j \frac{\partial y_k}{\partial x_q} D_{\frac{\partial}{\partial x_p}}((\nabla_{\frac{\partial}{\partial y_k}} e_j) \circ F)\right)\right) \end{aligned}$$

And the computation goes on

$$= \frac{\partial^2 a_j}{\partial x_p \partial x_q} e_j \circ F + \frac{\partial a_j}{\partial x_q}\frac{\partial y_\ell}{\partial x_p}(\nabla_{\frac{\partial}{\partial y_\ell}} e_j) \circ F$$
$$+ \left(\left(\frac{\partial a_j}{\partial x_p}\frac{\partial y_k}{\partial x_q} + a_j \frac{\partial^2 y_k}{\partial x_p \partial x_q} \right) (\nabla_{\frac{\partial}{\partial y_k}} e_j) \circ F + a_j \frac{\partial y_k}{\partial x_q}\frac{\partial y_\ell}{\partial x_p} \left(\nabla_{\frac{\partial}{\partial y_\ell}} \left(\nabla_{\frac{\partial}{\partial y_k}} e_j \right)\right) \circ F \right)$$

We use $[\frac{\partial}{\partial x_p}, \frac{\partial}{\partial x_q}] = 0$ and $[\frac{\partial}{\partial y_\ell}, \frac{\partial}{\partial y_k}] = 0$ to obtain

$$\begin{aligned}
& D_{\frac{\partial}{\partial x_p}}(D_{\frac{\partial}{\partial x_q}} f) - D_{\frac{\partial}{\partial x_q}}(D_{\frac{\partial}{\partial x_p}} f) \\
&= a_j \frac{\partial y_k}{\partial x_q}\frac{\partial y_\ell}{\partial x_p} \left(\left(\nabla_{\frac{\partial}{\partial y_\ell}} \left(\nabla_{\frac{\partial}{\partial y_k}} e_j \right)\right) \circ F - \left(\nabla_{\frac{\partial}{\partial y_k}} \left(\nabla_{\frac{\partial}{\partial y_\ell}} e_j \right)\right) \circ F \right) \\
&= a_j \frac{\partial y_k}{\partial x_q}\frac{\partial y_\ell}{\partial x_p} R_\nabla(\frac{\partial}{\partial y_\ell}, \frac{\partial}{\partial y_k}) e_j \circ F \\
&= R_\nabla \left(F_*(\frac{\partial}{\partial x_p}), F_*(\frac{\partial}{\partial x_q}) \right) f
\end{aligned}$$

where we denote R_∇ as our curvature.

Therefore the pullback curvature is defined as

$$(F^* R_\nabla)(\frac{\partial}{\partial x_p}, \frac{\partial}{\partial x_q})(f) = R_{F^*\nabla}(\frac{\partial}{\partial x_p}, \frac{\partial}{\partial x_q})(f) := R_\nabla \left(F_*(\frac{\partial}{\partial x_p}), F_*(\frac{\partial}{\partial x_q}) \right) f$$

Thus for general X, $Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M, F^*TN)$ we define

$$(F^* R_\nabla)(X,Y)(f) := D_X(D_Y f) - D_Y(D_X f) - D_{[Y,X]} f = R_\nabla(F_*(X), F_*(Y))(f) \tag{1.115}$$

Chapter 2

Riemannian Geometry

2.1 Riemannian Metric

Let M be C^∞ manifold of dimension n.

Definition 2.1.1 (Riemannian Metric). *A Riemannian Metric on M is a C^∞ $(0,2)$-tensor on M, which we denote as $g \in C^\infty(M, T_2^0 M)$*

$$\begin{aligned} g : M &\to T_2^0 M = (T^*M)^{\otimes 2} \\ p &\mapsto g(p) = g_p \end{aligned}$$

where

$$\begin{aligned} g_p : T_pM \times T_pM &\to \mathbb{R} \\ (v_1, v_2) &\mapsto g_p(v_1, v_2) \end{aligned}$$

defines an inner product on T_pM, i.e.

(a) Bilinear

$$\begin{aligned} g_p(\lambda v_1 + \mu v_2, v_3) &= \lambda g_p(v_1, v_3) + \mu g_p(v_2, v_3) \\ g_p(v_3, \lambda v_1 + \mu v_2) &= \lambda g_p(v_3, v_1) + \mu g_p(v_3, v_2) \qquad \forall\ \lambda, \mu \in \mathbb{R} \end{aligned}$$

(b) Symmetric

$$g_p(v_1, v_2) = g_p(v_2, v_1) \qquad \forall\ v_1,\ v_2 \in T_pM$$

(c) Positive-Definite

$$g_p(v, v) > 0 \qquad \forall\ v \neq 0$$

We call the pair (M, g) a Riemannian Manifold.

Sometimes we write

$$g(p)(u, v) = g_p(u, v) = \langle u, v \rangle_p$$

Symmetric Tensor Let $\dim M = n$. Then the tensor bundle splits into symmetric tensor bundle (of rank $\frac{n(n+1)}{2}$) and anti-symmetric tensor bundle (of rank $\frac{n(n-1)}{2}$)

$$T_2^0 M = \mathrm{Sym}^2 T^*M \otimes \Lambda^2 T^*M$$

defined for any $p \in M$ as

$$\begin{aligned} (\mathrm{Sym}^2 T^*M)_p &:= \{\text{symmetric bilinear forms on } T_pM\} \\ (\Lambda^2 T^*M)_p &:= \{\text{anti-symmetric bilinear forms on } T_pM\} \end{aligned}$$

Thus we notice a Riemannian metric $g \in C^\infty(M, \mathrm{Sym}^2 T^*M)$.

Local Coordinates In local coordinates, let (U, ϕ) be a C^∞ chart for M with $\phi = (x_1, \cdots, x_n)$. Define

$$dx_i dx_j := \frac{dx_i \otimes dx_j + dx_j \otimes dx_i}{2} \in C^\infty(U, \mathrm{Sym}^2\ T^*M|_U)$$

so $\{dx_i dx_j \mid 1 \leq i \leq j \leq n\}$ is C^∞ frame of $\mathrm{Sym}^2\ T^*M|_U = \mathrm{Sym}^2 T^*U$. On the other hand, note

$$dx_i \wedge dx_j := dx_i \otimes dx_j - dx_j \otimes dx_i \in C^\infty(U, \Lambda^2\ T^*M|_U)$$

so $\{dx_i \wedge dx_j \mid 1 \leq i < j \leq n\}$ is C^∞ frame of $\Lambda^2\, T^*M|_U$. Also denote for simplicity

$$dx_i^2 = dx_i dx_i = dx_i \otimes dx_i$$

Therefore on U, the Riemannian metric g has representation in local coordinates

$$g = \sum_{ij} g_{ij} dx_i \otimes dx_j = \sum_{ij} g_{ij} dx_i dx_j \qquad g_{ij} = g_{ji}$$

where the coefficients as smooth functions on U are given by

$$g_{ij}(p) = g_p\left(\frac{\partial}{\partial x_i}(p), \frac{\partial}{\partial x_j}(p)\right) = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_p$$

g_0 **Euclidean Metric and Spherical Coordinates** Consider the Euclidean metric on $M = \mathbb{R}^n$

$$g_0 = \sum_{i=1}^n dx_i^2 = \sum_{ij} g_{ij} dx_i dx_j, \qquad g_{ij} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

One may alternatively consider the Spherical Coordinates in $\mathbb{R}^n$. For $n = 2$ with

$$(x, y) = (r\cos(\theta), r\sin(\theta))$$

one may write

$$g_0 = dx^2 + dy^2 = (\cos(\theta)dr - r\sin(\theta)d\theta)^2 + (\sin(\theta)dr + r\cos(\theta)d\theta)^2 = dr^2 + r^2 d\theta^2$$

For $n = 3$ with

$$(x, y, z) = (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))$$

for $\rho > 0$, $\theta \in (0, 2\pi)$ and $\phi \in (0, \pi)$, one may write

$$\begin{aligned} g_0 &= dx^2 + dy^2 + dz^2 \\ &= (\sin(\phi)\cos(\theta)d\rho - \rho\sin(\theta)\sin(\phi)d\theta + \rho\cos(\phi)\cos(\theta)d\phi)^2 \\ &+ (\sin(\phi)\sin(\theta)d\rho + \rho\cos(\theta)\sin(\phi)d\theta + \rho\cos(\phi)\sin(\theta)d\phi)^2 \\ &+ (\cos(\phi)d\rho - \rho\sin(\phi)d\phi)^2 \\ &= d\rho^2 + \rho^2 d\phi^2 + \rho^2\sin^2(\phi)d\theta^2 \end{aligned}$$

Definition 2.1.2 (Orthonormal Frame)**.** *Let (M, g) be Riemannian manifold of $\dim M = n$. An orthonormal frame on $U \subseteq M$ is a collection $\{e_i\}_{1\leq i\leq n} \subseteq \mathfrak{X}(U)$ s.t. $\{e_i(p)\}_{1\leq i\leq n}$ are basis for T_pM and*

$$g_p(e_i(p), e_j(p)) = \delta_{ij} \qquad \forall\, p \in U,\ i, j = 1, \cdots, n \tag{2.1}$$

Now for Euclidean metric, $\{\frac{\partial}{\partial x_i}\}_{1\leq i\leq n}$ defines an orthonormal smooth frame of $T\mathbb{R}^n$. How about spherical coordinates? For $n = 2$

$$\frac{\partial}{\partial r} \qquad \frac{1}{r}\frac{\partial}{\partial \theta} \qquad \mathbb{R}^2 \setminus \{0\}$$

as orthonormal basis. For $n = 3$, one has

$$\frac{\partial}{\partial \rho}, \qquad \frac{1}{\rho}\frac{\partial}{\partial \phi}, \qquad \frac{1}{\rho\sin(\phi)}\frac{\partial}{\partial \theta} \qquad U \subset \mathbb{R}^3$$

as orthonormal basis.

2.1.1 Isometry and Pullback Metrc

Isometry

Definition 2.1.3. *Let (M, g), (N, h) be Riemannian manifolds. A smooth diffeomorphism $f : M \to N$ is an isometry if*

$$g_p(u, v) = h_{f(p)}(df_p(u), df_p(v)) \qquad \forall\, p \in M,\ u, v \in T_pM \tag{2.2}$$

A smooth map $f : M \to N$ is called a local isometry at $p \in M$ if there is a neighborhood $U \subseteq M$ of p s.t.

$$f : U \to f(U)$$

is a diffeomorphism satisfying (2.2).

f^*g **Pullback of Riemannian Metric** Let (N, g) Riemannian manifold, and $f : M \to N$ a C^∞ map from C^∞ manifold M to N.

Question: Given the metric on N, how to equip a metric on M?

Proposition 2.1.1 ([dC92] Example 2.5)**.** *Let $f : M \to N$ be a smooth immersion, i.e.,*

$$df_p : T_pM \to T_{f(p)}N$$

is injective for any $p \in M$.

Then the pullback (1.47)

$$\begin{aligned} f^*g : M &\to T_2^0M \\ p &\mapsto (f^*g)_p \\ (f^*g)_p : T_pM \times T_pM &\to \mathbb{R} \\ (u, v) &\mapsto g_{f(p)}(df_p(u), df_p(v)) \end{aligned} \tag{2.3}$$

defines a metric on M. In particular,

$$f : (M, f^*g) \to (N, g)$$

becomes an isometric immersion. We regard (M, f^*g) as an isometrically immersed submanifold of (N, g).

Proof. We want to check $(f^*g)_p$ defines an inner product at each $p \in M$. Symmetry is trivial. Bilinearity is preserved under linear isomorphism. To ensure $(f^*g)_p$ is positive-definite, if for $v \in T_pM$

$$g_{f(p)}(df_p(v), df_p(v)) = 0$$

Then that g is a metric ensures $df_p(v) = 0$. Now $v = 0$ iff df_p is injective iff f is an immersion. □

Example 2.1.1. *If (M, g) is Riemannian manifold and $M' \subset M$ is embedded C^∞ submanifold, so*

$$\begin{aligned} i : M' &\to M \\ p &\mapsto p \end{aligned}$$

inclusion map is C^∞ embedding (in particular, smooth immersion).

Thus (M', i^*g) is a Riemannian submanifold. For any $p \in M' \subseteq M$,

$$(i^*g)(p) : T_pM' \times T_pM' \to \mathbb{R}$$

is the restriction of $g(p) : T_pM \times T_pM \to \mathbb{R}$.

Recall the Preimage Theorem 1.5.1. For

$$h : M^{n+k} \to N^k$$

smooth map with $q \in N$ a regular value, i.e.

$$dh_p : T_pM \to T_{h(p)}N$$

is surjective for all $p \in h^{-1}(q)$. Then $h^{-1}(q) \subseteq M$ is a closed smooth submanifold of M of dimension $n \geq 0$.

Now if (M^{n+k}, g) is a Riemannian Manifold, one may put a metric on

$$(h^{-1}(q), i^*g)$$

as induced by the inclusion map.

$g_{\text{can}}^{\mathbb{S}^n(r)}$ **Canonical Metric** Recall the Sphere is defined via

$$\mathbb{S}^n(r) := \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = r^2\} \subseteq \mathbb{R}^{n+1}$$

Denote $i_r : \mathbb{S}^n(r) \hookrightarrow \mathbb{R}^{n+1}$ the inclusion map. Then the pullback metric, known as the *canonical metric* of $\mathbb{S}^n(r)$, is

$$g_{\text{can}}^{\mathbb{S}^n(r)} = i_r^*g_0 = i_r^*(dx_1^2 + \cdots dx_{n+1}^2)$$

For $n = 3$, the spherical coordinates on $\mathbb{R}^3$ is

$$g_0 = d\rho^2 + \rho^2 d\phi^2 + \rho^2 \sin^2(\phi) d\theta^2$$

One has

$$g_{\text{can}}^{\mathbb{S}^2(r)} = i_r^*g_0 = r^2(d\phi^2 + \sin^2(\phi)d\theta^2) \qquad \forall\ (\phi, \theta)$$

with local coordinates on $U \subseteq \mathbb{S}^2(r)$ open.

Now $i_r : (\mathbb{S}^n(r), g_{\text{can}}^{\mathbb{S}^n(r)}) \mapsto (\mathbb{R}^{n+1}, g_0)$ is an isometric embedding.

Isometry in $(\mathbb{R}^n, g_0)$ Take $A \in \mathrm{GL}(n, \mathbb{R})$. The corresponding linear isomorphism

$$L_A : \mathbb{R}^n \to \mathbb{R}^n$$
$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto Ax$$

gives a C^∞ diffeomorphism. For the Euclidean metric $g_0 = \sum_{i=1}^n dx_i^2$, one ask, when is L_A an isometry between $(\mathbb{R}^n, g_0)$? i.e., when is $L_A^* g_0 = g_0$?.

Note for $A = (a_{ij})$, $(Ax)_i = \sum_j a_{ij} x_j$. We compute the pullbacks

$$L_A^* x_i = x_i \circ L_A = (Ax)_i = \sum_j a_{ij} x_j$$
$$L_A^* dx_i \overset{(1.50)}{=} d(L_A^* x_i) = \sum_j a_{ij} dx_j$$
$$L_A^* g_0 = L_A^* (\sum_{i=1}^n dx_i^2) = \sum_{i,j,k} (a_{ij} dx_j)(a_{ik} dx_k) = \sum_{j,k=1}^n \left(\sum_{i=1}^n a_{ij} a_{ik} \right) dx_j dx_k$$
$$= \sum_{j,k=1}^n \left(A^T A\right)_{jk} dx_j dx_k$$

So $L_A^* g_0 = g_0$ iff $A^T A = T_n$ iff $A \in O(n)$.

Now for $b \in \mathbb{R}^n$, define translation

$$T_b : \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto x + b$$

Here $T_b^* x_i = x_i + b_i$, $T_b^* dx_i = dx_i$ and thus $T_b^* g_0 = g_0$.

We conclude the following.

Theorem 2.1.1. *$f : (\mathbb{R}^n, g_0) \to (\mathbb{R}^n, g_0)$ is an isometry iff*

$$f(x) = Ax + b \qquad A \in O(n), \qquad b \in \mathbb{R}^n$$

i.e., f is a rigid motion.

Observe that, for $A \in O(n+1)$, since $L_A : (\mathbb{R}^{n+1}, g_0) \to (\mathbb{R}^{n+1}, g_0)$ is an isometry and $L_A(\mathbb{S}^n) = \mathbb{S}^n$, one may conclude

$$L_A : (\mathbb{S}^n, g_{\text{can}}) \to (\mathbb{S}^n, g_{\text{can}})$$

is an isometry. In other words for the canonical metric $g_{\text{can}} = i^* g_0$

$$L_A^* g_{\text{can}} = g_{\text{can}}$$

Theorem 2.1.2. *$f : (\mathbb{S}^n,\, g_{can}) \to (\mathbb{S}^n,\, g_{can})$ is an isometry iff*

$$f : \mathbb{S}^n \to \mathbb{S}^n$$
$$x \mapsto Ax$$

for some $A \in O(n+1)$.

Product Metric

Definition 2.1.4 (Product Metric)**.** *Let (M_1, g_1), (M_2, g_2) be Riemannian manifolds and consider the Cartersian Product $M_1 \times M_2$ equipped with the product smooth structure.*

Denote the natural projections as

$$\pi_1 : M_1 \times M_2 \to M_1 \qquad \pi_2 : M_1 \times M_2 \to M_2$$
$$(p, q) \mapsto p \qquad (p, q) \mapsto q$$

Define

$$g_1 \times g_2 := \pi_1^* g_1 + \pi_2^* g_2 : M_1 \times M_2 \to (T^*(M_1 \times M_2))^{\otimes 2}$$
$$(p_1, p_2) \mapsto (g_1 \times g_2)_{(p_1, p_2)}$$

s.t.

$$(g_1 \times g_2)_{(p_1,p_2)} : T_{(p_1,p_2)}(M_1 \times M_2) \times T_{(p_1,p_2)}(M_1 \times M_2) \to \mathbb{R}$$
$$((u_1,u_2),(v_1,v_2)) \mapsto (g_1)_{p_1}(u_1,v_1) + (g_2)_{p_2}(u_2,v_2) \tag{2.4}$$

Or equivalently

$$\begin{aligned}\langle u,v\rangle_{(p_1,p_2)} &= (g_1 \times g_2)_{(p_1,p_2)}(u,v)\\ &= (g_1)_{p_1}((d\pi_1)_{(p_1,p_2)}(u),(d\pi_1)_{(p_1,p_2)}(v)) + (g_2)_{p_2}((d\pi_2)_{(p_1,p_2)}(u),(d\pi_2)_{(p_1,p_2)}(v))\\ &= \langle d\pi_1 \cdot u, d\pi_1 \cdot v\rangle_{p_1} + \langle d\pi_2 \cdot u, d\pi_2 \cdot v\rangle_{p_2} \qquad \forall\ (p_1,p_2) \times M_1 \times M_2, \qquad u,\, v \in T_{(p_1,p_2)}(M_1 \times M_2)\end{aligned}$$

Consider for example

$$f : \mathbb{R} \to \mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \overset{i^*}{\subseteq} \mathbb{R}^2$$
$$t \mapsto (\cos(t),\, \sin(t))$$

So

$$f^* g_{\text{can}}^{\mathbb{S}^1} = f^* i^* (dx^2 + dy^2) = (d(\cos(t)))^2 + (d(\sin(t)))^2 = (-\sin(t)dt)^2 + (\cos(t)dt)^2 = dt^2$$

And thus

$$f : (\mathbb{R}, dt^2) \to (\mathbb{S}^1, g_{\text{can}})$$
$$t \mapsto (\cos(t),\, \sin(t))$$

is a local isometry, and in fact a covering map.

Now let's consider n products of the above, i.e. the flat n-torus.

$$f : (\mathbb{R}^n, g_0 = dt_1^2 + \cdots + dt_n^2) \to \left(\mathbb{T}^n := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1, g_{\text{can}} \times \cdots \times g_{\text{can}}\right) \subseteq (\mathbb{R}^{2n}, g_0)$$
$$(t_1, \cdots, t_n) \mapsto (\cos(t_1),\, \sin(t_1), \cdots, \cos(t_n),\, \sin(t_n))$$

Now f is a local isometry.

2.1.2 Volume, Length and Distance

Volume

Definition 2.1.5 (Volume Form)**.** *Let M be C^∞ manifold of dimension n. A volume form on M is a nowhere-vanishing smooth n-form*

$$\nu \in \Omega^n(M) = C^\infty(M, \Lambda^n T^* M), \qquad \nu(p) \neq 0 \qquad \forall\ p \in M$$

One has equivalence criterion for existence of volume form.

Theorem 2.1.3. *Let M be C^∞ manifold. Then the following are equivalent*

1. *There exists a volume form ν on M.*
2. *$\Lambda^n T^* M$ is trivial.*
3. *M is orientable.*

Existence of Volume Form implies Orientability (1) implies (3). Suppose $\nu \in \Omega^n(M)$ is a volume form on M. We may choose C^∞ atlas $\{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$ with coordinates $\phi_\alpha = (x_1^\alpha, \cdots, x_n^\alpha)$, s.t. on U_α

$$\nu = a_\alpha dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha \qquad a_\alpha \in C^\infty(U_\alpha) \qquad a_\alpha > 0$$

Now on the intersections with any other chart $U_\alpha \cap U_\beta$ with transition functions

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n \to \phi_\beta(U_\alpha \cap U_\beta)$$
$$(x_1^\alpha, \cdots, x_n^\alpha) \mapsto (x_1^\beta(x_1^\alpha, \cdots, x_n^\alpha), \cdots, x_n^\beta(x_1^\alpha, \cdots, x_n^\alpha))$$

ν takes the form

$$\nu = a_\beta dx_1^\beta \wedge \cdots \wedge dx_n^\beta = a_\alpha dx_1^\alpha \wedge \cdots \wedge x_n^\alpha$$

Notice

$$dx_1^\beta \wedge \cdots \wedge dx_n^\beta = \left(\sum_{j_1} \frac{\partial x_1^\beta}{\partial x_{j_1}^\alpha} dx_{j_1}^\alpha\right) \wedge \cdots \wedge \left(\sum_{j_n} \frac{\partial x_n^\beta}{\partial x_{j_n}^\alpha} dx_{j_n}^\alpha\right)$$
$$\det(d(\phi_\beta \circ \phi_\alpha^{-1})) = \det(\frac{\partial x_i^\beta}{\partial x_j^\alpha})$$

So that

$$\begin{aligned} a_\beta dx_1^\beta \wedge \cdots \wedge dx_n^\beta &= a_\beta \det(d(\phi_\beta \circ \phi_\alpha^{-1}))dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha \\ &= a_\alpha dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha \\ \det(d(\phi_\beta \circ \phi_\alpha^{-1})) &= \frac{a_\beta}{a_\alpha} > 0 \end{aligned}$$

And therefore M is orientable.

Construction of Volume Form from the Metric

Proposition 2.1.2 (Orientable implies Existence of compatible volume form)**.** *Suppose (M, g) is an oriented Riemannian manifold.*

Then there exists a unique volume form $\nu \in \Omega^n(M)$ where $n = \dim M$ which is compatible with g and the orientation. In fact, in local coordinates

$$\nu_g(p) = \sqrt{\det(g_{ij})}(dx_1 \wedge \cdots \wedge dx_n)(p)$$

For any $p \in M$, using one has the metric g, let $(e_1, \cdots, e_n)$ be an *ordered orthonormal basis of* $(T_pM, \langle \cdot, \cdot \rangle_p)$ where $\langle e_i, e_j \rangle_p = \delta_{ij}$ is the inner product defined by $g(p)$. In particular

$$\langle e_j, e_j \rangle_p = g(p)(e_i, e_j) = \delta_{ij}$$

Let $\{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$ be the atlas defining the given *orientation.* For $p \in U_\alpha$, one has coordinates $\phi_\alpha = (x_1^\alpha, \cdots, x_n^\alpha)$.

We say $(e_1, \cdots, e_n)$ is *compatible with the orientation* if

$$e_i = \sum_{j=1}^n a_{ij} \frac{\partial}{\partial x_j^\alpha}(p), \qquad A = (a_{ij}) \qquad \det(A) > 0$$

Given compatibility, one has

$$(dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha)_p (e_1, \cdots, e_n) = \det(dx_i^\alpha(e_j)) = \det(A) > 0$$

Now let $(e_1^*, \cdots, e_n^*)$ be ordered basis of T_p^*M dual to $(e_1, \cdots, e_n)$, i.e.

$$g(p) = \sum_{i=1}^n e_i^* \otimes e_i^*$$

Then

$$\nu(p) = e_1^* \wedge \cdots \wedge e_n^* \in \Lambda^n T_p^* M$$

iff $\nu(p)(e_1, \cdots, e_n) = 1$ for any ordered orthonormal basis $(e_1, \cdots, e_n)$ of $(T_pM, \langle \cdot, \cdot \rangle_p)$ compatible with the orientation.

Proof of 2.1.2. Let (U, ϕ) be C^∞ chart on M that gives the orientation by local coordinates $\phi = (x_1, \cdots, x_n)$. On U, the metric takes the form $g_{ij} = \sum_{ij} g_{ij} dx_i dx_j$ for $g_{ij} = g_{ji} \in C^\infty(U)$. Now for any $p \in U$, let $(e_1, \cdots, e_n)$ be the orthonormal basis of T_pM compatible with the orientation.

One define

$$\nu(p) := e_1^* \wedge \cdots \wedge e_n^*$$

as above.

Notice

$$\frac{\partial}{\partial x_i}(p) = \sum_{j=1}^n b_{ij} e_j \qquad B = (b_{ij}) \in \mathrm{GL}(n, \mathbb{R}) \qquad \det(B) > 0$$

One first compute using that $(e_1, \cdots, e_n)$ gives an orthonormal basis

$$\begin{aligned} g_{ij}(p) &= \langle \frac{\partial}{\partial x_i}(p), \frac{\partial}{\partial x_j}(p) \rangle \\ &= \langle \sum_k b_{ik} e_k, \sum_\ell b_{j\ell} e_\ell \rangle \\ &= \sum_{k,\ell} b_{ik} b_{j\ell} \delta_{k\ell} \\ &= \sum_k b_{ik} b_{jk} = (BB^T)_{ij} \end{aligned}$$

Now we evaluate $\nu(p)$ via the definition

$$\nu(p)(\frac{\partial}{\partial x_1}(p), \cdots, \frac{\partial}{\partial x_n}(p)) = \nu(p)\left(\sum_j b_{1j}e_1, \cdots, \sum_j b_{nj}e_n\right)$$
$$= \det(B)\nu(p)(e_1, \cdots, e_n) = \det(B)$$

so that using $\det(g_{ij}(p)) = \det(BB^T) = (\det B)^2$

$$\nu(p) = \det(B)(dx_1 \wedge \cdots \wedge dx_n)$$
$$= \sqrt{\det(g_{ij})}(dx_1 \wedge \cdots \wedge dx_n)(p)$$

Now on U with $g = \sum_{ij} g_{ij}dx_i dx_j$, one has defined volume form

$$\nu = \sqrt{\det(g_{ij})}dx_1 \wedge \cdots \wedge dx_n$$

We write $\nu_g = \nu$. □

Volume Vol(R)

Definition 2.1.6 (Volume)**.** *Now let $R \subseteq M$ be an open connected subset whose closure is compact. Suppose R is contained in U open neighborhood with coordinates $\phi = (x_1, \cdots, x_n)$. We define the volume* Vol(R) *of R by the integral in $\mathbb{R}^n$*

$$\text{Vol}(R) := \int_{\phi(R)} \sqrt{\det(g_{ij})}dx_1 \cdots dx_n$$

Note the expression is well-defined. Indeed, if R is contained in another coordinate neighborhood V with parametrization $\psi = (y_1, \cdots, y_n)$ that is compatible with the orientation, and denote

$$h_{ij}(p) = h_p(\frac{\partial}{\partial y_i}(p), \frac{\partial}{\partial y_j}(p))$$

Then

$$\sqrt{\det(g_{ij})}(p) = \nu(p)(\frac{\partial}{\partial x_1}(p), \cdots, \frac{\partial}{\partial x_n}(p))$$
$$= \det(\frac{\partial x_i}{\partial y_j})\nu(p)(\frac{\partial}{\partial y_1}(p), \cdots, \frac{\partial}{\partial y_n}(p))$$
$$= \det(\frac{\partial x_i}{\partial y_j})\sqrt{\det(h_{ij})}(p)$$

so that

$$\int_{\phi(R)} \sqrt{\det(g_{ij})}dx_1 \cdots dx_n = \int_{\psi(R)} \sqrt{\det(h_{ij})}dy_1 \cdots dy_n$$

Example 2.1.2. *Consider $\mathbb{S}^2(r) = r^2(d\phi^2 + \sin^2(\phi)d\theta^2)$ with $(\phi, \theta) = (x_1, x_2)$. Here*

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2(\phi) \end{pmatrix} \implies \det(g) = r^4\sin^2(\phi)$$

So $\nu = \sqrt{\det(g)}d\phi \wedge d\theta = r^2\sin(\phi)d\phi \wedge d\theta$. Hence

$$\text{Vol}(\mathbb{S}^2(r), g_{can}^{\mathbb{S}^2(r)}) = \int_0^{2\pi}\int_0^{\pi} r^2 \sin\phi \, d\phi d\theta = 4\pi r^2$$

Length A Riemannian Metric can be used to calculate the lengths of curves.

Definition 2.1.7 (Length)**.** *For (M, g) Riemannian manifold, let $\gamma : [a, b] \to M$ be a C^∞ curve for $-\infty < a < b < \infty$. Notice for any $t \in (a, b)$, $\gamma'(t) \in T_{\gamma(t)}M$.*

We define the length of the curve γ on the segment $[a, b]$ as

$$\ell_g(r) := \int_a^b |\gamma'(t)| \, dt = \int_a^b \sqrt{\langle\gamma'(t), \gamma'(t)\rangle_{\gamma(t)}}dt \tag{2.5}$$

where

$$|\gamma'(t)|_{g(\gamma(t))} = \sqrt{\langle\gamma'(t), \gamma'(t)\rangle_{\gamma(t)}} = \sqrt{g(\gamma(t))(\gamma'(t), \gamma'(t))}$$

Recall $f : (M, g) \to (N, h)$ is isometric immersion if for any $p \in M$,

$$\langle v_1, v_2 \rangle_p = \langle df_p(v_1),\, df_p(v_2) \rangle_{f(p)}$$

Here the former inner product is defined by $g(p)$, while and the latter is defined by $h(f(p))$.

Then for any $\gamma : [a, b] \to M$ a C^∞ curve, $f \circ \gamma : [a, b] \to N$ is also C^∞ curve. Moreover

$$\ell_g(\gamma) = \ell_h(f \circ \gamma)$$

Example 2.1.3. *Consider the upper half plane*

$$H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

Consider $g_0 = dx^2 + dy^2$ Euclidean metric and $h = \frac{dx^2+dy^2}{y^2}$ the hyperbolic metric.

Consider the two curves

$$\begin{aligned} \gamma_1 : [x_0, x_1] &\to H & \gamma_2 : [y_0, y_1] &\to H \\ t &\mapsto (t, y_0) & t &\mapsto (x_0, t) \end{aligned}$$

with curve velocity

$$\gamma_1'(t) = \frac{\partial}{\partial x}(\gamma_1(t)) \qquad \gamma_2'(t) = \frac{\partial}{\partial y}(\gamma_2(t))$$

Then

$$\begin{aligned}
g_0(x, y)\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}, c\frac{\partial}{\partial x} + d\frac{\partial}{\partial y}\right) &= ac + bd \\
h(x, y)\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}, c\frac{\partial}{\partial x} + d\frac{\partial}{\partial y}\right) &= \frac{ac + bd}{y^2} \\
|\gamma_1'(t)|_{g_0} &= 1 = |\gamma_2'(t)|_{g_0} \\
|\gamma_1'(t)|_h &= \sqrt{\frac{1}{y_0^2}} = \frac{1}{y_0} \\
|\gamma_2'(t)|_h &= \frac{1}{t} \\
\ell_{g_0}(\gamma_1) &= \int_{x_0}^{x_1} |\gamma_1'(t)|_{g_0}\, dt = \int_{x_0}^{x_1} dt = x_1 - x_0 \\
\ell_{g_0}(\gamma_2) &= \int_{y_0}^{y_1} |\gamma_2'(t)|_{g_0}\, dt = \int_{y_0}^{y_1} dt = y_1 - y_0 \\
\ell_h(\gamma_1) &= \int_{x_0}^{x_1} \frac{dt}{y_0} = \frac{x_1 - x_0}{y_0} \\
\ell_h(\gamma_2) &= \int_{y_0}^{y_1} \frac{dt}{t} = \log(y_1) - \log(y_0) = \log(\frac{y_1}{y_0})
\end{aligned}$$

Let $\lambda > 0$. Define rescaling

$$\begin{aligned} \phi_\lambda : H &\to H \\ (x, y) &\mapsto (\lambda x,\, \lambda y) \end{aligned}$$

Compute the pullback of g_0

$$\begin{aligned}
\phi^* x &= \lambda x \\
\phi^* dx &= \lambda dx \\
\phi_\lambda^* g_0 &= \phi_\lambda^*(dx^2 + dy^2) = \lambda^2(dx^2 + dy^2) = \lambda^2 g_0 \\
\ell_{g_0}(\phi_\lambda \circ \gamma) &= \lambda \ell_{g_0}(\gamma)
\end{aligned}$$

On the other hand

$$\phi_\lambda^* h = \phi_\lambda^*\left(\frac{dx^2 + dy^2}{y^2}\right) = \frac{\lambda^2 dx^2 + \lambda^2 dy^2}{\lambda^2 y^2} = h$$

Hence for any $\lambda > 0$, $\phi_\lambda : (H, h) \to (H, h)$ is an isometry.

Distance More generally if $\gamma : [a, b] \to [a, b]$ is a piecewise C^∞ curve s.t. $\gamma : [a, b] \to M$ is continuous. i.e., let $a = t_0 < t_1 < \cdots < t_{k-1} < t_k = b$ we have

$$\gamma|_{[t_i,t_{i+1}]} \in C^\infty([t_i, t_{i+1}], M) \qquad i = 0, \cdots, k$$

Then $\gamma'(t_i^+)$ and $\gamma'(t_i^-)$ exist. One may define the piecewise curve length

$$\ell_g(\gamma) := \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} |\gamma'(t)|_g \, dt$$

Definition 2.1.8 (Distance)**.** *Let (M, g) be a connected Riemannian manifold.*

Then for any p, $q \in M$, there exists $\gamma : [0, 1] \to M$ piecewise C^∞ curve s.t.

$$\gamma(0) = p \qquad \gamma(1) = q$$

We define the distance between p, q determined by g to be

$$d_g(p, q) := \inf\{\ell_g(t) \mid \gamma : [0, 1] \to M \text{ piecewise smooth with } \gamma(0) = p, \gamma(1) = q\} \in [0, \infty)$$

Then

- $d_g(p, q) = d_g(q, p)$
- $d_g(p, p) = 0$
- $d_g(p, q) + d_g(q, r) \geq d_g(p, r)$.

In fact, if M is Hausdorff, then $d_g(p, q) = 0$ implies that $p = g$. In that case (M, d_g) is a metric space.

We examine the notorious example.

Example 2.1.4 (Bugged-eyed Line)**.** *Let*

$$M = (\mathbb{R} \times \{0, 1\}) \, / \, ((x, 0) \sim (x, 1) \text{ except for } x = 0)$$

Consider the Euclidean metric dx^2 on $\mathbb{R}$. Define $\pi : \mathbb{R} \times \{0, 1\} \to M$ as the projection. Then there exists a unique metric g on M s.t. $\pi^* g = dx^2$.

Now $[0, 0] \neq [0, 1]$ in M, whereas $d_g([0, 0], [0, 1]) = 0$.

We notice isometry preserves length.

Lemma 2.1.1. *If $f : (M_1, g_1) \to (M_2, g_2)$ is an isometry, then*

$$d_{g_2}(f(p), f(q)) = d_{g_1}(p, q) \qquad \forall\, p, q \in M_1$$

Proposition 2.1.3. *For x, $y \in \mathbb{R}^n$ with $g_0 = dx_1^2 + \cdots + dx_n^2$*

$$d_{g_0}(x, y) = |x - y| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Proof. $d_{g_0}(x, x) = 0$. Suppose $x \neq y$, let $d = |x - y| > 0$. Then there exists $A \in O(n)$ s.t. upon rotation, $A(x - y) = (d, 0, \cdots, 0)$. Then since translation by y is an isometry and that rotation by $O(n)$ is isometry

$$\begin{aligned} d_{g_0}(x, y) &= d_{g_0}(x - y, 0) = d_{g_0}(A(x - y), 0) = d_{g_0}((d, 0, \cdots, 0), 0) \\ &= d_{g_0}((0, \cdots, 0), (d, 0, \cdots, 0)) \end{aligned}$$

It remains to show that $d_{g_0}((0, \cdots, 0), (d, 0, \cdots, 0)) = d$. Consider smooth curve

$$\begin{aligned} \gamma : [0, 1] &\overset{C^\infty}{\to} \mathbb{R}^n \\ t &\mapsto (x_1(t), \cdots, x_n(t)) \end{aligned}$$

s.t.

$$\gamma(0) = (0, \cdots, 0), \qquad \gamma(1) = (d, 0, \cdots, 0)$$

Then

$$\begin{aligned}
\ell_{g_0}(\gamma) &= \int_0^1 |\gamma'(t)|_{g_0}\, dt = \int_0^1 \sqrt{x_1'(t)^2 + \cdots + x_n'(t)^2}\, dt \geq \int_0^1 |x_1'(t)|\, dt \\
&\geq \int_0^1 x_1'(t)\, dt = d - 0 = d \\
&= \ell_{g_0}(\gamma_0)
\end{aligned}$$

where $\gamma_0(t) = (dt, 0, \cdots, 0)$, so $\gamma(0) = 0$ and $\gamma(1) = (d, 0, \cdots, 0)$.

In fact if $\phi : (\mathbb{R}^n, g_0) \to (\mathbb{R}^n, g_0)$ is any isometry, then

$$|\phi(x) - \phi(y)| = |x - y|$$

□

2.2 Riemannian Connection

Let (M, g) be a Riemannian Manifold of dimension n. Consider the tangent bundle. Recall Affine Connection as in Definition 1.21.2.

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$
$$(X, Y) \mapsto \nabla_X Y$$

2.2.1 Symmetry and Compatibility with the Metric

Symmetric Affine Connection

Definition 2.2.1 (Symmetric affine connection). *An affine connection ∇ on a smooth manifold M is symmetric if for any X, $Y \in \mathfrak{X}(M)$*

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

Symmetric Affine Connection in Local Coordinates. Recall as in (1.66) with $e_j = \frac{\partial}{\partial x_j}$. The Christoffel symbols are defined as

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

Now that ∇ is symmetric implies

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} - \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} = [\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$$

Thus expanding the LHS gives

$$\sum_k \left(\Gamma_{ij}^k - \Gamma_{ji}^k\right) \frac{\partial}{\partial x_k} = 0$$

Hence that ∇ is symmetric is equivalent to saying

$$\Gamma_{ij}^k = \Gamma_{ji}^k \tag{2.6}$$

Notice the above has nothing to do with the metric g.

Compatibility with the Metric

Definition 2.2.2 (Compatible with metric). *An affine connection ∇ on a Riemannian manifold (M, g) is compatible with the Riemannian metric g if for any X, Y, $Z \in \mathfrak{X}(M)$ we have*

$$Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \tag{2.7}$$

where $g(X, Y) \in C^\infty(M)$. In fact, compatibility with the metric in equivalent to

$$\nabla_Z g = 0 \qquad \forall\ Z \in \mathfrak{X}(M) \tag{2.8}$$

Proposition 2.2.1 (Equivalence with Compatibility with Metric; [dC92] Proposition 3.2). *Let $\frac{D}{dt}$ be defined along $c : I \to M$ smooth curve by an affine connection ∇ on M, which is compatible with a Riemannian metric g on M.*

For V, W smooth vector fields along $c : I \to M$, i.e., V, $W \in C^\infty(I, c^*TM)$, the metric inner product writes

$$\langle V, W\rangle(t) = (g(c(t)) (V(t), W(t))$$

where $\langle V, W\rangle \in C^\infty(I)$. Then we have

$$\frac{d}{dt}\langle V, W\rangle(t) = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle \tag{2.9}$$

In fact

(i) ∇ is compatible with g iff (2.9) holds.

(ii) In particular, ∇ is compatible with g implies whenever V, W are parallel, we have

$$\langle V, W\rangle = constant \tag{2.10}$$

The converse holds as well.

Proof. Choose orthonormal basis $\{P_1(t_0), \cdots, P_n(t_0)\}$ of $T_{c(t_0)}M$ for $t_0 \in I$. Using Proposition 1.22.1, one may extend the vectors $P_i(t_0)$ along c via parallel transport. Assume (2.10) holds, then

$$\{P_1(t), \cdots, P_n(t)\} \qquad \text{is an orthonormal basis of } T_{c(t)}M \text{ for any } t \in I$$

Therefore one may write

$$V = \sum_i v^i P_i$$
$$W = \sum_i w^i P_i$$

where v^i, w^i are differentiable functions on I. It follows that

$$\frac{DV}{dt} = \sum_{i=1}^n \frac{dv^i}{dt} P_i$$
$$\frac{DW}{dt} = \sum_{i=1}^n \frac{dw^i}{dt} P_i$$

This is because using the Leibniz rule (1.89) and that P_i are parallel

$$\frac{DV}{dt} = \frac{D}{dt}\left(\sum_i v^i P_i\right) = \sum_i \frac{dv^i}{dt} P_i + \cancel{\sum_i v^i \frac{DP_i}{dt}}$$

Therefore

$$\begin{aligned}\langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle &= \sum_{i=1}^n \left(\frac{dv^i}{dt} w^i + \frac{dw^i}{dt} v^i\right) \\ &= \frac{d}{dt}\left(\sum_{i=1}^n v^i w^i\right) = \frac{d}{dt}\langle V, W\rangle = 0\end{aligned}$$

□

Symmetry and Compatibility under Pullback Let M be smooth manifold of dimension n and (N, h) be Riemannian manifold of dimension m. Let $F : M \overset{\infty}{\to} (N, h)$. Recall its pushforward (1.81) is defined via

$$\begin{aligned} F_* : \mathfrak{X}(M) &\to C^\infty(M, F^*TN) \\ X &\mapsto F_* X \end{aligned}$$

where

$$\begin{aligned} F_* X : M &\to F^*TN \\ p &\mapsto dF_p(X(p)) \in T_{F(p)}N = (F^*TN)_p \end{aligned}$$

Let ∇ be affine connection on N. Denote $D := F^*\nabla$

$$\begin{aligned} D : \mathfrak{X}(M) &\to C^\infty(M, F^*TN) \\ X &\mapsto D_X \end{aligned}$$

as the pullback connection on M in F^*TN as in (1.84).

We first need to study what are the objects

$$D_X(F_*Y) = (F^*\nabla)_X(F_*Y) \qquad \forall\ X,\ Y \in \mathfrak{X}(M)$$

or for $Y \in \mathfrak{X}(M)$

$$D(F_*Y) = (F^*\nabla)(F_*Y) \in C^\infty(M, T^*M \otimes F^*TN) = \Omega^1(M, F^*TN)$$

Proposition 2.2.2. *Under the above setting.*

(i) If ∇ is symmetric , then

$$D_X(F_*Y) - D_Y(F_*X) = F^*\nabla_X(F_*Y) - F^*\nabla_Y(F_*X) = F_*([X, Y]) \qquad \forall\ X,\ Y \in \mathfrak{X}(M) \tag{2.11}$$

(ii) If ∇ is compatible with the Riemannian metric h then

$$X\langle V, W\rangle = \langle D_X V, W\rangle + \langle V, D_X W\rangle \qquad \forall\ X \in \mathfrak{X}(M), \quad \forall\ W, V \in C^\infty(M, F^*TN) \tag{2.12}$$

Proof. Let $\phi : U \to \mathbb{R}^n$ be a chart on M with coordinates $(x_1, \dots, x_n)$ and let $(V, (y_1, \dots, y_m))$ be a chart on N with $F(U) \subset V$. Write

$$X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}, \qquad Y = \sum_{i=1}^n b_i \frac{\partial}{\partial x_i}.$$

For each i, for any $p \in U$ (in analog with (1.90))

$$\left(F_* \frac{\partial}{\partial x_i}\right)(p) = dF_p\left(\frac{\partial}{\partial x_i}(p)\right) = \sum_{j=1}^m \frac{\partial y_j}{\partial x_i}(p)\, \frac{\partial}{\partial y_j}(F(p)).$$

On the other hand, for any $V \in C^\infty(M, F^*TN)$ we write in local coordinates (in analog with (1.91))

$$V(p) = \sum_{j=1}^m V^j(p) \frac{\partial}{\partial y_j}(F(p)) = \sum_{j=1}^m V^j(p) \left(F^* \frac{\partial}{\partial y_j}\right)(p)$$

Hence, using $Y = \sum_i b_i \frac{\partial}{\partial x_i} \in \mathfrak{X}(M)$, so $F_*Y \in C^\infty(M, F^*TN)$, one write

$$\begin{aligned} F_*Y(p) &= \sum_{i=1}^n b_i(p)\, F_*\Big(\frac{\partial}{\partial x_i}\Big)(p) \\ &= \sum_{i=1}^n \sum_{j=1}^m b_i(p) \frac{\partial y_j}{\partial x_i}(p)\, \frac{\partial}{\partial y_j}(F(p)). \end{aligned} \tag{2.13}$$

Therefore the coordinate components are

$$(F_*Y)^j(p) = \sum_{i=1}^n b_i(p)\, \frac{\partial y_j}{\partial x_i}(p) = Y(y_j \circ F)(p). \tag{2.14}$$

Now let $\Gamma^k_{\ell r}$ be the Christoffel symbols of ∇ in the $(y_1, \dots, y_m)$ chart, i.e.

$$\nabla_{\frac{\partial}{\partial y_\ell}} \frac{\partial}{\partial y_r} = \sum_{k=1}^m \Gamma^k_{\ell r} \frac{\partial}{\partial y_k}.$$

Then by the defining property of pullback connection (1.84) for any $X = \sum_i a_i \frac{\partial}{\partial x_i}$,

$$D_X V = \sum_{k=1}^m \left(X(V^k) + \sum_{\ell, r=1}^m \Gamma^k_{\ell r}(F(p))\, X(y_\ell \circ F)\, V^r \right) \frac{\partial}{\partial y_k}(F(p)). \tag{2.15}$$

(Here $X(y_\ell \circ F) = \sum_i a_i \frac{\partial}{\partial x_i}(y_\ell \circ F)$.)

Symmetry. Apply (2.15) to $V = F_*Y$. Using (2.14), we have $(F_*Y)^k = Y(y_k \circ F)$, hence

$$D_X(F_*Y) = \sum_{k=1}^m \left(X\big(Y(y_k \circ F)\big) + \sum_{\ell, r=1}^m \Gamma^k_{\ell r}(F)\, X(y_\ell \circ F)\, Y(y_r \circ F) \right) \frac{\partial}{\partial y_k}(F). \tag{2.16}$$

Similarly,

$$D_Y(F_*X) = \sum_{k=1}^m \left(Y\big(X(y_k \circ F)\big) + \sum_{\ell, r=1}^m \Gamma^k_{\ell r}(F)\, Y(y_\ell \circ F)\, X(y_r \circ F) \right) \frac{\partial}{\partial y_k}(F). \tag{2.17}$$

Subtract (2.17) from (2.16):

$$\begin{aligned} D_X(F_*Y) - D_Y(F_*X) = \sum_{k=1}^m \Big(& X(Y(y_k \circ F)) - Y(X(y_k \circ F)) \\ & + \sum_{\ell, r=1}^m \Gamma^k_{\ell r}(F) \big[X(y_\ell \circ F)Y(y_r \circ F) - Y(y_\ell \circ F)X(y_r \circ F)\big] \Big) \frac{\partial}{\partial y_k}(F). \end{aligned}$$

If ∇ is symmetric, then $\Gamma^k_{\ell r} = \Gamma^k_{r \ell}$, so the Γ–term vanishes because the bracketed factor is antisymmetric in (ℓ, r). Therefore,

$$D_X(F_*Y) - D_Y(F_*X) = \sum_{k=1}^m \big([X, Y](y_k \circ F)\big)\, \frac{\partial}{\partial y_k}(F) = F_*([X, Y]),$$

because $F_*([X,Y])$ has components $([X,Y](y_k \circ F))_{k=1}^m$ in the $\frac{\partial}{\partial y_k}$ frame. This proves (2.11).

Compatibility. Let h be a Riemannian metric on N with local components $h_{kr}(y)$ in $(y_1,\dots,y_m)$:

$$h\Big(\frac{\partial}{\partial y_k},\frac{\partial}{\partial y_r}\Big) = h_{kr}(y).$$

The induced (pullback) inner product on F^*TN is

$$\langle V,W\rangle(p) = h_{kr}(F(p))\, V^k(p)\, W^r(p) \qquad \text{(Einstein summation)}.$$

Differentiate:

$$\begin{aligned} X\langle V,W\rangle &= X\big(h_{kr}(F)\, V^k W^r\big) \\ &= X(h_{kr}(F))\, V^k W^r + h_{kr}(F)\, X(V^k)\, W^r + h_{kr}(F)\, V^k\, X(W^r). \end{aligned} \tag{2.18}$$

Next compute $\langle D_X V,W\rangle + \langle V, D_X W\rangle$ using (2.15):

$$\begin{aligned} \langle D_X V,W\rangle &= h_{kr}(F)\,(D_X V)^k\, W^r \\ &= h_{kr}(F)\Big(X(V^k) + \Gamma^k_{\alpha\beta}(F)\, X(y_\alpha\circ F)\, V^\beta\Big)W^r, \end{aligned} \tag{2.19}$$

$$\begin{aligned} \langle V,D_X W\rangle &= h_{kr}(F)\, V^k\,(D_X W)^r \\ &= h_{kr}(F)\, V^k\Big(X(W^r) + \Gamma^r_{\alpha\beta}(F)\, X(y_\alpha\circ F)\, W^\beta\Big). \end{aligned} \tag{2.20}$$

Adding (2.19) and (2.20) gives

$$\begin{aligned} \langle D_X V,W\rangle + \langle V,D_X W\rangle &= h_{kr}(F)\, X(V^k)\, W^r + h_{kr}(F)\, V^k\, X(W^r) \\ &\quad + X(y_\alpha\circ F)\Big(h_{kr}(F)\Gamma^k_{\alpha\beta}(F)\, V^\beta W^r + h_{kr}(F)\Gamma^r_{\alpha\beta}(F)\, V^k W^\beta\Big). \end{aligned} \tag{2.21}$$

So (2.12) will follow if we show the remaining terms match:

$$X(h_{kr}(F))\, V^k W^r = X(y_\alpha\circ F)\Big(h_{sr}(F)\Gamma^s_{\alpha k}(F)\, V^k W^r + h_{ks}(F)\Gamma^s_{\alpha r}(F)\, V^k W^r\Big).$$

But $X(h_{kr}(F)) = X(y_\alpha\circ F)\,\partial_\alpha h_{kr}(F)$ by the chain rule, hence it suffices to use the coordinate form of $\nabla h = 0$:

$$\partial_\alpha h_{kr} = h_{sr}\Gamma^s_{\alpha k} + h_{ks}\Gamma^s_{\alpha r}. \tag{2.22}$$

Substituting (2.22) into (2.18) shows that (2.18) equals (2.21), i.e.

$$X\langle V,W\rangle = \langle D_X V,W\rangle + \langle V,D_X W\rangle,$$

which is (2.12). □

Symmetry and Compatibility as ∇ acting on $(0,2)$-tensors

Lemma 2.2.1. *Let ∇ be affine connection on a smooth manifold M. Then ∇ is symmetric iff for any $f \in C^\infty(M)$, the $(0,2)$-tensor ∇df is symmetric, i.e.*

$$(\nabla df)(X,Y) = (\nabla df)(Y,X) \qquad \forall\ X,\ Y\in\mathfrak{X}(M) \tag{2.23}$$

Proof. Using (1.106), since $df\in\Omega^1(M)$ for any $f\in C^\infty(M)$, for any $X,\ Y\in\mathfrak{X}(M)$, using Definition (1.108)

$$\begin{aligned} (\nabla df)(Y,X) &:= \nabla_X df(Y) = X(df(Y)) - df(\nabla_X Y) \\ &\overset{(1.43)}{=} X(Y(f)) - (\nabla_X Y)(f) \end{aligned}$$

Now assume ∇ is symmetric.

$$\begin{aligned} (\nabla df)(Y,X) &= X(Y(f)) - (\nabla_X Y)(f) = X(Y(f)) - ((\nabla_Y X)(f) - [Y,X](f)) \\ &= X(Y(f)) - X(Y(f)) + Y(X(f)) - (\nabla_Y X)(f) \\ &= Y(X(f)) - (\nabla_Y X)(f) = (\nabla df)(X,Y) \end{aligned}$$

On the other hand assume $(\nabla df)(Y,X) = (\nabla df)(X,Y)$. Then

$$\begin{aligned} 0 &= (\nabla df)(Y,X) - (\nabla df)(X,Y) = (X(Y(f)) - (\nabla_X Y)(f)) - (Y(X(f)) - (\nabla_Y X)(f)) \\ &= [X,Y](f) + \nabla_Y X(f) - \nabla_X Y(f) \qquad \forall\ f\in C^\infty(M) \end{aligned}$$

□

Lemma 2.2.2. *Let (M, g) be Riemannian manifold with ∇ an affine connection. Then ∇ is compatible with g iff*

$$\nabla g = 0$$

as an answer to (2.8).

Proof. For any X, Y, $Z \in \mathfrak{X}(M)$

$$\begin{aligned}(\nabla g)(X, Y, Z) &\overset{(1.108)}{=} (\nabla_Z g)(X, Y) \\ &\overset{(1.104)}{=} Z(g(X, Y)) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) \overset{(2.7)}{=} 0 \\ \nabla g = 0\end{aligned}$$

□

2.2.2 Levi-Civita Connction

Levi-Civita Connection

Theorem 2.2.1 (Levi-Civita)**.** *Let (M, g) be a Riemannian manifold. Then there exists a unique affine connection ∇ on M which is symmetric and compatible with the metric g.*

Such connection is called the Levi-Civita Connection.

Proof. Assume such connection exists. Let's show uniqueness. Take any X, Y, $Z \in \mathfrak{X}(M)$, if we have compatibility with the metric g, then

$$\begin{aligned}X(g(Y, Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Y(g(Z, X)) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ Z(g(X, Y)) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y)\end{aligned}$$

Now add up first two and subtract the third, using g is symmetric tensor, and then using ∇ is symmetric affine connection

$$\begin{aligned}X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) &= g(\nabla_X Y + \nabla_Y X, Z) + g(Y, \nabla_X Z - \nabla_Z X) + g(X, \nabla_Y Z - \nabla_Z Y) \\ &= 2g(\nabla_Y X, Z) + g(Z, \nabla_X Y - \nabla_Y X) + g(Y, \nabla_X Z - \nabla_Z X) + g(X, \nabla_Y Z - \nabla_Z Y) \\ &= 2g(\nabla_Y X, Z) + g(Z, [X, Y]) + g(Y, [X, Z]) + g(X, [Y, Z])\end{aligned}$$

Then

$$g(\nabla_Y X, Z) = \frac{1}{2}\left(X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) - g(Y, [X, Z]) - g(X, [Y, Z]) - g(Z, [X, Y])\right)$$

This uniquely determines $\nabla_Y X$ for any X, $Y \in \mathfrak{X}(M)$.

Now for Existence, we define $\nabla_Y X$ as above and check that ∇ is symmetric and compatible with the Riemannian metric g. □

Here we record our usual expression for Levi-Civita Connection.

$$g(\nabla_X Y, Z) = \frac{1}{2}\left(X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) - g(Y, [X, Z]) - g(X, [Y, Z]) + g(Z, [X, Y])\right) \quad (2.24)$$

Levi-Civita in Local Coordinates Let $Y = \frac{\partial}{\partial x_i}$, $X = \frac{\partial}{\partial x_j}$ and $Z = \frac{\partial}{\partial x_k}$ as in (2.24). Then making use of (1.66) with $e_j = \frac{\partial}{\partial x_j}$ so that

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \Gamma_{ij}^{k} \frac{\partial}{\partial x_k} \quad (2.25)$$

One write

$$\text{LHS of (2.24)} = g(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}) = g(\sum_{\ell=1}^{n} \Gamma_{ij}^{\ell} \frac{\partial}{\partial x_\ell}, \frac{\partial}{\partial x_k}) = \sum_{\ell=1}^{n} \Gamma_{ij}^{\ell} g_{\ell k}$$

$$\begin{aligned}\text{RHS of (2.24)} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}) + \frac{\partial}{\partial x_i} g(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j}) - \frac{\partial}{\partial x_k} g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) - g(\frac{\partial}{\partial x_i}, 0) - g(\frac{\partial}{\partial x_j}, 0) - g(\frac{\partial}{\partial x_k}, 0)\right) \\ &= \frac{1}{2}\left(g_{ik,j} + g_{kj,i} - g_{ij,k}\right)\end{aligned}$$

where $g_{ij,k} := \frac{\partial g_{ij}}{\partial x_k}$. Hence matching both sides gives

$$\Gamma_{ij}^{\ell} = \frac{1}{2}\sum_{k=1}^{n} g^{\ell k}\left(g_{ik,j} + g_{kj,i} - g_{ij,k}\right) \tag{2.26}$$

where $(g^{\ell k}(x)) := (g_{\ell k}(x))^{-1}$ is understood as the inverse matrix. The matrix is invertible because

$$g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$$

is symmetric, positive definite.

Christoffel Symbol Examples for Levi-Civita

Example 2.2.1. *Consider the flat metric* $(\mathbb{R}^n, g = dx_1^2 + \cdots dx_n^2)$ *where* $g_{ij} = \delta_{ij}$. *Then* $g_{ij,k} = 0$ *with*

$$\Gamma_{ij}^{\ell} = 0, \qquad \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0, \qquad \nabla \frac{\partial}{\partial x_j} = 0$$

Then for $c : I \to \mathbb{R}^n$ *smooth curve with* $c(t) = (x_1(t), \cdots, x_n(t))$, *and any* C^∞ *vector field along* c

$$V(t) = \sum_{j=1}^{n} V^j(t)\frac{\partial}{\partial x_j}(c(t))$$

Plugging in (1.92) *we see*

$$\frac{DV}{dt}(t) = \sum_{j=1}^{n} \frac{dV^j}{dt}(t)\frac{\partial}{\partial x_j}(c(t))$$

Thus V *is parallel* $\frac{DV}{dt} = 0$ *iff all coordinate components vanish* $\frac{dV^j}{dt}(t) = 0$.

Example 2.2.2. *Consider* $(\mathbb{S}^2, g_{\text{can}} = d\phi^2 + \sin^2(\phi)d\theta^2)$. *For spherical coordinates* $\theta \in (0, 2\pi)$ *and* $\phi \in (0, \pi)$.

$$(x, y, z) = (\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$$

And $(x_1, x_2) = (\phi, \theta)$.

Metric, its Inverse and the derivatives. *We have*

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\phi) \end{pmatrix}$$

Inverting gives

$$(g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2(\phi)} \end{pmatrix}$$

Thus $g_{ij} = 0$ *for any* $i \neq j$ *and* $g^{kk} = \frac{1}{g_{kk}}$. *In particular* $g_{22,1} = 2\sin(\phi)\cos(\phi)$ *and* $g_{ij,k} = 0$ *otherwise.*

Compute Christoffel Symbols Γ_{ij}^k. *Using* (2.26) *we compute*

$$\begin{aligned}
\Gamma_{11}^1 &= \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{21}^1 = \Gamma_{22}^2 = 0 \\
\Gamma_{12}^2 &= \frac{1}{2}\sum_{k=1}^{2}\left(g^{2k}(g_{1k,2} + g_{k2,1} - g_{12,k})\right) = \frac{1}{2g_{22}}\frac{\partial}{\partial \phi}g_{22} \\
&= \frac{1}{2}\frac{\partial}{\partial \phi}\log(\sin^2(\phi)) = \frac{\cos(\phi)}{\sin(\phi)} = \cot(\phi) = \Gamma_{21}^2 \\
\Gamma_{22}^1 &= \frac{1}{2}g^{11}(0 + 0 - g_{22,1}) = -\frac{1}{2}\frac{\partial}{\partial \phi}(\sin^2(\phi)) = -\sin(\phi)\cos(\phi)
\end{aligned}$$

Compute $\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}$. *Using* (2.25) *we derive relations*

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial \phi}}\frac{\partial}{\partial \phi} &= \Gamma_{11}^1\frac{\partial}{\partial \phi} + \Gamma_{11}^2\frac{\partial}{\partial \theta} \\
\nabla_{\frac{\partial}{\partial \phi}}\frac{\partial}{\partial \theta} &= \nabla_{\frac{\partial}{\partial \theta}}\frac{\partial}{\partial \phi} = \Gamma_{12}^1\frac{\partial}{\partial \phi} + \Gamma_{12}^2\frac{\partial}{\partial \theta} \\
\nabla_{\frac{\partial}{\partial \theta}}\frac{\partial}{\partial \theta} &= \Gamma_{22}^1\frac{\partial}{\partial \phi} + \Gamma_{22}^2\frac{\partial}{\partial \theta}
\end{aligned}$$

Thus

$$\nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\phi} = \Gamma^1_{11}\frac{\partial}{\partial\phi} + \Gamma^2_{11}\frac{\partial}{\partial\theta} = 0$$
$$\nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\theta} = \nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\phi} = \Gamma^1_{12}\frac{\partial}{\partial\phi} + \Gamma^2_{12}\frac{\partial}{\partial\theta} = \cot(\phi)\frac{\partial}{\partial\theta}$$
$$\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\theta} = \Gamma^1_{22}\frac{\partial}{\partial\phi} + \Gamma^2_{22}\frac{\partial}{\partial\theta} = -\sin(\phi)\cos(\phi)\frac{\partial}{\partial\phi}$$

Compute connection 1-forms ω^k_j. *With our Christoffel Symbols, we use local coordinates* (1.68)

$$\omega^k_j = \sum_{i=1}^n \Gamma^k_{ij} dx_i$$

to read off $\omega^1_1 = 0$, $\omega^2_1 = \cot(\phi)d\theta$, $\omega^1_2 = -\sin(\phi)\cos(\phi)d\theta$ *and* $\omega^2_2 = \cot(\phi)d\phi$*. The connection* 1*-form writes*

$$\begin{pmatrix} \omega^1_1 & \omega^1_2 \\ \omega^2_1 & \omega^2_2 \end{pmatrix} = \begin{pmatrix} 0 & -\sin(\phi)\cos(\phi)d\theta \\ \cot(\phi)d\theta & \cot(\phi)d\phi \end{pmatrix} \in \Omega^1(U, \mathfrak{gl}(2,\mathbb{R}))$$

Compute $\nabla\frac{\partial}{\partial x_j}$. *Hence for* (1.67) *with* $e_j = \frac{\partial}{\partial x_j}$

$$\nabla\frac{\partial}{\partial x_j} = \sum_{k=1}^2 \omega^k_j \otimes \frac{\partial}{\partial x_k}$$

we have

$$\nabla\frac{\partial}{\partial\phi} = d\phi\otimes\nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\phi} + d\theta\otimes\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\phi} = (\cot(\phi)d\theta)\otimes\frac{\partial}{\partial\theta}$$
$$\nabla\frac{\partial}{\partial\theta} = d\phi\otimes\nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\theta} + d\theta\otimes\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\theta} = (\cot(\phi)d\phi)\otimes\frac{\partial}{\partial\theta} - \sin(\theta)\cos(\theta)d\theta\otimes\frac{\partial}{\partial\phi}$$

Connection one-form for Orthonormal Frame We can use the Orthonormal frame (2.1) to compute connection one-form for $\mathbb{S}^2$.

$$\{\frac{\partial}{\partial\phi}, \frac{1}{\sin(\phi)}\frac{\partial}{\partial\theta}\}$$

Using Leibniz rule (1.61)

$$\nabla_1 = \nabla_{\frac{\partial}{\partial x_1}} = \nabla_{\frac{\partial}{\partial\phi}}$$
$$\nabla_2 = \nabla_{\frac{\partial}{\partial x_2}} = \nabla_{\frac{\partial}{\partial\theta}}$$
$$e_1 := \frac{\partial}{\partial\phi}$$
$$e_2 := \frac{1}{\sin(\phi)}\frac{\partial}{\partial\theta}$$
$$\nabla_1 e_1 = \nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\phi} = 0$$
$$\nabla_1 e_2 = \nabla_{\frac{\partial}{\partial\phi}}\left(\frac{1}{\sin(\phi)}\frac{\partial}{\partial\theta}\right) \overset{(1.61)}{=} -\frac{\cos(\phi)}{\sin^2(\phi)}\frac{\partial}{\partial\theta} + \frac{1}{\sin(\phi)}\nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\theta} = 0$$
$$\nabla_2 e_1 = \nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\phi} = \cot(\phi)\frac{\partial}{\partial\theta} = \cos(\phi)e_2$$
$$\nabla_2 e_2 = \nabla_{\frac{\partial}{\partial\theta}}\left(\frac{1}{\sin(\phi)}\frac{\partial}{\partial\theta}\right) \overset{(1.61)}{=} \frac{1}{\sin(\phi)}\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\theta} = \frac{1}{\sin(\phi)}(-\sin(\phi)\cos(\phi)\frac{\partial}{\partial\phi}) = -\cos(\phi)e_1$$

For $\nabla e_j = \sum_{k=1}^2 \tilde{\omega}^k_j \otimes e_k$, since

$$\nabla e_1 = d\phi\otimes\nabla_{\frac{\partial}{\partial\phi}}e_1 + d\theta\otimes\nabla_{\frac{\partial}{\partial\theta}}e_1 = d\theta\otimes\nabla_2 e_1 = \cos(\phi)d\theta\otimes e_2$$
$$\nabla e_2 = d\phi\otimes\nabla_1 e_2 + d\theta\otimes\nabla_2 e_2 = -\cos(\phi)d\theta\otimes e_1$$

One has

$$[\nabla e_1, \nabla e_2] = [e_1, e_2]\begin{pmatrix} 0 & -\cos(\phi) \\ \cos(\phi) & 0 \end{pmatrix} d\theta$$

and so our connection one-form $\tilde{\omega}$ writes

$$\begin{pmatrix} \tilde{\omega}_1^1 & \tilde{\omega}_2^1 \\ \tilde{\omega}_1^2 & \tilde{\omega}_2^2 \end{pmatrix} = \begin{pmatrix} 0 & -\cos(\phi)d\theta \\ \cos(\phi)d\theta & 0 \end{pmatrix} \in \Omega^1(U, \mathfrak{so}(2))$$

Proposition 2.2.3. *In general if $e_1, \cdots, e_n$ are local* ***orthonormal frame*** (2.1) *of $TM|_U = TU$, and ∇ is an affine connection compatible with the Riemannian metric, then*

$$\omega_j^k = -\omega_k^j$$

and $\omega \in \Omega^1(U, \mathfrak{so}(n))$.

Proof. Since e_i are orthonormal, $\langle e_i, e_j \rangle = \delta_{ij}$. Thus

$$0 = d\langle e_i, e_j \rangle = \langle \nabla e_i, e_j \rangle + \langle e_i, \nabla e_j \rangle$$

But $\nabla e_i = \sum_{k=1}^n \omega_i^k \otimes e_k$, so plugging in yields

$$0 = \omega_i^j + \omega_j^i$$

□

2.3 Geodesics

Let (M, g) be a Riemannian manifold of dimension n.

Definition 2.3.1 (Geodesic). *Let $\gamma : I \subseteq \mathbb{R} \to M$ be C^∞ curve. We say γ is geodesic at $t_0 \in I$ if*

$$\frac{D}{dt}\frac{d\gamma}{dt}(t_0) = 0 \in T_{\gamma(t_0)}M$$

where $\frac{D}{dt} = \gamma^*\nabla_{\frac{\partial}{\partial t}}$ is the covariant derivative defined by the pullback of Levi-Civita connection on (M, g) under γ itself.

We say γ is geodesic if

$$\frac{D}{dt}(\frac{d\gamma}{dt}) = \nabla_{\frac{d\gamma}{dt}}\frac{d\gamma}{dt} = \nabla_{\dot\gamma}\dot\gamma \equiv 0$$

In other words, geodesics are curves whose curve velocity are parallel w.r.t. the covariant derivative along itself (via Levi-Civita Connection). In particular, the size of curve velocity remains constant.

Lemma 2.3.1. *If $\gamma : I \to M$ is a geodesic in a Riemannian manifold (M, g) then*

$$|\gamma'| := |\frac{d\gamma}{dt}| = \sqrt{g(t)(\frac{d\gamma}{dt}(t), \frac{d\gamma}{dt}(t))} = constant$$

Proof. Using $\frac{D}{dt}$ defined by Levi-civita connection, which is compatible with the metric, (2.9)

$$\frac{d}{dt}\langle\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle = \langle\frac{D}{dt}\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle + \langle\frac{d\gamma}{dt}, \frac{D}{dt}\frac{d\gamma}{dt}\rangle = 0$$

□

Geodesic in Local Coordinates and Local Existence and Uniqueness Let (U, ϕ) for $\phi = (x_1, \cdots, x_n)$ be C^∞ chart on M. On U we have

$$\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} = \sum_k \Gamma^k_{ij}\frac{\partial}{\partial x_k}$$

where (2.26)

$$\Gamma^\ell_{ij} = \frac{1}{2}\sum_k g^{\ell k}\left(g_{ik,j} + g_{kj,i} - g_{ij,k}\right)$$

WLOG assume

$$\gamma : I \to U \stackrel{\phi}{\to} \mathbb{R}^n$$

Then in view of local coordinates representation (1.90) and (1.91), for any $V \in C^\infty(I, \gamma^*TM)$

$$\begin{aligned}
\phi\circ\gamma(t) &= (x_1(t), \cdots, x_n(t)) \\
\gamma'(t) &= \sum_k \frac{dx_k}{dt}(t)\frac{\partial}{\partial x_k}(\gamma(t)) \\
V(t) &= \sum_{k=1}^n V^k(t)\frac{\partial}{\partial x_k}(\gamma(t)) \\
\frac{DV}{dt}(t) &\stackrel{(1.92)}{=} \sum_{k=1}^n\left(\frac{dV^k}{dt}(t) + \sum_{i,j=1}^n \Gamma^k_{ij}(\gamma(t))\frac{dx_i}{dt}(t)V^j(t)\right)\frac{\partial}{\partial x_k}(\gamma(t))
\end{aligned}$$

Now in particular, we pick the smooth vector field along γ to be its curve velocity

$$V(t) = \gamma'(t) \equiv \frac{d\gamma}{dt}$$

By matching coefficients we have $V^k(t) = \frac{dx_k}{dt}(t)$. Thus the equations for covariant derivative in local coordinates write

$$\frac{D}{dt}\frac{d\gamma}{dt} = 0 \iff \frac{d^2x_k}{dt^2} + \sum_{i,j=1}^n \Gamma^k_{ij}\circ\gamma\frac{dx_i}{dt}\frac{dx_j}{dt} = 0 \qquad \forall\ k = 1\cdots n \tag{2.27}$$

This is a system of 2nd order ODEs in $x_1(t), \cdots, x_n(t)$. Denote

$$y_i(t) := \frac{dx_i}{dt}(t)$$

Then they satisfy

$$\begin{aligned} \frac{dx_k}{dt} &= y_k \\ \frac{dy_k}{dt} &= -\sum_{i,j=1}^{n} \Gamma_{ij}^k \circ \gamma y_i y_j \end{aligned} \tag{2.28}$$

This is a system of 1st order ODE in $x_1(t), \cdots, x_n(t)$ and $y_1(t), \cdots, y_n(t)$. Hence there exists unique solution if given initial data $a_i, b_i \in \mathbb{R}$

$$\begin{aligned} x_i(t_0) &= a_i \\ y_i(t_0) &= b_i = \frac{dx_i}{dt}(t_0) \end{aligned}$$

or in other words

$$\begin{aligned} \gamma(t_0) &= \phi^{-1}(a_1, \cdots, a_n) =: p \\ \gamma'(t_0) &= \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}(p) \end{aligned}$$

Theorem 2.3.1 (Existence and Uniqueness Theory for Geodesic)**.** *Let (M, g) be a Riemannian manifold. Given any $p \in M$ and $v \in T_pM$*

- *There exists a geodesic $\gamma : I \to M$ s.t. $0 \in I$, $\gamma(0) = p$ and $\gamma'(0) = v$.*
- *If $\beta : I' \to M$ is a geodesic s.t. $\beta(0) = p$, $\beta'(0) = v$ then we must have*

$$I' \subseteq I, \qquad \beta = \gamma|_{I'}$$

Examples for Geodesic

Example 2.3.1. *Let $(\mathbb{R}^n, g_0 = dx_1^2 + \cdots dx_n^2)$ be flat metric. Then*

$$g_{ij} = \delta_{ij} \qquad \Gamma_{ij}^k = 0$$

Hence using (2.27)

$$\frac{D}{dt}\gamma'(t) = 0 \iff \frac{d^2 x_k}{dt^2} = 0$$

so for

$$\begin{aligned} \gamma : I &\to \mathbb{R}^n \\ t &\mapsto (x_1(t), \cdots, x_n(t)) \end{aligned}$$

Given any $a \in \mathbb{R}^n$ and $b \in T_a\mathbb{R}^n \cong \mathbb{R}^n$ the unique geodesic $\gamma(t)$ with $\gamma(0) = a$ and $\gamma'(0) = b$ writes

$$\gamma(t) = a + bt \qquad t \in \mathbb{R}$$

Example 2.3.2. *Consider canoncial metric $(\mathbb{S}^n, g_{\text{can}})$. Given $p \in \mathbb{S}^n$ and $v \in T_p\mathbb{S}^n$. Recall*

$$(p, v) \in T\mathbb{S}^n \subseteq T\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$$

for $|p| = 1$ and $\langle p, v\rangle = 0$. The unique geodesic $\gamma(t)$ (unit-speed great circle through p with initial velocity v) in $(\mathbb{S}^n, g_{can})$ is given by

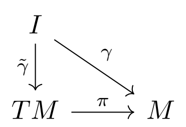
$$\gamma(t) = \begin{cases} p & \text{if } v = 0 \\ \cos(|v|t)p + \sin(|v|t)\frac{v}{|v|} & \text{if } v \neq 0 \end{cases} \tag{2.29}$$

2.3.1 Geodesic Field and Geodesic Flow

For $\gamma : I \to M$ smooth curve in M and V a C^∞ vector field along γ, the tuple

$$\begin{aligned} \tilde{\gamma} : I &\to TM \\ t &\mapsto (\gamma(t), V(t)) \end{aligned} \tag{2.30}$$

defines a smooth curve in TM s.t. the diagram commutes



We prescribe initial data $\gamma(0) = p$ and $\gamma'(0) = v$ for $(p, v) \in TM$. Now γ is a geodesic in (M, g), i.e.,

$$\frac{D}{dt}\frac{d}{dt}\gamma = 0$$

iff $\gamma(t)$ and $V(t)$ satisfy

$$\begin{aligned} \gamma'(t) &= V(t) \\ \frac{DV}{dt}(t) &= 0 \\ \tilde{\gamma}(0) &= (p, v) \end{aligned}$$

2.3.1.1 Geodesic Field

Now for any $(p, v) \in TM$, define $G(p, v) \in T_{(p,v)}(TM)$ as follows.

Definition 2.3.2 (Geodesic Field). *Take any $(p, v) \in TM$. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be the unique geodesic in (M, g) s.t. $\gamma(0) = p$, $\gamma'(0) = v$. Let $\tilde{\gamma}$ denote the lifted curve (2.30)*

$$\begin{aligned} \tilde{\gamma} : (-\varepsilon, \varepsilon) &\to TM \\ t &\mapsto (\gamma(t), \gamma'(t)) \end{aligned}$$

Now we define the geodesic field as

$$\begin{aligned} G : TM &\to T(TM) \\ (p, v) &\mapsto \tilde{\gamma}'(0) \in T_{\tilde{\gamma}(0)}(TM) = T_{(p,v)}(TM) \end{aligned} \tag{2.31}$$

where γ is the unique geodesic in (M, g) with $\gamma(0) = p$, $\gamma'(0) = v$.

In the following we make sense of the definition (2.31).

Well-definedness of Geodesic Field $G \in \mathfrak{X}(TM)$ Let (U, ϕ) with coordinates $\phi = (x_1, \cdots, x_n)$ be C^∞ chart for M.

We first lift coordinate chart into TM. We lift the chart to TM via $(\pi^{-1}(U), \tilde{\phi})$

$$\begin{aligned} \tilde{\phi} : \pi^{-1}(U) \subseteq TM &\to \phi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n} \\ (p, v) &\mapsto (x_1, \cdots, x_n, y_1, \cdots, y_n) \end{aligned}$$

where $p \in U$ and $\{y_i\}$ denotes the components w.r.t. basis fixed by $\{\frac{\partial}{\partial x_i}\}$

$$v = \sum_{i=1}^n y_i \frac{\partial}{\partial x_i}(p) \in T_pM$$

In particular, $\tilde{\phi}$ is related to ϕ via

$$\tilde{\phi}(p, v) = (\phi(p), y_1, \cdots, y_n)$$

Also note

$$\phi \circ \gamma(t) = (x_1(t), \cdots, x_n(t))$$

implies

$$\tilde{\phi} \circ \tilde{\gamma}(t) = (x_1(t), \cdots, x_n(t), y_1(t), \cdots, y_n(t))$$

Geodesic Field in Local Coordinates. Hence writing into equations

$$\begin{aligned} G(\tilde{\gamma}(t)) &\overset{(2.31)}{:=} \frac{d\tilde{\gamma}}{dt}(t) \overset{(1.90)}{=} \sum_{i=1}^n \frac{dx_i}{dt}(t) \frac{\partial}{\partial x_i}(\tilde{\gamma}(t)) + \sum_{k=1}^n \frac{dy_k}{dt}(t) \frac{\partial}{\partial y_k}(\tilde{\gamma}(t)) \\ &\overset{(2.28)}{=} \sum_{i=1}^n \frac{dx_i}{dt}(t) \frac{\partial}{\partial x_i}(\tilde{\gamma}(t)) - \sum_{i,j,k=1}^n (\Gamma_{ij}^k \circ \gamma)(t) y_i(t) y_j(t) \frac{\partial}{\partial y_k}(\tilde{\gamma}(t)) \end{aligned}$$

On $\pi^{-1}(U)$ we have

$$\left\{ \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_n} \right\}$$

as C^∞ frame of $T(TM)|_{\pi^{-1}(U)}$.

Hence

$$G = \sum_{k=1}^n y_k \frac{\partial}{\partial x_k} - \sum_{i,j,k=1}^n (\Gamma_{ij}^k \circ \phi^{-1}(x_1, \cdots, x_n)) y_i y_j \frac{\partial}{\partial y_k} \tag{2.32}$$

Since all coefficients are smooth functions over $\pi^{-1}(U)$, this justifies that G is indeed a C^∞ vector field on TM.

Notice that the geodesic field G is unique using $\tilde{\gamma}$ is a unique solution of the system of differential equations.

2.3.1.2 Geodesic Flow

Given our Geodesic Field G as in (2.31). For any $(p, v) \in TM$, using Theorem 1.15.1, there exists $\delta > 0$ and an open neighborhood TU of (p, v) in TM s.t. a flow ϕ (1.29) exists

$$\begin{aligned}\phi : (-\delta, \delta) \times TU &\overset{C^\infty}{\to} TM \\ (t, q, w) &\mapsto \phi(t, q, w)\end{aligned}$$

s.t.

$$\begin{cases} \frac{\partial}{\partial t}\phi(t, q, w) = G(\phi(t, q, w)) \\ \phi(0, q, w) = (q, w) \end{cases} \qquad \forall\ (t, q, w) \in (-\delta, \delta) \times TU \tag{2.33}$$

Such ϕ is called a *Geodesic Flow.*

Recover Geodesic via Geodesic Flow Using the geodesic flow, one may construct geodesics in M using any initial data in the neighborhood $\mathcal{U}$ of (p, v). This is the same procedure as recovering the integral curve from the flow.

Denote π as the projection for our tangent bundle

$$\begin{aligned}\pi : TM &\to M \\ (p, v) &\mapsto p\end{aligned}$$

Let ϕ be our geodesic flow (2.33). Then define

$$\begin{aligned}\gamma := \pi \circ \phi : (-\delta, \delta) \times \mathcal{U} &\to M \\ (t, q, w) &\mapsto \pi(\phi(t, q, w))\end{aligned} \tag{2.34}$$

Now for fixed $(q, w) \in \mathcal{U} \subseteq TM$ s.t. $q \in M$ and $w \in T_qM$, from γ we may recover the geodesic

$$\begin{aligned}\gamma_{q,w} : (-\delta, \delta) &\to M \\ t &\mapsto \gamma(t, q, w)\end{aligned} \tag{2.35}$$

with $\gamma_{q,w}(0) = q$ as initial position and $\gamma'_{q,w}(0) = w$ as initial velocity.

In fact γ and ϕ are related via

$$\phi(t, p, v) = (\gamma(t, p, v), \frac{\partial}{\partial t}\gamma(t, p, v)) \qquad \forall\ (t, p, v) \in (-\delta, \delta) \times \mathcal{U}$$

Abuse of notation, sometimes we call γ as in (2.34) also as geodesics.

Local Existence and Uniqueness of Geodesic under Geodesic Flow Fix $p \in M$. It is in fact able to choose the neighborhood $\mathcal{U} \subseteq TM$ in (2.34)(as the domain of definition for geodesic flow) in the form

$$\mathcal{U}_{V,\varepsilon_1} = \{(q, v) \in TM \mid q \in V,\ v \in T_qM,\ |v| < \varepsilon_1\} \tag{2.36}$$

where V is an open neighborhood of $p \in M$. This controls the size of velocity within ε_1.

Proposition 2.3.1 ([dC92] Proposition 3.2.5)**.** *Given $p \in M$, there exists*

1. *an open set $V \subseteq M$ s.t. $p \in V$*
2. *an open interval $(-\delta, \delta) \subseteq \mathbb{R}$ around 0*
3. *a parameter $\varepsilon_1 > 0$*

and a C^∞ mapping as in (2.34)

$$\begin{aligned}\gamma = \pi \circ \phi : (-\delta, \delta) \times \mathcal{U}_{V,\varepsilon_1} &\to M \\ (t, q, v) &\mapsto \gamma(t, q, v)\end{aligned} \tag{2.37}$$

s.t. the restriction as in (2.35)

$$\begin{aligned}\gamma_{q,v} : (-\delta, \delta) &\to M \\ t &\mapsto \gamma(t, q, v)\end{aligned}$$

is the unique geodesic of M, which at $t = 0$ passes through q with velocity v, for each $(q, v) \in \mathcal{U}_{V,\varepsilon_1}$ as in (2.36).

Proposition 2.3.2. *If (M, g) is compact Riemannin manifold. Then the geodesic flow ϕ and geodesic γ is globally defined on $\mathbb{R} \times TM$.*

$$\begin{aligned}\phi &: \mathbb{R} \times TM \to TM \\ \gamma &: \mathbb{R} \times TM \to M\end{aligned}$$

Examples of Geodesic Field and Geodesic Flow

Example 2.3.3. *Consider* $(\mathbb{R}^n, g = dx_1^2 + \cdots dx_n^2)$ *flat metric. We know* $\Gamma_{ij}^k = 0$*. One identify* $T\mathbb{R}^n \cong \mathbb{R}^{2n}$ *so the geodesic field writes*

$$\begin{aligned} G : T\mathbb{R}^n = \mathbb{R}^{2n} &\to T(T\mathbb{R}^n) \\ (x, y) &\mapsto \sum_{k=1}^n y_k \frac{\partial}{\partial x_k} \end{aligned}$$

Solving ODEs give the geodesic flow

$$\begin{aligned} \phi : \mathbb{R} \times T\mathbb{R}^n &\to T\mathbb{R}^n \\ (t, x, y) &\mapsto (x + ty, y) \end{aligned}$$

along with nearby geodesics (2.34) *in* $\mathbb{R}^n$

$$\begin{aligned} \gamma : \mathbb{R} \times T\mathbb{R}^n &\to \mathbb{R}^n \\ (t, x, y) &\mapsto x + ty \end{aligned}$$

Example 2.3.4. *Consider* $(\mathbb{S}^n, g_{can})$*. We have* γ *geodesics*

$$\begin{aligned} \gamma : \mathbb{R} \times T\mathbb{S}^n &\to \mathbb{S}^n \\ (t, x, y) &\mapsto \begin{cases} x & \text{if } y = 0 \\ \cos(|y|t)x + \sin(|y|t)\frac{y}{|y|} & \text{if } y \neq 0 \end{cases} \end{aligned}$$

For geodesic flows, making use of

$$\phi(t, q, w) = (\gamma(t, q, w), \frac{\partial \gamma}{\partial t}(t, q, w))$$

so $|\frac{\partial \gamma}{\partial t}(t, q, w)| = |w|$*, we have*

$$\begin{aligned} \phi : \mathbb{R} \times T\mathbb{S}^n &\to T\mathbb{S}^n \\ (t, x, y) &\mapsto \begin{cases} (x, 0) & \text{if } y = 0 \\ (\cos(|y|t)x + \sin(|y|t)\frac{y}{|y|}, -\sin(|y|t)|y|x + \cos(|y|t)y) & \text{if } y \neq 0 \end{cases} \end{aligned}$$

Geodesic Flow preserves the sphere bundle

$$S_{|v|}(TM) = \{(p, v) \in TM \mid |v| = r\}$$

with $r > 0$*. The geodesic field* $G(p, v)$ *is tangent to* $S_{|v|}(TM)$*.*

2.3.2 Exponential Map

Homogeneity of Geodesic Proposition 2.3.1 asserts that for $|v| < \varepsilon_1$, the geodesic $\gamma(t, q, v)$ exists in an interval $(-\delta, \delta)$ and is unique.

In fact it is possible to **increase the velocity of a geodesic, by decreasing its interval of definition, and vice versa**. This is known as homogeneity.

Lemma 2.3.2 ([dC92] Lemma 3.2.6)**.** *Given* $p \in M$*, let* γ *be as in* (2.37)*. In particular*

$$\begin{aligned} \gamma : (-\delta, \delta) \times \mathcal{U}_{V,\varepsilon_1} &\to M \\ (t, q, v) &\mapsto \gamma(t, q, v) \end{aligned}$$

defines the geodesic

$$\begin{aligned} \gamma_{q,v} : (-\delta, \delta) &\to M \\ t &\mapsto \gamma(t, q, v) \end{aligned}$$

Then for any $a > 0$*,*

$$\begin{aligned} \gamma_{q,av} : (-\frac{\delta}{a}, \frac{\delta}{a}) &\to M \\ t &\mapsto \gamma(t, q, av) \end{aligned}$$

is defined on the interval $(-\frac{\delta}{a}, \frac{\delta}{a})$ *and*

$$\gamma_{q,av}(t) \equiv \gamma(t, q, av) = \gamma(at, q, v) \equiv \gamma_{q,v}(at) \qquad \forall\ (t, q, v) \in (-\frac{\delta}{a}, \frac{\delta}{a}) \times \mathcal{U}_{V,\varepsilon_1} \tag{2.38}$$

In particular, the domain of definition for γ *can be modified from* $(-\delta, \delta) \times \mathcal{U}_{V,\varepsilon_1}$ *to* $(-\frac{\delta}{a}, \frac{\delta}{a}) \times \mathcal{U}_{V,a\varepsilon_1}$*.*

Proof. Fix $(q,v) \in \mathcal{U}_{V,\varepsilon_1}$. Define a curve

$$\begin{aligned} h : (-\frac{\delta}{a}, \frac{\delta}{a}) &\to M \\ t &\mapsto \gamma(at,q,v) \end{aligned}$$

so that $h(0) = q$ and $\frac{dh}{dt}(0) = av$. Since $h'(t) = a\gamma'(at,q,v)$,

$$\frac{D}{dt}(\frac{dh}{dt}) = \nabla_{h'(t)} h'(t) = a^2 \nabla_{\gamma'(at,q,v)} \gamma'(at,q,v) = 0 \qquad \forall\, t \in (-\frac{\delta}{a}, \frac{\delta}{a})$$

using that γ is geodesic. Thus h is a geodesic over $(-\frac{\delta}{a}, \frac{\delta}{a})$, passing through q with velocity av at the instant $t = 0$. By uniqueness of geodesics Proposition 2.3.1 (or in fact Theorem 2.3.1)

$$h(t) = \gamma(at,q,v) = \gamma(t,q,av)$$

□

Extending Interval of Definition Uniformly Large Leveraging homogeneity, one is able to make the interval of existence for the geodesic uniformly large in a neighborhood of p.

Proposition 2.3.3 ([dC92] Proposition 3.2.7)**.** *For any $p \in M$, there exists*

1. *an open set $V \subseteq M$ s.t. $p \in V$*
2. *a number $\varepsilon > 0$*

and a C^∞ mapping as in (2.34)

$$\begin{aligned} \gamma : (-2,2) \times \mathcal{U}_{V,\varepsilon} &\to M \\ (t,q,w) &\mapsto \gamma(t,q,w) \end{aligned} \tag{2.39}$$

where

$$\mathcal{U}_{V,\varepsilon} := \{(q,w) \in TM \mid q \in V,\ w \in T_qM,\ |w| < \varepsilon\} \tag{2.40}$$

s.t. the restriction as in (2.35)

$$\begin{aligned} \gamma_{q,w} : (-2,2) &\to M \\ t &\mapsto \gamma(t,q,w) \end{aligned}$$

is the unique geodesic of M, which at $t = 0$ passes through q with velocity w, for any $(q,w) \in \mathcal{U}$.

Proof. The geodesic γ defined in Proposition 2.3.1 exists for $|t| < \delta$ and $|v| < \varepsilon_1$. By homogeneity Lemma 2.3.2, one modify the domain from

$$(-\delta,\delta) \times \mathcal{U}_{V,\varepsilon_1} \qquad \text{to} \qquad (-2,2) \times \mathcal{U}_{V,\frac{\delta}{2}\varepsilon_1}$$

so $\gamma(t,q,\frac{\delta v}{2})$ is defined for $t \in (-2,2)$, $(q,v) \in \mathcal{U}_{V,\varepsilon_1}$. Choosing $\varepsilon < \frac{\delta}{2}\varepsilon_1$, we obtain that the geodesic $\gamma(t,q,w)$ is defined for $|t| < 2$ and $|w| < \varepsilon$. □

Via a similar approach, one could make the curve velocity 1 by homogeneity

$$\gamma(1,q,v) = \gamma(v,q,\frac{v}{|v|})$$

Exponential Map Proposition 2.3.3 permits us to introduce the concept of exponential map.

Definition 2.3.3 (Exponential Map)**.** *Let $p \in M$. There exists $\mathcal{U}_{V,\varepsilon}$* (2.40) *and γ* (2.39) *as in Proposition 2.3.3. We define the exponential map on $\mathcal{U}_{V,\varepsilon}$ as*

$$\begin{aligned} \exp : \mathcal{U}_{V,\varepsilon} \subseteq TM &\to M \\ (q,v) &\mapsto \gamma(1,q,v) = \gamma(|v|,q,\frac{v}{|v|}) \end{aligned} \tag{2.41}$$

One may also consider its restriction to an open subset of the tangent space T_qM.

$$\begin{aligned} \exp_q : B_\varepsilon(0) \subseteq T_qM &\to M \\ v &\mapsto \exp(q,v) = \gamma(1,q,v) \end{aligned} \qquad \forall\, q \in V \tag{2.42}$$

It follows that both $\exp$, $\exp_q$ *are smooth functions, and that*

$$\exp_q(0) = q$$

Geometrically, $\exp_q(v)$ **is the point of M obtained by going out the length equal to $|v|$, starting from q, along a geodesic which passes through q with velocity $\frac{v}{|v|}$.**

Exponential Map as Diffeomorphism

Proposition 2.3.4 ([dC92] Proposition 3.2.9). *For any $q \in M$, there exists an $\varepsilon > 0$ s.t.* (2.42)

$$\begin{aligned} \exp_q : B_\varepsilon(0) \subseteq T_qM &\to M \\ v &\mapsto \exp(q, v) = \gamma(1, q, v) \end{aligned}$$

is a diffeomorphism of $B_\varepsilon(0)$ onto an open subset of M.

Proof. We want to compute the differential of $\exp_q$ at the origin.

$$(d\exp_q)_0 : T_0(T_qM) \cong T_qM \to T_qM$$

Once it's a linear isomorphism, by Inverse Function Theorem we can conclude that $\exp_q$ is a local diffeomorphism.

To do so, we adopt the definition of differential (1.28) as mapping between curve velocity. For $q \in M$, and for any $v \in T_qM$ with $|v| < \varepsilon$ (here ε is as in $\mathcal{U}_{V,\varepsilon}$ (2.40)) fixed, consider the curve $c(t) = tv$ with $c'(0) = v$.

$$\begin{aligned} c : (-2, 2) &\to T_qM \\ t &\mapsto tv \end{aligned}$$

Now $\exp_q$ as a smooth map between T_qM and M composed with the curve c gives

$$\begin{aligned} \exp_q(tv) : (-2, 2) &\to M \\ t &\mapsto \exp_q(tv) = \gamma(1, q, tv) \end{aligned}$$

whose curve velocity is computed via

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \exp_q(tv) &= \frac{\partial}{\partial t}\, \gamma(1, q, tv)|_{t=0} \\ &= \left.\frac{\partial}{\partial t}\gamma(t, q, v)\right|_{t=0} = v \end{aligned}$$

Now

$$(d\exp_q)_0(v) = (d\exp_q)_0(c'(0)) = \left.\frac{d}{dt}\right|_{t=0} \exp_q(tv) = v \tag{2.43}$$

which is to say, the differential of $\exp_q$ at the origin 0 is the identity map. This is indeed linear isomorphism between two tangent spaces. Hence $\exp_p : B_\varepsilon(0) \to M$ is a local diffeomorphism at the origin 0, i.e., there exists $\varepsilon > 0$ sufficiently small s.t.

$$\exp_q : B_\varepsilon(0) \subseteq T_qM \to \exp_q(B_\varepsilon(0)) \subseteq M$$

is a diffeomorphism. □

Examples for Exponential Map

Example 2.3.5. *For $M = \mathbb{R}^n$, the exponential maps are the translations*

$$\begin{aligned} \exp_p : T_p\mathbb{R}^n &\to \mathbb{R}^n \\ v &\mapsto p + v \end{aligned}$$

Example 2.3.6. *For $M = \mathbb{S}^n$, recall* (2.29)

$$\exp_p(v) = \begin{cases} p & v = 0 \\ \cos(|v|)p + \sin(|v|)\frac{v}{|v|} & v \neq 0 \end{cases}$$

This is diffeomorphism of $B_\pi(0)$ onto $\mathbb{S}^n \setminus \{-p\}$.

2.3.3 Gauss Lemma

We first define some preliminaries.

Piecewise Smooth curve

Definition 2.3.4. *A piecewise smooth curve is*

$$c : [a, b] \subseteq \mathbb{R} \to M$$
$$t \mapsto c(t)$$

s.t. there exists a partition $a = t_0 < t_1 < \cdots < t_{k-1} < t_k = b$ *where the restriction are smooth curve*

$$c|_{[t_i, t_{i+1}]} \qquad i = 0, \cdots, k-1$$

We say c joins points $c(a)$ *and* $c(b)$. $c(t_i)$ *is called a vertex of c, and the angle formed by* $\lim_{t \to t_i^+} c'(t)$, $\lim_{t \to t_i^-} c'(t)$ *is called the vertex angle at* $c(t_i)$.

One may extend idea of parallel transport to piecewise smooth curves. Given $V_0 \in T_{c(t)}M$, for $t \in [t_i, t_{i+1}]$, one may extend V_0 to obtain a parallel field $V(t)$ for $t \in [t_i, t_{i+1}]$. Now taking $V(t_i)$ and $V(t_{i+1})$ as new initial values, one extend $V(t)$ in a similar manner to $[t_{i-1}, t_{i+2}]$.

Parametrized Surface Let $A \subseteq \mathbb{R}^2$ be connected open subset. A *parametrized surface in M* is a smooth function

$$s : A \subseteq \mathbb{R}^2 \to M$$
$$(u, v) \mapsto s(u, v)$$

Let (u, v) be global coordinates on $\mathbb{R}^2$, then $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\} \in \mathfrak{X}(A)$.

A *vector field V along s* is a smooth mapping s.t. $V(q) \in T_{s(q)}M$ for any $q \in A$, as in Definition (1.83). We denote $\frac{\partial s}{\partial u}, \frac{\partial s}{\partial v} \in C^\infty(A, s^*TM)$ as the pullback vector fields of $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ along the parametrized surface s

$$\frac{\partial s}{\partial u} := s^* \frac{\partial}{\partial u} = ds(\frac{\partial}{\partial u})$$
$$\frac{\partial s}{\partial v} := s^* \frac{\partial}{\partial v} = ds(\frac{\partial}{\partial v})$$

so that for any $(u, v) \in A$

$$\frac{\partial s}{\partial u}(u, v), \ \frac{\partial s}{\partial v}(u, v) \ \in T_{s(u,v)}M = (s^*TM)_{(u,v)}$$

For ∇ an affine connection on M, we denote $D := s^*\nabla$ as the pullback connection (1.84). Denote $\frac{D}{du} := D_{\frac{\partial}{\partial u}}$, $\frac{D}{dv} := D_{\frac{\partial}{\partial v}}$ the covariant derivative as in Definition (1.88).

$$\frac{D}{du} : C^\infty(A, s^*TM) \to C^\infty(A, s^*TM)$$
$$V \mapsto \frac{DV}{du}$$

One may understand $\frac{DV}{du}(u, v_0)$ as the covariant derivative along the curve

$$u \mapsto s(u, v_0)$$

of the restriction of V to this curve. This defines $\frac{DV}{du}(u, v)$ for any $(u, v) \in A$.

Lemma 2.3.3 ([dC92] Lemma 3.3.4)**.** *If* ∇ *is a symmetric affine connection on M, then*

$$\frac{D}{dv}\frac{\partial s}{\partial u} = \frac{D}{du}\frac{\partial s}{\partial v} \tag{2.44}$$

Proof. Using (2.11)

$$\begin{aligned}\frac{D}{dv}\frac{\partial s}{\partial u} - \frac{D}{du}\frac{\partial s}{\partial v} &= D_{\frac{\partial}{\partial v}} s_* \frac{\partial}{\partial u} - D_{\frac{\partial}{\partial u}} s_* \frac{\partial}{\partial v} \\ &= s^*\nabla_{\frac{\partial}{\partial v}} s_* \frac{\partial}{\partial u} - s^*\nabla_{\frac{\partial}{\partial u}} s_* \frac{\partial}{\partial v} \\ &= s_* \left([\frac{\partial}{\partial v}, \frac{\partial}{\partial u}] \right) = 0\end{aligned}$$

Or we directly prove by hand. Let (U, ϕ) be C^∞ chart around point of $s(A)$ with coordinates

$$\phi \circ s(u, v) = (x_1(u, v), \cdots, x_n(u, v))$$

Then

$$
\begin{aligned}
\frac{D}{dv}(\frac{\partial s}{\partial u})(u,v) &= \frac{D}{dv}\left(\sum_i \frac{\partial x_i}{\partial u}(u,v)\frac{\partial}{\partial x_i}(s(u,v))\right) \\
&\overset{(1.89)}{=} \sum_i \left(\frac{\partial^2 x_i}{\partial v \partial u}(u,v)\frac{\partial}{\partial x_i}(s(u,v)) + \frac{\partial x_i}{\partial u}(u,v)\frac{D}{dv}\left(\frac{\partial}{\partial x_i}(s(u,v))\right)\right) \\
&= \sum_i \left(\frac{\partial^2 x_i}{\partial v \partial u}(u,v)\frac{\partial}{\partial x_i}(s(u,v)) + \frac{\partial x_i}{\partial u}(u,v)\nabla_{\sum_j \frac{\partial x_j}{\partial v}\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_i}(s(u,v))\right) \\
&\overset{(1.58)}{=} \sum_i \left(\frac{\partial^2 x_i}{\partial v \partial u}(u,v)\frac{\partial}{\partial x_i}(s(u,v)) + \sum_j \frac{\partial x_i}{\partial u}(u,v)\frac{\partial x_j}{\partial v}(u,v)\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_i}(s(u,v))\right)
\end{aligned}
$$

where we've used essentially (1.90)

$$ds_{(u,v)}(\frac{\partial}{\partial v}) = \sum_j \frac{\partial x_j}{\partial v}\frac{\partial}{\partial x_j}(s(u,v))$$

Now using that ∇ is symmetric thus

$$\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_i} = \nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}$$

one conclude

$$\frac{D}{dv}(\frac{\partial s}{\partial u})(u,v) = \frac{D}{du}(\frac{\partial s}{\partial v})(u,v)$$

□

Gauss Lemma In what follows we identify the tangent space to T_pM at $v \in T_pM$ with T_pM itself, and write $T_pM \cong T_v(T_pM)$.

Lemma 2.3.4 ([dC92] Lemma 3.3.5). *Let (M,g) be Riemannian manifold. Let $p \in M$, and $v \in T_pM$ so that $\exp_p(v)$ as in (2.42) is defined.*

Then

$$\langle (d\exp_p)_v(v), (d\exp_p)_v(w)\rangle_{\exp_p(v)} = \langle v, w\rangle_p \qquad \forall\ w \in T_pM \cong T_v(T_pM) \tag{2.45}$$

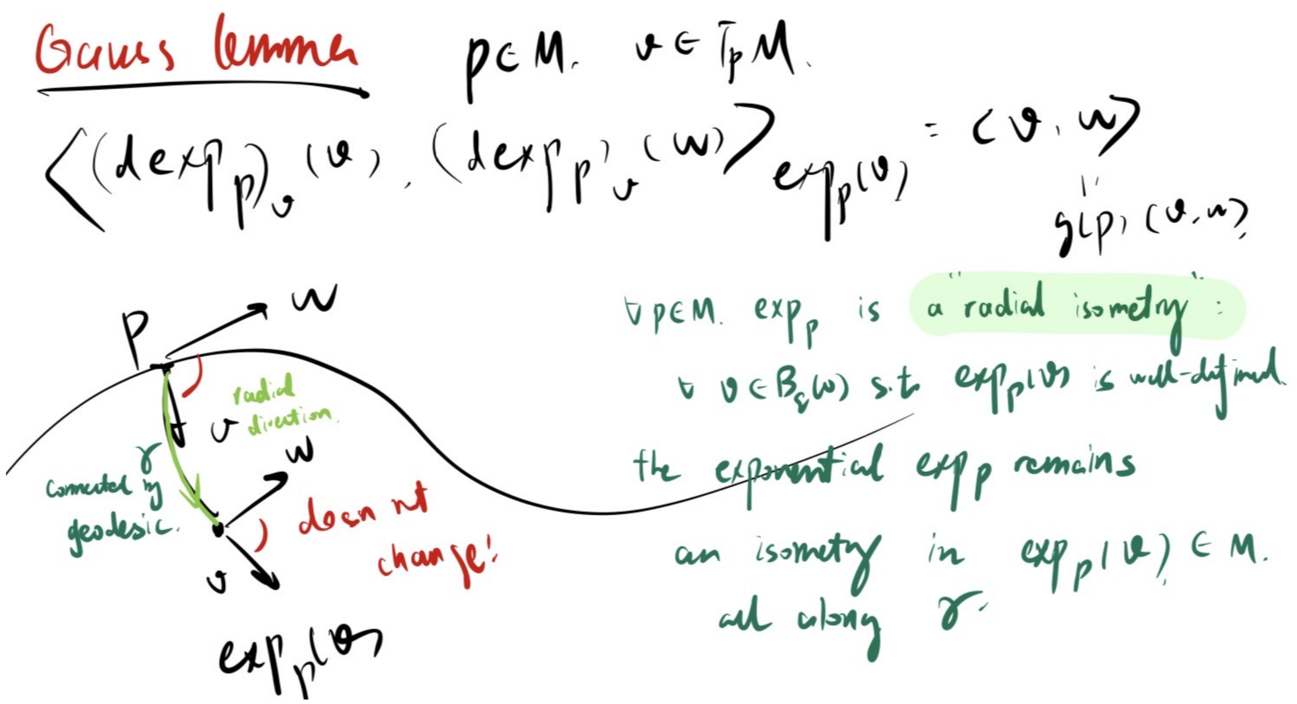


Figure 2.1: Gauss Lemma

Proof. Define for δ, $\varepsilon > 0$ sufficiently small.

$$f : (-\varepsilon, \varepsilon) \times (-\delta, 1+\delta) \to M$$
$$(s,t) \mapsto \exp_p(t(v+sw))$$

For any $s \in (-\varepsilon, \varepsilon)$ define f_s

$$f_s : (-\delta, 1+\delta) \to M$$
$$t \mapsto f(s,t) = \exp_p(t(v+sw))$$

Here f_s is geodesic with initial position $f_s(0) = p$ and initial velocity $f_s'(0) = v + sw$. Now using f_s is geodesic

$$\frac{D}{dt}\frac{\partial f}{\partial t}(s,t) = \frac{D}{dt} f_s'(t) = 0 \tag{2.46}$$

Also

$$\begin{aligned} \left\| \frac{\partial f}{\partial t}(s,t) \right\|^2 = \langle \frac{\partial f}{\partial t}(s,t), \frac{\partial f}{\partial t}(s,t) \rangle = \langle f_s'(t), f_s'(t) \rangle &\overset{(2.10)}{=} \langle f_s'(0), f_s'(0) \rangle \\ &= \langle v+sw, v+sw \rangle \\ &= \langle v,v \rangle + 2s\langle v,w \rangle + s^2 \langle w,w \rangle \end{aligned} \tag{2.47}$$

We differentiate

$$\begin{aligned} f(t,s) &= \exp_p(t(v+sw)) \\ \frac{\partial f}{\partial t}(t,s) &= (d\exp_p)_{t(v+sw)}(v+sw) \\ \frac{\partial f}{\partial s}(t,s) &= (d\exp_p)_{t(v+sw)}(tw) \\ \frac{\partial f}{\partial t}(t,0) &= (d\exp_p)_{tv}(v) \\ \frac{\partial f}{\partial s}(t,0) &= (d\exp_p)_{tv}(tw) \end{aligned}$$

Thus to recover LHS of (2.45), we care about

$$\frac{\partial f}{\partial t}(1,0) = (d\exp_p)_v(v), \qquad \frac{\partial f}{\partial s}(1,0) = (d\exp_p)_v(w)$$

We differentiate using compatibility with the Riemannian metric g , and that metric is symmetric (2.44)

$$\begin{aligned} \frac{\partial}{\partial t}\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \rangle (t,s) &\overset{(2.9)}{=} \langle \frac{D}{dt}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \rangle + \langle \frac{\partial f}{\partial t}, \frac{D}{dt}\frac{\partial f}{\partial s} \rangle \overset{(2.46)}{=} \langle \frac{\partial f}{\partial t}, \frac{D}{dt}\frac{\partial f}{\partial s} \rangle \\ &\overset{(2.44)}{=} \langle \frac{\partial f}{\partial t}, \frac{D}{ds}\frac{\partial f}{\partial t} \rangle \\ &\overset{(2.9)}{=} \frac{1}{2}\frac{\partial}{\partial s}\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \rangle \overset{(2.47)}{=} \frac{1}{2}\frac{\partial}{\partial s}\left(\langle v,v \rangle + 2s\langle v,w \rangle + s^2\langle w,w \rangle \right) \\ &= \langle v,w \rangle + s|w|^2 \\ \frac{\partial}{\partial t}\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \rangle (t,0) &= \langle v,w \rangle \end{aligned}$$

Thus upon integration

$$\langle \frac{\partial f}{\partial t}(1,0), \frac{\partial f}{\partial s}(1,0) \rangle - \langle \frac{\partial f}{\partial t}(0,0), \frac{\partial f}{\partial s}(0,0) \rangle = \int_0^1 \frac{\partial}{\partial t}\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \rangle (t,0)\, dt = \int_0^1 \langle v,w \rangle \, dt = \langle v,w \rangle$$

Conclude by noticing

$$\begin{aligned} \langle \frac{\partial f}{\partial t}(1,0), \frac{\partial f}{\partial s}(1,0) \rangle &= \langle (d\exp_p)_v(v), (d\exp_p)_v(w) \rangle \\ \langle \frac{\partial f}{\partial t}(0,0), \frac{\partial f}{\partial s}(0,0) \rangle &= 0 \end{aligned}$$

□

Normal(Geodesic) Ball If $\exp_p$ is a diffeomorphism of a neighborhood V of the origin in T_pM, then

$$U = \exp_p V$$

is called a *normal neighborhood of* p. For $B_\varepsilon(0)$ whose closure lies in V, we call

$$\exp_p B_\varepsilon(0) = B_\varepsilon(p)$$

the *normal ball* or *geodesic ball* with center p and radius $\varepsilon > 0$.

By Gauss Lemma (2.45), the boundary of a normal ball is a hypersurface (submanifold of codimension 1) in M orthogonal to the geodesics that start from p. We denote the boundary as

$$S_\varepsilon(p) := \partial B_\varepsilon(p) = \exp_p(\partial B_\varepsilon(0))$$

and we call it the *normal sphere* or *geodesic sphere* at p. The geodesics in $B_\varepsilon(p)$ that begin at p are referred to as *radial geodesics*.

Lemma 2.3.5. *$S_\varepsilon(p)$ is orthogonal to the geodesics that start from p.*

Proof. Fix $v \in \partial B_\varepsilon(0) \subseteq T_pM$ and set $q := \exp_p(v) \in \partial B_\varepsilon(p)$. Consider the radial geodesic

$$\gamma(t) := \exp_p(tv), \qquad \gamma(1) = q,$$

so that

$$\gamma'(1) = (d\exp_p)_v(v) \in T_qM.$$

To describe $T_q(\partial B_\varepsilon(p))$, let $v(s) \subseteq \partial B_\varepsilon(0)$ be any smooth curve with $v(0) = v$, and define

$$\alpha(s) := \exp_p(v(s)) \subseteq \partial B_\varepsilon(p), \qquad \alpha(0) = q.$$

By the chain rule,

$$\alpha'(0) = (d\exp_p)_v\big(v'(0)\big) \in T_q(\partial B_\varepsilon(p)).$$

Since $|v(s)| \equiv \varepsilon$, differentiating $|v(s)|^2 = \varepsilon^2$ at $s = 0$ gives

$$0 = \frac{d}{ds}\Big|_{s=0} \langle v(s), v(s)\rangle = 2\langle v, v'(0)\rangle, \qquad \text{hence} \;\; \langle v, v'(0)\rangle = 0.$$

Now apply Gauss Lemma (2.45) at v

$$\big\langle (d\exp_p)_v(v),\; (d\exp_p)_v(w)\big\rangle_{\exp_p(v)} = \langle v, w\rangle_p \qquad \forall\, w \in T_pM.$$

Taking $w = v'(0)$ yields

$$\langle \gamma'(1), \alpha'(0)\rangle_q = \big\langle (d\exp_p)_v(v),\; (d\exp_p)_v(v'(0))\big\rangle_q = \langle v, v'(0)\rangle_p = 0.$$

Since vectors $\alpha'(0)$ span $T_q(\partial B_\varepsilon(p))$, we conclude

$$\gamma'(1) \perp T_q(\partial B_\varepsilon(p)).$$

Thus $\partial B_\varepsilon(p)$ is orthogonal to the geodesics issuing from p. □

2.3.4 Minimizing Properties of Geodesics

Geodesics Locally Minimize Arclength Recall Arc-length ℓ is defined via (2.5)

Definition 2.3.5 (Minimizing Geodesic)**.** *A segment of the geodesic*

$$\gamma : [a, b] \to M$$

is called minimizing if

$$\ell(\gamma) \le \ell(c)$$

for any c piecewise smooth curve joining $\gamma(a)$ to $\gamma(b)$.

Recall the length is defined via (2.5).

Let (M, g) be Riemannian manifold.

Proposition 2.3.5 ([dC92] Proposition 3.3.6)**.** *For any $p \in M$, let U be a normal neighborhood of p in M, and $B_\delta(p) \subseteq U$ be a geodesic ball of radius $\delta > 0$.*

Let

$$\gamma : [0,1] \to B_\delta(p)$$

be a geodesic segment s.t.

$$\gamma(0) = p, \qquad \gamma(1) = q \neq p$$

Now for any $c : [0,1] \to M$ piecewise smooth curve in M joining $\gamma(0) = c(0) = p$ to $\gamma(1) = c(1) = q$,

$$\ell(\gamma) \leq \ell(c)$$

and if equality holds, $\gamma([0,1]) = c([0,1])$.

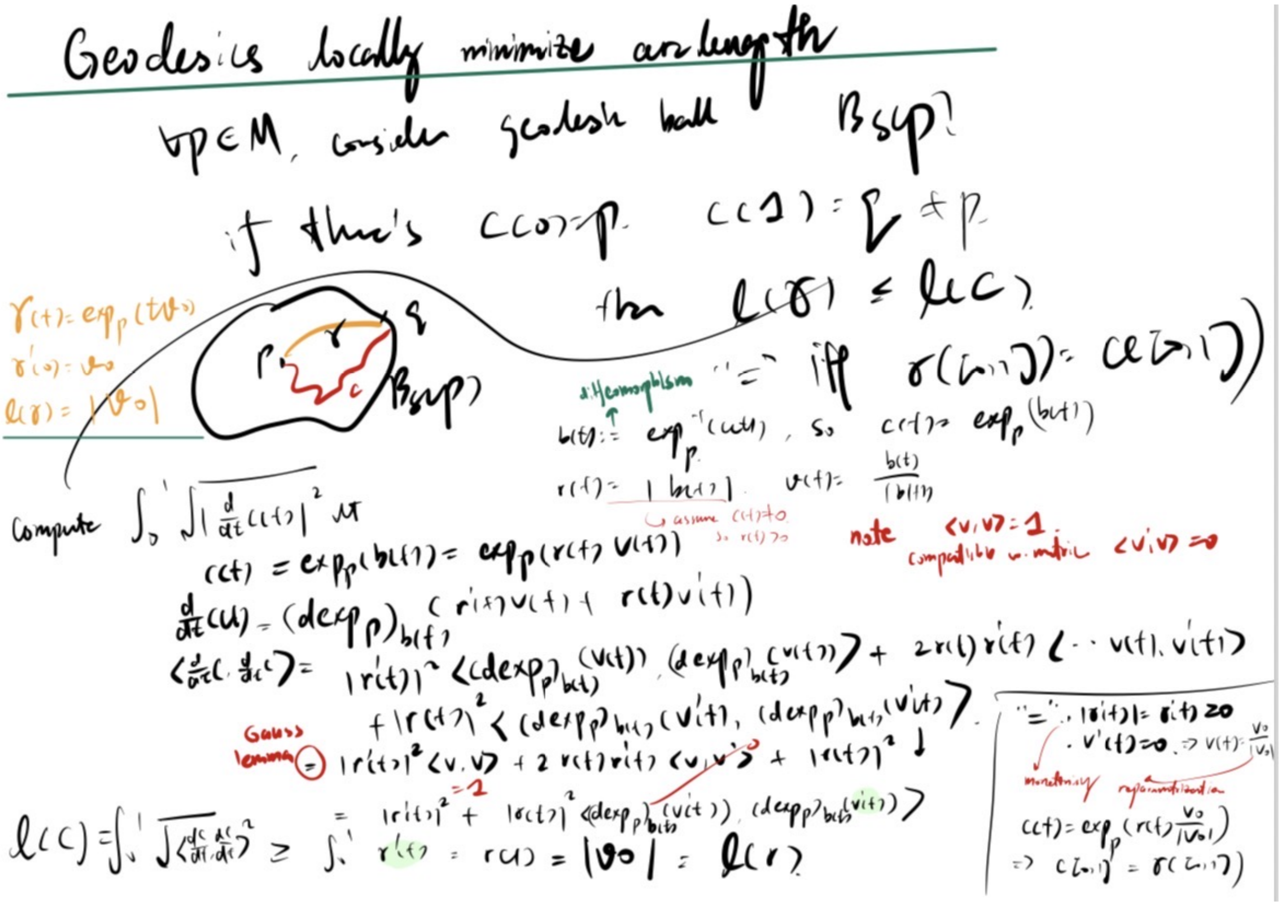


Figure 2.2: Geodesics Locally Minimize Arc-length

Proof. For the geodesic segment γ, if denote

$$\gamma'(0) = v_0 \in T_pM$$

we know

$$\gamma(t) = \exp_p(tv_0), \qquad \gamma(1) = q = \exp_p(v_0), \qquad \ell(\gamma) = |v_0|$$

WLOG, we conduct the following simplifications.

- Assume $c([0,1]) \subseteq B_\delta(p)$. Otherwise consider the smallest $t_1 \in [0,1]$ s.t. $c(t_1) \in \partial B_\delta(p)$ and show that $\ell(c) \geq \ell(c|_{[0,t_1]}) \geq \delta > \ell(\gamma)$.
- Assume $c(t) \neq p$ for any $t > 0$. Otherwise consider the largest $t_2 \in (0,1)$ s.t. $c(t_2) = p$. Consider $c|_{[t_2,1]}$ and show $\ell(c) \geq \ell(c|_{[t_2,1]}) \geq \ell(\gamma)$.

Define

$$b : [0,1] \to B_\delta(0) \subseteq T_pM$$
$$t \mapsto \exp_p^{-1}(c(t))$$

so $b(t)$ is piecewise smooth curve in T_pM. In particular $c(t) = \exp_p(b(t))$. By our assumption, $b(t) \neq 0$ for $t > 0$.

Thus in the following we may decompose $b(t) = |b(t)|\frac{b(t)}{|b(t)|}$. Let $r(t) = |b(t)|$ so

$$r : [0,1] \to \mathbb{R}_{\geq 0}$$

is piecewise C^∞. We have $r(t) > 0$ for any $t > 0$.

For $t > 0$, also define

$$v(t) := \frac{b(t)}{|b(t)|}$$

so $v : (0,1] \to T_pM$ is piecewise C^∞. Hence using Compatibility with the metric

$$\langle v(t), v(t)\rangle = 1 \implies \langle v(t), v'(t)\rangle = 0$$

Then for $0 < t \leq 1$

$$\begin{aligned}
c(t) &= \exp_p(b(t)) = \exp_p(r(t)v(t)) \\
\frac{d}{dt}c(t) &= (d\exp_p)_{b(t)}(r'(t)v(t) + r(t)v'(t)) \\
|\frac{d}{dt}c(t)|^2 &= \langle (d\exp_p)_{r(t)v(t)}(r'(t)v(t) + r(t)v'(t)), (d\exp_p)_{r(t)v(t)}(r'(t)v(t) + r(t)v'(t))\rangle \\
&= (r'(t))^2\langle (d\exp_p)_{r(t)v(t)}(v(t)), (d\exp_p)_{r(t)v(t)}(v(t))\rangle \\
&+ 2r(t)r'(t)\langle (d\exp_p)_{r(t)v(t)}(v(t)), (d\exp_p)_{r(t)v(t)}(v'(t))\rangle \\
&+ (r(t))^2\langle (d\exp_p)_{r(t)v(t)}(v'(t)), (d\exp_p)_{r(t)v(t)}(v'(t))\rangle \\
&\overset{(2.45)}{=} r'(t)^2\langle v(t), v(t)\rangle + 2r(t)r'(t)\langle v(t), v'(t)\rangle + (r(t))^2|(d\exp_p)_{r(t)v(t)}(v'(t))|^2 \\
&= r'(t)^2 + (r(t))^2|(d\exp_p)_{r(t)v(t)}(v'(t))|^2
\end{aligned}$$

Hence

$$|\frac{dc(t)}{dt}| = \sqrt{r'(t)^2 + (r(t))^2|(d\exp_p)_{r(t)v(t)}(v'(t))|^2} \geq |r'(t)| \geq r'(t)$$

so

$$\ell(c) \geq \int_0^1 |\frac{dc(t)}{dt}|dt \geq \int_\varepsilon^1 r'(t)dt = r(1) - r(\varepsilon)$$

for any $\varepsilon > 0$. Note $\lim_{\varepsilon\to 0} r(\varepsilon) = 0$ so using $r(1) = |v_0| = \ell(\gamma)$ yields

$$\ell(c) \geq \ell(\gamma)$$

Furthermore $\ell(c) = \ell(\gamma)$ implies $v'(t) = 0$ and $|r'(t)| = r'(t) \geq 0$. Then

$$v(t) = \frac{v_0}{|v_0|}$$

is constant unit vector. Now c is a monotonic reparametrization of γ

$$c(t) = \exp_p(r(t)\frac{v_0}{|v_0|}) \qquad r'(t) \geq 0 \qquad r(0) = 0 \qquad r(1) = |v_0|$$

where

$$\gamma(t) = \exp_p(tv_0) \qquad c(0) = \gamma(0) = p \qquad c(1) = \gamma(1) = \exp_p(v_0) = q$$

Hence

$$c([0,1]) = \gamma([0,1])$$

□

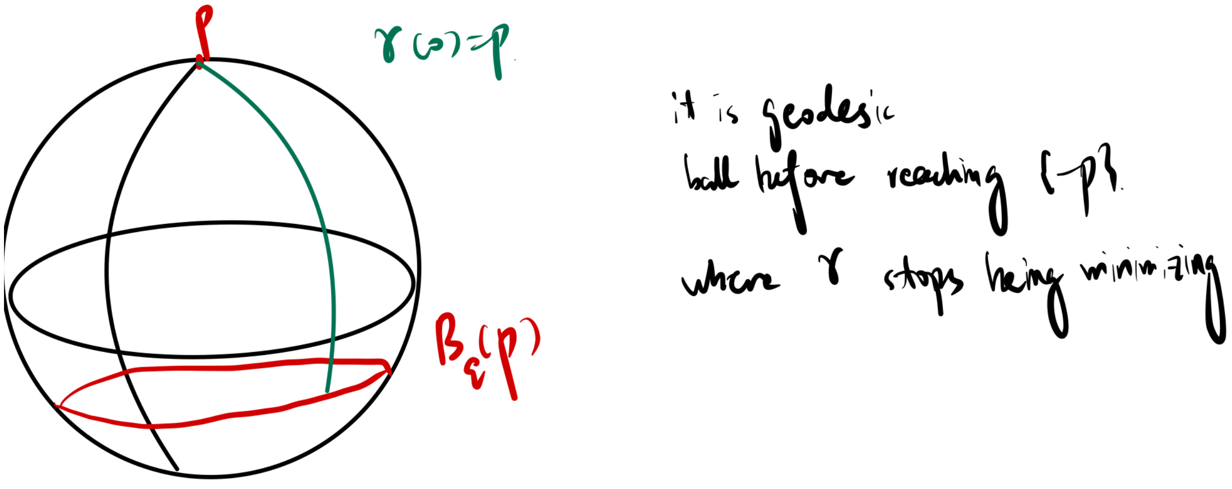


Figure 2.3: Geodesic ball on $\mathbb{S}^n$

Totally Normal Neighborhoods Let (M, g) be Riemannian.

Definition 2.3.6. *We say $W \subseteq M$ is a totally normal neighborhood of p if for each $q \in W$, W is also normal neighborhood of q.*

Theorem 2.3.2 ([dC92] Theorem 3.3.7)**.** *For any $p \in M$, there exists a neighborhood W of p and a number $\delta > 0$ s.t.*

for every $q \in W$,

$$\exp_q : B_\delta(0) \subseteq T_qM \to \exp_q(B_\delta(0)) \subseteq W$$

is a diffeomorphism, that is , W is a normal neighborhood of each of its points.

As a consequence of Proposition 2.3.5, for any two points $q_1, q_2 \in W$, there exists a unique minimizing geodesic γ of length $< \delta$ joining q_1 to q_2.

Proposition 2.3.6 ([dC92] Corollary 3.3.9)**.** *If a piecewise smooth curve*

$$\gamma : [a, b] \to M$$

with parameter proportional to arc length, has length less or equal to the length of any other piecewise smooth curve joining $\gamma(a)$ to $\gamma(b)$, then γ is a geodesic.

2.3.5 Killing Vector Fields

Let (M, g) be a Riemannian manifold with metric g. Let $X \in \mathfrak{X}(M)$. Let $p \in M$ and $U \subseteq M$ be open neighborhood of p.

Denote the local flow (1.29) that is trajectory of X passing through q at $t = 0$

$$\begin{aligned} \varphi : (-\varepsilon, \varepsilon) \times U &\to M \\ (t, q) &\mapsto \varphi(t, q) \end{aligned} \tag{2.48}$$

Definition 2.3.7 (Killing Vector Field)**.** *$X \in \mathfrak{X}(M)$ is called a Killing Vector Field if for each $t_0 \in (-\varepsilon, \varepsilon)$, the mapping*

$$\begin{aligned} \varphi(t_0, \cdot) : U \subseteq M &\to M \\ q &\mapsto \varphi(t_0, q) \end{aligned}$$

is an isometry (2.2), i.e.

$$\varphi(t_0, \cdot)^* g = g \qquad \forall\ t_0 \in (-\varepsilon, \varepsilon) \tag{2.49}$$

Killing Equation

Proposition 2.3.7 (Killing Equation)**.** *$X \in \mathfrak{X}(M)$ is a Killing vector field iff*

$$\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0 \qquad \forall\ Y,\ Z \in \mathfrak{X}(M) \tag{2.50}$$

Proposition 2.3.8. *Given Riemannian manifold (M, g). $X \in \mathfrak{X}(M)$ is Killing Field if the Lie-Derivative of the metric g w.r.t. X vanishes*

$$L_X g = 0$$

Proof. Let $L_X g = 0$. Then

$$\begin{aligned} 0 = L_X g(Y, Z) &= X(g(Y, Z)) - g(L_X Y, Z) - g(Y, L_X Z) \\ &= X(g(Y, Z)) - g([X, Y], Z) - g(Y, [X, Z]) \end{aligned}$$

Note for ∇ Levi-Civita connection that is compatible with the metric

$$0 = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = \nabla_X g(Y, Z)$$

and substitute using 'symmetric'

$$\nabla_Y Z - \nabla_Z Y = [Y, Z]$$

we conclude

$$0 = L_X g(Y, Z) = \langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle$$

□

Proposition 2.3.9 ([dC92] Exercise 5.8)**.** *Let X be a Killing vector field on a connected Riemannian Manifold M. If there exists point $q \in M$ s.t.*

$$X(q) = 0 \qquad \textit{and} \qquad (\nabla_Y X)(q) = 0 \qquad \forall\ Y(q) \in T_q M$$

Then $X \equiv 0$ identically vanishes.

2.4 Riemannian Curvature

2.4.1 Riemannian Curvature and Riemannain Curvaure Tensor

2.4.1.1 R_∇ Riemannian Curvature

Let $\pi : TM \to M$ be tangent bundle, thus we consider ∇ as affine connection (1.60)

$$\begin{aligned}\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M)\\ (X, Y) &\mapsto \nabla_X Y\end{aligned}$$

R_∇ Riemannian Curvature Recall the definition for Curvature for Smooth vector bundles (1.95).

Definition 2.4.1 (Riemannian Curvature)**.** *A Riemannian curvature over M is a smooth curvature on TM, that is, a $\mathbb{R}$-linear map*

$$\begin{aligned}R_\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M)\\ (X, Y, Z) &\mapsto R_\nabla(X, Y)(Z) := -F_\nabla(X, Y)(Z)\\ &= \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z\end{aligned} \tag{2.51}$$

One recall the defining properties

1. R_∇ is anti-symmetric in the first two arguments, i.e.

$$R_\nabla(X, Y) = -R_\nabla(Y, X) \qquad \forall\ X,\ Y \in \mathfrak{X}(M) \tag{2.52}$$

This follows from (1.96).

2. R_∇ is $C^\infty(M)$-linear in all three arguments $\mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M)$, i.e.

$$R_\nabla(fX, Y)(Z) = fR_\nabla(X, Y)(Z) \tag{2.53}$$
$$R_\nabla(X, Y)(fZ) = fR_\nabla(X, Y)(Z) \tag{2.54}$$

for any $X,\ Y,\ Z \in \mathfrak{X}(M)$ and for any $f \in C^\infty(M)$. This follows from (1.97) and (1.98).

Interpretation of R_∇ as $(1,3)$-tensor As inherited from F_∇, using $C^\infty(M)$-linearity and anti-symmetry we know from (1.99)

$$R_\nabla \in C^\infty(M, (\Lambda^2 T^*M) \otimes \mathrm{End}(TM)) = \Omega^2(M, \mathrm{End}(TM))$$

But on the other hand

$$\begin{aligned}R_\nabla &\in C^\infty(M, (\Lambda^2 T^*M) \otimes \mathrm{End}(TM))\\ &= C^\infty(M, (\Lambda^2 T^*M) \otimes T^*M \otimes TM)\\ &\subseteq C^\infty(M, (T^*M)^{\otimes 3} \otimes TM) = C^\infty(M, T^1_3 M)\end{aligned}$$

First Bianchi Identity For symmetric affine connections, the First Bianchi Identity follows by Jacobi Identity for Lie Bracket.

Proposition 2.4.1 (First Bianchi Identity)**.** *If ∇ is a symmetric affine connection on M, i.e.,* (2.6)

$$\nabla_X Y - \nabla_Y X = [X, Y] \qquad \forall\ X,\ Y \in \mathfrak{X}(M)$$

Then

$$R_\nabla(X, Y)Z + R_\nabla(Y, Z)X + R_\nabla(Z, X)Y = 0 \qquad \forall\ X,\ Y,\ Z \in \mathfrak{X}(M) \tag{2.55}$$

Proof.

$$\begin{aligned}R_\nabla(X, Y)Z + R_\nabla(Y, Z)X + R_\nabla(Z, X)Y = &\nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z\\ &+ \nabla_Z \nabla_Y X - \nabla_Y \nabla_Z X - \nabla_{[Z,Y]} X\\ &+ \nabla_X \nabla_Z Y - \nabla_Z \nabla_X Y - \nabla_{[X,Z]} Y\end{aligned}$$

Now using that the connection is symmetric, we group them via

$$\begin{aligned}&\nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_Z \nabla_Y X - \nabla_Y \nabla_Z X + \nabla_X \nabla_Z Y - \nabla_Z \nabla_X Y\\ &= \nabla_Y (\nabla_X Z - \nabla_Z X) + \nabla_X (\nabla_Z Y - \nabla_Y Z) + \nabla_Z (\nabla_Y X - \nabla_X Y)\\ &\overset{(2.6)}{=} \nabla_Y([X, Z]) + \nabla_X([Z, Y]) + \nabla_Z([Y, X])\end{aligned}$$

Thus using connection is symmetric again, we reduce to

$$R_\nabla(X,Y)Z + R_\nabla(Y,Z)X + R_\nabla(Z,X)Y = \nabla_Y[X,Z] + \nabla_Z[Y,X] + \nabla_X[Z,Y] - \nabla_{[X,Z]}Y - \nabla_{[Y,X]}Z - \nabla_{[Z,Y]}X \overset{(2.6)}{=} [Y,[X,Z]] + [Z,[Y,X]] + [X,[Z,Y]] \overset{(1.24)}{=} 0$$

where the last step follows from Jacobi Identity. □

2.4.1.2 R Riemannian Curvature Tensor

R **Riemannian Curvature Tensor** Now we define Riemannian Curvature Tensor using Riemannian Curvature. Recall in the previous definition, we state nothing about the metric g.

In the following, let (M,g) be a Riemannian Manifold. Let ∇ be the Levi-Civita Connection (2.24).

Definition 2.4.2 (Riemannian Curvature Tensor). *A Riemannian Curvature Tensor over (M,g) with Levi-Civita Connection ∇ is a $\mathbb{R}$-linear map*

$$\begin{aligned} R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) &\to C^\infty(M) \\ (X,Y,Z,T) &\mapsto g\left(R_\nabla(X,Y)(Z),T\right) \end{aligned} \tag{2.56}$$

One has a rich family of properties as inherited from the previous definitions.

$C^\infty(M)$-linearity and R as $(0,4)$-tensor

Proposition 2.4.2 ($C^\infty(M)$-linearity). *For any $f \in C^\infty(M)$*

$$fR(X,Y,Z,T) = R(fX,Y,Z,T) = R(X,fY,Z,T) = R(X,Y,fZ,T) = R(X,Y,Z,fT) \tag{2.57}$$

Proof. The first two equalities follows from (2.53), and the third eqaulity follows from (2.53). The last equality follows from the fact that smooth sections are $C^\infty(M)$-modules and the metric is one such smooth section (Lemma 1.10.1). □

Thus with $C^\infty(M)$-linearity in all 4 arguments, $R \in C^\infty(M, T^0_4 M)$ via the characterisation Lemma 1.19.1.

First Bianchi Identity

Proposition 2.4.3 ([dC92] Proposition 4.2.5). *For any X, Y, Z, $T \in \mathfrak{X}(M)$*

$$R(X,Y,Z,T) + R(Y,Z,X,T) + R(Z,X,Y,T) = 0 \tag{2.58}$$

Proof. Using ∇ is symmetric affine connection, one directly apply First Bianchi Identity (2.55)

$$R(X,Y,Z,T) + R(Y,Z,X,T) + R(Z,X,Y,T) = g(R_\nabla(X,Y)(Z),T) + g(R_\nabla(Y,Z)(X),T) + g(R_\nabla(Z,X)(Y),T) \overset{(2.55)}{=} g(0,T) = 0$$

□

Symmetries and $R \in C^\infty(M, \mathrm{Sym}^2(\Lambda^2 T^* M))$

Proposition 2.4.4 ([dC92] Proposition 4.2.5). *For any X, Y, Z, $T \in \mathfrak{X}(M)$*

1. *R is anti-symmetric in the first two coordinates*

$$R(X,Y,Z,T) = -R(Y,X,Z,T) \tag{2.59}$$

2. *R is anti-symmetric in the last two coordinates*

$$R(X,Y,Z,T) = -R(X,Y,T,Z) \tag{2.60}$$

3. *R is symmetric in the first and last **two sets** of coordinates*

$$R(X,Y,Z,T) = R(Z,T,X,Y) \tag{2.61}$$

Proof. (2.59) follows from (2.52).

$$R(X,Y,Z,T) = g\left(R_\nabla(X,Y)(Z),T\right) \overset{(2.52)}{=} g(-R_\nabla(Y,X)(Z),T) = -g(R_\nabla(Y,X)(Z),T) = -R(Y,X,Z,T)$$

(2.60) follows from that ∇ is compatible with the metric (1.101)

$$R(X,Y,Z,T) = g\left(R_\nabla(X,Y)(Z),T\right) \overset{(1.101)}{=} -g(R_\nabla(X,Y)(T),Z) = -R(X,Y,T,Z)$$

We show (2.61) from First Bianchi Identity (2.58) by fixing each vector field

$$\begin{aligned}
0 &= R(X,Y,Z,T) + R(Y,Z,X,T) + R(Z,X,Y,T)\\
0 &= R(Y,Z,T,X) + R(Z,T,Y,X) + R(T,Y,Z,X)\\
0 &= R(Z,T,X,Y) + R(T,X,Z,Y) + R(X,Z,T,Y)\\
0 &= R(T,X,Y,Z) + R(X,Y,T,Z) + R(Y,T,X,Z)
\end{aligned}$$

Summing up and using (2.59), (2.60) to permute yields

$$\begin{aligned}
0 = {}& \cancel{R(X,Y,Z,T)} + \cancel{R(Y,Z,X,T)} - R(X,Z,Y,T)\\
& \cancel{-R(Y,Z,X,T)} \cancel{-R(Z,T,X,Y)} + R(Y,T,X,Z)\\
& + \cancel{R(Z,T,X,Y)} + \cancel{R(X,T,Y,Z)} - R(X,Z,Y,T)\\
& \cancel{-R(X,T,Y,Z)} \cancel{-R(X,Y,Z,T)} + R(Y,T,X,Z)\\
R(X,Z,Y,T) = {}& R(Y,T,X,Z)
\end{aligned}$$

□

Now (2.59) and (2.60) together gives $R \in C^\infty(M, (\Lambda^2 T^*M) \otimes (\Lambda^2 T^*M))$. (2.61) gives $R \in C^\infty(M, \mathrm{Sym}^2(\Lambda^2 T^*M))$.

2.4.1.3 R_∇ and R in Local Coordinates

Let (U,ϕ) be C^∞ chart on M. Let $(x_1,\cdots,x_n)$ be local coordinates on U.

Let T be any (r,s)-tensor on M. Then locally on U, T takes the form (1.45)

$$T = \sum_{\substack{1\le i_1,\cdots,i_r\le n\\ 1\le j_1,\cdots,j_s\le n}} T^{i_1,\cdots,i_r}_{j_1,\cdots,j_s} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s} \qquad \text{for} \qquad T^{i_1,\cdots,i_r}_{j_1,\cdots,j_s} \in C^\infty(U)$$

Our Riemannian metric g in local coordinate writes

$$g = \sum_{i,j} g_{ij} dx_i dx_j$$

where

$$g_{ij} := g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \in C^\infty(U)$$

Our Levi-Civita connection ∇ acts locally via (1.66)

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x_k}$$

where the Christoffel Symbols take the form (2.26)

$$\Gamma^\ell_{ij} = \frac{1}{2} \sum_k g^{\ell k} \left(g_{ik,j} + g_{kj,i} - g_{ij,k}\right) \qquad g_{\ell j,i} := \frac{\partial}{\partial x_i} g_{\ell j}$$

Riemannian Curvature R_∇ in Local Coordinates Recall $R_\nabla \in C^\infty(M, T^1_3 M)$. Thus on U, R_∇ takes the form as $(1,3)$-tensor

$$R_\nabla = \sum_{i,j,k,m} R^m_{ijk} dx_i \otimes dx_j \otimes dx_k \otimes \frac{\partial}{\partial x_m}$$

where the components $R^m_{ijk} \in C^\infty(U)$ are defined by

$$R_\nabla\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \sum_m R^m_{ijk} \frac{\partial}{\partial x_m} \tag{2.62}$$

We want to compute precisely what the components R^m_{ijk} are. To do so, we use definition (2.51) for the LHS

$$R_\nabla(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k}=\nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_k}-\nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k}-\nabla_{[\frac{\partial}{\partial x_j},\frac{\partial}{\partial x_i}]}\frac{\partial}{\partial x_k}$$

Now by computations using (1.66) and the Leibniz Rule

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_k} &\overset{(1.66)}{=} \nabla_{\frac{\partial}{\partial x_j}}(\sum_\ell \Gamma^\ell_{ik}\frac{\partial}{\partial x_\ell})\\
&\overset{(1.61)}{=} \sum_\ell \frac{\partial}{\partial x_j}\Gamma^\ell_{ik}\frac{\partial}{\partial x_\ell}+\sum_\ell \Gamma^\ell_{ik}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_\ell}\\
&\overset{(1.66)}{=} \sum_m\left(\frac{\partial}{\partial x_j}\Gamma^m_{ik}+\sum_\ell\Gamma^\ell_{ik}\Gamma^m_{j\ell}\right)\frac{\partial}{\partial x_m}\\
\nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k} &= \nabla_{\frac{\partial}{\partial x_i}}\left(\sum_\ell \Gamma^\ell_{jk}\frac{\partial}{\partial x_\ell}\right)\\
&= \sum_\ell \frac{\partial}{\partial x_i}\Gamma^\ell_{jk}\frac{\partial}{\partial x_\ell}+\sum_\ell \Gamma^\ell_{jk}\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_\ell}\\
&= \sum_m\left(\frac{\partial}{\partial x_i}\Gamma^m_{jk}+\sum_\ell\Gamma^\ell_{jk}\Gamma^m_{i\ell}\right)\frac{\partial}{\partial x_m}\\
\nabla_{[\frac{\partial}{\partial x_j},\frac{\partial}{\partial x_i}]}\frac{\partial}{\partial x_k} &= 0
\end{aligned}$$

Hence we have local coordinate representations

$$R^m_{ijk}:=\frac{\partial}{\partial x_j}\Gamma^m_{ik}-\frac{\partial}{\partial x_i}\Gamma^m_{jk}+\sum_\ell\Gamma^\ell_{ik}\Gamma^m_{j\ell}-\sum_\ell\Gamma^\ell_{jk}\Gamma^m_{i\ell} \tag{2.63}$$

Riemannian Curvature Tensor R in Local Coordinates Let (U,ϕ) again be C^∞ chart on M with coordinates $\phi=(x_1,\cdots,x_n)$. The metric writes

$$g=\sum_{ij} g_{ij}\,dx_i dx_j$$

with Γ^k_{ij} Christoffel symbols (2.26).

On U, since $R\in C^\infty(M,T^0_4M)$ is $(0,4)$-tensor, one may write in local coordinates

$$R=\sum_{i,j,k,\ell=1}^n R_{i,j,k,\ell}dx_i\otimes dx_j\otimes dx_k\otimes dx_\ell$$

where the components are defined as

$$\begin{aligned}
R_{ijk\ell}:=R(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j},\frac{\partial}{\partial x_k},\frac{\partial}{\partial x_\ell}) &\overset{(2.56)}{=} g(R_\nabla(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k},\frac{\partial}{\partial x_\ell})\\
&\overset{(2.62)}{=} g(\sum_m R^m_{ijk}\frac{\partial}{\partial x_m},\frac{\partial}{\partial x_\ell})=\sum_m R^m_{ijk}g_{m\ell}\in C^\infty(U)
\end{aligned} \tag{2.64}$$

Moreover,

$$R_{ijk\ell}+R_{jki\ell}+R_{kij\ell}=0$$

follows from (2.58), and

$$R_{ijk\ell}=-R_{jik\ell}=-R_{ij\ell k}=R_{k\ell ij}$$

follows from (2.59), (2.60) and (2.61).

Examples for Riemannian Curvature and Riemannian Curvature Tensor

Example 2.4.1. *For* $\dim M=1$ *then*

$$R=R_{1111}(dx_1\otimes dx_1\otimes dx_1\otimes dx_1)$$

But this immediately implies $R_{1111}\equiv 0$ *via Bianchi identity* (2.58)*. Hence for* $\dim M=1$, $R=R_\nabla=0$.

2.4.2 Sectional Curvature

Given a vector space V, we denote by

$$|x \wedge y| := \sqrt{|x|^2|y|^2 - \langle x, y\rangle^2}$$

which represents the area of a two-dimensional parallelogram determined by the pair of vectors x, $y \in V$.

In general, an inner product on a vector space $V \cong \mathbb{R}^n$ induces an inner product on $\Lambda^2 V$ as follows: If $\{e_1, \cdots, e_n\} \subseteq V$ is an orthonormal basis of V, then

$$\{e_i \wedge e_j \mid 1 \leq i < j \leq n\}$$

is an orthonormal basis of $\Lambda^2 V$.

$K(p, \sigma)$ **Sectional Curvature** Let (M, g) be Riemannian manifold. Let ∇ be the Levi-Civita connection. Let $R \in C^\infty(M, T^0_4 M)$ be the Riemannian curvature tensor.

Definition 2.4.3 (Sectional Curvature)**.** *Let $p \in M$, and let σ be a 2* dim *subspace of T_pM, i.e.,*

$$\sigma \in \mathrm{Gr}(2, T_pM)$$

Let x, y be any two linearly independent vectors of T_pM.

We define the sectional curvature of σ at point p to be

$$K(p, \sigma) := \frac{R(p)(x, y, x, y)}{|x \wedge y|^2} = \frac{R(p)(x, y, x, y)}{\langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle^2} \tag{2.65}$$

Alternatively, one may define

$$K(p, \sigma) := R(p)(e_1, e_2, e_1, e_2) \tag{2.66}$$

where e_1, e_2 is an orthonormal basis of σ.

Well-definedness. We show that $K(p, \sigma) \in \mathbb{R}$ is well-defined independent of choice of vectors x, y, e_1, e_2. Indeed, given $\sigma \subseteq T_pM$ a two-dimensional subspace, let e_1, e_2 be orthonormal basis and x, y be any basis. Assume $\{x, y\}$ and $\{e_1, e_2\}$ are related via

$$\begin{aligned} x &= ae_1 + be_2 \\ y &= ce_1 + de_2 \end{aligned}$$

For $ad - bc \neq 0$. Then we compute using linearity

$$\begin{aligned} R(p)(x, y, x, y) &= R(p)(ae_1 + be_2, ce_1 + de_2, ae_1 + be_2, ce_1 + de_2) \\ &= a^2c^2R_{1111} + abc^2R_{2111} + a^2cdR_{1211} + abc^2R_{1121} + a^2cdR_{1112} + abcdR_{2211} + b^2c^2R_{2121} + abcdR_{2112} \\ &+ abcdR_{1221} + a^2d^2R_{1212} + abcdR_{1122} + b^2cdR_{2221} + abd^2R_{2212} + b^2cdR_{2122} + abd^2R_{1222} + b^2d^2R_{2222} \\ &= a^2c^2R_{1111} + (2abc^2 + 2a^2cd)R_{1211} + 2abcdR_{1122} + (a^2d^2 + b^2c^2 - 2abcd)R_{1212} \\ &+ (-b^2cd + abd^2 - b^2cd + abd^2)R_{1222} + b^2d^2R_{2222} \\ &= (ad - bc)^2R_{1212} \end{aligned}$$

where we're using any component with two consecutive repeated indices in either the first set or second set has to vanish, due to anti-symmetry. But

$$\begin{aligned} |x \wedge y|^2 &= |x|^2|y|^2 - \langle x, y\rangle^2 \\ &= (a^2 + b^2)(c^2 + d^2) - (ac + bd)^2 = a^2d^2 + b^2c^2 - 2acbd \\ &= (ad - bc)^2 \end{aligned}$$

□

Sectional Curvature $K(\sigma)$ **completely determines Riemannian Curvature Tensor** R We need an algebraic fact.

Lemma 2.4.1 ([dC92] Lemma 4.3.3 Linear Algebra)**.** *Let V be an inner product space over $\mathbb{R}$ with dimension n.*

Suppose that we have two maps $r, r' \in (V^*)^{\otimes 4}$

$$\begin{aligned} r, r' : V \times V \times V \times V &\to \mathbb{R} \\ (x, y, z, t) &\mapsto r(x, y, z, t), r'(x, y, z, t) \end{aligned}$$

$\mathbb{R}$-linear in x, y, z, t and both satisfy

(a) Bianchi identity

$$r(x, y, z, t) + r(y, z, x, t) + r(z, x, y, t) = 0$$

(b) $r \in \mathrm{Sym}^2(\Lambda^2 V^*)$*, i.e.*

$$\begin{aligned} r(x, y, z, t) &= -r(y, x, z, t) \\ r(x, y, z, t) &= -r(x, y, t, z) \\ r(z, t, x, y) &= r(x, y, z, t) \end{aligned}$$

Now define

$$\begin{aligned} K,\, K' : \mathrm{Gr}(2, V) &\to \mathbb{R} \\ \sigma &\mapsto \frac{r(x, y, x, y)}{|x \wedge y|^2},\ \frac{r'(x, y, x, y)}{|x \wedge y|^2} \end{aligned}$$

where $x,\, y \in \sigma$ *are two linearly independent vectors (that generates* σ*).*

If for all $\sigma \subseteq V$ *two-dimensional subspaces, one has* $K(\sigma) = K'(\sigma)$*, then* $r = r'$*.*

Proof. Let $\Delta = r - r' \in (V^*)^{\otimes 4}$ then Δ satisfies (a), (b) and

$$\Delta(x, y, x, y) = 0 \qquad \forall\ x,\, y \in V$$

We claim that

$$\Delta(x, y, z, t) = 0 \qquad \forall\ x, y, z, t \in V$$

Indeed for any $x,\ y,\ z \in V$ we have

$$\begin{aligned} 2\Delta(x, y, z, y) &= \Delta(x, y, z, y) + \Delta(z, y, x, y) \\ &= \Delta(x + z, y, x + z, y) - \Delta(x, y, x, y) - \Delta(z, y, z, y) = 0 \end{aligned}$$

Hence

$$\Delta(x, y, z, y) = 0 \qquad \forall\ x,\ y,\ z \in V$$

Now for any $x, y, z, t \in V$

$$\begin{aligned} 0 &= \Delta(x, y + t, z, y + t) - \Delta(x, y, z, y) - \Delta(x, t, z, t) \\ &= \Delta(x, y, z, t) + \Delta(x, t, z, y) \\ &= \Delta(x, y, z, t) + \Delta(z, y, x, t) \\ &= \Delta(x, y, z, t) - \Delta(y, z, x, t) \end{aligned}$$

using Bianchi we have

$$0 = \Delta(x, y, z, t) + \Delta(y, z, x, t) + \Delta(z, x, y, t) = 3\Delta(x, y, z, t)$$

□

Thus one has the characterisation.

Theorem 2.4.1. *The Riemannian curvature tensor* R *on a Riemannian manifold* (M, g) *is determined by its sectional curvature* $K(p, \sigma)$ *for any* $p \in M$ *and for any* $\sigma \in \mathrm{Gr}(2, T_pM)$*, i.e.*

$$\{R(X, Y, Z, T) \mid X,\ Y,\ Z,\ T \in \mathfrak{X}(M)\}$$

is determined completely by

$$\{R(X, Y, X, Y) \mid X,\ Y \in \mathfrak{X}(M)\}$$

Proof. Follows from the following lemma in linear algebra 2.4.1. □

Computation for R via Constant Sectional Curvature

Definition 2.4.4. *We say* (M, g) *have constant sectional curvature* K_0 *if for any* $p \in M$ *for any* $\sigma \in \mathrm{Gr}(2, T_pM)$

$$K(p, \sigma) = K_0$$

Theorem 2.4.2 ([dC92] Lemma 4.3.4)**.** (M, g) *has constant sectional curvature iff*

$$R(X, Y, Z, T) = K_0(g(X, Z)g(Y, T) - g(X, T)g(Y, Z)) \tag{2.67}$$

Proof. Define the RHS to be $K_0R_0(X, Y, Z, T)$ then for any e_1, e_2 orthonormal vectors

$$R_0(e_1, e_2, e_1, e_2) = g(e_1, e_2)g(e_1, e_2) - g(e_1, e_2)^2 = 1 \cdot 1 - 0^2 = 1$$

Hence

$$R_0(X, Y, Z, T) = g(X, Z)g(Y, T) - g(X, T)g(Y, Z)$$

satisfies (a) and (b) in Lemma 2.4.1. □

Flat Riemannian Manifold

Definition 2.4.5 (Flat). *We say a Riemannian manifold (M, g) is flat if it has constant sectional curvature $K_0 = 0$. This is equivalent to saying Riemannian curvature tensor $R \equiv 0$ due to Lemma 2.4.1.*

Example 2.4.2. *$(\mathbb{R}^n, g_0 = dx_1^2 + \cdots dx_n^2)$ is flat since $\Gamma_{ij}^k = 0$, and therefore $R_{ijk}^\ell = 0$.*

Sectional Curvature in Two Dimension Let Riemannian manifold (M, g) with $\dim M = 2$. Let (U, ϕ) be C^∞ chart on M and let (x_1, x_2) be coordinates on U.

On U

$$g = \sum_{i,j=1}^{2} g_{ij} dx_i dx_j = g_{11} dx_1^2 + 2g_{12} dx_1 dx_2 + g_{22} dx_2^2$$

We have Riemannian Curvature Tensor in local coordinates expressed as

$$\begin{aligned}
R = & \sum_{i,j,k,\ell=1}^{2} R_{ijk\ell} dx_i \otimes dx_j \otimes dx_k \otimes dx_\ell \\
&= R_{1212} dx_1 \otimes dx_2 \otimes dx_1 \otimes dx_2 + R_{2112} dx_2 \otimes dx_1 \otimes dx_1 \otimes dx_2 + R_{1221} dx_1 \otimes dx_2 \otimes dx_2 \otimes dx_1 + R_{2121} dx_2 \otimes dx_1 \otimes dx_2 \otimes dx_1 \\
&= R_{1212}(dx_1 \otimes dx_2 - dx_2 \otimes dx_1) \otimes (dx_1 \otimes dx_2 - dx_2 \otimes dx_1) \\
&= R_{1212}(dx_1 \wedge dx_2) \otimes (dx_1 \wedge dx_2)
\end{aligned}$$

The only 2-dim subspace of T_pM is itself. So sectional curvature

$$\begin{aligned}
K : M &\to \mathbb{R} \\
p &\mapsto K(p, T_pM)
\end{aligned}$$

writes

$$K = \frac{R(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})}{|\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}|^2} = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2}$$

Example 2.4.3 (Sectional Curvature for $\mathbb{S}^2$ is 1)**.** *Consider $(\mathbb{S}^2, g_{\text{can}} = d\phi^2 + \sin^2 \phi d\theta^2)$ for $(\phi, \theta) = (x_1, x_2)$.*

Recall Example 2.2.2

$$g_{11} = 1, \qquad g_{22} = \sin^2 \phi \qquad g_{12} = g_{21} = 0$$

Where

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial \phi}} \frac{\partial}{\partial \phi} &= 0 \\
\nabla_{\frac{\partial}{\partial \phi}} \frac{\partial}{\partial \theta} &= \nabla_{\frac{\partial}{\partial \theta}} \frac{\partial}{\partial \phi} = \cot(\phi) \frac{\partial}{\partial \theta} \\
\nabla_{\frac{\partial}{\partial \theta}} \frac{\partial}{\partial \theta} &= -\sin(\phi)\cos(\phi) \frac{\partial}{\partial \phi}
\end{aligned}$$

We want to compute

$$K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} = \frac{R_{1212}}{\sin^2(\phi)}$$

Let's compute by directly verifying by definition

$$\begin{aligned}
R_{1212} &= \langle R(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta}) \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta} \rangle \\
&= \langle \nabla_{\frac{\partial}{\partial \theta}} \nabla_{\frac{\partial}{\partial \phi}} \frac{\partial}{\partial \phi} - \nabla_{\frac{\partial}{\partial \phi}} \nabla_{\frac{\partial}{\partial \theta}} \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta} \rangle \\
&= -\langle \nabla_{\frac{\partial}{\partial \phi}} (\cot(\phi) \frac{\partial}{\partial \theta}), \frac{\partial}{\partial \theta} \rangle \\
&= -\langle -\csc^2(\phi) \frac{\partial}{\partial \theta} + \cot(\phi) \nabla_{\frac{\partial}{\partial \phi}} \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \rangle \\
&= -\langle -\csc^2(\phi) \frac{\partial}{\partial \theta} + \cot^2(\phi) \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \rangle \\
&= \langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \rangle = g_{22} = \sin^2(\phi)
\end{aligned}$$

Hence $K = 1$.

2.4.3 Ricci and Scalar Curvature

'Certain combinations of sectional curvature appear with such frequency that they deserve special names'.

2.4.3.1 Ricci Curvature and Ricci Curvature Tensor

Q **Trace of Riemannian Curvature** $z \mapsto R_\nabla(p)(x,z)(y)$ Consider (M,g) Riemannian manifold. Let ∇ denote its Levi-Civita connection. Recall Riemannian Curvature R_∇ as in (2.51). We define Q, the trace, as follows.

Definition 2.4.6. *Take any $p \in M$. Let $\{e_i\}$ be an orthonormal basis of T_pM. Define for any $x, y \in T_pM$ the bilinear form on T_pM as*

$$\begin{aligned} Q(p) &: T_pM \times T_pM \to \mathbb{R} \\ (x,y) &\mapsto \operatorname{tr}_g\left(z \mapsto R_\nabla(p)(x,z)(y)\right) := \sum_{i=1}^{n} \langle (R_\nabla)(p)(x,e_i)(y), e_i\rangle \end{aligned} \tag{2.68}$$

Lemma 2.4.2. *$Q(p)$ is symmetric bilinear form for any p, i.e.*

$$Q(p)(x,y) = Q(p)(y,x) \qquad \forall\ x,\ y \in T_pM$$

Proof. We compute for $\{e_i\}$ an orthonormal basis of T_pM

$$\begin{aligned} Q(p)(x,y) &= \sum_{i=1}^{n} \langle (R_\nabla)_p(x,e_i)(y), e_i\rangle = \sum_{i=1}^{n} R(p)(x,e_i,y,e_i) \\ &\overset{(2.61)}{=} \sum_{i=1}^{n} R(p)(y,e_i,x,e_i) = Q(p)(y,x) \end{aligned}$$

□

Thus

$$\begin{aligned} Q &: M \to \operatorname{Sym}^2 T^*M \\ p &\mapsto Q(p) \in \operatorname{Sym}^2 T_p^*M \end{aligned} \tag{2.69}$$

defines $Q \in C^\infty(M, \operatorname{Sym}^2 T^*M)$ as smooth symmetric $(0,2)$-tensor.

Ric **Ricci Curvature Tensor** Let (M,g) be Riemannian manifold. Let Q be trace of its Riemannian curvature tensor (2.69).

Definition 2.4.7 (Ricci Curvature Tensor)**.** *We define*

$$\begin{aligned} \operatorname{Ric} &: M \to \operatorname{Sym}^2 T^*M \\ p &\mapsto \frac{1}{n-1} Q(p) \in \operatorname{Sym}^2 T_p^*M \end{aligned} \tag{2.70}$$

As inherited from Q, $\operatorname{Ric} \in C^\infty(M, \operatorname{Sym}^2 T^*M)$.

Ric_p **Ricci Curvature in direction** $x \in T_pM$ In particular, for any $p \in M$, let $x = z_n$ be a unit vector in T_pM. We take an orthonormal basis $\{z_1, \cdots, z_{n-1}\}$ of the hyperplane orthogonal in T_pM w.r.t. $x = z_n$.

Definition 2.4.8 (Ricci Curvature)**.** *The Ricci Curvature in the direction x at p is defined via*

$$\begin{aligned} \operatorname{Ric}_p &: T_pM \to \mathbb{R} \\ x &\mapsto \operatorname{Ric}(p)(x,x) = \frac{1}{n-1} \sum_{i=1}^{n-1} \langle (R_\nabla)(p)(x,z_i)(x), z_i\rangle \end{aligned} \tag{2.71}$$

2.4.3.2 Scalar Curvature

$\tilde{Q}$ **Self-adjoint Mapping induced by** Q Let (M,g) be Riemannian manifold. Let Q be trace of its Riemannian curvature tensor (2.69).

Definition 2.4.9. *For any $p \in M$, define a linear mapping*

$$\begin{aligned} \tilde{Q}(p) : T_pM &\to T_pM \\ x &\mapsto \tilde{Q}(p)(x) \end{aligned}$$

where

$$\begin{aligned} \tilde{Q}(p)(x) : T_pM &\to \mathbb{R} \\ y &\mapsto \langle \tilde{Q}(p)(x), y\rangle := Q(p)(x,y) \end{aligned} \tag{2.72}$$

Lemma 2.4.3. *$\tilde{Q}(p)$ is self-adjoint operator for any p, i.e.*

$$\langle \tilde{Q}(p)(x), y\rangle = \langle x, \tilde{Q}(p)(y)\rangle \qquad \forall\ x,\, y \in T_pM$$

Proof. Compute

$$\langle \tilde{Q}(p)(x), y\rangle = Q(p)(x,y) = Q(p)(y,x) = \langle \tilde{Q}(p)(y), x\rangle = \langle x, \tilde{Q}(p)(y)\rangle$$

□

Thus

$$\begin{aligned} \tilde{Q} : M &\to TM \otimes T^*M \\ p &\mapsto \tilde{Q}(p) \in \mathrm{End}(T_pM) = T_pM \otimes T_p^*M \end{aligned} \tag{2.73}$$

defines $\tilde{Q} \in C^\infty(M, T_1^1M)$ as smooth $(1,1)$-tensor.

Trace of $\tilde{Q}(p)$ We compute the trace of (2.72). Take an orthonormal basis $\{e_i\}$ of T_pM. We compute

$$\begin{aligned} \mathrm{tr}_g(z \mapsto \tilde{Q}(p)(z)) &= \sum_{i=1}^n \langle \tilde{Q}(p)(e_i), e_i\rangle \overset{(2.72)}{=} \sum_{i=1}^n Q(p)(e_i,e_i) \\ &\overset{(2.68)}{=} \sum_{i,\,j=1}^n \langle (R_\nabla)(p)(e_i,e_j)(e_i), e_j\rangle \\ &\overset{(2.70)}{=} (n-1)\sum_{i=1}^n \mathrm{Ric}(p)(e_i,e_i) \overset{(2.71)}{=} (n-1)\sum_{i=1}^n \mathrm{Ric}_p(e_i) \end{aligned}$$

S Scalar Curvature Let (M,g) be Riemannian manifold.

Definition 2.4.10 (Scalar Curvature)**.** *The Scalar Curvature is defined at each $p \in M$ via*

$$\begin{aligned} S : M &\to \mathbb{R} \\ p &\mapsto \frac{1}{(n-1)n}\mathrm{tr}_g(z \mapsto \tilde{Q}(p)(z)) = \frac{1}{n}\sum_{i=1}^n \mathrm{Ric}_p(e_i) \\ &= \frac{1}{(n-1)n}\sum_{i,\,j=1}^n \langle (R_\nabla)(p)(e_i,e_j)(e_i), e_j\rangle \end{aligned} \tag{2.74}$$

Thus

$$S \in C^\infty(M)$$

2.4.3.3 Ricci and Scalar Curvature in Local Coordinates

Let (M,g) be Riemannian manifold with $\dim M = 2$. Let (U,ϕ) be C^∞ chart on M with coordinates $(x_1,\cdots,x_n)$ on U. Let $\{e_k\}$ be a local smooth orthonormal frame on U.

On U, our metric admits local expression

$$g = \sum_{i,j=1}^n g_{ij}dx_idx_j, \qquad g_{ij} := g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \in C^\infty(U)$$

Now notice one has two frames on U, $\{\frac{\partial}{\partial x_i}\}$ and $\{e_k\}$. How do they interact with each other? Assume

$$\frac{\partial}{\partial x_i} = \sum_k a_{ik}e_k \qquad \frac{\partial}{\partial x_j} = \sum_\ell a_{j\ell}e_\ell \tag{2.75}$$

Then g_{ij} in terms of coordinates in $\{e_k\}$ writes

$$\begin{aligned} g_{ij} &= \langle \sum_k a_{ik} e_k, \sum_\ell a_{j\ell} e_\ell \rangle = \sum_{k,\ell} a_{ik} a_{j\ell} \langle e_k, e_\ell \rangle = \sum_{k=1}^n a_{ik} a_{jk} \\ g_{ij} &= (aa^T)_{ij} \\ g^{ij} &= (a^{-T} a^{-1})_{ij} \end{aligned} \tag{2.76}$$

Trace Q in Local Coordinates Since Q is a $(0,2)$ symmetric tensor on U, locally

$$Q = \sum_{i,j=1}^n R_{ij} dx_i dx_j$$

where the components $R_{ij} \in C^\infty(U)$ are given by

$$R_{ij} := Q(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \overset{(2.68)}{=} \sum_{k=1}^n \langle R_\nabla(\frac{\partial}{\partial x_i}, e_k) \frac{\partial}{\partial x_j}, e_k \rangle$$

We need to understand the RHS.

Lemma 2.4.4 (Taking the Trace in Local Coordinates)**.** *For any x, $y \in T_pM$, and $\frac{\partial}{\partial x_i}(p)$, $\frac{\partial}{\partial x_j}(p) \in T_pM$*

$$Q(p)(x,y) = \sum_{i=1}^n R(p)(x, e_i, y, e_i) = \sum_{i,j=1}^n R(p)(x, \frac{\partial}{\partial x_i}(p), y, \frac{\partial}{\partial x_j}(p)) g^{ij}(p) \tag{2.77}$$

Proof. We compute using (2.75)

$$\begin{aligned} \sum_{i,j=1}^n R(p)(x, \frac{\partial}{\partial x_i}(p), y, \frac{\partial}{\partial x_j}(p)) g^{ij}(p) &= \sum_{i,j=1}^n R(p)(x, \sum_k a_{ik} e_k, y, \sum_\ell a_{j\ell} e_\ell) g^{ij} = \sum_{k,\ell} R(p)(x, e_k, y, e_\ell) \sum_{i,j=1}^n a_{ik} g^{ij} a_{j\ell} \\ &= \sum_{k,\ell} R(p)(x, e_k, y, e_\ell)(a^T g^{-1} a)_{k\ell} \overset{(2.76)}{=} \sum_{k,\ell} R(p)(x, e_k, y, e_\ell)(a^T a^{-T} a^{-1} a)_{k\ell} \\ &= \sum_{k,\ell} R(p)(x, e_k, y, e_\ell) \delta_{k\ell} = \sum_{k=1}^n R(p)(x, e_k, y, e_k) \end{aligned}$$

□

Now one may compute using (2.77)

$$\begin{aligned} R_{ij} &:= Q(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \sum_{k=1}^n \langle R_\nabla(\frac{\partial}{\partial x_i}, e_k) \frac{\partial}{\partial x_j}, e_k \rangle \\ &= \sum_{k=1}^n R(\frac{\partial}{\partial x_i}, e_k, \frac{\partial}{\partial x_j}, e_k) \overset{(2.77)}{=} \sum_{k,\ell=1}^n R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_\ell}) g^{k\ell} \\ &= \sum_{k\ell}^n R_{ikj\ell} g^{k\ell} \end{aligned} \tag{2.78}$$

Ric **in Local Coordinates** Consider Ricci Curvature Tensor

$$\text{Ric} = \frac{1}{n-1} \sum_{i,j=1}^n R_{ij} dx_i dx_j$$

The components write

$$\text{Ric}_{ij} = \frac{1}{n-1} R_{ij} = \frac{1}{n-1} \sum_k R^k_{ikj} \overset{(2.78)}{=} \frac{1}{n-1} \sum_{k,\ell} R_{ikj\ell} g^{k\ell} \tag{2.79}$$

Ric **in Local Coordinates with Constant Sectional Curvature** K_0. Let's explain why we want a normalization by $\frac{1}{n-1}$. If (M,g) has constant sectional curvature K_0, then

$$\begin{aligned} R(X,Y,Z,T) &\overset{(2.67)}{=} K_0(g(X,Z)g(Y,T) - g(X,T)g(Y,Z)) \\ R_{ijk\ell} &= K_0(g_{ik}g_{j\ell} - g_{i\ell}g_{jk}) \end{aligned}$$

Now

$$\begin{aligned} R_{ik} &= \sum_{j,\ell} R_{ijk\ell} g^{j\ell} = K_0 \left(\sum_{\ell} g_{ik} \sum_{j} g^{j\ell} g_{j\ell} - \sum_{\ell} g_{i\ell} \sum_{j} g_{jk} g^{j\ell} \right) \\ &= K_0 \left(g_{ik} \sum_{\ell} \delta_\ell^\ell - \sum_{\ell} g_{i\ell} \delta_k^\ell \right) \\ &= K_0 \left(g_{ik} n - g_{ik} \right) = (n-1) K_0 g_{ik} \end{aligned}$$

Hence $Q = (n-1)K_0 g$. The Ricci curvature tensor is simply a multiple of metric g via constant sectional curvature

$$\mathrm{Ric} = K_0 g \tag{2.80}$$

S Scalar Curvature in Local Coordinates Consider Scalar Curvature

$$S = \frac{1}{n} \sum_{i=1}^{n} \mathrm{Ric}(e_i, e_i)$$

where $\{e_i\}$ is an orthonormal basis of T_pM. Applying similar strategy (2.75) gives

$$\mathrm{Ric}(e_i, e_i) \overset{(2.77)}{=} \sum_{j=1}^{n} \mathrm{Ric}(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) g^{ij} = \sum_{j=1}^{n} \frac{1}{n} R_{ij} g^{ij}$$

Thus the Scalar Curvature writes

$$S = \frac{1}{n} \sum_{i,j} \mathrm{Ric}_{ij} g^{ij} = \frac{1}{n(n-1)} \sum_{i,j} R_{ij} g^{ij} = \frac{1}{n(n-1)} \sum_{i,j,k} R_{ikj}^k g^{ij} = \frac{1}{n(n-1)} \sum_{i,j,k,\ell} R_{ijk\ell} g^{ik} g^{j\ell} \tag{2.81}$$

S in Local Coordinates with Constant Sectional Curvature K_0. When (M, g) has constant sectional curvature K_0, directly using (2.80) gives

$$S = \frac{1}{n} \sum_{i,j} \mathrm{Ric}_{ij} g^{ij} \overset{(2.80)}{=} \frac{1}{n} \sum_{i,j} K_0 g_{ij} g^{ij} = K_0 \tag{2.82}$$

Examples

Example 2.4.4 (Two Dimension)**.** *For* $\dim M = 2$*, one has formula*

$$\begin{aligned} R &= R_{1212} (dx_1 \wedge dx_2) \otimes (dx_1 \wedge dx_2) \\ \mathrm{Ric} &= \frac{R_{1212}}{g_{11} g_{22} - g_{12}^2} g = K g \\ S &= \frac{R_{1212}}{g_{11} g_{22} - g_{12}^2} = K \end{aligned}$$

Proof. We carry out the calculation

$$\begin{aligned} S &= \frac{1}{2} \left(R_{1212} g^{11} g^{22} + R_{2112} g^{21} g^{12} + R_{1221} g^{12} g^{21} + R_{2121} g^{22} g^{11} \right) \\ &= \frac{1}{2} \left(R_{1212} g^{11} g^{22} - R_{1212} g^{21} g^{12} - R_{1212} g^{12} g^{21} + R_{1212} g^{22} g^{11} \right) \\ &= R_{1212} g^{11} g^{22} - R_{1212} (g^{12})^2 = \frac{R_{1212}}{g_{11} g_{22} - g_{12}^2} = K \end{aligned}$$

□

2.4.4 Levi-Civita Covariant Derivative on Tensors

Let (M, g) be Riemannian manifold. Let R be $(0, 4)$ Riemannian Curvature Tensor. Let ∇ denote the Levi-Civita Connection.

2.4.4.1 ∇R Covariant Derivative of Riemannian Curvature Tensor

From Covariant Derivative of (r, s)-tensors (1.108), we know ∇R defines a $(0, 5)$-tensor

$$\begin{aligned} \nabla R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) &\to C^\infty(M) \\ (X, Y, Z, T, W) &\mapsto (\nabla_W R)(X, Y, Z, T) \end{aligned} \tag{2.83}$$

where $\nabla_W R$ reads as in (1.102)

$$\begin{aligned} \nabla_W R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) &\to C^\infty(M) \\ (X, Y, Z, T) &\mapsto (\nabla_W R)(X, Y, Z, T) \end{aligned} \tag{2.84}$$

that is $\mathbb{R}$-linear, commutes with contractions (1.103) and satisfies Leibniz Rule (1.104).

Second Bianchi Identity

Proposition 2.4.5 (2nd Bianchi Identity). *∇R satisfies*

$$\nabla R(X, Y, Z, T, W) + \nabla R(X, Y, T, W, Z) + \nabla R(X, Y, W, Z, T) = 0 \tag{2.85}$$

Locally Symmetric Spaces Let (M, g) be Riemannian manifold. Let ∇ be the Levi-Civita connection on M. Let R be Riemannian curvature tensor (2.56) of M.

Definition 2.4.11 (Locally Symmetric Space). *We say M is locally symmetric space if*

$$\nabla R = 0$$

Proposition 2.4.6. *Let (M, g) be be Riemannian manifold.*

1. *Let M be locally symmetric space and let*

$$\gamma : [0, \ell) \to M$$

be geodesic of M. Then for any X, Y, Z parallel vector fields along γ

$$R(X, Y)Z$$

is also a parallel vector field along γ.

2. *If M is locally symmetric, connected, and $\dim M = 2$, then M has constant sectional curvature.*

3. *If M has constant sectional curvature, then M is locally symmetric space.*

2.4.4.2 Gradient, Divergence, Hessian, Laplacian

Let (M, g) be Riemannian manifold with ∇ Levi-Civita Connection.
Using that ∇ is compatible with the metric g, we know

$$g_{ij;k} = 0 \qquad \forall\ i,\ j,\ k \tag{2.86}$$

and that ∇ is symmetric, we get (2.23).
Let (U, ϕ) be a C^∞ chart for M, with coordinates $\phi = (x_1, \cdots, x_n)$. Denote

$$e_i = \frac{\partial}{\partial x_i}, \qquad e^j = dx_j$$

They're related under ∇ via (1.110)

$$\nabla_{e_i} e_j = \Gamma_{ij}^k e_k \qquad \nabla_{e_i} e^j = -\Gamma_{ik}^j e^k$$

We recall notations from Local Coordinates for ∇T (1.111)

$$T^{i_1, \cdots, i_r}_{j_1, \cdots, j_s; k} := (\nabla T)^{i_1, \cdots, i_r}_{j_1, \cdots, j_s, k} \overset{(1.108)}{=} (\nabla_{e_k} T)^{i_1, \cdots, i_r}_{j_1, \cdots, j_s}$$

$\nabla f = df$ **for** $f \in C^\infty(M)$ In particular, for $f \in C^\infty(M) = C^\infty(M, T_0^0 M)$, we denote

$$f_{;i} = \nabla_{e_i}(f) \overset{(1.105)}{=} e_i(f) = \frac{\partial f}{\partial x_i}$$

What is the covariant derivative ∇f? We check via (1.108)

$$\begin{aligned} \nabla f : \mathfrak{X}(M) &\to C^\infty(M) \\ X &\mapsto \nabla_X f = X(f) = df(X) \end{aligned}$$

so that

$$df \overset{(1.43)}{=} \nabla f = f_{;i} e^i = \sum_i \frac{\partial f}{\partial x_i} dx_i$$

Notice the differential as $df \in \Omega^1(M)$ agrees with $\nabla f \in \Omega^1(M)$.

In particular one may compute

$$|df|^2 = f_{;i} f_{;j} \langle e^i, e^j \rangle = f_{;i} f_{;j} g^{ij} \tag{2.87}$$

grad$f$ **Gradient and its local coordinates** We define the gradient operator

$$\begin{aligned} \text{grad} : C^\infty(M) &\to \mathfrak{X}(M) \\ f &\mapsto \text{grad}(f) \end{aligned}$$

where grad(f) makes sense via pairing w.r.t. the metric g

$$g\,(\text{grad} f, X) := \nabla_X(f) = X(f) = df(X) \qquad \forall\ X \in \mathfrak{X}(M) \tag{2.88}$$

In local coordinates

$$\text{grad} f = \sum_i (\text{grad} f)^i e_i$$

We would like to compute the coefficients $(\text{grad} f)^i$. Note

$$f_{;j} = e_j(f) = \frac{\partial f}{\partial x_j} = df(e_j) = \langle \text{grad} f, e_j \rangle = \sum_i (\text{grad} f)^i g_{ij}$$

Therefore the **gradient in local coordinates** are obtained by inverting the matrix g_{ij}

$$\text{grad} f = \sum_i (\text{grad} f)^i e_i = \sum_i f_;^i e_i = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i} \tag{2.89}$$

where

$$f_;^i := g^{ij} f_{;j}$$

One may compare with the differential in local coordinates

$$df = f_{;i} e^i = \sum_i \frac{\partial f}{\partial x_i} dx_i$$

div(Y) **Divergence and its local coordinates** Recall for any $Y \in \mathfrak{X}(M)$, the map $\nabla Y \in C^\infty(M, T_1^1 M)$ as (1.63)

$$\begin{aligned} \nabla Y : \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ X &\mapsto \nabla_X Y \end{aligned}$$

Now for any $p \in M$, it defines a linear operator

$$\begin{aligned} (\nabla Y)(p) : T_p M &\to T_p M \\ v &\mapsto \nabla_v Y(p) \end{aligned}$$

which takes the form of $(1,1)$-tensor. One is able to take its trace.

We define the divergence operator

$$\begin{aligned} \text{div} : \mathfrak{X}(M) &\to C^\infty(M) \\ Y &\mapsto \text{div}(Y) \end{aligned}$$

where

$$\begin{aligned} \text{div}(Y) : M &\to \mathbb{R} \\ p &\mapsto \text{div}(Y)(p) := \text{tr}(v \in T_p M \mapsto \nabla_v Y(p) \in T_p M) \end{aligned} \tag{2.90}$$

In local coordinates, we write

$$Y = Y^i e_i$$

Using

$$Y^i_{;j} = e_j(Y^i) + \Gamma^i_{jk} Y^k$$

as in (1.112), we also have in coordinates

$$\nabla Y = Y^i_{;j} e_i \otimes e^j$$

Therefore taking the trace (1.41) gives the **Divergence in local coordinates**

$$\operatorname{div}(Y) = Y^i_{;i} = e_i(Y^i) + \Gamma^i_{ik} Y^k = \sum_i \frac{\partial}{\partial x_i} Y^i + \sum_{i,k=1}^{n} \Gamma^i_{ik} Y^k \tag{2.91}$$

Lemma 2.4.5. *Given $Y \in \mathfrak{X}(M)$ and $\operatorname{div} Y$ as in (2.91)*

$$\operatorname{div} Y = \frac{1}{\sqrt{\det(g)}} \sum_i \frac{\partial}{\partial x_i} \left(\sqrt{\det(g)} Y^i \right) \tag{2.92}$$

Proof. Using Jacobi's Formula

$$\frac{\partial}{\partial x_i} (\det(g)) = \det(g) \operatorname{tr}(g^{-1} \frac{\partial g}{\partial x_i})$$

We look at

$$\begin{aligned}
\sum_{i=1}^{n} \Gamma^i_{ik} &= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} g^{ij} (g_{ij,k} + g_{kj,i} - g_{ik,j}) = \frac{1}{2} \sum_{i,j=1}^{n} g^{ij} \frac{\partial}{\partial x_k} g_{ij} + \frac{1}{2} \left(\sum_{ij} g^{ij} g_{kj,i} - g^{ji} g_{jk,i} \right) \\
&= \frac{1}{2} \sum_{i,j=1}^{n} g^{ij} \frac{\partial}{\partial x_k} g_{ij} = \frac{1}{2} \operatorname{tr}(g^{-1} \frac{\partial}{\partial x_k} g) = \frac{1}{2} \frac{1}{\det(g)} \frac{\partial}{\partial x_k} (\det(g)) \\
&= \frac{1}{2} \frac{\partial}{\partial x_k} \log(\det(g)) = \frac{\partial}{\partial x_k} \log(\sqrt{\det(g)}) = \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x_k} \left(\sqrt{\det(g)} \right)
\end{aligned}$$

Hence

$$\begin{aligned}
\operatorname{div}(Y) &= \sum_i \frac{\partial}{\partial x_i} Y_i + \sum_{i,k} \Gamma^i_{ik} Y^k \\
&= \sum_k \frac{\partial}{\partial x_k} Y_k + \sum_k \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x_k} \left(\sqrt{\det(g)} \right) Y^k \\
&= \frac{1}{\sqrt{\det(g)}} \sum_i \frac{\partial}{\partial x_i} \left(\sqrt{\det(g)} Y^i \right)
\end{aligned}$$

□

$\operatorname{Hess}(f)$ **Hessian and its local coordinates** Define the Hessian Operator

$$\begin{aligned}
\operatorname{Hess} : C^\infty(M) &\to C^\infty(M, T^0_2 M) \\
f &\mapsto \operatorname{Hess}(f) := \nabla \nabla f \equiv \nabla^2 f = \nabla df
\end{aligned}$$

where

$$\begin{aligned}
\operatorname{Hess}(f) : \mathfrak{X}(M) \times \mathfrak{X}(M) &\to C^\infty(M) \\
(X, Y) &\mapsto (\nabla df)(X, Y)
\end{aligned} \tag{2.93}$$

Using that Levi-Civita Connection is symmetric, and that (2.23), we know $\operatorname{Hess}(f)$ is symmetric $(0,2)$-tensor

$$\operatorname{Hess}(f) \in C^\infty(M, \operatorname{Sym}^2 T^* M)$$

In particular

$$\begin{aligned}
\operatorname{Hess}(f)(X, Y) = (\nabla df)(X, Y) &\overset{(1.108)}{=} (\nabla_Y df)(X) \overset{(1.106)}{=} Y(df(X)) - df(\nabla_Y X) \\
&\overset{(1.43)}{=} YX(f) - (\nabla_Y X) f \\
&\overset{(2.6)}{=} XY(f) - (\nabla_X Y) f \\
&= \operatorname{Hess}(f)(Y, X)
\end{aligned}$$

Define $f_{;ij}$ s.t.

$$\nabla\nabla f = \nabla df = \nabla(f_{;i}e^i) = \sum_{i,j} f_{;ij} e^i \otimes e^j$$

One may calculate **Hessian in local coordinates** as in (1.112)

$$f_{;ij} = e_j(f_{;i}) - \Gamma^k_{ij} f_k = \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_k \Gamma^k_{ij} \frac{\partial f}{\partial x_k} \tag{2.94}$$

Δf Laplacian and its local coordinates Define the Laplacian Operator

$$\begin{aligned} \Delta : C^\infty(M) &\to C^\infty(M) \\ f &\mapsto \Delta f := \mathrm{div}(\mathrm{grad} f) \end{aligned} \tag{2.95}$$

In local coordinates

$$\begin{aligned} \Delta f = \mathrm{div}(\mathrm{grad} f) &\overset{(2.89)}{=} \mathrm{div}(f_;^i e_i) \\ &= \mathrm{div}(g^{ij} f_{;j} e_i) = f^i_{;i} = f_{;ij} g^{ij} \end{aligned}$$

For $e_i = \frac{\partial}{\partial x_i}$ we have **Laplacian in local coordinates** as in (1.112)

$$\Delta f = \sum_{i,j} g^{ij} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_k \Gamma^k_{ij} \frac{\partial f}{\partial x_k} \right) \tag{2.96}$$

In particular for $g_{ij} = \delta_{ij}$ we recover

$$\Delta f = \sum_i \frac{\partial^2 f}{\partial {x_i}^2}$$

Lemma 2.4.6. *In local coordinates, for $f \in C^\infty(M)$*

$$\Delta f = \frac{1}{\sqrt{\det(g)}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(\sqrt{\det(g)} g^{ij} \frac{\partial f}{\partial x_j} \right) \tag{2.97}$$

Proof. Using $\Delta f = \mathrm{div}(\mathrm{grad} f)$ where

$$\mathrm{grad} f = \sum_{ij} g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}$$

plugging in (2.92) we have the result. □

2.5 Jacobi Fields

Let (M, g) be a Riemannian manifold. Consider any geodesic

$$\gamma : [0, a] \to M$$

Think of doing variations of the given geodesic γ

$$\begin{aligned} f : (-\varepsilon, \varepsilon) \times [0, a] &\to M \\ (s, t) &\mapsto f(s, t) \equiv f_s(t) \end{aligned}$$

where f_0 recovers our given geodesic

$$f_0(t) = \gamma(t)$$

Let's **impose the condition** that for any $s \in (-\varepsilon, \varepsilon)$

$$\begin{aligned} f_s : [0, a] &\to M \\ t &\mapsto f_s(t) \end{aligned}$$

also gives a geodesic.

We're interested in: how will our given geodesic γ vary to remain a geodesic under the given variation? In other words, we would like to study

$$\begin{aligned} J : [0, a] &\to TM \\ t &\mapsto \frac{\partial f}{\partial s}(0, t) \end{aligned}$$

We would like to derive an equation for $J \in C^\infty([0, a], \gamma^* TM)$, which is a C^∞ vector field along the geodesic γ.

2.5.1 Jacobi Equation

2.5.1.1 Derivation for Jacobi Equation

Parametrized Surface Let $A = (-\varepsilon, \varepsilon) \times [0, a] \subseteq \mathbb{R}^2$ be connected. Recall we've defined a parametrized surface in M

$$\begin{aligned} f : (-\varepsilon, \varepsilon) \times [0, a] \subseteq \mathbb{R}^2 &\to M \\ (s, t) &\mapsto f(s, t) \end{aligned}$$

and for $\frac{\partial}{\partial s}$, $\frac{\partial}{\partial t}$ smooth vector fields on A, we put

$$\frac{\partial f}{\partial s} := f_*(\frac{\partial}{\partial s}), \ \frac{\partial f}{\partial t} := f_*(\frac{\partial}{\partial t}) \ \in C^\infty(A, f^* TM)$$

so that for any $(s, t) \in A$,

$$\frac{\partial f}{\partial s}(s, t), \ \frac{\partial f}{\partial t}(s, t) \in T_{f(s,t)} M = (f^* TM)_{(s,t)}$$

Let ∇ now be the Levi-Civita connection. Denote $D = f^* \nabla$ the pullback connection on A, defined as (1.86) and (1.81)

$$D_X(f_* Y) = (f^* \nabla)_X (f_* Y)$$

which acts on $f_* Y$ via (1.114).

Hitting one Covariant Derivative. In view of (2.44), we've shown that

$$\frac{D}{ds} \frac{\partial f}{\partial t} = \frac{D}{dt} \frac{\partial f}{\partial s}$$

using that the Levi-Civita Connection ∇ is symmetric. In particular

$$\begin{aligned} \frac{D}{ds} \frac{\partial f}{\partial t} - \frac{D}{dt} \frac{\partial f}{\partial s} &= D_{\frac{\partial}{\partial s}} f_*(\frac{\partial}{\partial t}) - D_{\frac{\partial}{\partial t}} f_*(\frac{\partial}{\partial s}) \\ &= f^* \nabla_{\frac{\partial}{\partial s}} f_*(\frac{\partial}{\partial t}) - f^* \nabla_{\frac{\partial}{\partial t}} f_*(\frac{\partial}{\partial s}) \\ &= f_* \left([\frac{\partial}{\partial s}, \frac{\partial}{\partial t}] \right) = 0 \end{aligned} \tag{2.98}$$

Hitting two Covariant Derivatives. On the other hand, we differentiate once more to see the curvature term pops out. Recall the Riemannian curvature is defined via (2.51)

$$R_\nabla(X, Y)(Z) = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z$$

By definition of $R_{f^*\nabla} = f^* R_\nabla$ the pullback Riemannian Curvature (1.115)

$$(f^*R_\nabla)(\frac{\partial}{\partial s}, \frac{\partial}{\partial t})(f_*\frac{\partial}{\partial t}) = \frac{D}{dt}\frac{D}{ds}f_*(\frac{\partial}{\partial t}) - \frac{D}{ds}\frac{D}{dt}f_*(\frac{\partial}{\partial t}) - D_{[\frac{\partial}{\partial t},\frac{\partial}{\partial s}]}f_*(\frac{\partial}{\partial t})$$

From the previous computation (2.98), the First covariant derivative commutes

$$\frac{D}{ds}f_*(\frac{\partial}{\partial t}) = \frac{D}{dt}f_*(\frac{\partial}{\partial s})$$

Also, note $[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = 0$. Hence

$$R(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t})\frac{\partial f}{\partial t} = \frac{D^2}{dt^2}\frac{\partial f}{\partial s} - \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} \tag{2.99}$$

From now on we abuse of notation and write $R = R_\nabla$.
(2.98) and (2.99) are true for any C^∞ map $f : A \to M$.

Derivation of Jacobi Equation Now we impose our setup, **in addition that f_s is a geodesic for any** $s \in (-\varepsilon, \varepsilon)$, i.e.

$$\frac{D}{dt}\frac{\partial f_s}{\partial t} = \frac{D}{dt}\frac{\partial f}{\partial t}(s,t) = 0 \qquad \forall\ s \in (-\varepsilon, \varepsilon)$$

In particular the second term in (2.99) vanishes. Hence we're left with two terms

$$\frac{D^2}{dt^2}\frac{\partial f}{\partial s} + R(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s})\frac{\partial f}{\partial t} = 0 \tag{2.100}$$

If **set** $s = 0$, then since we've defined and required

$$J(t) := \frac{\partial f}{\partial s}(0,t)$$
$$\gamma'(t) = \frac{d}{dt}\gamma(t) = \frac{\partial f}{\partial t}(0,t)$$

Then (2.100) at $s = 0$ writes

$$\frac{D^2}{dt^2}J(t) + R(\gamma', J(t))\gamma' = 0 \qquad \forall\ t \in [0,a] \tag{2.101}$$

This is the Jacobi Equation.

Definition 2.5.1 (Jacobi Field)**.** *A C^∞ vector field $J(t)$ along a geodesic*

$$\gamma : [0,a] \to M$$

is called a Jacobi Field if it satisfies the Jacobi Equation (2.101)*.*

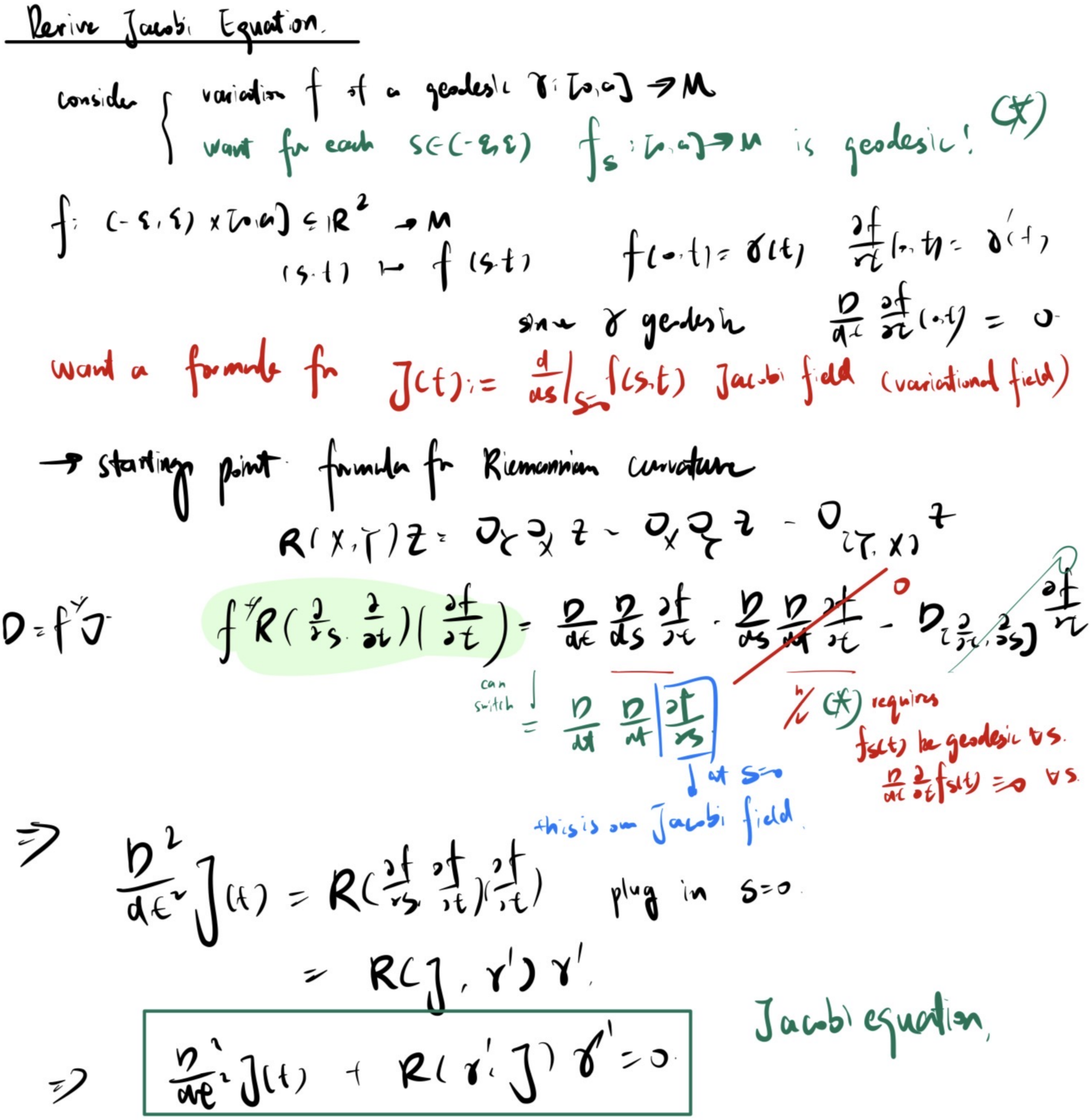


Figure 2.4: Jacobi Equation

2.5.1.2 Existence and Uniqueness of Jacobi Field

Proposition 2.5.1. *Let*

$$\gamma : [0, a] \to M$$

be a geodesic s.t.

$$\gamma(0) = p \qquad \gamma'(0) = v \in T_pM$$

In other words, the geodesic

$$\gamma(t) = \exp_p(tv)$$

is determined by the exponential map.

Then

1. *For any u, $w \in T_pM$, there exists a unique Jacobi Field J along γ s.t.*

$$J(0) = u, \qquad \frac{D}{dt}J(0) = w$$

In other words, a Jacobi Field is determined by initial conditions $J(0)$ and $\frac{D}{dt}J(0)$.

2. *If $J(t)$ is a Jacobi Field along γ, then there exists variational field* (2.103)

$$\begin{aligned} f : (-\varepsilon, \varepsilon) \times [0, a] &\to M \\ (s, t) &\mapsto f(s, t) = f_s(t) \end{aligned}$$

s.t.

(a) *for any $s \in (-\varepsilon, \varepsilon)$, $f_s : [0, a] \to M$ is a geodesic.*

(b) $f_0 = \gamma$.

(c) $\frac{\partial f}{\partial s}(0, t) = J(t)$.

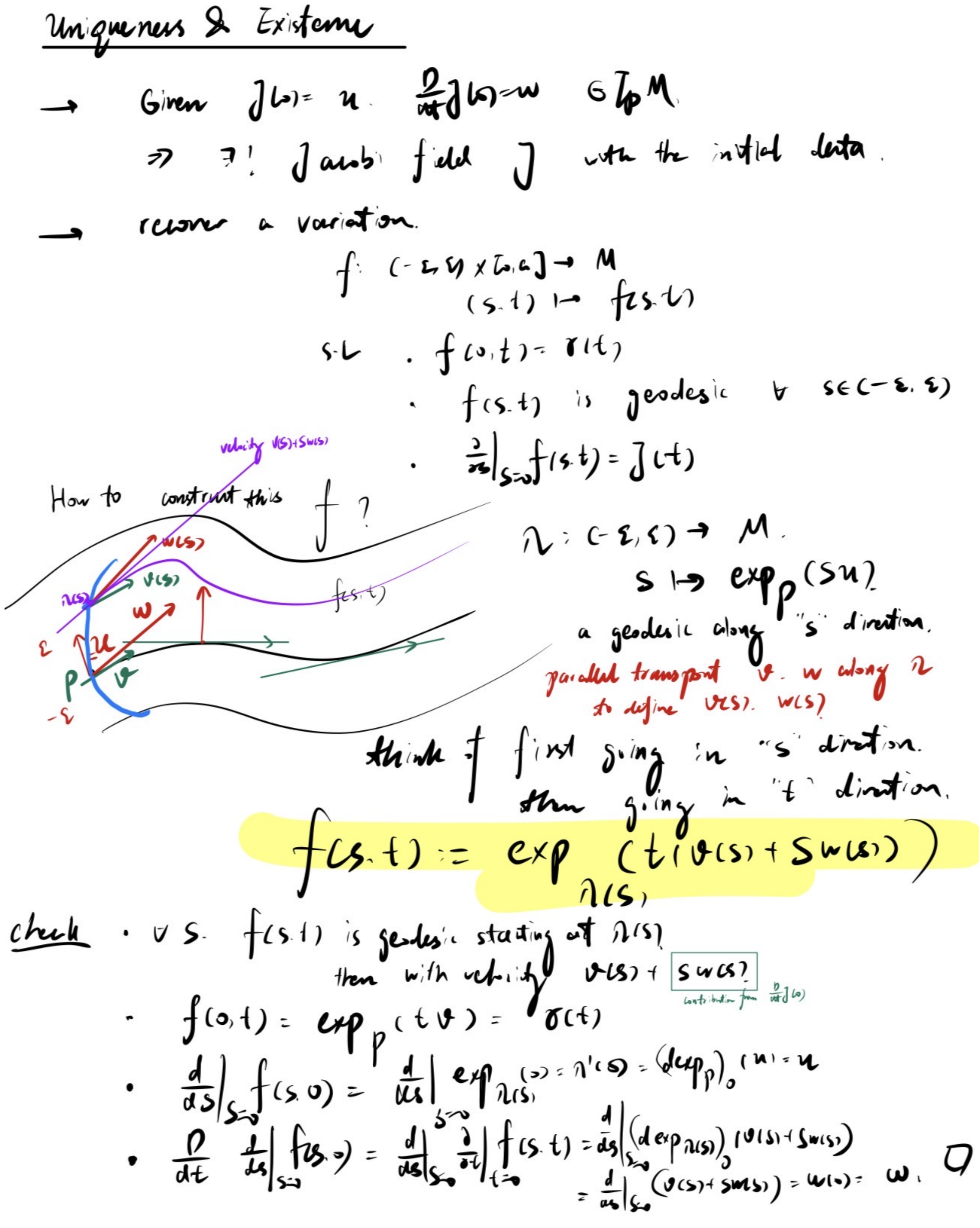


Figure 2.5: Jacobi Field Variation

Proof. 1. To get an ODE out of (2.101) we need to take $\{e_1, \cdots, e_n\}$ ONB of T_pM. Then we think of parallel transport. Let $e_1(t), \cdots, e_n(t)$ be the parallel transport of $e_1, \cdots, e_n$ along $\gamma(t)$, i.e.

$$\begin{cases} D_{\frac{\partial}{\partial t}} e_i(t) = 0 \\ e_i(0) = e_i \end{cases}$$

Hence $\{e_i(t)\}_{1 \le i \le n}$ forms an ONB of $T_{\gamma(t)}M$ for every $t \in [0, a]$. For any $J(t)$ as C^∞ vector fields along

$\gamma(t)$, we can write

$$J(t) = \sum_{i=1}^{n} f_i(t)e_i(t)$$

for $f_i : [0, a] \to \mathbb{R}$. Then $J(t)$ is a Jacobi Field iff (2.101) is satisfied iff

$$\sum_{i=1}^{n} f_i''(t)e_i(t) + f_i(t)R(\gamma'(t), e_i)\gamma'(t) = 0$$

We're using the fact that $e_i(t)$ are parallel transports.
Then we take inner product with $e_j(t)$ for each of these and selects

$$f_j''(t) + \sum_{i=1}^{n} f_i(t)R(\gamma'(t), e_j(t), \gamma'(t), e_j(t)) = 0 \qquad \forall\, j = 1, \cdots, n \tag{2.102}$$

Denote

$$A_{ij}(t) := R(\gamma'(t), e_j(t), \gamma'(t), e_j(t))$$

Then we write

$$f_j''(t) + \sum_{i=1}^{n} A_{ij}(t)f_i(t) = 0$$

Hence we have

$$\frac{d^2}{dt^2}\mathbf{f} + A\mathbf{f} = 0$$

where we apply Existence and Uniqueness of ODE.

2. Set $u := J(0)$ and $w := \frac{D}{dt}J(0)$. Let

$$\begin{aligned} \lambda : (-\varepsilon, \varepsilon) &\to M \\ s &\mapsto \lambda(s) := \exp_p(su) \end{aligned}$$

Let $v(s)$, $w(s)$ be parallel transport along $\lambda(s)$. Define

$$\begin{aligned} f : (-\varepsilon, \varepsilon) \times [0, a] &\to M \\ (s, t) &\mapsto \exp_{\lambda(s)}(t(v(s) + sw(s))) \end{aligned} \tag{2.103}$$

We need to check

(a) For each s, f_s is the unique geodesic that starts at $f_s(0) = \lambda(s)$ and with

$$f_s'(0) = v(s) + sw(s)$$

(b) $f_0(t) = \exp_{\lambda(0)}(t(v(0) + 0)) = \exp_p(tv) = \gamma(t)$.

(c) $\overline{J}(t) = \frac{\partial f}{\partial s}(0, t)$ is a Jacobi Field by our previous derivation. Check

$$\begin{aligned} \overline{J}(0) &= \frac{\partial f}{\partial s}(0, 0) = \lambda'(0) = u \\ \frac{D}{dt}\overline{J}(0) &= \frac{D}{dt}\frac{\partial f}{\partial s}(0, 0) = \frac{D}{ds}\frac{\partial f}{\partial t}(0, 0) = w(0) = w \end{aligned}$$

where the second line follows from

$$\begin{aligned} \frac{\partial f}{\partial t}(s, 0) &= (d\exp_{\lambda(s)})_0(v(s) + sw(s)) \overset{(2.43)}{=} v(s) + sw(s) \\ \frac{D}{ds}\frac{\partial f}{\partial t}(s, 0) &= w(s) \qquad \text{both } v(s) \text{ and } w(s) \text{ are parallel transports} \end{aligned}$$

Since they have same initial conditions, we conclude by uniqueness.

□

Example of Jacobi Field

Example 2.5.1. *In $(\mathbb{R}^n, g_0)$, the geodesics are*

$$\gamma(t) = p + tv \qquad p, v \in \mathbb{R}^n$$

Now our Jacobi Field writes

$$J(t) = u + tw \qquad \forall\ u, w \in \mathbb{R}^n$$

and f writes

$$f(s,t) = p + su + t(v + sw)$$

for fixed p, v, u, w. f is in fact

$$f(s,t) = \exp_{p+su}\left(t(v + sw)\right)$$

2.5.1.3 Properties of Jacobi Field

Jacobi Field with $J(0) = 0$ In the special case $u = 0$, $J(t)$ Jacobi field along $\gamma(t)$ with

$$\gamma(0) = p \qquad \gamma'(0) = v \qquad J(0) = 0 \qquad J'(0) = w$$

satisfies

$$\lambda(s) = p \qquad \text{and} \qquad f(s,t) = \exp_p(t(v(s) + sw(s)))$$

Thus

$$J(t) = \frac{\partial f}{\partial s}(0,t) = (d\exp_p)_{tv}(tw) \tag{2.104}$$

For λ fixed its easier to take derivatives.

In particular, $|(d\exp_p)_v(w)|$ denotes intuitively the rate of spreading of the geodesic

$$t \mapsto \exp_p(tv(s))$$

which starts from p.

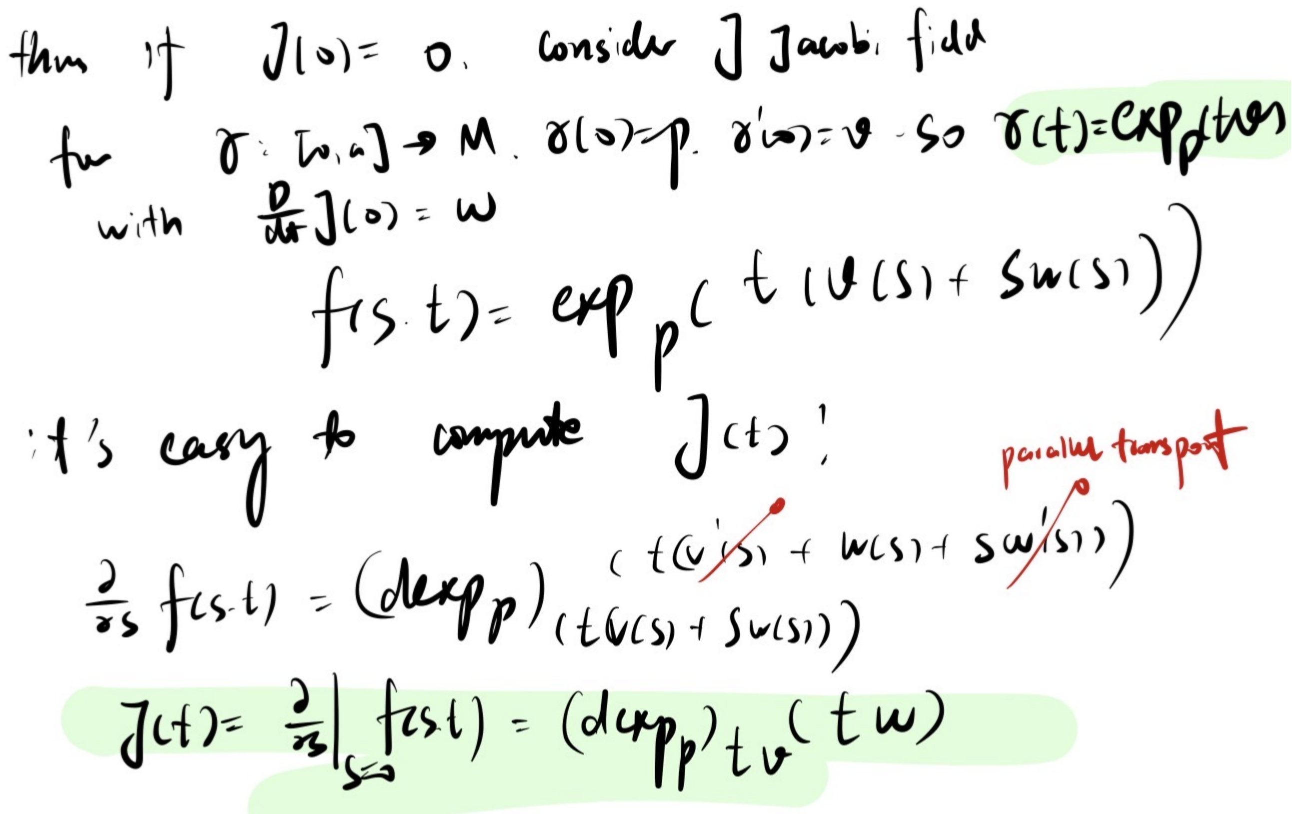


Figure 2.6: Jacobi Field for $J(0) = 0$.

$\langle J,\gamma'\rangle(t)$ as linear function

Lemma 2.5.1 ([dC92] Proposition 5.3.6)**.** *Let*

$$\gamma : [0,a] \to M$$

be geodesic in M, J Jacobi field along γ. Then $\langle J,\gamma'\rangle(t)$ is linear function in t

$$\langle J,\gamma'\rangle(t) = \langle J(0),\gamma'(0)\rangle + t\langle J'(0),\gamma'(0)\rangle \qquad J'(0) := \frac{D}{dt}J(0) \tag{2.105}$$

Proof. Let

$$\begin{aligned} f(t) &= \langle J,\gamma'\rangle(t) \\ f'(t) &= \langle J',\gamma'\rangle(t) \qquad \text{using } D_{\frac{\partial}{\partial t}}\gamma' = 0 \\ f''(t) &= \langle J'',\gamma'\rangle(t) \overset{(2.101)}{=} \langle -R(\gamma',J)\gamma',\gamma'\rangle = 0 \qquad \text{by anti-symmetry of } R \end{aligned}$$

Hence f is a linear function in t. □

Corollary 2.5.1 ([dC92] Corollary 5.3.7)**.** *If $\langle J,\gamma'\rangle(t_1) = \langle J,\gamma'\rangle(t_2)$ for t_1, $t_2 \in [0,a]$, $t_1 \neq t_2$, then $\langle J,\gamma'\rangle$ does not depend on t.*

In particular if $J(0) = J(a) = 0$, then

$$\langle J,\gamma'\rangle \equiv 0$$

Decomposition of J into γ' and $t\gamma'(t)$ Let γ be geodesic in M. In fact γ' and $t\gamma'(t)$ are examples of Jacobi Fields along $\gamma(t)$.

Indeed

$$\frac{D^2}{dt^2}\gamma'(t) + R(\gamma',\gamma')\gamma' = 0 \qquad \text{using } \frac{D}{dt}\gamma' = 0$$

and

$$\begin{aligned} \frac{D^2}{dt^2}(t\gamma'(t)) &= \frac{D}{dt}(\gamma'(t)) = 0 \\ R(\gamma',t\gamma')\gamma' &= 0 \end{aligned}$$

In particular given initial conditions we can explicitly write J using γ' and $t\gamma'$. For any J Jacobi field along γ we have decomposition

$$J(t) = (\langle J(0),\gamma'(0)\rangle + t\langle J'(0),\gamma'(0)\rangle)\frac{\gamma'(t)}{|\gamma'(0)|^2} + J^{\perp}(t) \qquad \text{where } \langle J^{\perp}(t),\gamma'(t)\rangle = 0 \tag{2.106}$$

Hence it suffices to consider Jacobi Fields normal w.r.t. $\gamma'(t)$.

Corollary 2.5.2 ([dC92] Corollary 5.3.8)**.** *Suppose $J(0) = 0$. Then $\langle J'(0),\gamma'(0)\rangle = 0$ iff*

$$\langle J,\gamma'\rangle \equiv 0$$

Application for Jacobi Field induced by Killing vector field

Proposition 2.5.2 ([dC92] Exercise 5.8)**.** *Let*

$$\gamma : [0,a] \to M$$

be a geodesic and let X be a Killing Vector field on M (2.49). Then

(a) The restriction $X(\gamma(s))$ of X to $\gamma(s)$ is a Jacobi Field along γ.

(b) As a consequence of above, if M is connected and there exists $p \in M$ s.t.

$$X(p) = 0 \qquad \textit{and} \qquad \nabla_Y X(p) = 0 \qquad \forall\ Y \in T_pM$$

Then $X \equiv 0$ on M.

Proof. 1. We first show (a). Since X is a Killing Vector Field, its flow

$$\varphi_t : U \subset M \to M \qquad q \mapsto \varphi(t,q) = \varphi_t(q) \qquad \forall\, t \in (-\varepsilon, \varepsilon)$$

is a 1-parameter subgroup of isometries on (M,g) with $\varphi_0 = \mathrm{Id}$. The flow φ_t relates to X via

$$\varphi_t(q) \text{ is the trajectory of } X \text{ passing through } q \text{ at } t=0 \text{ for any } q \in U$$

or in other words using X as integral curve

$$X(\varphi(t,\gamma(s))) = \frac{\partial}{\partial t}\varphi(t,\gamma(s)) = \frac{d}{dt}\varphi_t(\gamma(s))$$
$$\gamma(s) = \varphi(0,\gamma(s))$$

Since image of the geodesic γ by a family of isometries remains a geodesic,

$$\phi_t(s) = \varphi_t(\gamma(s)) \qquad \forall\, t \in (-\varepsilon, \varepsilon)$$

are a 1-parameter family of geodesics on (M,g). Thus restriction of X to γ is a variational field

$$X(\gamma(s)) = \left.\frac{d}{dt}\right|_{t=0} \phi_t(s)$$

of γ by geodesics. Hence $X(\gamma(s))$ is Jacobi Field along γ.

2. We prove (b). From (a) we know

$$J(s) := X(\gamma(s)) \qquad \forall\, s \in [0,a]$$

defines a Jacobi Field along γ. We first conduct a simple computation using Definition of pullback section

$$\frac{D}{ds}J(s) = \gamma^*\nabla_{\frac{d}{ds}}(X(\gamma(s))) = \gamma^*\nabla_{\frac{d}{ds}}(\gamma^* X(s))$$
$$= \nabla_{\gamma_*\frac{d}{ds}} X = \nabla_{\gamma'(s)} X$$

Notice assumptions imply

$$X(p) = 0 \implies X(\gamma(0)) = J(0) = 0$$

and

$$\nabla_Y X(p) = 0 \qquad \forall\, Y \in T_{\gamma(0)}M \implies \text{choosing } Y = \gamma'(0) \qquad \nabla_{\gamma'(0)}X = \frac{D}{ds}J(0) = 0$$

Hence by Existence and Uniqueness Theorem,

$$J(s) = X(\gamma(s)) \equiv 0$$

is the unique Jacobi Field along γ. But now since M is connected, for any other point $q \in M$, there exists smooth curve connecting p to q. Covering the curve by geodesic segments and applying previous argument, one obtain

$$X(q) = 0 \qquad \forall\, q \in M$$

□

2.5.2 Jacobi Fields on Constant Sectional Curvature Manifolds

Let (M,g) be Riemannian manifold with constant sectional curvature K. Let

$$\gamma : [0,a] \to M$$

be a normalized geodesic, i.e., $|\gamma'(t)| = 1$.

Derivation of Jacobi Field for Constant Sectional Curvature K Take a Jacobi field J along γ that is normal to γ', in particular

$$\langle J, \gamma' \rangle(t) = \langle J(0), \gamma'(0) \rangle + t \langle \frac{D}{dt} J(0), \gamma'(0) \rangle = 0 \qquad \forall\, t \in [0, a]$$

Let V be C^∞ vector field along γ. Then using an equivalent condition for constant sectional curvature and Riemannian Curvature

$$\begin{aligned} \langle R(\gamma', J)\gamma', V \rangle &= R(\gamma', J, \gamma', V) \\ &\overset{(2.67)}{=} K \left(\langle \gamma', \gamma' \rangle \langle J, V \rangle - \langle \gamma', J \rangle \langle \gamma', V \rangle \right) \\ &= \langle KJ, V \rangle \end{aligned}$$

where the last step uses $\langle \gamma', \gamma' \rangle = 1$ and $\langle \gamma', J \rangle = 0$.

Hence

$$R(\gamma', J)\gamma' = KJ$$

and our Jacobi Equation (2.101) writes

$$\frac{D^2}{dt^2} J + KJ = 0 \tag{2.107}$$

Solving Jacobi Equation for constant sectional curvature with Initial $\frac{D}{dt}J(0)$ Let $w(t)$ be parallel transport of w along $\gamma(t)$ with $w(0) = w$ so

$$\langle w(t), \gamma'(t) \rangle = 0 \qquad |w(t)| = 1$$

We look for solutions of the form

$$J(t) = f(t) w(t) \qquad f : [0, a] \to \mathbb{R}$$

Then equation (2.107) writes, given nontrivial initial condition $\frac{D}{dt}J(0) = w$

$$\begin{aligned} \frac{D^2}{dt^2} J + KJ &= 0 \\ J(0) &= 0 \\ \frac{D}{dt} J(0) &= w \end{aligned}$$

This is equivalent to system of equations on f

$$\begin{aligned} \frac{d^2}{dt^2} f(t) + Kf(t) &= 0 \\ f(0) &= 0 \\ f'(0) &= 1 \end{aligned}$$

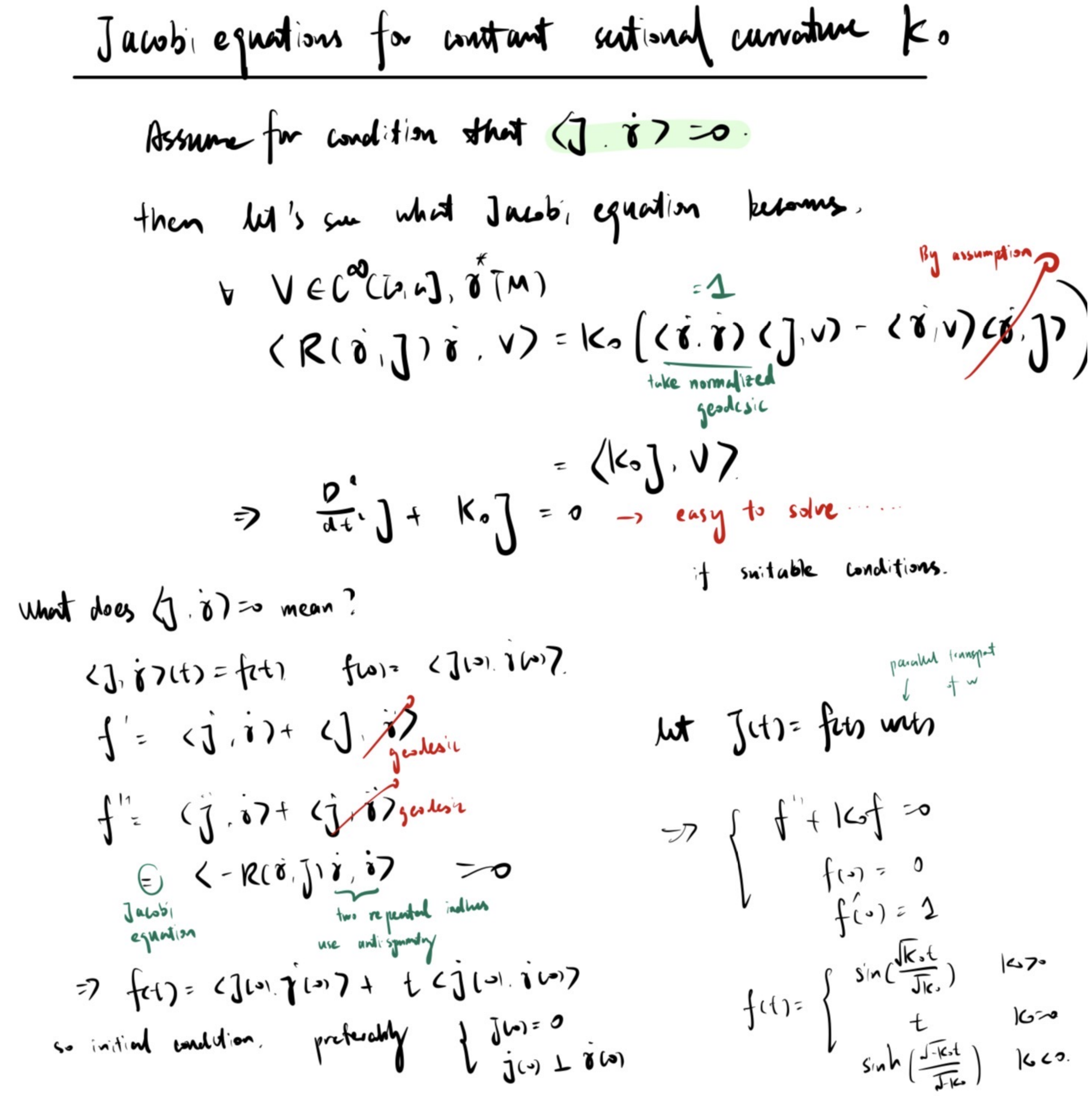


Figure 2.7: Jacobi Equation for Constant Sectional Curvature

Now this has unique solution. So the Jacobi field

$$J = fw$$

that we find this way is the unique solution of

$$\frac{D^2}{dt^2}J + KJ = 0$$

Solutions to system of equations in f and J are given by

$$f(t) = \begin{cases} \frac{\sin(\sqrt{K}t)}{\sqrt{K}} & K > 0 \\ t & K = 0 \\ \frac{\sinh(\sqrt{-K}t)}{\sqrt{-K}} & K < 0 \end{cases}$$

$$J(t) = \begin{cases} \frac{\sin(\sqrt{K}t)}{\sqrt{K}} w(t) & K > 0 \\ tw(t) & K = 0 \\ \frac{\sinh(\sqrt{-K}t)}{\sqrt{-K}} w(t) & K < 0 \end{cases} \tag{2.108}$$

Solving Jacobi Field for constant sectional curvature with Initial $J(0)$ Similarly, if write

$$J(t) = f(t)u(t)$$

for $u(t)$ the parallel transport of u along γ, it takes initial conditions

$$\frac{D^2}{dt^2} J + KJ = 0$$
$$J(0) = u$$
$$\frac{D}{dt} J(0) = 0$$

Then this corresponds to

$$\frac{d^2}{dt^2} f(t) + Kf(t) = 0$$
$$f(0) = 1$$
$$f'(0) = 0$$

Solutions write

$$f(t) = \begin{cases} \frac{\cos(\sqrt{K}t)}{\sqrt{K}} & K > 0 \\ 1 & K = 0 \\ \frac{\cosh(\sqrt{-K}t)}{\sqrt{-K}} & K < 0 \end{cases}$$

$$J(t) = \begin{cases} \frac{\cos(\sqrt{K}t)}{\sqrt{K}} u(t) & K > 0 \\ u(t) & K = 0 \\ \frac{\cosh(\sqrt{-K}t)}{\sqrt{-K}} u(t) & K < 0 \end{cases}$$

In general, it's a combination between these two solutions. What we did here is the orthogonal part in 2.106.

Geodesic Polar Coordinates We would like to define Geodesic Polar Coordinates on a manifold (M, g).

Sphere $\mathbb{S}^2$ Take $\mathbb{S}^2$ round sphere of radius 1. Take $p = (0, 0, 1)$ to be north pole. Consider $v \in T_p\mathbb{S}^2$. The exponential map sends

$$\exp_p : T_p\mathbb{S}^2 \to \mathbb{S}^2$$
$$\{\text{circles of radius } \rho \text{ centered at origin}\} \mapsto \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = \sin^2(\rho),\ z = \cos(\rho)\}$$

Then let (ρ, θ) be polar coordinates on $T_p\mathbb{S}^2 = \mathbb{R}^2$. By Gauss Lemma (2.45)

$$\exp_p^*(dx^2 + dy^2 + dz^2) = d\rho^2 + \sin^2\rho d\theta^2$$

Sphere $\mathbb{S}^2(\frac{1}{\sqrt{K}})$ More generally, given $K > 0$, consider sphere of radius $\frac{1}{\sqrt{K}}$

$$\mathbb{S}^2(\frac{1}{\sqrt{K}}) := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = \frac{1}{K}\}$$

This has constant sectional curvature K. Let $p = (0, 0, \frac{1}{\sqrt{K}})$ and the exponential map

$$\exp_p : T_p\mathbb{S}^2(\frac{1}{\sqrt{K}}) \to \mathbb{S}^2(\frac{1}{\sqrt{K}})$$
$$\{\text{circles of radius } \rho\} \mapsto \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = \frac{\sin^2(\sqrt{K}\rho)}{\sqrt{K}},\ z = \frac{\cos(\sqrt{K}\rho)}{\sqrt{K}}\}$$

Let (ρ, θ) be the polar coordinates on $\mathbb{R}^2 = T_p\mathbb{S}^2(\frac{1}{\sqrt{K}})$, then

$$\exp_p^*(dx^2 + dy^2 + dz^2) = d\rho^2 + \left(\frac{\sin(\sqrt{K}\rho)}{\sqrt{K}}\right)^2 d\theta^2 \tag{2.109}$$

Geodesic Polar Coordinates for Manifold with Constant Sectional Curvature.
In general, we want to define the geodesic polar coordinates on M as pullback metric g under the following map

$$\begin{aligned} F : (0, \delta) \times \mathbb{S}^{n-1} &\to B_\delta(p) = \{\text{normal ball centered at } p \text{ with radius } \delta > 0\} \subseteq M \\ (\rho, v) &\mapsto \exp_p(\rho v) \end{aligned}$$

We compute the differential, w.r.t. the basis vector $\{\frac{\partial}{\partial\rho}, w\}$ for $w \in T_v\mathbb{S}^{n-1}$

$$\begin{aligned} (dF_{(\rho,v)})(\frac{\partial}{\partial\rho}) &= (d\exp_p)_{\rho v}(v) \\ (dF_{(\rho,v)})(w) &= (d\exp_p)_{\rho v}(\rho w) \qquad \text{where } w \in T_v\mathbb{S}^{n-1} = \{w \in \mathbb{R}^n \mid \langle w, v\rangle = 0\} \end{aligned}$$

Then we can describe what the differential map does.

$$dF_{(\rho,v)} : T_{(\rho,v)}((0, \delta) \times \mathbb{S}^{n-1}) = \mathbb{R}\frac{\partial}{\partial\rho} \oplus T_v\mathbb{S}^{n-1} \to T_{F(\rho,v)}M$$

Recall special case $u = 0$ yields (2.104). Hence in fact $(dF_{(\rho,v)})(w)$ is the Jacobi Field with formula (2.108)

$$(dF_{(\rho,v)})(w) = (d\exp_p)_{\rho v}(\rho w) = f_K(\rho)w(\rho v)$$

In particular, we've used Gauss Lemma which says exponential map is isometry

$$\begin{aligned} \langle (d\exp_p)_v(v), (d\exp_p)_v(v)\rangle &= \langle v, v\rangle = 1 \\ \langle (d\exp_p)_v(v), (d\exp_p)_v(\rho w)\rangle &= \langle v, \rho w\rangle = 0 \end{aligned}$$

Let (M, g) be our manifold with metric g. We thus define the geodesic polar coordinates on M as

$$F^*g = (\exp_p)^*g = d\rho^2 + f_K^2(\rho)g_{\text{can}}^{\mathbb{S}^{n-1}} = \begin{cases} d\rho^2 + \frac{\sin^2(\sqrt{K}\rho)}{K}g_{\text{can}}^{\mathbb{S}^{n-1}} & K > 0 \\ d\rho^2 + \rho^2 g_{\text{can}}^{\mathbb{S}^{n-1}} & K = 0 \\ d\rho^2 + \frac{\sinh^2(\sqrt{-K}\rho)}{-K}g_{\text{can}}^{\mathbb{S}^{n-1}} & K < 0 \end{cases}$$

2.5.3 Taylor Expansion of g_{ij} in Local Coordinates

Let (M, g) be Riemannian Manifold. Given an orthonormal frame, the metric takes the form

$$g_{ij} = \delta_{ij}$$

Now using ∇ is compatible with the metric (2.86) one get

$$\frac{\partial g_{ij}}{\partial x_k} = 0$$

Taylor Expansion in $|J(t)|^2$ We want to look at its Taylor Expansion. Take $p \in M$ Let

$$\gamma : [0, a] \to M$$

be geodesic with

$$\gamma(0) = p \qquad \gamma'(0) = v$$

Let $J(t)$ be Jacobi Field along $\gamma(t)$ with

$$J(0) = 0 \qquad \frac{D}{dt}J(0) = w$$

Therefore in particular

$$\gamma(t) = \exp_p(tv) \qquad J(t) = (d\exp_p)_{tv}(tw)$$

Proposition 2.5.3 (Taylor Expansion of $|J(t)|^2$ in Riemannian curvature)**.**

$$
\begin{aligned}
|J(t)|^2 &= \langle w,w\rangle t^2 - \frac{1}{3}R(v,w,v,w)t^4 - \frac{1}{6}(\nabla_v R)(v,w,v,w)t^5 && (2.110)\\
&+ \left(\frac{2}{45}\langle R(v,w)v, R(v,w)v\rangle - \frac{1}{20}(\nabla_v\nabla_v R)(v,w,v,w)\right)t^6 + o(t^6) && (2.111)
\end{aligned}
$$

Proof. Let $f(t) = \langle J(t), J(t)\rangle$. Need to compute $f^{(k)}(0)$ for $0 \le k \le 6$.

$$
\begin{aligned}
f'(t) &= 2\langle J'(t), J(t)\rangle\\
f''(t) &= 2\langle J^{(2)}(t), J(t)\rangle + 2\langle J'(t), J'(t)\rangle\\
f^{(3)}(t) &= 2\langle J^{(3)}(t), J(t)\rangle + 6\langle J^{(2)}(t), J(t)\rangle\\
f^{(4)}(t) &= 2\langle J^{(4)}(t), J(t)\rangle + 8\langle J^{(3)}(t), J'(t)\rangle + 6\langle J^{(2)}(t), J^{(2)}(t)\rangle\\
f^{(5)}(t) &= 2\langle J^{(5)}(t), J(t)\rangle + 10\langle J^{(4)}(t), J'(t)\rangle + 20\langle J^{(3)}(t), J^{(2)}(t)\rangle\\
f^{(6)}(t) &= 2\langle J^{(6)}(t), J(t)\rangle + 12\langle J^{(5)}(t), J'(t)\rangle + 30\langle J^{(4)}(t), J^{(2)}(t)\rangle + 20\langle J^{(3)}(t), J^{(3)}(t)\rangle
\end{aligned}
$$

We have $J(0) = 0$, $J'(0) = w$. Notice we're about to compute

$$
\begin{aligned}
\frac{D}{dt}(R(\gamma', J)\gamma') &\equiv \nabla_{\gamma'}(R(\gamma', J)\gamma')\\
&= (\nabla_{\gamma'}R)(\gamma', J)\gamma' + R(\frac{D}{dt}\gamma', J)\gamma' + R(\gamma', \frac{D}{dt}J)\gamma' + R(\gamma', J)\frac{D}{dt}\gamma'\\
&= (\nabla_{\gamma'}R)(\gamma', J)\gamma' + R(\gamma', \frac{D}{dt}J)\gamma' \qquad \text{using that } \gamma \text{ is geodesic}
\end{aligned}
$$

where the second line follows from

$$
\begin{aligned}
\nabla_W(R(X,Y,Z,T)) &= (\nabla_W R)(X,Y,Z,T) + R(\nabla_W X, Y, Z, T) + \cdots + R(X,Y,Z,\nabla_W T) \qquad \forall\, T\\
\nabla_W(\langle R(X,Y)Z, T\rangle) &= (\nabla_W R)(X,Y,Z,T) + R(\nabla_W X, Y, Z, T) + \cdots + R(X,Y,Z,\nabla_W T)\\
\langle \nabla_W(R(X,Y)Z), T\rangle &= \nabla_W(\langle R(X,Y)Z,T\rangle) - R(X,Y,Z,\nabla_W T) \qquad \text{by definition}\\
&= \langle(\nabla_W R)(X,Y)Z, T\rangle + \langle R(\nabla_W X, Y)Z, T\rangle + \langle R(X, \nabla_W Y)Z, T\rangle + \langle R(X,Y)\nabla_W Z, T\rangle
\end{aligned}
$$

Thus

$$
\begin{aligned}
J'' &= -R(\gamma', J)\gamma' \implies J''(0) = -R(v,0)v = 0\\
J^{(3)} &= -R'(\gamma', J)\gamma' - R(\gamma', J')\gamma' \implies J^{(3)}(0) = -R(v,w)v\\
J^{(4)} &= -R''(\gamma', J)\gamma' - 2R'(\gamma', J')\gamma' - R(\gamma', J'')\gamma' \implies J^{(4)}(0) = -2(\nabla_v R)(v,w)v\\
J^{(5)} &= -R'''(\gamma', J)\gamma' - 3R''(\gamma', J')\gamma' - 3R'(\gamma', J'')\gamma' - R(\gamma', J''')\gamma' \implies J^{(5)}(0) = -3(\nabla_v\nabla_v R)(v,w)v + R(v, R(v,w)w)w
\end{aligned}
$$

So

$$
\begin{aligned}
f(0) &= 0\\
f'(0) &= 0\\
f''(0) &= 2\langle w, w\rangle\\
f^{(3)}(0) &= 0\\
f^{(4)}(0) &= 8\langle -R(v,w)v, w\rangle\\
f^{(5)}(0) &= 10\langle -2(\nabla_v R)(v,w)v, w\rangle\\
f^{(6)}(0) &= 12\langle -3\nabla_v\nabla_v R(v,w)v + R(v, R(v,w)v)v, w\rangle + 20\langle R(v,w)v, R(v,w)v\rangle\\
&= -36\langle \nabla_v\nabla_v R(v,w)v, w\rangle + 32\langle R(v,w)v, R(v,w)v\rangle
\end{aligned}
$$

Then

$$
\begin{aligned}
f(t) &= \frac{1}{2!}2\langle w,w\rangle t^2 - \frac{1}{4!}8\langle R(v,w)v, w\rangle t^4 - \frac{1}{5!}20\langle(\nabla_v R)(v,w)v, w\rangle t^5\\
&+ \frac{1}{6!}\left(-36\langle \nabla_v\nabla_v R(v,w)v, w\rangle + 32\langle R(v,w)v, R(v,w)v\rangle\right)t^6 + o(t^6)
\end{aligned}
$$

□

Corollary 2.5.3 (Taylor Expansion of $|J(t)|^2$ in Sectional Curvature)**.** *Take v, w orthonormal, i.e.*

$$|v| = |w| = 1 \qquad \langle v, w\rangle = 0$$

Let

$$\sigma = \mathbf{Span}(v, w)$$

Then for $t > 0$

$$|J(t)|^2 = t^2 - \frac{1}{3}K(\sigma)t^4 + o(t^4)$$

$$|J(t)| = t\left(1 - \frac{1}{3}K(\sigma)t^2 + o(t^2)\right)^{\frac{1}{2}} = t - \frac{1}{6}K(\sigma)t^3 + o(t^3) \qquad \text{using } (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + o(x^2) \tag{2.112}$$

Taylor Expansion for the Metric g_{ij} in Normal Coordinates Now write

$$J(t) = (d\exp_p)_{tv}(tw)$$

One has

$$f(t) = \langle J(t), J(t)\rangle = \langle (d\exp_p)_{tv}(tw), (d\exp_p)_{tv}(tw)\rangle = t^2\langle (d\exp_p)_{tv}(w), (d\exp_p)_{tv}(w)\rangle$$

Define

$$B(u, w) := \langle (d\exp_p)_{tv}(u), (d\exp_p)_{tv}(w)\rangle$$

This is symmetric bilinear form on T_pM. Now the diagonal quadratic form is

$$Q(w) = \langle (d\exp_p)_{tv}(w), (d\exp_p)_{tv}(w)\rangle$$

By polarizing for u, $w \in T_pM$, i.e., by polarized identity

$$B(u, w) = \frac{1}{2}\left(Q(u + w) - Q(u) - Q(w)\right)$$

One has

$$\begin{aligned}\langle (d\exp_p)_{tv}(u), (d\exp_p)_{tv}(w)\rangle &\overset{(2.111)}{=} \langle u, w\rangle - \frac{1}{3}R(v, w, v, u)t^2 - \frac{1}{6}(\nabla_v R)(v, w, v, u)t^3 \\ &+ \left(\frac{2}{45}\langle R(v, w)v, R(v, u)v\rangle - \frac{1}{20}(\nabla_v\nabla_v R)(v, w, v, u)\right)t^4 + o(t^4)\end{aligned}$$

Now for $|v|$ small. One can deduce via Taylor Expansion around 0

$$\begin{aligned}\langle (d\exp_p)_v(u), (d\exp_p)_v(w)\rangle &= \langle u, w\rangle - \frac{1}{3}R(v, w, v, u) - \frac{1}{6}(\nabla_v R)(v, w, v, u) \\ &+ \frac{2}{45}\langle R(v, w)v, R(v, u)v\rangle - \frac{1}{20}(\nabla_v\nabla_v R)(v, w, v, u) + o(|v|^4)\end{aligned}$$

Let $\{e_1, \cdots, e_n\}$ as ONB basis for T_pM. Consider normal ball $B_\delta(p) \subseteq M$ and point $q \in B_\delta(p)$. Then q is viewed as endpoint of geodesic starting from p with velocity as linear combination of e_i. In particular

$$q = \exp_p(\sum_k x_k e_k) \in B_\delta(p)$$

where $\sum_k x_k e_k \in T_pM$, and x_k are the normal coordinates associated to $\{e_1, \cdots e_n\}$.

Then

$$\left.\frac{\partial}{\partial x_i}\right|_q = (d\exp_p)_{\sum_k x_k e_k}(e_i)$$

So

$$g_{ij}(x_1, \cdots, x_n) = \langle \left.\frac{\partial}{\partial x_i}\right|_q, \left.\frac{\partial}{\partial x_j}\right|_q\rangle = \langle (d\exp_p)_{\sum_k x_k e_k}(e_i), (d\exp_p)_{\sum_k x_k e_k}(e_j)\rangle$$

Now apply with $v = \sum_k x_k e_k \in T_pM$ for $|x_k|$ small, and with $u = e_i$, $w = e_j$. We finally obtain using the formula

$$\begin{aligned}g_{ij}(x) &= \delta_{ij} - \frac{1}{3}\sum_{k,\ell} R(e_i, e_k, e_j, e_\ell)x_k x_\ell - \frac{1}{6}\sum_{\ell,m,k} R_{\ell imj,k}x_\ell x_m x_k \\ &+ \frac{2}{45}\sum_{\ell,k,r,s,m} R_{ik\ell m}R_{jrsm}x_k x_\ell x_r x_s - \frac{1}{20}\sum_{\ell,r,m,k} R_{\ell jri,mk}x_\ell x_r x_m x_k + o(|x|^4) \\ &= \delta_{ij} - \frac{1}{3}R_{ikj\ell}(p)x_k x_\ell - \frac{1}{6}R_{ikj\ell,m}(p)x_k x_\ell x_m \\ &+ \frac{2}{45}R_{ik\ell m}(p)R_{jrsm}(p)x_k x_\ell x_r x_s - \frac{1}{20}R_{ikj\ell,rs}(p)x_k x_\ell x_r x_s + o(|x|^4)\end{aligned}$$

Taylor Expansion for $\det(g_{ij})$ in Normal Coordinates Note

$$\det g_{ij} = \exp(\mathrm{tr}(\log(g_{ij})))$$

One has

$$g(x) = I + g^{(2)}(x) + g^{(3)}(x) + g^{(4)}(x) + o(|x|^4)$$
$$\log g(x) = g^{(2)}(x) + g^{(3)}(x) + g^{(4)}(x) - \frac{1}{2}(g^{(2)})^2 + o(|x|^4) \qquad \text{using } \log(1+t) = t - \frac{1}{2}t^2 + o(t^2)$$

Here

$$-\frac{1}{2}(g^{(2)})^2 = -\frac{1}{2}g^{(2)}_{i\ell}g^{(2)}_{\ell j} = -\frac{1}{18}\sum_{k,\ell,r,s,m} R_{ki\ell m}R_{rmsj}x_kx_\ell x_rx_s$$
$$= -\frac{1}{18}\sum_{k,\ell,r,s,m} R_{ik\ell m}R_{jrsm}x_kx_\ell x_rx_s$$

Now take the trace, i.e., contracting (gives Ricci curvature)

$$\mathrm{tr}\log g(x) = -\frac{1}{3}\sum_{k,\ell} R_{k\ell}x_kx_\ell - \frac{1}{6}\sum_{\ell,m,k} R_{\ell m,k}x_\ell x_m x_k - \frac{1}{90}\sum_{k,\ell,r,s,m,c} R_{ck\ell m}R_{crsm}x_kx_\ell x_rx_s$$
$$-\frac{1}{20}\sum_{\ell,r,m,k} R_{\ell r,mk}x_\ell x_r x_m x_k + o(|x|^4)$$

Now lifting to exponential

$$e^y = 1 + y + \frac{y^2}{2} + o(|y|^2)$$

One has

$$\det g(x) = 1 - \frac{1}{3}R_{k\ell}x_kx_\ell - \frac{1}{6}R_{\ell m,k}x_\ell x_m x_k - \frac{1}{90}R_{ck\ell m}R_{crsm}x_kx_\ell x_rx_s - \frac{1}{20}R_{\ell r,mk}x_\ell x_r x_m x_\ell - \frac{1}{18}R_{k\ell}R_{mr}x_kx_\ell x_m x_r + o(|x|^4)$$

Applications of Taylor Expansion

Guassian Curvature in Polar Coordinates This is [dC92] Exercise 5.6.
Let M be Riemannian manifold of dimension 2 (identified as surface). Let $B_\delta(p)$ be normal ball around $p \in M$ and consider the parametrized surface

$$f(\rho, \theta) = \exp_p(\rho v(\theta)) \qquad \forall\ 0 < \rho < \delta \qquad -\pi < \theta < \pi$$

for $v(\theta)$ circle of radius 1 in T_pM as parametrized by the central angle θ.

1. (ρ, θ) are coordinates in an open subset $U \subset M$ formed by the open ball minus the ray

$$U := B_\delta(p) \setminus \{\exp_p(-\rho v(0)) \mid 0 < \rho < \delta\}$$

These coordinates are polar coordinates at p.

Proof. It suffices to prove that

$$f : (0, \delta) \times (-\pi, \pi) \subset B_\delta(0) \subset \mathbb{R}^2 \to U \subset B_\delta(p) \subset M \qquad \text{defines a smooth diffeomorphism}$$

i.e., a bijection smooth map with smooth inverse.

(a) f as composition of smooth maps is indeed smooth in $(\rho,\ \theta)$ on $B_\delta(0)$.

(b) f is injective since $\exp_p$ is injective on $B_\delta(0) \subset \mathbb{R}^2$, which follows that

$$\exp_p(\rho_1 v(\theta_1)) = \exp_p(\rho_2 v(\theta_2)) \implies \rho_1 v(\theta_1) = \rho_2 v(\theta_2) \implies \rho_1 = \rho_2, \quad \theta_1 = \theta_2 \mod 2\pi$$

Since both $\theta_1,\ \theta_2 \in (-\pi,\ \pi)$ one has $\theta_1 = \theta_2$.

(c) f is surjective follows from the definition of the geodesic ball $B_\delta(p)$. By definition $\exp_p : B_\delta(0) \to B_\delta(p)$ is a diffeomorphism, hence for any

$$q \in U = B_\delta(p) \setminus \{\exp_p(-\rho v(0)) \mid 0 < \rho < \delta\}$$

There exists $w \in B_\delta(0)$ s.t.

$$\exp_p(w) = q$$

By injectivity of $\exp_p$ and excluding all possible points where $\rho v(0)$ ranging from $0 < \rho < \delta$ can map to, there must exists $\theta \neq 0 \mod 2\pi$ and $0 < \rho < \delta$ s.t.

$$\exp_p(\rho v(\theta)) = q$$

But θ has representative at $(-\pi, \pi)$.

(d) To show f is immersion, we need $\ker df_{(\rho,\theta)} = \{0\}$ for any $(\rho, \theta) \in (0, \delta) \times (-\pi, \pi)$ where

$$df_{(\rho,\theta)} : T_{(\rho,\theta)}\left((0, \delta) \times (-\pi, \pi)\right) = \mathbb{R}^2 \to T_{f(\rho,\theta)}U \cong \mathbb{R}^2$$

But using Chain rule

$$\left.\frac{\partial f}{\partial \rho}\right|_{(\rho,\theta)} = d(\exp_p \circ(\rho v(\theta)))_{(\rho,\theta)} = (d\exp_p)_{\rho v(\theta)}(v(\theta))$$
$$\left.\frac{\partial f}{\partial \theta}\right|_{(\rho,\theta)} = d(\exp_p \circ(\rho v(\theta)))_{(\rho,\theta)} = (d\exp_p)_{\rho v(\theta)}(\rho v'(\theta))$$

Yet $v(\theta)$ and $\rho v'(\theta)$ are orthogonal, hence they span $\mathbb{R}^2$. Under $(d\exp_p)_{\rho v(\theta)}$ as isomorphism between vector spaces

$$\left\{ \left.\frac{\partial f}{\partial \rho}\right|_{(\rho,\theta)}, \left.\frac{\partial f}{\partial \theta}\right|_{(\rho,\theta)} \right\} \qquad \text{indeed form a basis for } T_{f(\rho,\theta)}U$$

Thus the differential $df_{(\rho,\theta)}$ is injective, and hence f is immersion.

(e) By Inverse Function Theorem, and using f is bijection, f^{-1} inverse is defined everywhere on U and is smooth.

□

2. The coefficients g_{ij} of the Riemannian metric in polar coordinates are given by

$$g_{12} = 0, \qquad g_{11} = |\frac{\partial f}{\partial \rho}|^2 = |v(\theta)|^2 = 1, \qquad g_{22} = |\frac{\partial f}{\partial \theta}|^2$$

Proof. (a) Notice by setting $\rho = 0$, the initial radial velocity of our geodesic is one

$$\left.\frac{\partial f}{\partial \rho}\right|_{(0,\theta)} = (d\exp_p)_0(v(\theta)) = v(\theta) \implies |\left.\frac{\partial f}{\partial \rho}\right|_{(0,\theta)}| = |v| = 1$$

But a geodesic has constant speed. Hence in radial direction

$$g_{11} = \langle \left.\frac{\partial f}{\partial \rho}\right|_{(\rho,\theta)}, \left.\frac{\partial f}{\partial \rho}\right|_{(\rho,\theta)} \rangle = |v(\theta)|^2 = 1 \qquad \forall\, \theta \in (-\pi, \pi)$$

(b) Using Gauss Lemma

$$\begin{aligned} g_{12} &= \langle \left.\frac{\partial f}{\partial \rho}\right|_{(\rho,\theta)}, \left.\frac{\partial f}{\partial \theta}\right|_{(\rho,\theta)} \rangle = \langle (d\exp_p)_{\rho v(\theta)}(v(\theta)), (d\exp_p)_{\rho v(\theta)}(\rho v'(\theta)) \rangle \\ &= \langle v(\theta), \rho v'(\theta) \rangle = 0 \end{aligned}$$

Using that radial and angular velocity are orthogonal.

(c) By definition

$$g_{22} = \langle \left.\frac{\partial f}{\partial \theta}\right|_{(\rho,\theta)}, \left.\frac{\partial f}{\partial \theta}\right|_{(\rho,\theta)} \rangle = |\frac{\partial f}{\partial \theta}|^2$$

□

3. Along the geodesic $f(\rho, 0)$, we have

$$(\sqrt{g_{22}})_{\rho\rho} = -K(p)\rho + \tilde{R}(\rho) \qquad \text{for some } \tilde{R} \text{ where} \qquad \lim_{\rho\to 0} \frac{\tilde{R}(\rho)}{\rho} = 0 \tag{2.113}$$

and $K(p)$ the sectional curvature of M at p.

Proof. For $\theta = 0$, we make the observation that

$$\begin{aligned}\sqrt{g_{22}} &= |\left.\frac{\partial f}{\partial \theta}\right|_{(\rho,0)}| = |(d\exp_p)_{\rho v(0)}(\rho v'(0))| \\ &= |J(\rho)| \qquad \text{for Jacobi Field with } J(0) = 0 \text{ and } J'(0) = v'(0)\end{aligned}$$

Then directly apply (2.112), for the plane spanned by $v(0)$ and $v'(0)$

$$\sigma = \mathbf{Span}\{v(0),\, v'(0)\}$$

one obtain

$$J(\rho) = \rho - \frac{1}{6}K(\sigma)\rho^3 + o(\rho^3)$$

But $\dim M = 2$, the only 2-dim subspace of T_pM is itself, so $K(\sigma) = K(p)$ is indeed the sectional curvature of M at p. Thus

$$(\sqrt{g_{22}})_{\rho\rho} = -K(p)\rho + o(\rho)$$

□

4. In dimension 2, the sectional curvature coincides with the Gaussian Curvature.

$$\lim_{\rho\to 0} \frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}} = -K(p)$$

Proof.

$$\begin{aligned}\frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}} &= \frac{-K(p)\rho + o(\rho)}{\rho - \frac{1}{6}K(\sigma)\rho^3 + o(\rho^3)} \\ \lim_{\rho\to 0} \frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}} &= -K(p) + \lim_{\rho\to 0} \frac{o(1)}{\rho - \frac{1}{6}K(\sigma)\rho^3 + o(\rho^3)} = -K(p)\end{aligned}$$

where for both limits, we compute using L'Hôpital's rule. □

Sectional Curvature for $\dim = 2$

Corollary 2.5.4 ([dC92] Exercise 5.7.)**.** *Let M be Riemannian manifold of dimension 2. Let $p \in M$ and let $V \subset T_pM$ be a neighborhood of the origin where $\exp_p$ is a diffeomorphism. Let $S_r(0) \subset V$ be circle centered at the origin. Let L_r denote the length of the curve $\exp_p(S_r)$ in M. Then the sectional curvature at $p \in M$ is given by*

$$K(p) = \lim_{r\to 0} \frac{3}{\pi} \frac{2\pi r - L_r}{r^3}$$

Proof. Let $B_\delta(p)$ be normal ball around $p \in M$ s.t. $r < \delta$ and in the tangent space, $B_\delta(0) \subset V$. One parametrize the surface $\exp_p(B_\delta(0)) = B_\delta(p)$ using

$$f(\rho, \theta) = \exp_p(\rho v(\theta)) \qquad 0 < \rho < \delta, \qquad -\pi < \theta < \pi$$

Notice $f(r, \theta)$ therefore parametrizes the curve $\exp_p(S_r)$. In particular

$$|\frac{\partial}{\partial \theta} f(r,\theta)| = \sqrt{g_{22}(r,\theta)}$$

Hence the length L_r is computed via

$$L_r = \int_{-\pi}^{\pi} \sqrt{g_{22}(r,\theta)}\, d\theta$$

and since we're working with polar coordinates so the metric is radially symmetric, one obtain

$$L_r = 2\pi\sqrt{g_{22}(r)}$$

Now directly using (2.112)

$$\begin{aligned} \frac{3}{\pi}\frac{2\pi r - L_r}{r^3} &= \frac{3}{\pi}\frac{2\pi r - 2\pi\sqrt{g_{22}(r)}}{r^3} = 6\frac{r - (r - \frac{1}{6}K(p)r^3 + o(r^3))}{r^3} \\ &= K(p) + 6\frac{o(r^3)}{r^3} \\ \lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r - L_r}{r^3} &= K(p) \end{aligned}$$

□

2.5.4 Conjugate Points

We study relationship between singularities of the exponential map and Jacobi Fields. Conjugate points give degeneracy of the geodesics.

Definition 2.5.2 (Conjugate Point)**.** *Given geodesic.*

$$\gamma : [0, a] \to M$$

Let $t_0 \in (0, a]$. The point $\gamma(t_0)$ is conjugate to $\gamma(0)$ along γ if there exists Jacobi Field J along γ s.t.

1. *$J \neq 0$ nontrivial.*
2. *$J(0) = 0 = J(t_0)$.*

We call the multiplicity of the conjugate point $\gamma(t_0)$ as the maximum number of such linearly independent Jacobi fields, i.e.

$$\text{Multiplicity}(\gamma(t_0)) := \dim\{J(t) \mid \textit{Jacobi field along } \gamma(t) \textit{ s.t. } J(0) = 0 = J(t_0)\} \geq 1$$

Definition 2.5.3 (Conjugate Locus)**.** *Given $p \in M$, the set of first conjugate points to the point p, for all the geodesics that start at p, is the conjugate locus of p which we denote $C(p)$.*

$$C(p) := \{\gamma(t_0) \mid \gamma(t_0) \textit{ is the first conjugate point to } p \textit{ along } \gamma, \textit{ for all } \gamma \textit{ geodesic s.t. } \gamma(0) = p\}$$

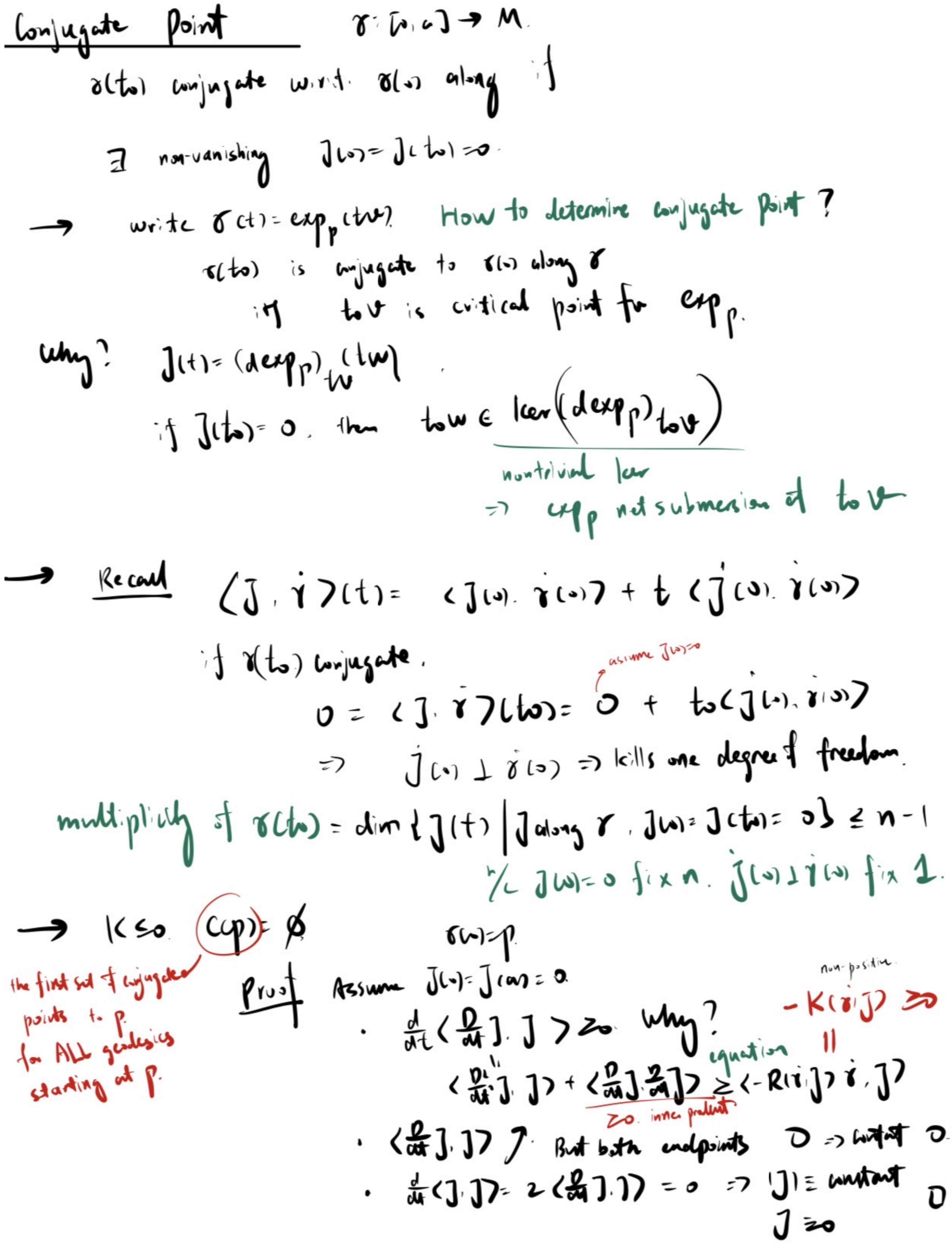


Figure 2.8: Jacobi Field Conjugate Points

Conjugate Points and singularities of the exponential map

Proposition 2.5.4 ([dC92] Proposition 5.3.5). *Let*

$$\gamma : [0, a] \to M$$

be geodesic with $\gamma(0) = p$ *and* $\gamma'(0) = v$*, hence*

$$\gamma(t) = \exp_p(tv)$$

Then the point $q = \gamma(t_0)$ *for* $t_0 \in (0, a]$ *is conjugate point to* $p = \gamma(0)$ *along* γ *iff*

$$t_0\gamma'(0) = t_0 v$$

is a critical point of $\exp_p$*, i.e.,* $(d\exp_p)_{t_0 v}$ *is not surjective (has non-trivial kernel). Moreover*

$$\textit{Multiplicity of } q \textit{ as a conjugate point of } p = \dim\ker((d\exp_p)_{t_0 v})$$

Proof. Any Jacobi Field $J(t)$ along $\gamma(t)$ s.t. $J(0) = 0$ is of the form

$$J(t) = (d\exp_p)_{tv}(tw) \qquad w := \frac{D}{dt}J(0)$$

Suppose $t_0 \neq 0$, then q is conjugate to p iff $J(t_0) = 0$ iff

$$(d\exp_p)_{t_0 v}(t_0 w) = 0$$

But $t_0 > 0$, so this vanishes iff

$$(d\exp_p)_{t_0 v}(w) = 0 \iff w \in \ker((d\exp_p)_{t_0 v})$$

Hence due to non-trivial kernel, $t_0 v$ is a critical point for $\exp_p$ via definition. □

Multiplicity of Conjugate Point never exceeds $n - 1$ Notice the multiplicity never exceeds $n - 1$. Recall that if

$$\gamma(t) = \exp_p(tv)$$

Then $J(0) = 0$ implies

$$J(t) = (d\exp_p)_{tv}(tw)$$

and (2.105)

$$\langle J, \gamma' \rangle(t) = \langle J(0), \gamma'(0)\rangle + t\langle J'(0), \gamma'(0)\rangle$$

Applying to $t = t_0$ yields

$$0 = 0 + t_0\langle J'(0), \gamma'(0)\rangle$$
$$0 = \langle J'(0), \gamma'(0)\rangle$$

so $\frac{D}{dt}J(0)$ is perpendicular to $\gamma'(0)$.

Thus the space

$$J(t) \in \{J(t) \mid \text{Jacobi Fields along } \gamma(t),\ J(0) = 0 \text{ and } \langle \frac{D}{dt}J(0), \gamma'(0)\rangle = 0\} \cong \mathbb{R}^{n-1} \tag{2.114}$$

since originally one has $2n$ initial conditions to determine $J(t)$, and $J(0) = 0$ kills n while $\langle w, v\rangle = 0$ kills 1, we're left with $n - 1$.

In fact, if $e_1, \cdots, e_n$ are ONB of T_pM

$$e_1 = \frac{v}{|v|} = \frac{\gamma'(0)}{|\gamma'(0)|}$$

Let $J_i(t)$ be Jacobi Fields with

$$J_i(0) = 0 \qquad \frac{D}{dt}J_i(0) = e_i \qquad i \in \{1, \cdots, n\}$$

and let $J_{n+i}(t)$ be Jacobi Field s.t.

$$J_{n+i}(0) = e_i, \qquad \frac{D}{dt}J_{n+i}(0) = 0 \qquad i \in \{1, \cdots, n\}$$

So look at the space

$$\{J(t) \mid \text{Jacobi Fields } J(0) = 0 = \langle \frac{D}{dt}J(0), \gamma'(0)\rangle\} = \text{Span}\{J_2(t), \cdots, J_n(t)\} \implies \text{Multiplicity}(\gamma(t_0)) \leq n - 1$$

Space of Jacobi Fields

Proposition 2.5.5 ([dC92] Proposition 5.3.9)**.** *Let*

$$\gamma : [0, a] \to M$$

be geodesic. Let $V_1 \in T_{\gamma(0)}M$ *and* $V_2 \in T_{\gamma(a)}M$*. If* $\gamma(a)$ *is not conjugate to* $\gamma(0)$ *along* γ*, there exists a unique Jacobi Field* J *along* γ *s.t.*

$$J(0) = V_1, \qquad J(a) = V_2$$

Corollary 2.5.5 ([dC92] Corollary 5.3.10)**.** *Let* (M, g) *be Riemannian manifold of dimension* n*. Let*

$$\gamma : [0, a] \to M$$

be geodesic.

Denote as in (2.114)

$$\mathcal{J}^{\perp} := \{J(t) \mid \textit{Jacobi Fields } J(0) = 0,\ J'(0) \perp \gamma'(0)\} \cong \mathbb{R}^{n-1}$$

Let $\{J_1, \cdots, J_{n-1}\}$ *be a basis of* $\mathcal{J}^{\perp}$*.*

Assume $\gamma(t)$*,* $t \in (0, a]$ *is not conjugate to* $\gamma(0)$*. Then* $\{J_1(t), \cdots, J_{n-1}(t)\}$ *is a basis for the orthogonal complement of* $\gamma'(t)$

$$\{\gamma'(t)\}^{\perp} \subseteq T_{\gamma(t)}M$$

2.5.4.1 Conjugate Points and Sectional Curvature

If (M, g) has constant sectional curvature K. Then

$$\{J(t) \mid \text{Jacobi Fields } J(0) = 0\} = \text{Span}\{t\gamma'(t), f_K e_2, f_K e_3, \cdots, f_K e_n\} \cong \mathbb{R}^n$$

where

$$e_1 = \frac{\gamma'(0)}{|\gamma'(0)|},\ e_2, \cdots, e_n$$

are ONB of T_pM and e_i are parallel transported along $\gamma(t)$.

The differential $(d\exp_p)_{tv}$ is singular exactly when there exits a non-zero Jacobi field with $J(0) = J(1) = 0$. Now from the spanning set, $t\gamma'(t)$ never vanishes for $t > 0$. The transverse ones are $f_K(t)e_i$, so zero happens exactly when $f_K(t) = 0$.

Negative Sectional Curvature gives $C(p) = \varnothing$

Proposition 2.5.6 ([dC92] Exercise 5.3)**.** *Let* M *be a Riemannian manifold with non-positive sectional curvature. Then for any* $p \in M$*, the conjugate locus* $C(p) = \varnothing$ *is empty.*

Proof. Fix any $p \in M$. Given a geodesic

$$\gamma : [0, a] \to M$$

s.t. $\gamma(0) = p$. Assume there exists nontrivial Jacobi Field J s.t.

$$J(0) = J(a) = 0$$

1. We first show that

$$\frac{d}{dt}\langle \frac{D}{dt}J, J\rangle \geq 0$$

One calculate using that the covariant derivative $\frac{D}{dt}$ corresponds to Levi-Civita Connection (hence compatible with the metric g), and the Jacobi Equation (2.101).

$$\begin{aligned} \frac{d}{dt}\langle \frac{D}{dt}J, J\rangle &= \langle \frac{D^2}{dt^2}J, J\rangle + \langle \frac{D}{dt}J, \frac{D}{dt}J\rangle \\ &= -\langle R(\gamma', J(t))\gamma', J\rangle + \langle \frac{D}{dt}J, \frac{D}{dt}J\rangle \end{aligned}$$

Notice the first term is essentially sectional curvature in the plane spanned by γ' and J with flipped sign so

$$-\langle R(\gamma', J(t))\gamma', J\rangle \geq 0$$

due to our assumption on non-positive sectional curvature. The second term is always non-negative due to inner product structure.

2. But then
$$\langle \frac{D}{dt}J, J\rangle(a) = 0 = \langle \frac{D}{dt}J, J\rangle(0)$$
yields
$$\langle \frac{D}{dt}J, J\rangle \equiv 0$$
3. Using compatible with the metric g again
$$\frac{d}{dt}\langle J, J\rangle = 2\langle \frac{D}{dt}J, J\rangle \equiv 0$$
Thus
$$|J|^2 = \langle J, J\rangle(t) = 0 \qquad \forall\, t \in [0, a]$$

We've reached a contradiction that J is assumed to be non-trivial. □

Spheres as Positive Sectional Curvature On the other hand, for positive sectional curvature, there could be conjugate points.

Let $K > 0$. Denote by
$$\mathbb{S}^n\left(\frac{1}{\sqrt{K}}\right)$$
the round n–sphere of radius $\frac{1}{\sqrt{K}}$ (equivalently, constant sectional curvature K). Fix $p \in \mathbb{S}^n\left(\frac{1}{\sqrt{K}}\right)$ and a unit-speed geodesic
$$\gamma : [0, \pi/\sqrt{K}] \to \mathbb{S}^n\left(\frac{1}{\sqrt{K}}\right)$$
with $\gamma(0) = p$. Let $W(t)$ be a parallel vector field along γ such that $W(t) \perp \dot\gamma(t)$ and $W(0) = w \neq 0$. Then the vector field (2.108)
$$J(t) \;=\; \frac{\sin(\sqrt{K}\,t)}{\sqrt{K}}\, W(t)$$
is a (nontrivial) Jacobi field orthogonal to $\dot\gamma$ and it satisfies
$$J(0) = 0, \qquad J\left(\frac{\pi}{\sqrt{K}}\right) = 0.$$

Hence $\gamma\left(\frac{\pi}{\sqrt{K}}\right)$ is conjugate to p along γ and the first conjugate time equals $\pi/\sqrt{K}$. Moreover, for every unit vector $v \in T_p\mathbb{S}^n\left(\frac{1}{\sqrt{K}}\right)$,
$$\exp_p\left(\frac{\pi}{\sqrt{K}}\, v\right) = -p,$$
so the cut locus of p is the singleton
$$C(p) = \{-p\}.$$

Negative Sectional Curvature Jacobi Field Formula

Proposition 2.5.7 ([dC92] Exercise 5.4)**.** *Let M be Riemannian manifold with constant negative sectional curvature $b < 0$. Let*
$$\gamma : [0, \ell] \to M$$
s.t. $\gamma(0) = p$ be normalized geodesics, and let $v \in T_{\gamma(\ell)}M$ s.t.
$$\langle v, \gamma'(\ell)\rangle = 0, \qquad |v| = 1$$

Then the Jacobi Field J along γ as determined by
$$J(0) = 0 \qquad J(\ell) = v$$
is given by
$$J(t) = \frac{\sinh(t\sqrt{-b})}{\sinh(\ell\sqrt{-b})} w(t) \tag{2.115}$$
where $w(t)$ is the parallel transport along γ of the vector
$$w(0) = \frac{u_0}{|u_0|} \qquad u_0 = (d\exp_p)^{-1}_{\ell\gamma'(0)}(v) \in T_{\gamma(0)}M \cong T_{\ell\gamma'(0)}(T_{\gamma(0)}M)$$

Proof. 1. Since M has constant negative curvature, using Proposition 2.5.6 we know $\gamma(\ell)$ is not conjugate point to $\gamma(0)$ along γ.

2. Using b is constant sectional curvature, our Jacobi Equation writes as in (2.107). Given initial data

$$J_1(0) = 0 \qquad J_1'(0) = w(0) = \frac{u_0}{|u_0|}$$

According to solution (2.108) with $b < 0$, one has

$$J_1(t) = \frac{\sinh(\sqrt{-b}t)}{\sqrt{-b}} w(t)$$

as the unique solution. But notice J_1 is not the solution J we seek for.

3. Since $J_1(0) = 0$, using (2.104) one may write J_1 as

$$J_1(t) = (d\exp_{\gamma(0)})_{t\gamma'(0)}(tw(0))$$

In particular one may evaluate at $t = \ell$ and obtain

$$J_1(\ell) = (d\exp_p)_{\ell\gamma'(0)}(\ell w(0)) = (d\exp_p)_{\ell\gamma'(0)}(\ell \frac{u_0}{|u_0|})$$

But making use of

$$u_0 = (d\exp_p)^{-1}_{\ell\gamma'(0)}(v) \implies (d\exp_p)_{\ell\gamma'(0)}(u_0) = v$$

so via linearity

$$J_1(\ell) = \ell \frac{v}{|u_0|}$$

4. Finally, notice both $J_1(\ell)$ and $J(\ell)$ are expected to be in the direction of $v \in T_{\gamma(\ell)}M$. We define scaling

$$\tilde{J}(t) := \frac{|u_0|}{\ell} J_1(t) = \frac{\sinh(\sqrt{-b}t)}{\sqrt{-b}} \frac{|u_0|}{\ell} w(t)$$

so that

$$\tilde{J}(0) = 0 \qquad \tilde{J}(\ell) = v$$

Indeed due to two boundary conditions, via Existence and Uniqueness of ODE solution

$$\tilde{J} \equiv J$$

5. It suffices to argue one has the correct scaling that matches (2.115). Using

$$1 = |v| = |J(\ell)| = |\frac{\sinh(\sqrt{-b}\ell)}{\sqrt{-b}} \frac{|u_0|}{\ell} w(\ell)| = |\frac{\sinh(\sqrt{-b}\ell)}{\sqrt{-b}} \frac{|u_0|}{\ell}|$$

Hence

$$J(t) = \frac{\sinh(\sqrt{-b}t)}{\sqrt{-b}} \frac{|u_0|}{\ell} w(t) = \frac{\sinh(t\sqrt{-b})}{\sinh(\ell\sqrt{-b})} w(t)$$

□

2.5.4.2 Jacobi Fields and Conjugate Points on Locally Symmetric Spaces

This is [dC92] Exercise 5.5.

Let M be locally symmetric space as in Definition 2.4.11. Let

$$\gamma : [0, \infty) \to M$$

be a geodesic in M and let

$$\gamma(0) = p, \qquad \gamma'(0) = v$$

Define

$$K_v : T_pM \to T_pM$$
$$x \mapsto K_v(x) := R(v, x)v$$

Then

1. K_v is self-adjoint.

Proof. For any x, $y \in T_pM$, using Symmetry of Riemannian Curvature Tensor (2.61)

$$\begin{aligned}\langle K_v(x), y\rangle =& \langle R(v,x)v, y\rangle = R(v,x,v,y) = R(v,y,v,x) \\ &= \langle R(v,y)v, x\rangle = \langle K_v(y), x\rangle = \langle x, K_v(y)\rangle\end{aligned}$$

where the last equality follows by symmetric of metric. □

2. Choose an ONB $\{e_1, \cdots, e_n\}$ of T_pM that diagonalizes K_v, i.e.

$$K_v(e_i) = \lambda_i e_i \qquad \forall\ i = 1, \cdots, n$$

We extend e_i to Vector fields along γ via Parallel Transport. Then (note λ_i does not depend on t)

$$K_{\gamma'(t)}(e_i(t)) = \lambda_i e_i(t) \qquad \forall\ t$$

Proof. Notice $\gamma'(t)$ is the parallel transport of $\gamma'(0) = v$ along γ. Since M is locally symmetric space, given $e_i(t)$ parallel transport of e_i along γ

$$K_{\gamma'(t)}(e_i(t)) = R(\gamma'(t), e_i(t))\gamma'(t) \qquad \text{is also parallel transport along } \gamma$$

Thus for any $e_j(t)$ where $j \neq i$, we take covariant derivative using ∇ is Levi-Civita Connection

$$\begin{aligned}\frac{d}{dt}\langle K_{\gamma'(t)}(e_i(t)), e_j(t)\rangle &= \langle \frac{D}{dt}K_{\gamma'(t)}(e_i(t)), e_j(t)\rangle + \langle K_{\gamma'(t)}(e_i(t)), \frac{D}{dt}e_j(t)\rangle \\ &= 0 \qquad \text{since both are parallel vector fields} \\ \langle K_{\gamma'(t)}(e_i(t)), e_j(t)\rangle &= \langle K_{\gamma'(0)}(e_i(0)), e_j(0)\rangle = \langle K_v(e_i), e_j\rangle = \lambda_i\langle e_i, e_j\rangle = 0 \qquad \forall\ t\end{aligned}$$

Then due to choice of ONB basis

$$K_{\gamma'(t)}(e_i(t)) = Ce_i(t) \qquad \forall\ t$$

To determine constant

$$\begin{aligned}\frac{d}{dt}\langle K_{\gamma'(t)}(e_i(t)), e_i(t)\rangle &= \langle \frac{D}{dt}K_{\gamma'(t)}(e_i(t)), e_i(t)\rangle + \langle K_{\gamma'(t)}(e_i(t)), \frac{D}{dt}e_i(t)\rangle = 0 \\ \langle K_{\gamma'(t)}(e_i(t)), e_i(t)\rangle &= \langle K_v(e_i), e_i\rangle = \lambda_i \\ C &= \lambda_i\end{aligned}$$

Thus

$$K_{\gamma'(t)}(e_i(t)) = \lambda_i e_i(t) \qquad \forall\ t$$

□

3. Let

$$J(t) := \sum_i x_i(t)e_i(t) \qquad \forall\ t \qquad \text{be Jacobi Field along } \gamma$$

Then the Jacobi Equation is equivalent to the system of ODEs

$$\frac{d^2}{dt^2}x_i(t) + \lambda_i x_i = 0 \qquad i = 1, \cdots, n \tag{2.116}$$

Proof. Recall (2.101) writes

$$\frac{D^2}{dt^2}J(t) + R(\gamma'(t), J(t))\gamma'(t) = 0 \qquad \forall\ t \in [0, \infty)$$

So plugging in, using product rule and using Linearity of Riemannian Curvature Tensor in $C^\infty(M)$

$$\begin{aligned}\sum_i \frac{D^2}{dt^2}(x_i(t)e_i(t)) + \sum_i R(\gamma'(t), x_i(t)e_i(t))\gamma'(t) &= 0 \\ \sum_i e_i(t)\frac{d^2}{dt^2}x_i(t) + \sum_i x_i(t)R(\gamma'(t), e_i(t))\gamma'(t) &= 0 \\ \sum_i e_i(t)\frac{d^2}{dt^2}x_i(t) + \sum_i x_i(t)K_{\gamma'(t)}(e_i(t)) &= 0 \\ \sum_i e_i(t)\frac{d^2}{dt^2}x_i(t) + \sum_i x_i(t)\lambda_i e_i(t) &= 0 \\ \frac{d^2}{dt^2}x_i(t) + \lambda_i x_i(t) &= 0 \qquad \forall\ t\end{aligned}$$

Using the fact that $\{e_1(t), \cdots, e_n(t)\}$ are ONB frames parallel w.r.t. γ. □

4. The conjugate points of p along γ are given by

$$\gamma(\frac{\pi k}{\sqrt{\lambda_i}}) \qquad \forall\, k \in \mathbb{Z},\ k \geq 1, \qquad \forall\, i \in \{1, \cdots, n\} \cap \{\lambda_i \text{ is a positive eigenvalue of } K_v\ \}$$

Proof. Solving system of ODEs for (2.116) with

$$\vec{x}(0) = (0, \cdots, 0)$$

We wish to look for t_k s.t.

$$\vec{x}(t_k) = (0, \cdots, 0)$$

(a) In the case $\lambda_i > 0$, the general solution

$$x_i(t) = A_i \sin(\sqrt{\lambda_i} t) \qquad x_i'(0) = A_i \sqrt{\lambda_i}$$

To set $x_i(t_k) = 0$, and to keep $A_i \neq 0$ we obtain

$$\sin(\sqrt{\lambda_i} t_k) = 0 \qquad \forall\, k \implies t_k = \frac{k\pi}{\sqrt{\lambda_i}} \qquad \forall\, k \in \mathbb{Z},\ k > 0$$

Notice we omit $k = 0$ for the simple reason that it coincides with the origin of γ.

(b) In the case $\lambda_i = 0$, the general solution

$$x_i(t) = C_i t \qquad x_i'(0) = C_i$$

Setting $x_i(t_k) = 0$ but keeping $C_i \neq 0$ yields $t_k = 0$, which we omit.

(c) In the case $\lambda_i < 0$, the general solution

$$x_i(t) = D_i \sinh(\sqrt{-\lambda_i} t) \qquad x_i'(0) = D_i \sqrt{-\lambda_i}$$

Setting $x_i(t_k) = 0$ but keeping $D_i \neq 0$ yields

$$\sinh(\sqrt{-\lambda_i} t_k) = 0 \qquad \forall\, k \implies t_k = 0$$

which we omit

Hence

$$\gamma(t_k) = \gamma(\frac{\pi k}{\sqrt{\lambda_i}})$$

are precisely the conjugate points of $\gamma(0) = p$ along γ. □

2.6 Isometric Immersion

In this section, we want to measure the way a manifold is immersed in another.

Let (M, g) and $(\overline{M}, \overline{g})$ be two Riemannian manifolds and ∇, $\overline{\nabla}$ their respective Levi-Civita connections.

2.6.1 Isometric Immersion and the Tangential Part

Isometric Immersion Let's recall the definition for isometric immersion (2.2) and pullback of Riemannian metric under an immersion (2.3).

Definition 2.6.1 (Isometric Immersion).

$$f : (M, g) \to (\overline{M}, \overline{g})$$

is an isometric immersion if

1. *f is an immersion, i.e., for any $p \in M$,*

$$\begin{aligned} df_p : T_pM &\to T_{f(p)}\overline{M} \\ v &\mapsto df_p(v) \end{aligned} \qquad \text{is injective}$$

   In particular, since both tangent spaces are finite-dimensional, df_p defines an linear isomorphism.

2. *f is an isometry, i.e.*

$$f^*\overline{g} = g$$

   which is to say (2.3)

$$g_p(u, v) = (f^*\overline{g})_p(u, v) = \overline{g}_{f(p)}(df_p(u), df_p(v)) \qquad \forall\ u, v \in T_pM$$

If such f exists, then

$$n := \dim M \leq \overline{n} := \dim \overline{M}$$

Let's consider some example.

Example 2.6.1 ([dC92] Exercise 6.2). *The map*

$$\begin{aligned} \mathbf{x} : \mathbb{R}^2 &\to \mathbb{R}^4 \\ (\theta, \varphi) &\mapsto \frac{1}{\sqrt{2}}(\cos(\theta), \sin(\theta), \cos(\varphi), \sin(\varphi)) \end{aligned}$$

is an immersion of $\mathbb{R}^2$ into the unit sphere $\mathbb{S}^3(1) \subseteq \mathbb{R}^4$, whose image $\mathbf{x}(\mathbb{R}^2)$ equipped with the pullback metric is a torus $\mathbb{T}^2$ with sectional curvature 0 in the induced metric.

Proof. 1. We first show $\mathbf{x}$ defines an immersion.

$$\begin{aligned} \mathbf{x}_\theta &= \frac{\partial \mathbf{x}}{\partial \theta} = \frac{1}{\sqrt{2}}(-\sin(\theta), \cos(\theta), 0, 0) \\ \mathbf{x}_\varphi &= \frac{\partial \mathbf{x}}{\partial \varphi} = \frac{1}{\sqrt{2}}(0, 0, -\sin(\varphi), \cos(\varphi)) \end{aligned}$$

   Due to linear independence of $\mathbf{x}_\theta$ and $\mathbf{x}_\varphi$, clearly $d\mathbf{x}_{\theta,\varphi} = (\mathbf{x}_\theta, \mathbf{x}_\varphi)$ is injective for any $(\theta, \varphi) \in \mathbb{R}^2$. Hence $\mathbf{x}$ defines an immersion.

2. We compute

$$|\mathbf{x}(\theta, \varphi)|^2 = \frac{1}{2}\left(\cos^2(\theta) + \sin^2(\theta) + \cos^2(\varphi) + \sin^2(\varphi)\right) = 1$$

   hence $\mathbf{x}$ defines an immersion into the unit sphere $\mathbb{S}^3(1) \subset \mathbb{R}^4$. Notice indeed

$$\mathbf{x}(\mathbb{R}^2) = \mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{2}}) = \mathbb{T}^2$$

   is the two dimensional torus as a set.

3. We compute the induced metric on $(\mathbf{x}(\mathbb{R}^2), \mathbf{x}^* g_0)$ where $(\mathbb{R}^4, g_0)$ is the Euclidean space.

$$\begin{aligned}\mathbf{x}^* g_0 &= \sum_{i=1}^{4} d\mathbf{x}_i^2 = \sum_{i=1}^{4} (\frac{\partial \mathbf{x}_i}{\partial \theta} d\theta + \frac{\partial \mathbf{x}_i}{\partial \varphi} d\varphi)^2 \\ &= (-\frac{1}{\sqrt{2}} \sin(\theta) d\theta)^2 + (\frac{1}{\sqrt{2}} \cos(\theta) d\theta)^2 + (-\frac{1}{\sqrt{2}} \sin(\varphi) d\varphi)^2 + (\frac{1}{\sqrt{2}} \cos(\varphi) d\varphi)^2 \\ &= d\theta^2 + d\varphi^2 = \text{Euclidean metric on } \mathbb{R}^2\end{aligned}$$

Hence $\mathbf{x}$ is in fact an isometric immersion.

4. Since we're in $(\mathbf{x}(\mathbb{R}^2), \mathbf{x}^* g_0)$ a Riemannian surface of dimension 2, the sectional curvature

$$K = \frac{R_{1212}}{g_{11} g_{22} - g_{12}^2}$$

But due to the fact that induced metric is flat, $R_{1212} = 0$ hence sectional curvature is 0

$$K = 0$$

□

First Fundamental Form Consider isometric immersion

$$f : (M, g) \to (\overline{M}, \overline{g})$$

For any $p \in M$.

Definition 2.6.2 (First Fundamental Form)**.** *The 1st fundamental form is the form we have on the tangent space of the manifold.*

$$(T_p M, \langle \cdot, \cdot \rangle_g)$$

where the pullback metric $g(\cdot, \cdot) = \langle \cdot, \cdot \rangle_g$ *is defined by*

$$g = f^* \overline{g}$$

as an inner product space. In other words

$$\begin{aligned}\mathbb{I} : T_p M \times T_p M &\to \mathbb{R} \\ (u, v) &\mapsto g_{f(p)}(df_p(u), df_p(v))\end{aligned}$$

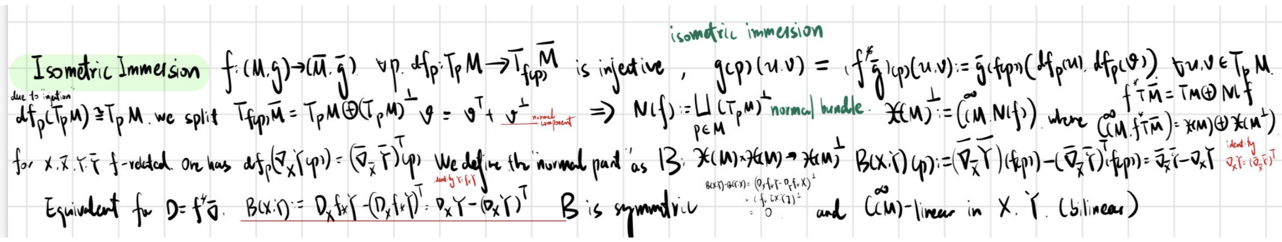


Figure 2.9: Isometric Immersion

Orthogonal Part of Tangent Space $(T_p M)^\perp$ We want to understand how $f(M)$ can be understood as part of $\overline{M}$. Both its structure coming from g and from $\overline{g}$, i.e., tangent part, that coincides due to isometry. What plays an important role is the normal part, which is known as the second fundamental form.

Let

$$f : M^n \to \overline{M}^{n+m}$$

be an isometric immersion. Then for each $p \in M$, there exits a neighborhood $U \subseteq M$ of p, s.t. $f(U) \subseteq \overline{M}$ is a submanifold of M.

In other words, there exists a neighborhood $\overline{U} \subseteq \overline{M}$ of $f(p)$ and a diffeomorphsim

$$\varphi : \overline{U} \subseteq \overline{M} \to V \subseteq \mathbb{R}^{n+m}$$

to an open subset V of $\mathbb{R}^{n+m}$, s.t. φ maps $f(U) \cap \overline{U}$ diffeomorphically onto an open set of a subspace of $\mathbb{R}^n \subseteq \mathbb{R}^{n+m}$.

We identify U with $f(U)$, and each $v \in T_q M$ for $q \in U$, with $df_q(v) \in T_{f(q)} \overline{M}$.

Under these identifications, one may extend a local vector field defined on $U \subseteq M$ to a local vector field on $\overline{U} \subseteq \overline{M}$. Such extension is always possible if U is sufficiently small.

Definition 2.6.3 ($(T_pM)^\perp$)**.** *Now for each $p \in M$, one define the subspace $(T_pM)^\perp \subseteq T_{f(p)}\overline{M}$ as the orthogonal complement*

$$T_{f(p)}\overline{M} = T_pM \oplus (T_pM)^\perp$$

where we identified $T_pM \cong df_p(T_pM)$.

Here $\perp$ is orthogonal complement defined through $\overline{g}_{f(p)}$. In particular

$$(T_pM)^\perp := \{v \in T_{f(p)}\overline{M} \mid \overline{g}(f(p))(v, w) = 0 \quad \forall\ w \in df_p(T_pM)\}$$

For any $v \in T_{f(p)}\overline{M}$, there exists unique decomposition

$$v = v^T + v^\perp \in T_pM \oplus (T_pM)^\perp$$

We call v^T the tangential component of v and $v^\perp$ the normal component of v.

Such decomposition is smooth in the sense that the mappings of $T\overline{M}$ into $T\overline{M}$

$$(p, v) \mapsto (p, v^T), \qquad (p, v) \mapsto (p, v^\perp)$$

are smooth.

Normal Bundle $N(f)$ Let

$$f : M^n \to \overline{M}^{n+m}$$

be an isometric immersion.

Definition 2.6.4 (Normal Bundle)**.** *The normal bundle of an isometric immersion of f is*

$$N(f) := \bigsqcup_{p \in M} (T_pM)^\perp$$

The vector bundle

$$N(f) \to M$$

is of rank $n + m - n = m$.

Alternatively, the normal bundle is achieved via pullback and decomposing

$$f^*T\overline{M} = TM \oplus N(f)$$

Similarly

$$\begin{aligned} C^\infty(M, f^*T\overline{M}) &= C^\infty(M, TM) \oplus C^\infty(M, N(f)) \\ &= \mathfrak{X}(M) \oplus \mathfrak{X}(M)^\perp \end{aligned}$$

where we denote

$$\mathfrak{X}(M)^\perp := C^\infty(M, N(f))$$

Pushforward and Pullback Vector Fields Recall our old friend (1.81), (1.82).

For smooth map

$$f : M \to \overline{M}$$

We've defined

$$\begin{aligned} f_* : \mathfrak{X}(M) &\to C^\infty(M, f^*T\overline{M}) \\ X &\mapsto (f_*X)(p) := df_p(X(p)) \qquad \forall\ p \in M \end{aligned}$$

and

$$\begin{aligned} f^* : \mathfrak{X}(\overline{M}) &\to C^\infty(M, f^*T\overline{M}) \\ Y &\mapsto (f^*Y)(p) := Y(f(p)) \qquad \forall\ p \in M \end{aligned}$$

Also recall f-related as in Definition 1.20.1

Definition 2.6.5 (f-related)**.** *$X \in \mathfrak{X}(M)$ is f-related to $\overline{X} \in \mathfrak{X}(\overline{M})$ if*

$$f_*X = f^*\overline{X}$$

In other words

$$df_p(X(p)) = \overline{X}(f(p)) \qquad \forall\ p \in M \tag{2.117}$$

Tangential Part $(\overline{\nabla}_{\overline{X}}\overline{Y}(f(p)))^T$

Lemma 2.6.1. *Suppose*

$$f : (M, g) \to (\overline{M}, \overline{g})$$

is an isometric immersion.

Then for any $X \in \mathfrak{X}(M)$ *f-related to* $\overline{X} \in \mathfrak{X}(V)$ *and* $Y \in \mathfrak{X}(M)$ *f-related to* $\overline{Y} \in \mathfrak{X}(V)$ *where* $V \subseteq \overline{M}$ *is any open neighborhood of* $f(p)$

$$df_p(\nabla_X Y(p)) = (\overline{\nabla}_{\overline{X}}\overline{Y}(f(p)))^T$$

Recall we've identified

$$df_p(\nabla_X Y(p)) = \nabla_X Y(p)$$

thus the tangential part essentially carries from the manifold M itself

$$\nabla_X Y(p) = (\overline{\nabla}_{\overline{X}}\overline{Y}(f(p)))^T$$

2.6.2 Second Fundamental Form

But what is the normal part? It is not contained in previous information.

Second Fundamental Form $B(X, Y)(p) = (\overline{\nabla}_{\overline{X}}\overline{Y}(f(p)))^{\perp}$ We want to define a bilinear capturing the normal behavior.

Definition 2.6.6 (Bilinear Form B)**.**

$$\begin{aligned} B : \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M)^{\perp} \\ (X, Y) &\mapsto B(X, Y)(p) := (\overline{\nabla}_{\overline{X}}\overline{Y}(f(p)))^{\perp} \qquad \forall\, p \in M \end{aligned} \tag{2.118}$$

whenever $X \in \mathfrak{X}(M)$ *f-related to* $\overline{X} \in \mathfrak{X}(V)$ *and* $Y \in \mathfrak{X}(M)$ *f-related to* $\overline{Y} \in \mathfrak{X}(V)$ *where* $V \subseteq \overline{M}$ *is any open neighborhood of* $f(p)$*.*

In particular

$$\begin{aligned} B(X, Y)(p) &= (\overline{\nabla}_{\overline{X}}\overline{Y}(f(p)))^{\perp} \\ &= \overline{\nabla}_{\overline{X}}\overline{Y}(f(p)) - (\overline{\nabla}_{\overline{X}}\overline{Y}(f(p)))^T \\ &= \overline{\nabla}_{\overline{X}}\overline{Y}(f(p)) - \nabla_X Y(p) \end{aligned}$$

Equivalently, if one define

$$D = f^*\overline{\nabla}$$

as the pullback connection on $f^*T\overline{M}$.

Then one has equivalent definition for $B(X, Y)$

$$\begin{aligned} B(X, Y)(p) &= (D_X f_* Y)^{\perp}(p) \\ &= D_X f_* Y(p) - (D_X f_* Y)^T(p) \qquad \forall\, X,\, Y \in \mathfrak{X}(M) \end{aligned} \tag{2.119}$$

This justifies why B is well-defined. In particular, this also shows B is $C^\infty(M)$-linear in X.

In fact B is symmetric.

Proposition 2.6.1 (Symmetric Bilinear Form)**.**

$$B(X, Y) = B(Y, X) \qquad \forall\, X,\, Y \in \mathfrak{X}(M)$$

Proof.

$$\begin{aligned} B(X, Y) - B(Y, X) &= (D_X f_* Y)^{\perp} - (D_Y f_* X)^{\perp} \\ &= (D_X f_* Y - D_Y f_* X)^{\perp} \\ &= (f_*([X, Y]))^{\perp} = 0 \end{aligned}$$

Since $[X, Y] \in \mathfrak{X}(M)$, then its orthogonal part is 0. □

Now since B is $C^\infty(M)$-linear in both X and Y. In fact

$$B \in C^\infty(M, \mathrm{Sym}^2 T^*M \otimes N(f))$$

Sometimes the 'second fundamental form' is used to designated the mapping B, which at each $p \in M$ gives a symmetric bilinear mapping, taking values in $(T_pM)^{\perp}$.

Second Fundamental Form of f along normal η One has 3 one-to-one correspondence of second fundamental forms for fixed $\eta \in (T_pM)^\perp$

1. symmetric bilinear forms
$$H_\eta : T_pM \times T_pM \to \mathbb{R}$$
2. quadratic form
$$\mathbb{II}_\eta : T_pM \to \mathbb{R}$$
3. Self adjoint operators
$$S_\eta(x) : T_pM \to T_pM$$

One may define using one from another

$$\begin{aligned} \mathbb{II}_\eta(x) &:= H_\eta(x,x) \\ H_\eta(x,y) &:= \frac{1}{2}\left(\mathbb{II}_\eta(x+y) - \mathbb{II}_\eta(x) - \mathbb{II}_\eta(y)\right) \\ \langle S_\eta(x), y\rangle_g &:= H_\eta(x,y) = \langle x, S_\eta(y)\rangle_g \qquad \forall\ x,\, y \in T_pM \end{aligned}$$

Let's define them one by one.
Let
$$f : (M,g) \to (\overline{M}, \overline{g})$$
be an isometric immersion. Fix a vector on the orthogonal by Letting $\eta \in (T_pM)^\perp = (N(f))_p$.

Definition 2.6.7 (Second Fundamental Form)**.** *For the symmetric bilinear form B as in* (2.118) *(or* (2.119)*), one define*
$$\begin{aligned} H_\eta : T_pM \times T_pM &\to \mathbb{R} \\ (x,y) &\mapsto H_\eta(x,y) := \langle B(x,y), \eta\rangle_{\overline{g}(f(p))} \end{aligned} \tag{2.120}$$
Alternatively one may define the quadratic form
$$\begin{aligned} \mathbb{II}_\eta : T_pM &\to \mathbb{R} \\ x &\mapsto H_\eta(x,x) \end{aligned}$$
One may also define the operator
$$\begin{aligned} S_\eta : T_pM &\to T_pM \\ x &\mapsto S_\eta(x) \end{aligned}$$
s.t.
$$\langle S_\eta(x), y\rangle_{\overline{g}(f(p))} = H_\eta(x,y) \qquad \forall\ x,\, y \in T_pM \tag{2.121}$$
These are called the second fundamental form of f at p along η.

One may write in general $\eta \in \mathfrak{X}(M)^\perp$.
Let
$$f : (M^n, g) \to (\overline{M}^{n+k}, \overline{g})$$
be isometric immersion.

Given $\eta \in \mathfrak{X}(M)^\perp$ normal fields w.r.t. $N(f)$. Let S_η be the operator associated to the second fundamental form of f along η as in (2.121)
$$\begin{aligned} S_\eta : \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ X &\mapsto S_\eta(X) \end{aligned}$$
s.t. S_η is viewed as a tensor of order 2
$$\langle S_\eta(X), Y\rangle_{\overline{g}} = H_\eta(X,Y) \qquad \forall\ X,\, Y \in \mathfrak{X}(M) \tag{2.122}$$

Note S_η is self-adjoint is equivalent to the tensor H_η being symmetric
$$H_\eta(X,Y) = H_\eta(Y,X)$$
which is indeed the case following Proposition 2.6.1.

Now one view
$$H_\eta \in C^\infty(M, \mathrm{Sym}^2 T^*M)$$
and
$$S_\eta \in C^\infty(M, T^*M \otimes TM) = C^\infty(M, \mathrm{End}(TM))$$

Lemma 2.6.2 ($\nabla_V S_\eta$ is self-adjoint)**.** *For any $V \in \mathfrak{X}(M)$*

$$\begin{aligned}\nabla_V S_\eta : \mathfrak{X}(M) &\to \mathfrak{X}(M)\\ X &\mapsto \nabla_V S_\eta(X)\end{aligned}$$

is self-adjoint

Proof. We differentiate the equation

$$\langle S_\eta(X), Y\rangle = \langle X, S_\eta(Y)\rangle$$

w.r.t. $V \in \mathfrak{X}(M)$ using that ∇ is the Levi-Civita connection

$$\langle \nabla_V(S_\eta(X)), Y\rangle + \langle S_\eta(X), \nabla_V Y\rangle = \langle \nabla_V X, S_\eta Y\rangle + \langle X, \nabla_V(S_\eta(Y))\rangle$$

Notice, again by Leibniz rule

$$\nabla_V(S_\eta(X)) = (\nabla_V S_\eta)(X) + S_\eta(\nabla_V X)$$

Hence

$$\langle (\nabla_V S_\eta)(X), Y\rangle + \langle S_\eta(\nabla_V X), Y\rangle + \langle S_\eta(X), \nabla_V Y\rangle = \langle \nabla_V X, S_\eta Y\rangle + \langle X, (\nabla_V S_\eta)(Y))\rangle + \langle X, S_\eta(\nabla_V Y)\rangle$$
$$\langle (\nabla_V S_\eta)(X), Y\rangle = \langle X, (\nabla_V S_\eta)(Y))\rangle$$

Using the fact that S_η is self-adjoint, applied to vector fields $\nabla_V X$, Y, and $\nabla_V Y$, X. □

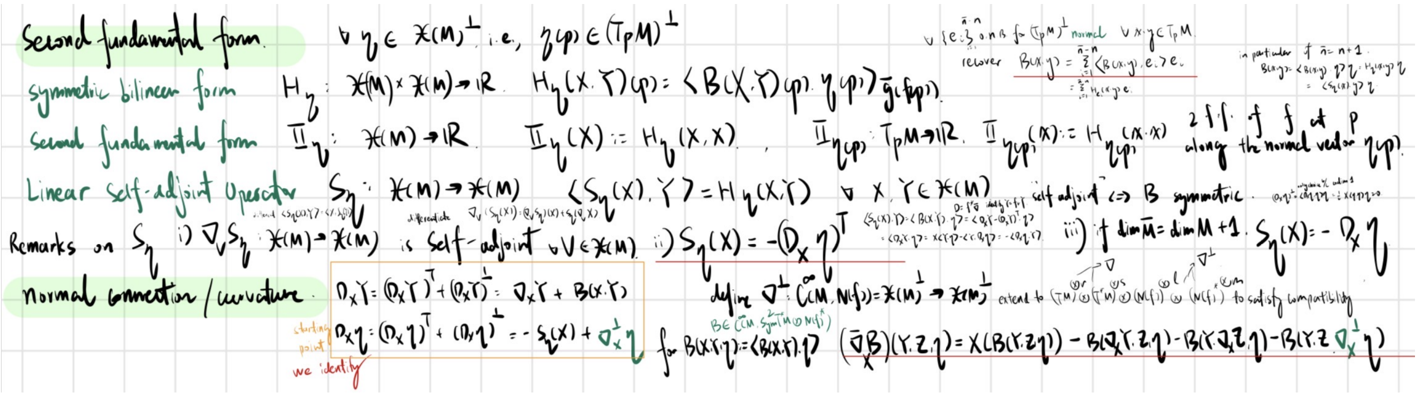


Figure 2.10: Second Fundamental Form

Explicit Expression of S_η We want to write all above. Denote $D = f^*\overline{\nabla}$.

$$\begin{aligned}H_\eta(X,Y) &= \langle B(X,Y), \eta\rangle = \langle D_X Y - D_X Y^T, \eta\rangle = \langle D_X Y, \eta\rangle\\ &= X\langle Y, \eta\rangle - \langle Y, D_X\eta\rangle\\ &= -\langle Y, D_X \eta\rangle\\ \mathrm{I\!I}_\eta(X) = H_\eta(X,X) &= \langle D_X X, \eta\rangle\end{aligned}$$

We incorporate this particularly important example as below.

Lemma 2.6.3 (Explicit Expression)**.**

$$S_\eta(X) = -(D_X\eta)^T \qquad \forall\ X \in \mathfrak{X}(M),\ \eta \in (T_pM)^\perp \tag{2.123}$$

This is the tangent part of how η changes along X. We identify η as the local extension normal to M.

Proof.

$$\begin{aligned}\langle S_\eta(x), y\rangle &= H_\eta(x,y) = \langle B(x,y), \eta\rangle\\ &= \langle (D_X Y)^\perp, \eta\rangle = \langle D_X Y, \eta\rangle \qquad \text{since } \eta \text{ is orthogonal already}\\ &= X(\langle Y, \eta\rangle) - \langle Y, D_X\eta\rangle \qquad \text{now we use that } D \text{ is compatible with the metric}\\ &= -\langle Y, (D_X\eta)^T\rangle = -\langle (D_X\eta)^T, y\rangle \qquad \text{using } Y \in \mathfrak{X}(M) \text{ and } \eta \in (T_pM)^\perp \text{ so only tangential part is preserved}\end{aligned}$$

□

Corollary 2.6.1 (Shape Operator)**.** *Let* $\dim \overline{M} = \dim M + 1$*. Now there exists unique* η *s.t.* $\|\eta\| = 1$*. In this case* $f(M) \subseteq \overline{M}$ *is called a hypersurface.*

Then one has the shape operator

$$S_\eta(X) = -D_X\eta \tag{2.124}$$

Proof.

$$\begin{aligned}(D_X\eta)^\perp &= \langle D_X\eta, \eta\rangle\eta \\ &= \frac{1}{2}X(\langle\eta,\eta\rangle)\eta \qquad D \text{ is compatible with the metric} \\ &= 0\end{aligned}$$

□

Examples of S_η

Example 2.6.2 ($\mathbb{S}^n$)**.** *Consider*

$$f : (\mathbb{S}^n, g_{can}) \hookrightarrow (\mathbb{R}^{n+1}, g)$$

For any $p \in \mathbb{S}^n$, $p = (x_1, \cdots, x_n)$ *satisfies* $\sum_{i=1}^n x_i^2 = 1$*.*

Consider inward unit normal on the sphere $\mathbb{S}^n$

$$\eta(p) = -p \qquad s.t. \qquad \eta \in \mathfrak{X}(\mathbb{S}^n)^\perp \tag{2.125}$$

In particular

$$\eta(p) = -\sum_{i=1}^{n+1} x^i \left.\frac{\partial}{\partial x_i}\right|_p$$

In fact, for any $p \in \mathbb{S}^n$

$$S_\eta(p) : T_p\mathbb{S}^n \to T_p\mathbb{S}^n$$

is the identity.

Proof. We do computation in local coordinates. For any $v \in T_p\mathbb{S}^n$ s.t.

$$v = \sum_{i=1}^{n+1} a^i \left.\frac{\partial}{\partial x_i}\right|_p$$

using coordinates from the ambient manifold $\mathbb{R}^{n+1}$. What is then the shape operator? For $\overline{\nabla}$ the Levi-Civita connection on $\mathbb{R}^{n+1}$ and $D = f^*\overline{\nabla}$, we define $\overline{\eta} \in \mathfrak{X}(\mathbb{R}^{n+1})$ s.t.

$$\overline{\eta} := -\sum_{i=1}^{n+1} x^i \frac{\partial}{\partial x_i} \qquad \text{so} \qquad \overline{\eta}(p) := \eta(p) \qquad \forall\ p \in \mathbb{S}^n$$

Thus

$$\begin{aligned}S_\eta(p)(v) &= -D_v\eta = -\overline{\nabla}_v\overline{\eta} \qquad \text{where } \overline{\eta} \text{ is } f\text{-related to } \eta \text{ defined in the full ambient space } \mathbb{R}^{n+1} \text{ that restricts to } \eta \text{ on } \mathbb{S}^n \\ &= -\overline{\nabla}_{\sum_i a^i \frac{\partial}{\partial x_i}} \left.\left(-\sum_j x^j \frac{\partial}{\partial x_j}\right)\right|_p \\ &= \sum_i a^i \overline{\nabla}_{\frac{\partial}{\partial x_i}} \left.\left(\sum_j x^j \frac{\partial}{\partial x_j}\right)\right|_p \\ &= \sum_{ij} a^i \frac{\partial}{\partial x_i}(x^j) \left.\frac{\partial}{\partial x_j}\right|_p + \sum_{ij} a^i x^j\, \overline{\nabla}_{\frac{\partial}{\partial x^i}} \left.\frac{\partial}{\partial x^j}\right|_p \\ &= \sum_{ij} a^i \frac{\partial}{\partial x_i}(x^j) \left.\frac{\partial}{\partial x_j}\right|_p \qquad \text{where } \overline{\nabla}_{\frac{\partial}{\partial x^i}} \left.\frac{\partial}{\partial x^j}\right|_p = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k} = 0 \text{ due to } \Gamma = 0 \text{ on } (\mathbb{R}^{n+1}, g_0) \\ &= \sum_{ij} a^i \delta_{ij} \left.\frac{\partial}{\partial x_j}\right|_p = \sum_j a^j \left.\frac{\partial}{\partial x_j}\right|_p = v\end{aligned}$$

Therefore

$$\begin{aligned}H_\eta(x,y) &= \langle B(x,y), \eta\rangle = \langle S_\eta(x), y\rangle = \langle x, y\rangle = g_{\text{can}}(x,y) \qquad \forall\ x,\, y \in \mathfrak{X}(\mathbb{S}^n) \\ B(x,y) &= \langle x, y\rangle\eta\end{aligned}$$

□

One may compute normals for graphs.

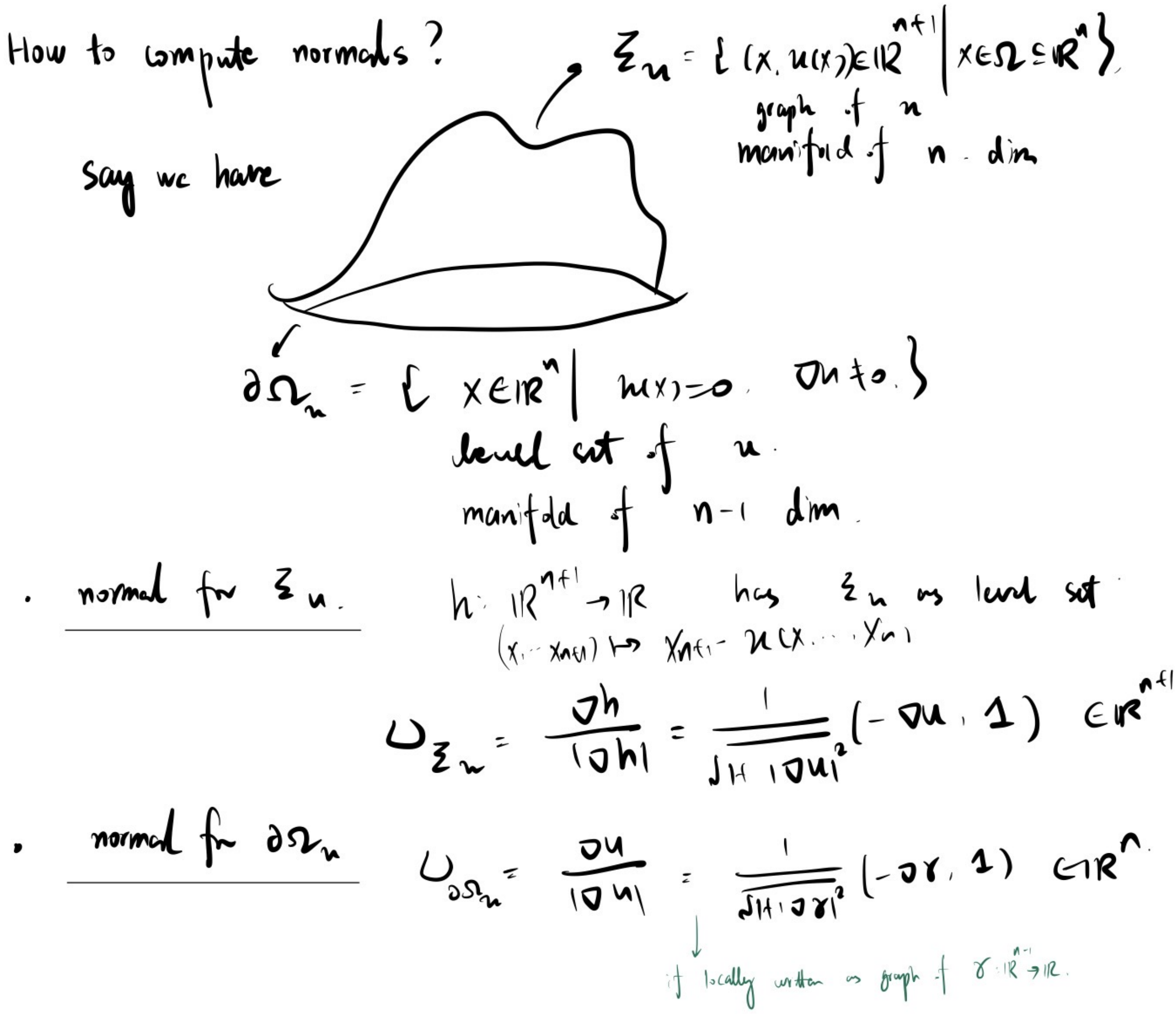


Figure 2.11: Normal for Graphs

2.6.3 Local Coordinates

Let

$$f : M^n \to \overline{M}^{n+m}$$

be isometric immersion. Let M^n have local coordinates $(x_1, \cdots, x_n)$, while $\overline{M}^{n+m}$ have coordinates $(y_1, \cdots, y_n, \cdots, y_{n+m})$. Let's write our smooth map f in coordinates

$$f(x_1, \cdots, x_n) = (f^1(x), \cdots, f^{n+m}(x))$$

We denote coordinate vector fields as

$$\partial_i = \frac{\partial}{\partial x_i}, \qquad \overline{\partial_\alpha} = \frac{\partial}{\partial y_\alpha}, \qquad 1 \le i \le n,\ 1 \le \alpha \le n+m$$

Thus

$$f_*(\partial_i) = \sum_{\alpha=1}^{n+m} \partial_i f^\alpha \overline{\partial}_\alpha$$

The induced metric is given by

$$g_{ij} = \overline{g}(f_*(\frac{\partial}{\partial x_i}), f_*(\frac{\partial}{\partial x_j})) = \overline{g}\left(\partial_i f^\alpha \overline{\partial}_\alpha, \partial_j f^\beta \overline{\partial}_\beta\right) = \overline{g}_{\alpha\beta} \partial_i f^\alpha \partial_j f^\beta \tag{2.126}$$

The smooth frame for $f^*T\overline{M}$ is

$$\{\overline{\partial}_\alpha \circ f\}_{\alpha=1}^{n+m} \subseteq C^\infty(M, f^*T\overline{M})$$

The tangent frame along f is

$$\{f_*(\partial_i)\}_{1\le i\le n} = \{\partial_i f^\alpha \overline{\partial}_\alpha\}_{1\le i\le n} \subseteq C^\infty(M, f_*(TM)) \cong \mathfrak{X}(M)$$

Smooth Frame for Normal Bundle We would like to make sense of the frame for normal bundle, i.e.,

$$\mathfrak{X}(M)^\perp = C^\infty(M, N(f))$$

Recall

$$\eta \in (T_pM)^\perp \qquad \Longleftrightarrow \qquad \overline{g}_{f(p)}(\eta, df_p(u)) = 0 \qquad \forall\ u \in T_pM$$

Thus

$$\eta \in \mathfrak{X}(M)^\perp \qquad \Longleftrightarrow \qquad \overline{g}(\eta, f_*(\partial_i)) = 0 \qquad \forall\ i = 1, \cdots, n$$

Since we're working on $f^*T\overline{M}$ with frame $\{\overline{\partial}_\alpha \circ f\}_{\alpha=1}^{n+m}$. We first assume for η of the form

$$\eta_a = \eta_a^\alpha(\overline{\partial}_\alpha \circ f)$$

A local normal frame $\{\eta_a\}_{a=1}^m \subseteq C^\infty(M, N(f))$ are expected to satisfy

$$\overline{g}(\eta_a, \eta_b) = \delta_{ab}, \qquad \overline{g}(\eta_a, f_*(\partial_i)) = 0 \qquad \forall\ i = 1, \cdots, n \tag{2.127}$$

Thus we solve for

$$\begin{aligned} 0 &= \overline{g}(\eta_a^\alpha(\overline{\partial}_\alpha \circ f), \partial_i f^\beta(\overline{\partial}_\beta \circ f)) \\ &= (\overline{g}_{\alpha\beta} \circ f)\eta_a^\alpha \partial_i f^\beta \end{aligned}$$

Now there are n linear equations with $n+m$ unknowns η_a^α for $\alpha = 1, \cdots, n+m$. Thus the solution space would be m dimensional, which spans $(T_pM)^\perp$.

Define the $n \times (n+m)$ matrix to be

$$A_{i\alpha}(p) = (\overline{g}_{\alpha\beta} \circ f)\partial_i f^\beta$$

so that η^α (we drop a) solve

$$\sum_{\alpha=1}^{n+m} A_{i\alpha}(p)\eta^\alpha(p) = 0 \qquad \forall\ i = 1, \cdots, n$$

Because f is immersion so df_p has full rank n, $A(p)$ has rank n, so

$$(T_pM)^\perp = \ker(A(p))$$

and thus $\dim(T_pM)^\perp = m$.

Now we pick an orthonormal frame $\{\eta_a\}_{a=1}^m$ that satisfies (2.127).

Coordinates for Second Fundamental Form B Recall B (2.118)

$$\begin{aligned} B : \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M)^\perp \\ (X, Y) &\mapsto D_X Y - (D_X Y)^T \end{aligned}$$

To understand the above in local coordinates, we need the pullback connection.

Now let $D = f^*\overline{\nabla}$. For any $V \in C^\infty(M, f^*T\overline{M})$ the pullback connection is defined via

$$D_{\partial_i} V = \overline{\nabla}_{f_*(\partial_i)} \tilde{V}$$

where $\tilde{V}$ is now a local extension of V to a vector field on $\overline{M}$.

Thus for the section

$$V = V^\alpha(\overline{\partial}_\alpha \circ f)$$

We compute

$$\begin{aligned} D_{\partial_i} V = D_{\partial_i}(V^\alpha(\overline{\partial}_\alpha \circ f)) &\overset{(1.114)}{=} \partial_i V^\alpha(\overline{\partial}_\alpha \circ f) + V^\alpha(\overline{\nabla}_{f_*(\partial_i)}\overline{\partial}_\alpha) \circ f \\ &= \partial_i V^\alpha(\overline{\partial}_\alpha \circ f) + V^\alpha \partial_i f^\beta(\overline{\nabla}_{\overline{\partial}_\beta}\overline{\partial}_\alpha) \circ f \\ &= \partial_i V^\alpha(\overline{\partial}_\alpha \circ f) + V^\alpha \partial_i f^\beta \Gamma_{\beta\alpha}^\gamma \overline{\partial}_\gamma \circ f \\ &= \left(\partial_i V^\alpha + V^\gamma \partial_i f^\beta \Gamma_{\beta\gamma}^\alpha\right)(\overline{\partial}_\alpha \circ f) \end{aligned}$$

In particular

$$D_{\partial_i}(\overline{\partial}_\alpha \circ f) = \partial_i f^\beta (\Gamma^\gamma_{\beta\alpha} \circ f)(\overline{\partial}_\gamma \circ f)$$

Moreover

$$\begin{aligned} D_{\partial_i}(f_*(\partial_j)) &= D_{\partial_i}(\partial_j f^\alpha \overline{\partial}_\alpha \circ f) = \partial_i\partial_j f^\alpha \overline{\partial}_\alpha \circ f + \partial_j f^\alpha D_{\partial_i}(\overline{\partial}_\alpha \circ f) \\ &= \partial_i\partial_j f^\alpha \overline{\partial}_\alpha \circ f + \partial_j f^\alpha \partial_i f^\beta (\Gamma^\gamma_{\beta\alpha} \circ f)(\overline{\partial}_\gamma \circ f) \\ &= \Big(\partial_i\partial_j f^\gamma + \partial_j f^\alpha \partial_i f^\beta (\Gamma^\gamma_{\beta\alpha} \circ f)\Big)(\overline{\partial}_\gamma \circ f) \end{aligned}$$

Now

$$D_{\partial_i}(f_*(\partial_j)) \in C^\infty(M, f^* T\overline{M})$$

This is essentially what we want to decompose!

$$D_{\partial_i}(f_*(\partial_j)) = D_{\partial_i}(f_*(\partial_j))^T + D_{\partial_i}(f_*(\partial_j))^\perp$$

The second part gives our desired

$$B_{ij} = B(\partial_i, \partial_j) = D_{\partial_i}(f_*(\partial_j))^\perp$$

The tangential part. Since the tangent part lies in $C^\infty(M, f_*(TM)) \cong \mathfrak{X}(M)$. We assume

$$D_{\partial_i}(f_*(\partial_j))^T = T^k_{ij} f_*(\partial_k)$$

How to find the coefficients T^k_{ij}? We use orthogonal projection

$$\overline{g}(D_{\partial_i}(f_*(\partial_j)) - T^k_{ij} f_*(\partial_k), f_*(\partial_\ell)) = 0 \qquad \forall\ \ell = 1, \cdots, n$$

Thus

$$\overline{g}(D_{\partial_i}(f_*(\partial_j)), f_*(\partial_\ell)) = T^k_{ij}\overline{g}(f_*(\partial_k), f_*(\partial_\ell)) \overset{(2.126)}{=} T^k_{ij} g_{k\ell}$$

So inverting gives

$$\begin{aligned} T^k_{ij} &= g^{k\ell}\overline{g}(D_{\partial_i}(f_*(\partial_j)), f_*(\partial_\ell)) \qquad \forall\ k = 1, \cdots, n \\ &= g^{k\ell}\overline{g}\Big(\Big(\partial_i\partial_j f^\gamma + \partial_j f^\alpha \partial_i f^\beta (\Gamma^\gamma_{\beta\alpha} \circ f)\Big)(\overline{\partial}_\gamma \circ f), \partial_\ell f^\delta (\overline{\partial}_\delta \circ f)\Big) \\ &= g^{k\ell}\partial_\ell f^\delta \Big(\partial_i\partial_j f^\gamma + \partial_j f^\alpha \partial_i f^\beta (\Gamma^\gamma_{\beta\alpha} \circ f)\Big)\overline{g}_{\gamma\delta} \circ f \end{aligned}$$

Hence

$$\begin{aligned} D_{\partial_i}(f_*(\partial_j))^T &= \Big(g^{k\ell}\partial_\ell f^\delta \Big(\partial_i\partial_j f^\gamma + \partial_j f^\alpha \partial_i f^\beta (\Gamma^\gamma_{\beta\alpha} \circ f)\Big)\overline{g}_{\gamma\delta} \circ f\Big) f_*(\partial_k) \\ &= \Big(g^{k\ell}\partial_k f^\varepsilon \partial_\ell f^\delta \Big(\partial_i\partial_j f^\gamma + \partial_j f^\alpha \partial_i f^\beta (\Gamma^\gamma_{\beta\alpha} \circ f)\Big)\overline{g}_{\gamma\delta} \circ f\Big)\overline{\partial}_\varepsilon \circ f \end{aligned}$$

The normal part.
Now we can compute

$$B_{ij} = B(\partial_i, \partial_j) = D_{\partial_i}(f_*(\partial_j))^\perp = D_{\partial_i}(f_*(\partial_j)) - D_{\partial_i}(f_*(\partial_j))^T$$

in terms of frame $\{\overline{\partial}_\gamma \circ f\}$.

But this is messy. In fact one does not need to compute the tangential part at all! Instead we write in coordinates

$$B(\partial_i, \partial_j) = h^a_{ij}\eta_a$$

for local frame of normal bundle as in (2.127), where

$$\begin{aligned} h^a_{ij} &= \overline{g}(B(\partial_i, \partial_j), \eta_a) \\ &= \overline{g}(D_{\partial_i}(f_*(\partial_j)), \eta_a) \\ &= \overline{g}\Big(\Big(\partial_i\partial_j f^\gamma + \partial_j f^\alpha \partial_i f^\beta (\Gamma^\gamma_{\beta\alpha} \circ f)\Big)(\overline{\partial}_\gamma \circ f), \eta_a\Big) \end{aligned}$$

On the other hand one has the form

$$\eta_a = \eta^\delta_a (\overline{\partial}_\delta \circ f)$$

so

$$\begin{aligned} h^a_{ij} &= \overline{g}\Big(\Big(\partial_i\partial_j f^\gamma + \partial_j f^\alpha \partial_i f^\beta (\Gamma^\gamma_{\beta\alpha} \circ f)\Big)(\overline{\partial}_\gamma \circ f),\ \eta^\delta_a (\overline{\partial}_\delta \circ f)\Big) \\ &= \Big(\partial_i\partial_j f^\gamma + \partial_j f^\alpha \partial_i f^\beta (\Gamma^\gamma_{\beta\alpha} \circ f)\Big)\eta^\delta_a \overline{g}_{\gamma\delta} \circ f \end{aligned}$$

2.6.4 Gauss-Codazzi-Ricci Equations

This section is about how the curvature splits into tangent and normal parts. Let

$$f : (M, g) \to (\overline{M}, \overline{g})$$

be an isometric immersion. In particular,

$$\begin{aligned} f_* : \mathfrak{X}(M) &\to C^\infty(M, f^*T\overline{M}) \\ X &\mapsto f_*(X)(p) := df_p(X(p)) \qquad \forall\, p \in M \end{aligned}$$

is an injective map.

From now on we identify X with f_*X.

Denote the pullback connection as $D := f^*\overline{\nabla}$. For any $X, Y \in \mathfrak{X}(M)$ and $\eta \in \mathfrak{X}(M)^\perp$, we look at the quantities

$$\begin{aligned} D_X Y &= (D_X Y)^T + (D_X Y)^\perp \\ &= \nabla_X Y + B(X, Y) \end{aligned} \tag{2.128}$$

$$\begin{aligned} D_X \eta &= (D_X \eta)^T + (D_X \eta)^\perp \\ &= -S_\eta(X) + (D_X \eta)^\perp \end{aligned} \tag{2.129}$$

We denote $\nabla^\perp_X \eta := (D_X\eta)^\perp$. From this, we interpret $\nabla^\perp$ as connection on the normal bundle $N(f) \to M$. This allows us to extend the definition of covariant derivative and connection on the normal bundle.

Covariant Derivative and Curvature on Normal Bundle In particular, we're given connections

1. ∇ on TM, T^*M
2. and $\nabla^\perp$ on $N(f)$, $N(f)^*$.

We obtain **covariant derivative $\overline{\nabla}$ acting on**

$$(TM)^{\otimes r} \otimes (T^*M)^{\otimes s} \otimes (N(f))^{\otimes \ell} \otimes (N(f)^*)^{\otimes m}$$

In particular, we're interested in hitting the connection on

$$B(X, Y, \eta) := \langle B(X, Y), \eta \rangle$$

which is an object

$$B \in C^\infty(M, \mathrm{Sym}^2 T^*M \otimes N(f)^*)$$

For any $X \in \mathfrak{X}(M)$, define using the compatibility condition (Leibniz Rule (1.104))

$$\begin{aligned} \overline{\nabla}_X B : \mathfrak{X}(M) \times \mathfrak{X}(M) \times N(f) &\to C^\infty(M) \\ (Y, Z, \eta) &\mapsto X(B(Y, Z, \eta)) - B(\nabla_X Y, Z, \eta) - B(Y, \nabla_X Z, \eta) - B(Y, Z, \nabla^\perp_X \eta) \end{aligned}$$

In other words

$$(\overline{\nabla}_X B)(Y, Z, \eta) := X(B(Y, Z, \eta)) - B(\nabla_X Y, Z, \eta) - B(Y, \nabla_X Z, \eta) - B(Y, Z, \nabla^\perp_X \eta) \qquad \forall\, Y,\, Z \in \mathfrak{X}(M),\, \eta \in \mathfrak{X}(M)^\perp \tag{2.130}$$

Similarly, for the curvature, we have

1. full curvature
$$\overline{R} \in \Omega^2(\overline{M}, \mathrm{End}(T\overline{M})) \qquad \text{curvature of } \overline{\nabla}$$
2. and pullback curvature
$$f^*\overline{R} \in \Omega^2(M, \mathrm{End}(f^*T\overline{M})) \qquad \text{curvature of } f^*\overline{\nabla} = D$$
3. and the exact curvature of the submanifold
$$R \in \Omega^2(M, \mathrm{End}(TM)) \qquad \text{curvature of } \nabla$$
4. and the curvature of the normal
$$R^\perp \in \Omega^2(M, \mathrm{End}(N(f))) \qquad \text{curvature of } \nabla^\perp$$

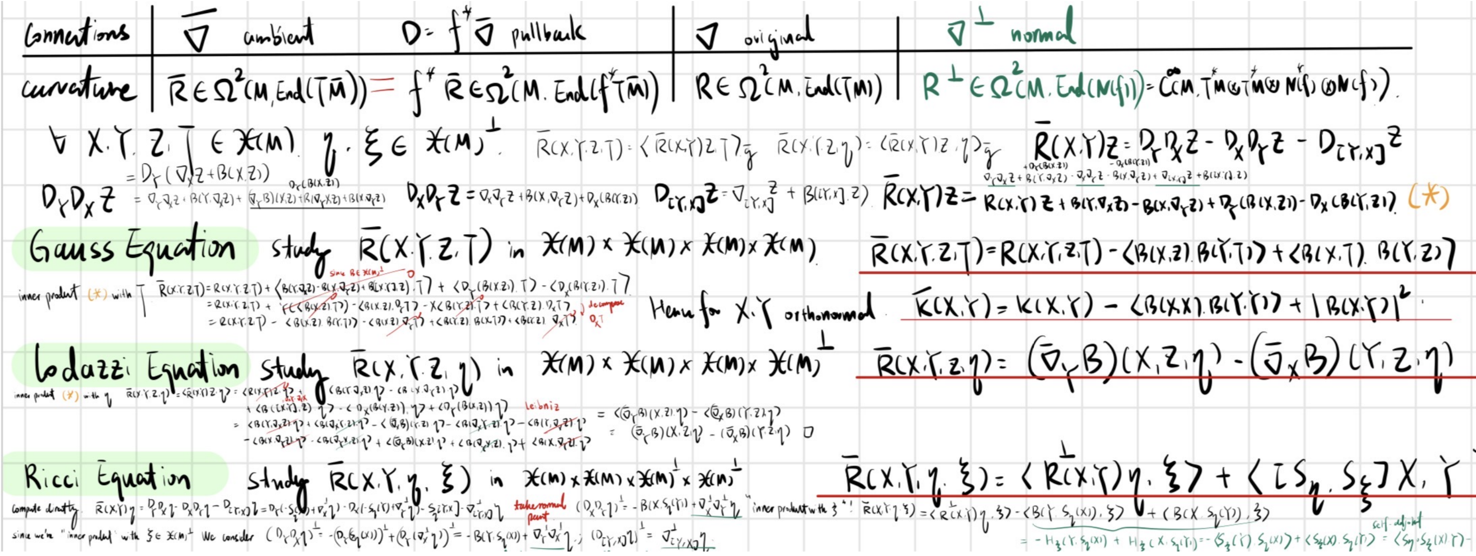


Figure 2.12: Gauss-Codazzi-Ricci

Gauss-Codazzi-Ricci Equations For $\overline{R}(X,Y,\cdot,\cdot)$ where $X,\,Y\in\mathfrak{X}(M)$ but with the last two variables free

1. In $TM\times TM$ ($\mathfrak{X}(M)\times\mathfrak{X}(M)$) this gives Gauss Equation.
2. In $TM\times N(f)$ or $N(f)\times TM$ ($\mathfrak{X}(M)\times\mathfrak{X}^{\perp}(M)$) this gives Codazzi Equation.
3. In $N(f)\times N(f)$ ($\mathfrak{X}^{\perp}(M)\times\mathfrak{X}^{\perp}(M)$) this gives Ricci Equation.

Proposition 2.6.2 (Gauss Equation)**.** *Let $X,\,Y,\,Z,\,T\in\mathfrak{X}(M)$. We have Riemannian Curvature of the Ambient Manifold.*

$$\overline{R}(X,Y,Z,T)=R(X,Y,Z,T)-\langle B(X,Z),B(Y,T)\rangle+\langle B(X,T),B(Z,Y)\rangle \tag{2.131}$$

Proposition 2.6.3 (Codazzi Equation)**.** *For $\eta\in\mathfrak{X}(M)^{\perp}$*

$$\overline{R}(X,Y,Z,\eta)=(\overline{\nabla}_Y B)(X,Z,\eta)-(\overline{\nabla}_X B)(Y,Z,\eta) \tag{2.132}$$

where

$$B(X,Y,\eta):=\langle B(X,Y),\eta\rangle$$

Proposition 2.6.4 (Ricci Equation)**.** *For $\eta,\,\xi\in\mathfrak{X}(M)^{\perp}$*

$$\overline{R}(X,Y,\eta,\xi)=\langle R^{\perp}(X,Y)\eta,\xi\rangle+\langle[S_\eta,S_\xi]X,Y\rangle \tag{2.133}$$

where $R^{\perp}\in\Omega^2(M,\mathrm{End}(N(f)))$ is the curvature of $\nabla^{\perp}$.

Proof of three equations (2.131), (2.132), (2.133)*.* By Definition

$$\overline{R}(X,Y,Z,T)=\langle\overline{R}(X,Y)Z,T\rangle$$
$$\overline{R}(X,Y,Z,\eta)=\langle\overline{R}(X,Y)Z,\eta\rangle$$

And

$$\overline{R}(X,Y)Z=D_YD_XZ-D_XD_YZ+D_{[X,Y]}Z$$

We want to write

$$\begin{aligned}
D_YD_XZ&=D_Y(\nabla_XZ+B(X,Z))\\
&=D_Y(\nabla_XZ)+D_Y(B(X,Z))\\
&=\nabla_Y\nabla_XZ+B(Y,\nabla_XZ)+D_Y(B(X,Z))\\
D_XD_YZ&=\nabla_X\nabla_YZ+B(X,\nabla_YZ)+D_X(B(Y,Z))\\
D_{[X,Y]}Z&=\nabla_{[X,Y]}Z+B([X,Y],Z)
\end{aligned}$$

Now

$$\begin{aligned}
\overline{R}(X,Y)Z&=\nabla_Y\nabla_XZ+B(Y,\nabla_XZ)+D_Y(B(X,Z))\\
&\quad-\nabla_X\nabla_YZ-B(X,\nabla_YZ)-D_X(B(Y,Z))\\
&\quad+\nabla_{[X,Y]}Z+B([X,Y],Z)\\
&=R(X,Y)Z+(B(Y,\nabla_XZ)-B(X,\nabla_YZ)+B([X,Y],Z))+D_Y(B(X,Z))-D_X(B(Y,Z))
\end{aligned}$$

so we take away

$$\overline{R}(X,Y)Z = R(X,Y)Z + (B(Y,\nabla_X Z) - B(X,\nabla_Y Z) + B([X,Y],Z)) + D_Y(B(X,Z)) - D_X(B(Y,Z)) \quad (2.134)$$

Gauss Equations.
Let's prove Gauss Equation (2.131) first. We contract it with T, something tangent

$$\begin{aligned}
\overline{R}(X,Y,Z,T) &\overset{(2.134)}{=} R(X,Y,Z,T) + \underbrace{\cancel{\langle B(Y,\nabla_X Z) - B(X,\nabla_Y Z) + B([X,Y],Z),T\rangle}}_{0 \text{ because } B(\cdot,\cdot)\in\mathfrak{X}(M)^\perp} \\
&+ \langle D_Y(B(X,Z)),T\rangle - \langle D_X(B(Y,Z)),T\rangle \\
&= R(X,Y,Z,T) + 0 \\
&+ \underbrace{\cancel{Y(\langle B(X,Z),T\rangle)}}_{0 \text{ because } B(\cdot,\cdot)\in\mathfrak{X}(M)^\perp} - \langle B(X,Z),D_Y T\rangle - \underbrace{\cancel{X(\langle B(Y,Z),T\rangle)}}_{0 \text{ because } B(\cdot,\cdot)\in\mathfrak{X}(M)^\perp} + \langle B(Y,Z),D_X T\rangle \\
&= R(X,Y,Z,T) - \langle B(X,Z),D_Y T\rangle + \langle B(Y,Z),D_X T\rangle \\
&\overset{(2.128)}{=} R(X,Y,Z,T) - \underbrace{\cancel{\langle B(X,Z),\nabla_Y T\rangle}}_{B(\cdot,\cdot)\in\mathfrak{X}(M)^\perp} - \langle B(X,Z),B(Y,T)\rangle + \underbrace{\cancel{\langle B(Y,Z),\nabla_X T\rangle}}_{B(\cdot,\cdot)\in\mathfrak{X}(M)^\perp} + \langle B(Y,Z),B(X,T)\rangle \\
&= R(X,Y,Z,T) - \langle B(X,Z),B(Y,T)\rangle + \langle B(Y,Z),B(X,T)\rangle
\end{aligned}$$

Codazzi Equation.
Now let's prove Codazzi (2.132). We contract it with η

$$\begin{aligned}
\overline{R}(X,Y,Z,\eta) &\overset{(2.134)}{=} \underbrace{\cancel{\langle R(X,Y)Z,\eta\rangle}}_{0 \text{ because } R(X,Y)Z\in\mathfrak{X}(M)} + \langle B(Y,\nabla_X Z),\eta\rangle - \langle B(X,\nabla_Y Z),\eta\rangle + \langle B(\nabla_X Y - \nabla_Y X, Z),\eta\rangle \\
&+ \langle D_Y(B(X,Z)),\eta\rangle - \langle D_X(B(Y,Z)),\eta\rangle \\
&= 0 + \langle B(Y,\nabla_X Z),\eta\rangle - \langle B(X,\nabla_Y Z),\eta\rangle + \langle B(\nabla_X Y - \nabla_Y X, Z),\eta\rangle \\
&+ \langle D_Y(B(X,Z)),\eta\rangle - \langle D_X(B(Y,Z)),\eta\rangle \\
&= \langle B(Y,\nabla_X Z),\eta\rangle + \langle B(\nabla_X Y, Z),\eta\rangle - \langle D_X(B(Y,Z)),\eta\rangle \qquad \text{we put together all the } X \text{ derivatives} \\
&- \langle B(X,\nabla_Y Z),\eta\rangle - \langle B(\nabla_Y X, Z),\eta\rangle + \langle D_Y(B(X,Z)),\eta\rangle \qquad \text{and the } Y \text{ derivatives} \\
&= \langle B(Y,\nabla_X Z),\eta\rangle + \langle B(\nabla_X Y, Z),\eta\rangle - (X(\langle B(Y,Z),\eta\rangle) - \langle B(Y,Z),D_X\eta\rangle) \qquad \text{Leibniz} \\
&- \langle B(X,\nabla_Y Z),\eta\rangle - \langle B(\nabla_Y X, Z),\eta\rangle + (Y(\langle B(X,Z),\eta\rangle) - \langle B(X,Z),D_Y\eta\rangle) \\
&\overset{(2.129)}{=} \langle B(Y,\nabla_X Z),\eta\rangle + \langle B(\nabla_X Y, Z),\eta\rangle - \big(X(\langle B(Y,Z),\eta\rangle) - \langle B(Y,Z),\nabla^\perp_X\eta\rangle\big) \\
&- \langle B(X,\nabla_Y Z),\eta\rangle - \langle B(\nabla_Y X, Z),\eta\rangle + \big(Y(\langle B(X,Z),\eta\rangle) - \langle B(X,Z),\nabla^\perp_Y\eta\rangle\big) \\
&= -(\overline{\nabla}_X B)(Y,Z,\eta) + (\overline{\nabla}_Y B)(X,Z,\eta) \qquad \text{using definition of (2.130)}
\end{aligned}$$

Ricci Equation.
Finally for Ricci Equation (2.133), by definition

$$\overline{R}(X,Y)\eta = D_Y D_X\eta - D_X D_Y\eta + D_{[X,Y]}\eta$$

We compute

$$\begin{aligned}
D_X\eta &\overset{(2.129)}{=} (D_X\eta)^T + (D_X\eta)^\perp = -S_\eta(X) + \nabla^\perp_X\eta \\
(D_Y D_X\eta)^\perp &= (D_Y(-S_\eta(X) + \nabla^\perp_X\eta))^\perp \\
&= -B(Y,S_\eta(X)) + \nabla^\perp_Y\nabla^\perp_X\eta \\
(D_X D_Y\eta)^\perp &= -B(X,S_\eta(Y)) + \nabla^\perp_X\nabla^\perp_Y\eta \\
(D_{[X,Y]}\eta)^\perp &= \nabla^\perp_{[X,Y]}\eta
\end{aligned}$$

Now we contract with $\xi\in\mathfrak{X}(M)^\perp$ so that

$$\begin{aligned}
\overline{R}(X,Y,\eta,\xi) &= -\langle B(Y,S_\eta(X)),\xi\rangle + \langle\nabla^\perp_Y\nabla^\perp_X\eta,\xi\rangle + \langle B(X,S_\eta(Y)),\xi\rangle - \langle\nabla^\perp_X\nabla^\perp_Y\eta,\xi\rangle + \langle\nabla^\perp_{[X,Y]}\eta,\xi\rangle \\
&= \langle R^\perp(X,Y)\eta,\xi\rangle + \langle B(X,S_\eta(Y)),\xi\rangle - \langle B(Y,S_\eta(X)),\xi\rangle
\end{aligned}$$

But Recall that

$$\langle S_\xi(X), Y\rangle = H_\xi(X,Y) = \langle B(X,Y), \xi\rangle$$

So here the terms

$$\langle B(X, S_\eta(Y)), \xi\rangle = \langle S_\xi(X), S_\eta(Y)\rangle$$
$$\langle B(Y, S_\eta(X)), \xi\rangle = \langle S_\xi(Y), S_\eta(X)\rangle$$

Therefore

$$\begin{aligned}\overline{R}(X,Y,\eta,\xi) &= R^\perp(X,Y,\eta,\xi) + \langle S_\xi(X), S_\eta(Y)\rangle - \langle S_\xi(Y), S_\eta(X)\rangle \\ &= R^\perp(X,Y,\eta,\xi) + \langle S_\eta \circ S_\xi(X), Y\rangle - \langle Y, S_\xi \circ S_\eta(X)\rangle \qquad \text{using } S \text{ is self-adjoint} \\ &= R^\perp(X,Y,\eta,\xi) + \langle [S_\eta, S_\xi](X), Y\rangle\end{aligned}$$

□

Sectional Curvature Gauss Formula As an immediate corollary, one may relate the Sectional Curvature of M and $\overline{M}$.

Corollary 2.6.2 (Gauss [dC92] Theorem 6.2.5)**.** *If X, Y are orthonormal, then the sectional curvature of* Span$\{X, Y\}$

$$K(X,Y) = R(X,Y,X,Y)$$

satisfies

$$\overline{K}(X,Y) - K(X,Y) = -\langle B(X,X), B(Y,Y)\rangle + |B(X,Y)|^2 \tag{2.135}$$

In the case of a hypersurface

$$f : M^n \to \overline{M}^{n+1}$$

the Gauss formula (2.135) has simple expression. Let $p \in M$ and $\eta \in (T_pM)^\perp$. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of T_pM in which S_η is diagonal, i.e.

$$S_\eta(e_i) = \sum_{i=1}^n \lambda_i e_i \qquad \forall\ i = 1, \cdots, n$$

where λ_i are the eigenvalues of S_η. Then using definition (2.121)

$$H(e_i, e_i) = \lambda_i \qquad H(e_i, e_j) = 0 \qquad i \neq j$$

Therefore with $X = e_i$ and $X = e_j$ in (2.135) one obtain

$$K(e_i, e_j) - \overline{K}(e_i, e_j) = \lambda_i \lambda_j$$

Example 2.6.3. *Consider isometric immersion of the sphere*

$$f : (\mathbb{S}^n, g_{can}) \hookrightarrow (\mathbb{R}^{n+1}, g_0)$$

Recall (2.125) *the unit inward normal*

$$\eta(p) = -p \in \mathfrak{X}(\mathbb{S}^n)^\perp \qquad |\eta| = 1$$

Then

$$B(X,Y) = \langle X, Y\rangle \eta$$

On our tangent space we pick X, Y orthonormal. Using (2.135)

$$\begin{aligned}\overline{K}(X,Y) - K(X,Y) &= -\langle B(X,X), B(Y,Y)\rangle + |B(X,Y)|^2 \\ 0 - K(X,Y) &= -\langle \langle X,X\rangle\eta, \langle Y,Y\rangle\eta\rangle + |B(X,Y)|^2 \\ &= -\langle \eta, \eta\rangle + |\langle X,Y\rangle\eta|^2 = -1 \\ K(X,Y) &= 1\end{aligned}$$

Hence the sectional curvature of $\mathbb{S}^n$ is 1.

2.6.5 Principal Curvature

Consider the codimension 1 case, i.e., let

$$f : (M^n, g) \hookrightarrow (\overline{M}^{n+1}, \overline{g})$$

be isometric immersion, $f(M) \subseteq \overline{M}$ the hypersurface.

Let $p \in M$ and $\eta \in (T_pM)^\perp$, $|\eta| = 1$. Since

$$S_\eta : T_pM \to T_pM$$

is symmetric, by spectral decomposition, there exists an orthonormal basis of eigenvectors $\{e_1, \cdots, e_n\}$ of T_pM with real eigenvalues $\lambda_1, \cdots, \lambda_n$, i.e.

$$S_\eta(e_i) = \lambda_i e_i \qquad \forall\ i = 1, \cdots, n$$

If M and $\overline{M}$ are both orientable and oriented, i.e. orientations chosen on M and $\overline{M}$, then the vector η is uniquely determined, if we require that both

1. $\{e_1, \cdots, e_n\}$ is basis in the orientation of M
2. and $\{e_1, \cdots, e_n, \eta\}$ is basis in the orientation of $\overline{M}$.

In this case, we say e_i are principal directions and that

$$\{\lambda_i\}_{1 \le i \le n}$$

are *principal curvatures* of f.

The symmetric functions of $\lambda_1, \cdots, \lambda_n$ are invariants of the immersion. Of particular interest are

1. Gauss-Kronecker Curvature of f
$$\det(S_\eta) = \lambda_1 \cdots \lambda_n$$
2. Mean curvature of f
$$\frac{1}{n}\operatorname{tr}(S_\eta) = \frac{1}{n}\sum_{i=1}^n \lambda_i$$

Proof. Let's check this coincides with (2.139). Indeed the mean curvature writes

$$\begin{aligned} H(p) &\overset{(2.139)}{=} \frac{1}{n}\operatorname{tr}(S_\eta)\eta \\ &= \frac{1}{n}\sum_{i=1}^n \langle S_\eta(e_i), e_i\rangle \eta \\ &= \frac{1}{n}\sum_{i=1}^n \langle \lambda_i e_i, e_i\rangle \eta = \frac{1}{n}\sum_{i=1}^n \lambda_i \eta \end{aligned}$$

But we're in codimension 1 so the average of principal curvatures recover the mean curvature. □

Principal Curvature in $n = 2$

Example 2.6.4 ([dC92] Remark 6.2.7)**.** *Consider two dimensional*

$$f : (M^2, g) \hookrightarrow (\mathbb{R}^3, g_0)$$

surface in $\mathbb{R}^3$.

Then the sectional curvature

$$K = \lambda_1\lambda_2$$

Proof. Choose $n \in T_pM^\perp$. Notice

$$\begin{aligned} H(e_1, e_1) = \langle B(e_1, e_1), n\rangle = \langle S_n(e_1), e_1\rangle = \langle \lambda_1 e_1, e_1\rangle = \lambda_1 \\ \langle B(e_1, e_2), n\rangle = \langle S_n(e_1), e_2\rangle = \lambda_1\langle e_1, e_2\rangle = 0 \end{aligned}$$

and again since we're in codimension 1

$$\langle B(e_1, e_1), B(e_2, e_2)\rangle = \langle \langle B(e_1, e_1), n\rangle n, \langle B(e_2, e_2), n\rangle n\rangle = \langle \lambda_1 n, \lambda_2 n\rangle$$

Thus the intrinsic sectional curvature is in fact recovered by the second fundamental form

$$\begin{aligned} K(p) = K(e_1, e_2) &\overset{(2.135)}{=} \langle B(e_1, e_1), B(e_2, e_2)\rangle - |B(e_1, e_2)|^2 \\ &= \langle \lambda_1 n, \lambda_2 n\rangle - 0 = \lambda_1\lambda_2 \end{aligned}$$

□

Now the product $\lambda_1\lambda_2$ of the principal curvature coincides with Gaussian curvature of the surface. In this case, the Gaussian Curvature coincides with the sectional curvature of the surface.

In short, for $\dim M = 2$

$$\text{Ricci} = \text{Sectional Curvature} = \text{Gaussian Curvature} = \text{product of Principal Curvature}$$

This is the famous Theorem Egregium of Gauss: Gaussian Curvature of $M^2 \subseteq \mathbb{R}^3$ is invariant under isometries.

2.6.6 Gauss Map

In general, if

$$M^n \hookrightarrow (\mathbb{R}^{n+1}, g_0)$$

and M^n is two–sided (oriented) hypersurface, then there exists a unit global normal vector $N \in \mathfrak{X}(M)^\perp$.

Then for any $p \in M$

$$N(p) \in (T_pM)^\perp \subseteq T_p\mathbb{R}^{n+1} = \mathbb{R}^{n+1}$$

and

$$N(p) \in \mathbb{S}^n \qquad \text{due to} \qquad |N| = 1$$

Definition 2.6.8 (Gauss Map)**.** *We define the unit global normal as*

$$\begin{aligned} N : M &\to \mathbb{S}^n \\ p &\mapsto N(p) \end{aligned} \tag{2.136}$$

This is known as the Gauss Map (Gauss spherical mapping).

Of particular interest is its differential, known as the *Weingarten Map.*

$$\begin{aligned} dN_p : T_pM &\to T_{N(p)}\mathbb{S}^n = T_pM \\ v &\mapsto dN_p(v) = (\overline{\nabla}_v N)(p) \overset{(2.124)}{=} -S_{N(p)}(v) \end{aligned}$$

By identifying $T_{N(p)}\mathbb{S}^n = T_pM$ as both orthogonal to $\mathbb{R}N(p)$ in $\mathbb{R}^{n+1}$.

Second Fundamental form H_N in Local Coordinates For $p = \mathbf{x}(u_1, \cdots, u_n)$ with coordinate chart

$$\mathbf{x}(u_1, \cdots, u_n) = (x_1(u_1, \cdots, u_n), \cdots, x_{n+1}(u_1, \cdots, u_n))$$

We compute its differential

$$\begin{aligned} d\mathbf{x} : T_uV &\to T_{\mathbf{x}(u)}M \subset T_{\mathbf{x}(u)}\mathbb{R}^{n+1} = \mathbb{R}^{n+1} \\ \frac{\partial}{\partial u_i} &\mapsto \frac{\partial \mathbf{x}}{\partial u_i} \equiv \mathbf{x}_i \end{aligned}$$

where

$$\frac{\partial}{\partial u_i} \cong \frac{\partial \mathbf{x}}{\partial u_i} = (\frac{\partial x_1}{\partial u_i}, \cdots, \frac{\partial x_n}{\partial u_i}, \frac{\partial x_{n+1}}{\partial u_i}) = \sum_{k=1}^{n+1} \frac{\partial x_k}{\partial u_i}\frac{\partial}{\partial x_k}$$

Now we look at Gauss Map

$$\begin{array}{ccc} V \subset \mathbb{R}^n \text{ with coordinates } (u_1, \cdots, u_n) & & \\ \mathbf{x}\text{ chart}\downarrow & \overset{\mathbf{y}\text{ chart}}{\searrow} & \\ p \in M & \xrightarrow{\quad N \quad} & \mathbb{S}^n \end{array}$$

The good thing about Gauss Map is that then we do computation for second fundamental form (2.120). Write

$$H_N = \sum h_{ij}\, du_i du_j \tag{2.137}$$

using that $H_N \in C^\infty(M, \text{Sym}^2 T^*M)$. Now the coefficients are

$$h_{ij} = \langle B(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}), N\rangle = \langle (\overline{\nabla}_{\frac{\partial}{\partial u_i}} \frac{\partial \mathbf{x}}{\partial u_j})^\perp, N\rangle = \langle \overline{\nabla}_{\frac{\partial}{\partial u_i}} \frac{\partial \mathbf{x}}{\partial u_j}, N\rangle$$

Notice in $\mathbb{R}^{n+1}$ with standard coordinates

$$\frac{\partial \mathbf{x}}{\partial u_j} = \sum_{k=1}^{n+1} \frac{\partial x_k}{\partial u_j} \frac{\partial}{\partial x_k}$$

and

$$\overline{\nabla}_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial x_k} = 0$$

Thus

$$\overline{\nabla}_{\frac{\partial}{\partial u_i}} \frac{\partial \mathbf{x}}{\partial u_j} = \overline{\nabla}_{\frac{\partial}{\partial u_i}} \left(\sum_{k=1}^{n+1} \frac{\partial x_k}{\partial u_j} \frac{\partial}{\partial x_k} \right) = \sum_{k=1}^{n+1} \frac{\partial^2 x_k}{\partial u_i \partial u_j} \frac{\partial}{\partial x_k} = \frac{\partial^2 \mathbf{x}}{\partial u_i \partial u_i} \equiv \mathbf{x}_{ij}$$

Therefore

$$h_{ij} = \langle \overline{\nabla}_{\frac{\partial}{\partial u_i}} \left(\sum_k \frac{\partial x_k}{\partial u_j} \frac{\partial}{\partial x_k} \right), N \rangle = \langle \sum_k \frac{\partial^2 x_k}{\partial u_i \partial u_j} \frac{\partial}{\partial x_k}, N \rangle = \langle \mathbf{x}_{ij}, N \rangle$$

Weingarten Map in Coordinates Now what is dN_p in local coordinates? Using $dN_p(v) = -S_{N(p)}(v)$ we get

$$\begin{aligned} dN_p &: T_pM \to T_pM \\ \mathbf{x}_i &= \frac{\partial \mathbf{x}}{\partial u_i} \mapsto -S_{N(p)}(\mathbf{x}_i) \end{aligned}$$

For if we're able to fully characterize $-S_{N(p)}$ acting on a basis of T_pM, we fully characterize the Differential of Gauss Map. But the latter is done via the computations for h_{ij}.

$$\underbrace{\langle \mathbf{x}_{ij}, N \rangle}_{\text{something we can compute}} = h_{ij} = H_N\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i}\right) = \langle S_N\left(\frac{\partial}{\partial u_i}\right), \frac{\partial}{\partial u_j} \rangle$$

By aligning with equations $j = 1, \cdots, n$ one solve for $S_N(\frac{\partial}{\partial u_i})$. Repeat for $i = 1, \cdots, n$ we one recover dN_p as $-S_{N(p)}$ acting on the set $\{\frac{\partial \mathbf{x}}{\partial u_i}\}_{1 \le i \le n}$

Example: Catenoid Let $S \subseteq \mathbb{R}^3$ be the surface of revolution of $y = \cosh(z)$. Now, to do rotation we need sine and cosine.

$$\mathbf{x}(u, v) = (\cos(u)\cosh(v), \sin(u)\cosh(v), v)$$

Then using local coordinates

$$\begin{aligned} \mathbf{x}_u &= \frac{\partial \mathbf{x}}{\partial u} = (-\sin(u)\cosh(v), \cos(u)\cosh(v), 0) \\ \mathbf{x}_v &= \frac{\partial \mathbf{x}}{\partial v} = (\cos(u)\sinh(v), \sin(u)\sinh(v), 1) \end{aligned}$$

This is basis for tangent space.

Let's compute the first fundamental form $g = \mathbf{x}^* g_0$ (the induced metric on S).

$$\begin{aligned} g &= Edu^2 + 2Fdudv + Gdv^2 \\ &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle du^2 + 2\langle \mathbf{x}_u, \mathbf{x}_v \rangle dudv + \langle \mathbf{x}_v, \mathbf{x}_v \rangle dv^2 \\ &= \cosh^2(v) du^2 + (\sinh^2(v) + 1) dv^2 = \cosh^2(v)(du^2 + dv^2) \end{aligned}$$

Next we compute the second fundamental form. In $\mathbb{R}^3$, normal vector is given by the cross product.

$$\begin{aligned} N &= \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} = \frac{(\cos(u)\cosh(v), \sin(u)\cosh(v), -\sinh(v)\cosh(v))}{\sqrt{\cosh^2(v) + \sinh^2(v)\cosh^2(v)}} \\ &= \frac{(\cos(u), \sin(u), -\sinh(v))}{\cosh(v)} \end{aligned}$$

The second fundamental form writes

$$\begin{aligned} H_N &= edu^2 + 2fdudv + gdv^2 \\ &= \langle \mathbf{x}_{uu}, N \rangle du^2 + 2\langle \mathbf{x}_{uv}, N \rangle dudv + \langle \mathbf{x}_{vv}, N \rangle dv^2 \end{aligned}$$

Notice

$$\begin{aligned}
e = \langle \mathbf{x}_{uu}, N\rangle &= \langle -\cos(u)\cosh(v)\vec{i} - \sin(u)\cosh(v)\vec{j}, \frac{\cos(u)}{\cosh(v)}\vec{i} + \frac{\sin(u)}{\cosh(v)}\vec{j} - \frac{\sinh(v)}{\cosh(v)}\vec{k}\rangle \\
&= -\cos^2(u) - \sin^2(u) = -1 \\
f = \langle \mathbf{x}_{uv}, N\rangle &= \langle -\sin(u)\sinh(v)\vec{i} + \cos(u)\sinh(v)\vec{j}, \frac{\cos(u)}{\cosh(v)}\vec{i} + \frac{\sin(u)}{\cosh(v)}\vec{j} - \frac{\sinh(v)}{\cosh(v)}\vec{k}\rangle = 0 \\
g = \langle \mathbf{x}_{vv}, N\rangle &= \langle \cos(u)\cosh(v)\vec{i} + \sin(u)\cosh(v)\vec{j}, \frac{\cos(u)}{\cosh(v)}\vec{i} + \frac{\sin(u)}{\cosh(v)}\vec{j} - \frac{\sinh(v)}{\cosh(v)}\vec{k}\rangle = 1
\end{aligned}$$

Thus the second fundamental form is

$$H_N = -du^2 + dv^2$$

Now to fully write down the differential of Gauss map, we need to determine the shape operator. Assume for (recall we've identified $\frac{\partial \mathbf{x}}{\partial u}$ with $\frac{\partial}{\partial u}$)

$$S_N(\frac{\partial}{\partial u}) = a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}$$

We first determine a by plugging into (here one needs the first fundamental form)

$$\begin{aligned}
e = -1 = H_N(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}) &= \langle S_N(\frac{\partial}{\partial u}), \frac{\partial}{\partial u}\rangle = \langle a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}, \frac{\partial}{\partial u}\rangle = a\cosh^2(v) \\
a = -\frac{1}{\cosh^2(v)}
\end{aligned}$$

Now determine b by

$$\begin{aligned}
f = 0 = H_N(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}) &= \langle S_N(\frac{\partial}{\partial u}), \frac{\partial}{\partial v}\rangle = \langle a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\rangle = b\cosh^2(v) \\
b = 0
\end{aligned}$$

Thus the Shape Operator acting on $\frac{\partial}{\partial u}$ is explicit

$$S_N(\frac{\partial}{\partial u}) = -\frac{1}{\cosh^2(v)}\frac{\partial}{\partial u}$$

and thus

$$dN_p(\mathbf{x}_u) = \frac{1}{\cosh^2(v)}\mathbf{x}_u$$

Doing the same for $\frac{\partial}{\partial v}$ gives

$$dN_p(\mathbf{x}_v) = -\frac{1}{\cosh^2(v)}\mathbf{x}_v$$

Therefore

$$dN_p = \frac{1}{\cosh^2(v)}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

w.r.t. basis $\{\mathbf{x}_u, \mathbf{x}_v\}$.

2.6.7 Totally Geodesic

Let M be dimension n, $\overline{M}$ be dimension $n+1$. Let

$$f : (M, g) \to (\overline{M}, \overline{g})$$

be an isometric immersion.

Definition 2.6.9 (Totally Geodesic)**.** *Let $p \in M$. We say that f is geodesic at p if the second fundamental form is* 0

$$S_\eta = 0 \qquad \forall\, \eta \in (T_pM)^\perp$$

or equivalently

$$H_\eta = 0 \qquad \forall\, \eta \in (T_pM)^\perp$$

or equivalently

$$B(p) : T_pM \times T_pM \to (T_pM)^\perp \qquad \text{is zero map}$$

The immersion f is totally geodesic if it is geodesic at any $p \in M$.

Characterisation of Totally Geodesic

Proposition 2.6.5 ([dC92] Proposition 6.2.9)**.** *An isometric immersion*

$$f : M \to \overline{M}$$

is geodesic at $p \in M$ iff every geodesic γ of M starting from p is a geodesic of $\overline{M}$ at p.

Proof. Let $\gamma(0) = p$, $\gamma'(0) = v$. Let $\eta \in (T_pM)^{\perp}$ and denote N as local extension. Let X be local extension of $\gamma'(t)$ to a vector field on M.

Since

$$\langle X, N\rangle = 0$$

we obtain at p

$$H_\eta(v, v) = \langle S_\eta(v), v\rangle = -\langle D_X N, X\rangle = -X\cancel{\langle N, X\rangle} + \langle N, D_X X\rangle$$

Now f is geodesic at p iff for all $v \in T_pM$, the geodesic γ of M that is tangent to v at p satisfies $D_X X$ does not have normal component. This is to say γ remains a geodesic in $\overline{M}$. □

We also record the version from class. They're just the same...

Proposition 2.6.6. *Let*

$$f : (M, g) \to (\overline{M}, \overline{g})$$

be an isometric immersion. Then f is geodesic at $p \in M$, i.e. the second fundamental form

$$B(x, y) = 0 \qquad \forall\ x,\ y \in T_pM$$

iff for any

$$\gamma : (-\varepsilon, \varepsilon) \to M \qquad \text{geodesic} \qquad \gamma(0) = p$$

one has

$$\tilde{\gamma} := f \circ \gamma : (-\varepsilon, \varepsilon) \xrightarrow{\gamma} M \xrightarrow{f} \overline{M} \qquad \text{is a geodesic on } \overline{M}$$

Proof. Assume f is geodesic at $p \in M$. Suppose γ is a geodesic in M, we want to prove $\gamma = f \circ \gamma$ is geodesic in $\overline{M}$. What is the covariant derivative of γ'?

$$\begin{aligned}(\frac{\overline{D}}{dt}\gamma')(0) &= (\frac{D}{dt}\gamma')(0) + B(\gamma'(0), \gamma'(0))\\ &= (\frac{D}{dt}\gamma')(0) = 0 \qquad \text{since } \gamma \text{ is geodesic in } M\end{aligned}$$

On the converse, it suffices to show that $B(x, x) = 0$ for any $x \in T_pM$. Let

$$\gamma(t) = \exp_p(tx)$$

Then

$$\begin{aligned}(\frac{D}{dt}\gamma')(t) &= 0\\ (\frac{\overline{D}}{dt}\gamma')(t) &= 0\\ B(x, x) = (\frac{\overline{D}}{dt}\gamma')(0) - (\frac{D}{dt}\gamma')(0) &= 0\end{aligned}$$

□

Recall $(\overline{M}, \overline{g})$ is Riemannian manifold of dimension $\overline{n}$ with $p \in \overline{M}$, then there exists $\varepsilon > 0$ s.t.

$$\overline{\exp}_p : B_\varepsilon(0) \subset T_p\overline{M} \to \overline{M}$$

is a C^∞ embedding. Now let

$$M := \overline{\exp}_p(W \cap B_\varepsilon(0))$$

where W is a subspace of $T_p\overline{M}$ of dimension n. Then M is a n-dim submanifold of $\overline{M}$. This gives you a canonical local n-dimensional submanifold through p whose tangent space at p is exactly the chosen plane W.

Corollary 2.6.3. *If $f : (M, g) \to (\overline{M}, \overline{g})$ is totally geodesic, i.e, f is geodesic for any $p \in M$. Then*

$$\exp_p = \overline{\exp}_p\big|_{V \cap T_pM}$$

where V is a neighborhood of origin of $T_p\overline{M}$ on which $\overline{\exp}_p$ is defined.

Geometric Interpretation of Sectional Curvature Now one has a good geometric interpretation of sectional curvature. Let M be Riemannian manifold and $p \in M$. Let $B \subseteq T_pM$ be an open ball in T_pM on which $\exp_p$ is a diffeomorphism.

Let $\sigma \subseteq T_pM$ be a subspace of dimension 2. Then

$$\exp_p(\sigma \cap B) = S$$

is a submanifold of dimension two of M passing through p.

S is the surface formed by 'small' geodesics that start from p and are tangent to σ at p. By Proposition 2.6.5 S is geodesic at p. Hence the second fundamental forms of the inclusion

$$i : S \hookrightarrow M$$

vanish at p.

As a submanifold of M, S has an induced Riemannian metric whose Gaussian curvature at p will be denoted as K_S. It follows from Guass Formula (2.135) that

$$K_S(p) = K(p, \sigma)$$

In other words, the sectional curvature $K(p, \sigma)$ is the Gaussian curvature at p of a small surface formed by geodesics of M that start from p and are tangent to σ.

Transitivity of Totally geodesic

Proposition 2.6.7 ([dC92] Exercise 6.3)**.** *Let M be a Riemannian manifold and let $N \subset K \subset M$ be submanifolds of M. Suppose N is totally geodesic in K and that K is totally geodesic in M. Then N is totally geodesic in M.*

Proof. Since N and K are Riemannian submanifolds, consider f and g as isometric immersions

$$N \overset{f}{\hookrightarrow} K \overset{g}{\hookrightarrow} M$$

We're given that f is totally geodesic in K and g is totally geodesic in M, and want to show $g \circ f$ is totally geodesic in M.

1. For any $p \in N$, by identifying $p \cong f(p) \cong g \circ f(p)$, we observe that the normal splits w.r.t. both K and M

$$\begin{aligned} T_pK &= T_pN \oplus (T_pN)^{\perp}_K \\ T_pM &= T_pK \oplus (T_pK)^{\perp}_M \\ &= T_pN \oplus (T_pN)^{\perp}_M \\ \implies (T_pN)^{\perp}_M &= (T_pN)^{\perp}_K \oplus (T_pK)^{\perp}_M \end{aligned}$$

Furthermore, for connection ∇ on N, $\overline{\nabla}$ on K and $\overline{\overline{\nabla}}$ on M, one can write bilinear forms

$$\begin{aligned} B^K_N &: \mathfrak{X}(N) \times \mathfrak{X}(N) \to \mathfrak{X}(N)^{\perp}_K \\ &(X, Y) \mapsto B^K_N(X, Y)(p) := (\overline{\nabla}_{\overline{X}}\overline{Y})(f(p)) - df_p(\nabla_X Y(p)) \\ B^M_K &: \mathfrak{X}(K) \times \mathfrak{X}(K) \to \mathfrak{X}(K)^{\perp}_M \\ &(\overline{X}, \overline{Y})) \mapsto B^M_K(\overline{X}, \overline{Y})(f(p)) := (\overline{\overline{\nabla}}_{\overline{\overline{X}}}\overline{\overline{Y}})(g \circ f(p)) - dg_{f(p)}(\overline{\nabla}_{\overline{X}}\overline{Y}(f(p))) \\ B^M_N &: \mathfrak{X}(N) \times \mathfrak{X}(N) \to \mathfrak{X}(N)^{\perp}_M \\ &(X, Y) \mapsto B^M_N(X, Y)(g \circ f(p)) := (\overline{\overline{\nabla}}_{\overline{\overline{X}}}\overline{\overline{Y}})(g \circ f(p)) - d(g \circ f)_p(\nabla_X Y(p)) \end{aligned}$$

Now using f, g-related vector fields and Chain rule

$$\begin{aligned} df_p(\nabla_X Y(p)) &= (\overline{\nabla}_{\overline{X}}\overline{Y})(f(p))^T \\ dg_{f(p)}(\overline{\nabla}_{\overline{X}}\overline{Y}(f(p))) &= (\overline{\overline{\nabla}}_{\overline{\overline{X}}}\overline{\overline{Y}})(g \circ f(p))^T \\ d(g \circ f)_p(\nabla_X Y(p)) &= dg_{f(p)}(df_p(\nabla_X Y(p))) \\ &= dg_{f(p)}((\overline{\nabla}_{\overline{X}}\overline{Y})(f(p))^T) \\ &= dg_{f(p)}(\overline{\nabla}_{\overline{X}}\overline{Y}(f(p))) - dg_{f(p)}(\overline{\nabla}_{\overline{X}}\overline{Y}(f(p))^{\perp}) \\ &= (\overline{\overline{\nabla}}_{\overline{\overline{X}}}\overline{\overline{Y}})(g \circ f(p))^T - dg_{f(p)}(\overline{\nabla}_{\overline{X}}\overline{Y}(f(p))^{\perp}) \\ &= (\overline{\overline{\nabla}}_{\overline{\overline{X}}}\overline{\overline{Y}})(g \circ f(p)) - (\overline{\overline{\nabla}}_{\overline{\overline{X}}}\overline{\overline{Y}})(g \circ f(p))^{\perp} - dg_{f(p)}(\overline{\nabla}_{\overline{X}}\overline{Y}(f(p))^{\perp}) \\ B^M_N(X, Y)(g \circ f(p)) &= B^M_K(\overline{X}, \overline{Y})(f(p)) + dg_{f(p)}(B^K_N(X, Y)(p)) \end{aligned}$$

2. Then since f is totally geodesic in K, $B_N^K(p) \equiv 0$, and since g is totally geodesic in M, $B_K^M(f(p)) \equiv 0$, one conclude

$$B_N^M(g \circ f(p)) \equiv 0$$

Hence by definition N is totally geodesic in M.

□

Totally Geodesic under Product

Proposition 2.6.8 ([dC92] Exercise 6.4)**.** *Let $N_1 \subset M_1$ and $N_2 \subset M_2$ be totally geodesic submanifolds of the Riemannian manifolds M_1 and M_2 respectively. Then $N_1 \times N_2$ is a totally geodesic submanifold of the product $M_1 \times M_2$ with the product metric.*

Example 2.6.5 ([dC92] Exercise 6.5)**.** *The sectional curvature of the Riemannian manifold $\mathbb{S}^2 \times \mathbb{S}^2$ equipped with the product metric, where $\mathbb{S}^2 \subseteq \mathbb{R}^3$ is the unit sphere, is non-negative.*

Moreover, there exists a totally geodesic, flat torus $\mathbb{T}^2$ embedded in $\mathbb{S}^2 \times \mathbb{S}^2$.

Proof. 1. Recall $(\mathbb{S}^2, g_{\text{can}})$ is equipped with the round metric

$$g_{\text{can}}^{\mathbb{S}^2}(\phi, \theta) = d\phi^2 + \sin^2(\phi) d\theta^2$$

Hence the product metric g_{prod} on $\mathbb{S}^2 \times \mathbb{S}^2$ writes

$$\begin{aligned} g_{\text{prod}}((\phi_1, \theta_1), (\phi_2, \theta_2)) &:= g_{\text{can}}^{\mathbb{S}^2}(\phi_1, \theta_1) \oplus g_{\text{can}}^{\mathbb{S}^2}(\phi_2, \theta_2) \\ &= d\phi_1^2 + \sin^2(\phi_1) d\theta_1^2 + d\phi_2^2 + \sin^2(\phi_2) d\theta_2^2 \end{aligned}$$

Notice that

(a) When a 2-plane Π is tangent to one common copy of $\mathbb{S}^2$, then $K(\Pi) = K(\mathbb{S}^2) = 1$ equal to the sectional curvature of the sphere, which we know to be 1.

(b) When a 2-plane Π contains fixed tangent vectors from both factors of $\mathbb{S}^2$, say $X \in T\mathbb{S}^2 \times \{p\}$ and $Y \in \{p\} \times T\mathbb{S}^2$, then X and Y are orthogonal, hence independent to each other due to the product metric. Thus

$$R(X, Y, X, Y) = 0$$

and $K(\Pi) = 0$.

2. Consider

$$\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$$

define embedding

$$\begin{aligned} \mathbb{T}^2 &\hookrightarrow \mathbb{S}^2 \times \mathbb{S}^2 \\ (\theta_1,\, \theta_2) &\mapsto ((\frac{\pi}{2}, \theta_1), (\frac{\pi}{2}, \theta_2)) \end{aligned}$$

In view of Proposition 2.6.8, it suffices to prove

$$\mathbb{S}^1 \subset \mathbb{S}^2$$

is totally geodesic. Philosophically this is true because $\mathbb{S}^1$, the great circle, is preserved by the geodesic flow on $\mathbb{S}^2$. In particular, let (ϕ, θ) denote coordinates on $\mathbb{S}^2$ and let embedding be

$$\begin{aligned} f : \mathbb{S}^1 &\hookrightarrow \mathbb{S}^2 \\ \theta &\mapsto (\frac{\pi}{2}, \theta) \end{aligned}$$

where

$$g_{\text{round}}^{\mathbb{S}^1} = d\theta^2 = f^* g_{\text{round}}^{\mathbb{S}^2}$$

Thus f is isometric immersion. But

$$\begin{aligned} B : \mathfrak{X}(\mathbb{S}^1) \times \mathfrak{X}(\mathbb{S}^1) &\to \mathfrak{X}(\mathbb{S}^1)^\perp \\ (X, Y) &\mapsto B(X, Y)(p) := (\nabla_{\overline{X}} \overline{Y}(f(p)))^\perp \end{aligned}$$

and observe

$$\Gamma^{\phi}_{\theta\theta} = -\sin(\phi)\cos(\phi)$$
$$\Gamma^{\theta}_{\theta\theta} = \cot(\phi)$$
$$\nabla_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\theta} = \Gamma^{\phi}_{\theta\theta}\frac{\partial}{\partial\phi} + \Gamma^{\theta}_{\theta\theta}\frac{\partial}{\partial\theta}$$

But evaluating at $\phi = \frac{\pi}{2}$ yields

$$\nabla_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\theta} = 0$$

Hence

$$B(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}) \equiv 0$$

But this is the only chance for B to be non-zero, hence we obtain bilinear form B as a zero map, and f is thus a totally geodesic in $\mathbb{S}^2$.

□

2.6.8 Minimality and Mean Curvature

A much weaker notion than totally geodesic is minimality.

Let

$$f : M^n \to \overline{M}^{n+m}$$

be isometric immersion.

Definition 2.6.10 (Minimal)**.** *We say f is minimal if trace of the second fundamental form is* 0

$$\operatorname{tr}(S_\eta) = 0 \qquad \forall\ \eta \in (T_pM)^\perp \qquad \forall\ p \in M$$

What is the trace of S_η? Note

$$S_\eta : T_pM \to T_pM$$

If we take $\{e_i\}_{i=1}^n$ an orthonormal basis of T_pM then

$$\operatorname{tr}(S_\eta) := \sum_{i=1}^n \langle S_\eta(e_i), e_i\rangle \overset{(2.121)}{=} \sum_{i=1}^n H_\eta(e_i, e_i) \overset{(2.120)}{=} \sum_{i=1}^n \langle B(e_i, e_i), \eta\rangle$$

Note this is independent of basis chosen, hence well-defined.

Since minimality only requires such sum to be 0, this is indeed a weaker condition than requiring all second fundamental form vanish.

2.6.8.1 Mean Curvature Vector

Definition 2.6.11 (Mean Curvature)**.** *We define the mean curvature vector*

$$H(p) := \frac{1}{n}\sum_{i=1}^n B(e_i, e_i) \in (T_pM)^\perp \qquad \textit{for } \{e_i\}_{i=1}^n \textit{ as orthonormal basis of } T_pM \tag{2.138}$$

Thus $H \in C^\infty(M, N(f))$.

Immediately notice that f is minimal iff $H \equiv 0$.

On the other hand, choosing a local orthonormal frame $\{E_j\}$ of vectors in $\mathfrak{X}(M)^\perp = C^\infty(M, f^*T\overline{M})$, one may write at p

$$B(x, y) = \sum_j H_{E_j}(x, y)E_j \qquad \forall\ x, y \in T_pM, \qquad j = 1, \cdots, m$$

Indeed

$$H_{E_j}(x, y) = \langle B(x, y), E_j\rangle$$

Note the normal vector given by

$$\frac{1}{n}\sum_{j=1}^m \operatorname{tr}(S_{E_j})E_j$$

does not depend on the choice of frame E_j. In fact

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{m}\mathrm{tr}(S_{E_j})E_j &= \frac{1}{n}\sum_{j=1}^{m}\sum_{i=1}^{n}\langle B(e_i,e_i),E_j\rangle E_j \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{m}\langle B(e_i,e_i),E_j\rangle E_j\right) \\ &= \frac{1}{n}\sum_{i=1}^{n}B(e_i,e_i) = H(p)\end{aligned}$$

Hence one has equivalent definitions for Mean Curvature vector

$$H(p) = \frac{1}{n}\sum_{i=1}^{n}B(e_i,e_i) = \frac{1}{n}\sum_{j=1}^{m}\mathrm{tr}(S_{E_j})E_j \tag{2.139}$$

where $\{e_i\}_{i=1}^n$ is orthonormal basis for T_pM and $\{E_j\}_{j=1}^m$ is orthonormal basis for $(T_pM)^\perp$.

We use the word ‘minimal’ because such immersions minimize the volume in the induced metric, in the same way that geodesics minimize arclength.

More precisely, if $M \subseteq \overline{M}$ is a minimal submanifold and $D \subseteq M$ is a sufficiently domain of M with regular boundary ∂D, then the volume of D in the induced metric is less than or equal to the volume of any other submanifold of $\overline{M}$ with the same boundary.

Example 2.6.6. *Consider sphere*

$$(\mathbb{S}^n, g_{\mathrm{can}}) \hookrightarrow (\mathbb{R}^{n+1}, g_0)$$

Recall $\eta(p) = -p$, and

$$B(X,Y) = \langle X,Y\rangle\eta$$

Then

$$H(p) = \frac{1}{n}\sum_{i=1}^{n}\langle e_i,e_i\rangle\eta(p) = \frac{1}{n}\sum_{i=1}^{n}\eta(p) = \eta(p) = -p$$

the mean curvature vector of the sphere is also pointing inwards, the same as normal.

Mean Curvature for Codimension 1 If $m = 1$, then take η as the normal (whichever direction). Consider parametrization

$$\begin{aligned}\mathbf{x} : U \subseteq \mathbb{R}^n &\to \mathbb{R}^{n+1} \\ (u_1,\cdots,u_n) &\mapsto (x_1(u_1,\cdots,u_n),\cdots,x_{n+1}(u_1,\cdots,u_n))\end{aligned}$$

The second fundamental form takes the form (2.137)

$$H_\eta = \sum_{ij} h_{ij}du_idu_j$$

where

$$h_{ij} = \langle S_\eta(\frac{\partial}{\partial u_i}),\frac{\partial}{\partial u_j}\rangle = \langle B(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}),\eta\rangle = \langle D_{\frac{\partial}{\partial u_i}}\mathbf{x}_*(\frac{\partial}{\partial u_j}),\eta\rangle = \langle \mathbf{x}_{ij},\eta\rangle$$

Now we compute the Mean Curvature via

$$H = \frac{1}{n}\mathrm{tr}(S_\eta)\eta$$

It suffice to compute

$$\mathrm{tr}(S_\eta) = \sum_{i,j}\langle S_\eta(\frac{\partial}{\partial u_i}),\frac{\partial}{\partial u_j}\rangle g^{ij} = \sum_{ij}\langle \mathbf{x}_{ij},\eta\rangle g^{ij}$$

Mean Curvature for Level Set Let's recall the definition for Hessian.

Definition 2.6.12 (Hessian)**.** *Let*

$$f : \overline{M}^{n+1} \to \mathbb{R}$$

be a differentiable function. Define the Hessian $\mathrm{Hess}(f)$ *of* f *at* $p \in \overline{M}$ *as the linear operator*

$$\begin{aligned} \mathrm{Hess}(f) : T_p\overline{M} &\to T_p\overline{M} \\ Y &\mapsto (\mathrm{Hess}(f))(Y) := \overline{\nabla}_Y \mathrm{grad}(f) \qquad \forall\ Y \in T_p\overline{M} \end{aligned} \tag{2.140}$$

where $\overline{\nabla}$ *is the Riemannian connection of* $\overline{M}$*.*

Lemma 2.6.4 (Laplacian)**.** *The Laplacian* $\overline{\Delta} f$ *is given by*

$$\overline{\Delta} f = \mathrm{tr}(\mathrm{Hess}(f)) \tag{2.141}$$

Proof. By definition

$$\begin{aligned} \overline{\Delta} f &:= \mathrm{div}(\mathrm{grad}(f)) \\ &:= \mathrm{tr}(\text{linear mapping } Y(p) \to \overline{\nabla}_Y \mathrm{grad}(f)(p) \text{ for any } p \in M) \\ &= \mathrm{tr}(\mathrm{Hess}(f)) \end{aligned}$$

□

Lemma 2.6.5 (Hessian as symmetric bilinear form)**.** *For any* $X, Y \in \mathfrak{X}(\overline{M})$

$$\langle \mathrm{Hess}(f)Y, X\rangle = \langle Y, (\mathrm{Hess}(f)X\rangle \tag{2.142}$$

Hence $\mathrm{Hess}(f)$ *is self-adjoint, and determines a symmetric bilinear form on* $T_p\overline{M}$ *for any* $p \in \overline{M}$ *via*

$$\mathrm{Hess}(f)(X, Y) := \langle(\mathrm{Hess}(f))X, Y\rangle \qquad \forall\ X,\ Y \in T_p\overline{M} \tag{2.143}$$

Proof.

$$\begin{aligned} \langle(\mathrm{Hess}(f))Y, X\rangle &= \langle \overline{\nabla}_Y \mathrm{grad}(f), X\rangle = Y(\langle \mathrm{grad}(f), X\rangle) - \langle \mathrm{grad}(f), \overline{\nabla}_Y X\rangle \\ &= Y(X(f)) - (\overline{\nabla}_Y X)(f) \qquad \text{using definition of grad}(f) \text{ and Levi-Civita is compatible with metric} \\ &= [Y, X](f) + X(Y(f)) - (\overline{\nabla}_Y X)(f) \qquad \text{using definition of Lie Bracket} \\ &= (\overline{\nabla}_X Y)(f) + X(Y(f)) \qquad \text{using Levi-Civita is symmetric} \\ &= \langle \mathrm{grad}(f), \overline{\nabla}_X Y\rangle + X(\langle \mathrm{grad}(f), Y\rangle) \\ &= \langle \overline{\nabla}_X \mathrm{grad}(f), Y\rangle = \langle Y, \overline{\nabla}_X \mathrm{grad}(f)\rangle = \langle Y, (\mathrm{Hess}(f))X\rangle \end{aligned}$$

□

Proposition 2.6.9 ([dC92] Exercise 6.11)**.** *Let 'a' be a regular value of* f*, i.e., for any* $p \in f^{-1}(a)$*,* f *is a submersion at* p*. Let* $M^n \subseteq \overline{M}^{n+1}$ *be the hypersurface in* $\overline{M}$ *defined by*

$$M := \{p \in \overline{M} \mid f(p) = a\} = f^{-1}(a)$$

1. *The mean curvature* H *of* $M \subseteq \overline{M}$ *is given by*

$$nH = -\mathrm{div}(\frac{\mathrm{grad}(f)}{|\mathrm{grad}(f)|}) \tag{2.144}$$

Proof. First of all, we claim $\mathrm{grad}(f)$ is normal to the level surface M. Let any $p \in M$ and $v \in T_pM$, since $f \equiv a$ on M, the tangential derivatives all vanish

$$v(f) = 0$$

Thus by definition of gradient (w.r.t. ambient metric $\overline{g}$), for any $X \in \mathfrak{X}(M)$

$$\overline{g}(\mathrm{grad}(f), X) = df(X) = X(f)$$

which vanishes for any X tangent to M at p. Thus $\mathrm{grad}(f)$ has to point in the normal direction.

Take an Orthonormal frame $E_1, \cdots, E_n$ and our normal vector

$$E_{n+1} := \frac{\text{grad}(f)}{|\text{grad}(f)|} = \eta$$

in a neighborhood p of M in $\overline{M}$. Recall H as in (2.139) and S_η as in (2.121). Since codimension is 1, the mean curvature is essentially a scalar

$$\begin{aligned} nH = \text{tr}(S_\eta) &= \sum_{i=1}^{n} \langle S_\eta(E_i), E_i \rangle_{\overline{g}} \\ &\overset{(2.123)}{=} -\sum_{i=1}^{n} \langle (D_{E_i}\eta)^T, E_i \rangle_{\overline{g}} \\ &\overset{(2.124)}{=} -\sum_{i=1}^{n} \langle D_{E_i}\eta, E_i \rangle_{\overline{g}} \\ &= -\sum_{i=1}^{n} \langle \overline{\nabla}_{E_i}\eta, E_i \rangle_{\overline{g}} - \langle \overline{\nabla}_\eta \eta, \eta \rangle_{\overline{g}} \\ &= -\sum_{i=1}^{n+1} \langle \overline{\nabla}_{E_i}\eta, E_i \rangle \\ &\overset{(2.91)}{=} -\text{div}_{\overline{M}}\eta \qquad \text{using definition of divergence} \\ &= -\text{div}\left(\frac{\text{grad}(f)}{|\text{grad}(f)|}\right) \end{aligned}$$

where we used that $\langle \eta, \eta \rangle = 1$ so

$$\eta(\langle \eta, \eta \rangle) = 2\langle \overline{\nabla}_\eta \eta, \eta \rangle = 0$$

□

2. *Notice Every Embedded hypersurface $M^n \subseteq \overline{M}^{n+1}$ is locally the inverse image of a regular value. Moreover, the mean curvature H of such a hypersurface is given by*

$$H = -\frac{1}{n}\text{div}(N)$$

where N is an appropriate local extension of the unit normal vector field on $M^n \subset \overline{M}^{n+1}$.

Proof. (a) Since $M \hookrightarrow \overline{M}$, there exists a smooth immersion

$$f : M \to \overline{M}$$

s.t. $f(M) \subset \overline{M}$ is homeomorphism w.r.t. subspace topology. Or using the alternative definition, for any $q \in M$, there exists a neighborhood U of q in $\overline{M}$ and a coordinate chart $\phi = (x_1, \cdots, x_{n+1})$ on U s.t.

$$\phi(M \cap U) = \phi(M) \cap \{x_{n+1} = 0\}$$

In other words

$$M \cap U = \{q \in U \mid x_{n+1}(q) = 0\}$$

It suffices to see 0 is a regular value for $f = x_{n+1}$. But for any $p \in \overline{M}$

$$df_p : T_p\overline{M} \to \mathbb{R} \qquad df_p = \frac{\partial f}{\partial x_{n+1}} dx_{n+1}$$

Then

$$\frac{\partial f}{\partial x_{n+1}} = 1$$

Hence df_p is surjective for any $p \in M \cap U$ so 0 is a regular value for $f = x_{n+1}$.

(b) For any $q \in M$, there exists neighborhood U of q in $\overline{M}$ and $a \in \mathbb{R}$ s.t.

$$U \cap M = f_U^{-1}(a)$$

for some smooth f_U and a as its regular value. Applying (2.144), the mean curvature H of $M \cap U \subset \overline{M}$ is

$$nH = -\text{div}(\frac{\text{grad}(f_U)}{|\text{grad}(f_U)|})$$

However one can extend the formula to neighborhood U in $\overline{M}$ because f_U is submersion on $U \cap M$, hence has non-vanishing gradient. By continuity of f_U one can extend smoothly to open neighborhood in $\overline{M}$. Now one can define a unit normal vector field N as the local extension s.t.

$$N_U := \frac{\text{grad}(f_U)}{|\text{grad}(f_U)|} \qquad \forall\, U \subset \overline{M} \qquad \text{local neighborhood s.t. } N_U \text{ is well-defined}$$

□

2.6.8.2 Minimal Graph

This is taken from Lecture notes by Xin Zhou.

Consider a domain

$$\Omega \subseteq \mathbb{R}^{n-1}$$

Let

$$u : \Omega \subseteq \mathbb{R}^{n-1} \to \mathbb{R}$$

be a smooth function with $|\nabla u| \neq 0$. The graph of u is defined via

$$\Sigma_u = \{(x, u(x)) \mid x = (x_1, \cdots, x_{n-1}) \in \Omega\}$$

Denote the smooth embedding of Ω to $\mathbb{R}^n$ as

$$\begin{aligned} F : \Omega \subseteq \mathbb{R}^{n-1} &\to \mathbb{R}^n \\ x &\mapsto (x, u(x)) \end{aligned}$$

Denote $g = F^* g_0$ as the pullback metric of Euclidean metric in $\mathbb{R}^n$ on Σ_u. Then

$$g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \rangle = \delta_{ij} + \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}$$

i.e., as a matrix

$$g = I + \nabla u (\nabla u)^T \tag{2.145}$$

Eigenvalues of g We extend $g = g \oplus dx_n^2$ when necessary.

We observe that g as a matrix acting on ∇u gives

$$\begin{aligned} g\nabla u = (I + \nabla u(\nabla u)^T)\nabla u &= \nabla u + \nabla u(\nabla u^T \nabla u) \\ &= \nabla u + |\nabla u|^2 \nabla u = (1 + |\nabla u|^2)\nabla u \end{aligned} \tag{2.146}$$

so that $1 + |\nabla u|^2$ is an eigenvalue of g with eigenvector ∇u.

Moreover, if $v \perp \nabla u$ (i.e., in the tangential direction), then

$$gv = (I + \nabla u(\nabla u)^T)v = v + \nabla u(\nabla u^T v) = v \tag{2.147}$$

so $\{\nabla u^\perp\}$ is the $(n-1)$-dimensional eigenspace for the eigenvalue 1.

Volume form and Area of Σ_u Let $dx = dx_1 \cdots dx_{n-1}$ we have the volume form of Σ as

$$d\text{vol} = \sqrt{\det(g)}dx = \sqrt{1 + |\nabla u|^2}dx$$

Now we study the critical points of the area functional

$$\text{Area}(\Sigma_u) = \int_\Omega \sqrt{1 + |\nabla u|^2}dx$$

For any $\eta \in C_0^\infty(\Omega)$, we consider the change of area under perturbations. If the graph Σ_u is minimal, one must have

$$\frac{d}{dt}\Big|_{t=0} \text{Area}(\Sigma_{u+t\eta}) = 0 \qquad \forall\, \eta \in C_0^\infty(\Omega)$$

We compute

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\text{Area}(\Sigma_{u+t\eta}) &= \frac{d}{dt}\Big|_{t=0}\int_\Omega \sqrt{1+|\nabla(u+t\eta)|^2}dx \\ &= \int_\Omega \frac{\nabla(u+t\eta)\cdot\nabla\eta}{\sqrt{1+|\nabla(u+t\eta)|^2}}\Big|_{t=0} dx = \int_\Omega \frac{\nabla u\cdot\nabla\eta}{\sqrt{1+|\nabla u|^2}}dx \\ &= -\int_\Omega \eta\text{div}(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}})\end{aligned}$$

We set this to 0 for any $\eta\in C_0^\infty(\Omega)$. In other words we recovered the minimal surface equation

$$\text{div}(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = 0$$

Graphical Mean Curvature We're going to show that (in the not normalized version)

$$H = \text{div}(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}})$$

First calculate the inverse matrix of g_{ij}. Since

$$\begin{aligned}g(\nabla u(\nabla u)^T) &= (g\nabla u)\nabla u^T \\ &\overset{(2.146)}{=} (1+|\nabla u|^2)\nabla u\nabla u^T \\ &\overset{(2.145)}{=} (1+|\nabla u|^2)(g-I) \\ g(1+|\nabla u|^2-\nabla u(\nabla u)^T) &= (1+|\nabla u|^2)I \\ (g^{ij}) = g^{-1} &= I - \frac{\nabla u(\nabla u)^T}{1+|\nabla u|^2}\end{aligned}$$

Thus

$$g^{ij} = \delta_{ij} - \frac{\partial_i u\partial_j u}{1+|\nabla u|^2} \tag{2.148}$$

One may regard Σ_u as the level set given by

$$\begin{aligned}h:\mathbb{R}^n &\to \mathbb{R} \\ (x_1,\cdots,x_n) &\mapsto x_n - u(x_1,\cdots,x_{n-1})\end{aligned}$$

Hence the unit normal of ∇u is given by

$$\nu = \frac{\nabla h}{|\nabla h|} = \frac{1}{\sqrt{1+|\nabla u|^2}}(-\nabla u, 1)$$

Thus if define

$$H = \text{tr}(S_\nu) = \sum_{i=1}^{n-1}\langle S_\nu(e_i), e_i\rangle = \langle S_\nu(\frac{\partial}{\partial x_i}), \frac{\partial}{\partial x_j}\rangle g^{ij}$$

we notice that, writing $D = F^*\nabla$

$$\begin{aligned}\langle S_\nu(\frac{\partial}{\partial x_i}), \frac{\partial}{\partial x_j}\rangle &= \langle B(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}), \nu\rangle \\ &\overset{(2.119)}{=} \langle D_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} - (D_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j})^T, \nu\rangle \\ &= \langle D_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}, \nu\rangle\end{aligned}$$

Now

$$\begin{aligned}D_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} &= F^*\nabla_{\frac{\partial}{\partial x_i}}(F^*\frac{\partial}{\partial x_j}) = (\nabla_{\frac{\partial F}{\partial x_i}}\frac{\partial}{\partial x_j})\circ F = \partial_{ij}F \\ &= (0,\cdots,0,\partial_{ij}u)\end{aligned}$$

Thus continuing by using the expression for unit normal

$$\begin{aligned} H &= \frac{\partial_{ij} u}{\sqrt{1+|\nabla u|^2}} g^{ij} \\ &\overset{(2.148)}{=} \frac{\partial_{ij} u}{\sqrt{1+|\nabla u|^2}} \left(\delta_{ij} - \frac{\partial_i u \partial_j u}{1+|\nabla u|^2} \right) \\ &= \frac{\partial_{ii} u}{\sqrt{1+|\nabla u|^2}} - \frac{\partial_i u \partial_j u \partial_{ij} u}{(1+|\nabla u|^2)^{\frac{3}{2}}} \end{aligned}$$

Notice that

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \partial_i \left(\frac{\partial_i u}{\sqrt{1+|\nabla u|^2}} \right) = H$$

2.7 Global Differential Geometry

One of the interesting aspects of differential geometry is the interplay that exists between the local properties and the global properties of a Riemannian manifold.

2.7.1 Complete Riemannian Manifold

We always assume M is Hausdorff.

Distance d

Definition 2.7.1 (Path-Connected). *M is path-connected if for any p, $q \in M$, there exists continuous*

$$c : [0, 1] \to M \qquad s.t. \qquad c(0) = p \qquad c(1) = q$$

Lemma 2.7.1. *If M is a connected topological manifold, then M is path connected.*

Lemma 2.7.2. *If M is a connected C^k manifold, there exists a C^K map*

$$c : [0, 1] \to M \qquad s.t. \qquad c(0) = p \qquad c(1) = q$$

We revisit our definition for distance.

Definition 2.7.2 (Distance). *Let (M, g) be a connected Riemannian manifold. For every p, $q \in M$, we define the distance between p and q as infimum of the length of all curves connecting p and q*

$$d_g(p, q) := \inf\{\ell(c) \mid c : [0, 1] \to M \quad \text{piecewise smooth} \quad s.t. \quad c(0) = p \qquad c(1) = q\}$$

1. *The set is non-empty due to M is connected. Hence $d_g(p, q) \geq 0$.*
2. *We fix the metric g and denote $d(p, q)$.*

(M, d) **metric space**

Proposition 2.7.1. *(M, d) defines a metric space.*

Proof. 1. Triangle Inequality. For any p, q, $m \in M$.

$$d(p, q) + d(q, m) \geq d(p, m)$$

due to composition of curves.

2. d is symmetric trivially by reversing the curve parametrization.
3. $d(p, q) \geq 0$ due to nonempty set. It suffices to check $d(p, q) = 0 \iff p = q$. We need to check $d(p, q) = 0$ implies $p = q$. We prove the contrapositive, i.e., for $p \neq q$, we want to show $d(p, q) > 0$. For this we need to use our manifold M is Hausdorff. There exists an open neighborhood U of $p \in M$ s.t. $q \notin U$. There exists r s.t. the normal ball

$$B_r(p) \subset U$$

But then $d(p, q) \geq r$ because all points at distance $\leq r$ from p are in $B_r(p)$, otherwise $q \in B_r(p)$.

□

Example 2.7.1. 1. *On $\mathbb{R}^n$, $d(x, y) = |x - y|$.*

2. *Line with two origins. Let $M = (\mathbb{R} \times \{0, 1\}) \,/\, [(x, 0) \sim (x, 1)$ except for $x = 0]$. Then*

$$\begin{aligned} d([x, 0], [y, 1]) &= |x - y| \qquad \forall\ x,\ y \neq 0 \\ d([x, 0], [0, 1]) &= |x| \\ d([0, 0], [0, 1]) &= 0 \end{aligned}$$

Hence we indeed need Hausdorff condition.

Also we remark the following.

1. If there exists a minimizing geodesic γ between p and q, then

$$\ell(\gamma) = d(p, q)$$

2. The topology induced by d is the same as the original topology, i.e., the one with basis

$$\{B_r(p) \mid r > 0,\ p \in M\}$$

3. Fix $p_0 \in M$, then

$$\begin{aligned} f : (M, d) &\to (\mathbb{R}, |\cdot|) \\ q &\mapsto d(p_0, q) \end{aligned}$$

is continuous. In fact

$$|f(q) - f(p)| = |d(p_0, q) - d(p_0, p)| \leq |d(p, q)|$$

Then f is Lipschitz continuous.

Geodescially Complete

Definition 2.7.3 (Geodesically Complete)**.** *A Riemannian manifold (M, g) is geodesically complete if for any $p \in M$,*

$$\exp_p(v) \qquad \textit{is defined for all } v \in T_pM$$

i.e., all geodesics $\gamma(t)$ are defined for all $t \in \mathbb{R}$.

2.7.2 Hopf-Rinow Theorem

Hopf-Rinow says (M, g) is geodesically complete iff (M, d) is complete metric space.

Theorem 2.7.1 (Hopf-Rinow)**.** *The following are $(a) - (e)$ equivalent and all imply (f)*

(a) $\exp_p(v)$ *is defined for all $v \in T_pM$ at a particular point $p \in M$.*

(b) Closed and Bounded sets of (M, d) are compact.

(c) (M, d) is a complete metric space.

(d) (M, g) is geodesically complete, i.e., $\exp_q(v)$ *is defined for all $v \in T_qM$ for any $q \in M$.*

(e) There exists a sequence of compact sets $\{K_n\}$

$$K_n \subset K_{n+1} \qquad \bigcup_n K_n = M$$

s.t. if $q_n \notin K_n\ \forall\ n$ then $d(p, q_n) \to \infty$.

(f) In the above cases, for any $q \in M$ fixed there exists minimizing geodesic γ between any $p \in M$ and q, i.e.

$$\ell(\gamma) = d(p, q)$$

Example 2.7.2 (Counter example for (f) does not imply (a))**.** *Take $B_1(0)$ open ball in $\mathbb{R}^n$. (f) is satisfied. But* $\exp_0(v)$ *is not defined for $|v| \geq 1$. In particular $B_1(0)$ is not complete.*

2.7.2.1 Proof and Consequences of Hopf-Rinow

Proof of Hopf-Rinow

Proof of Theorem 2.7.1 $(a) \implies (f)$. We want to find the initial velocity $v \in T_pM$ s.t. $|v| = 1$ of the geodesic γ where $\gamma(0) = p$ and $\gamma(1) = q$. In this case we want

$$\gamma(t) = \exp_p(tv) \qquad \gamma(r) = q \qquad \text{for } r = d(p, q)$$

There exists r_0 s.t. $B_{r_0}(p)$ is a normal ball at p.

1. Case one. If $r < r_0$, and $q \in B_{r_0}(p)$, then there exists a minimizing geodesic connecting p and q.

2. Case two. If $r > r_0$. The idea is to construct the curve with initial velocity step by step. Consider the map
$$f : M \to \mathbb{R} \qquad x \mapsto d(q, x)$$
which is continuous. There exists $x_0 \in S_{r_0}(p)$ the sphere of the normal ball s.t.
$$x_0 = \min_{x \in S_{r_0}(p)} f(x)$$
Note x_0 may not be unique. In particular,
$$x_0 = \exp_p(r_0 v) \qquad \text{for some unit tangent vector } v$$
Finally we can use the assumption that $\exp_p$ is defined for all $v \in T_pM$. So we define the curve
$$\gamma(t) := \exp_p(tv)$$
and I want to show that $\gamma(r) = q$. To do so we use the continuity method. We define a set
$$A = \{s \in [0, r] \mid d(\gamma(s), q) = r - s\}$$
If one can prove A is non-empty, closed and open, then since A is connected, we have $A = [0, r]$. In particular we conclude $r \in A$, and finally $d(\gamma(r), q) = r - r = 0 \implies \gamma(r) = q$.

- A is non-empty since $s = 0$ lies inside
$$d(\gamma(0), q) = d(p, q) = r - 0 = r$$
- A is closed due to closed condition.
- We're left to prove A open. We show that if $s \in A$, then there exists $\delta > 0$ s.t. $s + \delta \in A$. Since $s \in A$, one has
$$d(\gamma(s), q) = r - s$$
Consider the normal ball centered at $\gamma(s)$, of some radius δ, which is between $(0, r - s)$. Now we consider x' that minimizes the distance between the ball and q, i.e.
$$x' = \min_{x \in S_\delta(\gamma(s))} f(x)$$
Then
$$\begin{aligned} r - s = d(\gamma(s), q) &= \delta + \min\{d(q, x) \mid x \in S_\delta(\gamma(s))\} \\ &= \delta + d(q, x') \\ d(x', q) &= r - (s + \delta) \end{aligned}$$
Now by triangle inequality
$$\begin{aligned} s + \delta \geq d(p, x') &\geq d(p, q) - d(q, x') = r - r + (s + \delta) = s + \delta \\ d(p, x') &= s + \delta \\ \implies x' &= \gamma(s + \delta) \end{aligned}$$
Thus
$$d(\gamma(s + \delta), q) = r - (s + \delta) \implies s + \delta \in A$$

□

Proof of Theorem 2.7.1 (*a*) $\implies$ (*b*). Let $A \subset M$ closed and bounded. Then there exists $r > 0$ s.t.
$$A \subset \{x \in M \mid d(x, p) \leq r\} = \overline{B_r(p)} \subset \exp_p(\overline{B_r(0)})$$
where the latter is indeed a compact set. Hence using A closed subset of a compact set and Hausdorff topology, one knows that A is compact. □

Proof of Theorem 2.7.1 (*b*) $\implies$ (*c*). Start with a Cauchy Sequence $\{x_n\}$. Let $A = \overline{\{x_n\}}$ be closed and bounded. Then A is compact, and there exists a subsequence $x_{n_k} \to p_0 \in M$. Hence $x_n \to p_0$ since it's Cauchy. □

Proof of Theorem 2.7.1 (c) $\implies$ (d). Let $q \in M$, we want to show that $\exp_q$ is defined on T_qM. Suppose for contradiction that

$$\gamma : (a, s_0) \to M$$

is a normalized geodesic, and is not defined for s_0. Then we prove that γ can be extended to

$$\gamma : (a, s_0 + \delta) \to M$$

How do we prove? Remember we assume Cauchy sequence converges. We take a sequence that converges to s_0. Let s_n be an increasing sequence s.t. $s_n \nearrow s_0$. Then since we have normalized geodesic

$$d(\gamma(s_n), \gamma(s_m)) \leq |s_n - s_m|$$

Since $\{s_n\}$ is Cauchy, $\gamma(s_n)$ are Cauchy, and using our assumption, $\gamma(s_n) \to p_0$ in M. Then there exists $\delta > 0$ and a totally normal neighborhood V of p_0 s.t.

1. for any p_1, $p_2 \in V$, there is a minimizing geodesic between p_1 and p_2
2. for every $q \in V$,

$$\exp_q : B_{2\delta}(0) \subset T_qM \to V$$

   is defined.

What is remarkable is that the δ is uniform in $q \in V$. If $\gamma(s_n)$ and $\gamma(s_m) \in V$, then γ coincides with the minimizing geodesic between $\gamma(s_n)$ and $\gamma(s_m)$. Choose s_n s.t. $\gamma(s_n) \in V$ and

$$s_0 - \delta < s_n < s_0$$

Then for the exponential map at $\gamma(s_n)$, we again center a ball at $\gamma(s_n)$ with radius 2δ, i.e.

$$\exp_{\gamma(s_n)} : B_{2\delta}(0) \to V$$

is defined. Hence $\gamma(t)$ is defined for $t \in (s_n - 2\delta, s_n + \delta)$. But $s_n + 2\delta > s_0$ by our choice. Hence γ is extended. □

Proof of Theorem 2.7.1 (d) $\implies$ (a). Trivial. □

Proof of Theorem 2.7.1 (b) $\implies$ (e). Let $K_n = \overline{B_n(p)}$. They satisfy (e). If $q_n \notin K_n$ for any n, then $d(p, q_n) \geq n$. □

Proof of Theorem 2.7.1 (e) $\implies$ (b). Let A be a closed and bounded set. Then there exists n s.t. $A \subset K_n$, hence A is compact. □

Corollaries and Examples of Hopf-Rinow We discuss certain examples and immediate consequences of Hopf-Rinow.

Compact Manifolds are Complete

Corollary 2.7.1 ([dC92] Corollary 7.2.9)**.** *Any Riemannian metric on a compact manifold gives a complete manifold.*

Proof. Property (e) is always verified. □

Closed Submanifold of Complete Manifolds are Complete

Corollary 2.7.2. *Let (M, g) be complete Riemannian manifold. Let N be a closed submanifold. Denote*

$$i : N \to M \qquad \textit{as inclusion}$$

Then

$$(N, i^*g) \qquad \textit{is complete}$$

Proof. By Theorem 2.7.1 property (b), we need to show closed and bounded sets of N are compact. Here closed and bounded sets are w.r.t. the distance d_N given by i^*g. But

$$d_N(p, q) \geq d_M(p, q)$$

so any closed and bounded sets of N are also closed and bounded in M. So they're compact. □

Example 2.7.3. *1. $\mathbb{S}^n$, $\mathbb{T}^n$ are complete.*

2. $(\mathbb{R}^n, g_0)$ *is complete.*
3. $(B_1^n(0), g_0)$ *is not complete.*

Proof. Let

$$\begin{aligned} \phi : \mathbb{R}^n &\to B_1^n \\ x &\mapsto \frac{x}{\sqrt{1+|x|^2}} \end{aligned}$$

Then the inverse writes

$$\begin{aligned} \phi^{-1} : B_1^n &\to \mathbb{R}^n \\ y &\mapsto \frac{y}{\sqrt{1-|y|^2}} \end{aligned}$$

The diffeomorphism $(\mathbb{R}^n, \phi^* g_0)$ is not complete since the ball is not complete. □

Criterion for Completeness

Proposition 2.7.2 ([dC92] Exercise 7.7)**.** *Let M and $\overline{M}$ be Riemannian manifolds and let*

$$f : M \to \overline{M} \qquad \text{be a diffeomorphism}$$

Let $\overline{M}$ be complete, and assume there exists a constant $c \geq 0$ s.t.

$$|v| \geq c|df_p(v)| \qquad \forall\, p \in M \qquad v \in T_pM$$

Then M is complete.

Proof. Let $\{p_n\}$ be a Cauchy sequence in M. By Hopf-Rinow 2.7.1 (c), it suffices to prove p_n converges. Notice $\{f(p_n)\}_n$ is a sequence in $\overline{M}$, and since $\overline{M}$ is complete, if we're able to show $\{f(p_n)\}_n$ is Cauchy, we have convergence of $f(p_n)$ to some point $q \in \overline{M}$. Indeed, for any $p_n \in M$, there exists totally normal neighborhood V of p_n s.t. for any $p_m \in V$, there exists γ a minimizing geodesic joining p_n and p_m, i.e.

$$\gamma : [0,1] \to M \qquad \gamma(0) = p_n \qquad \gamma(1) = p_m$$

One has

$$\begin{aligned} d_{\overline{M}}(f(p_n), f(p_m)) &\leq \int_0^1 |df_{\gamma(t)}(\gamma'(t))|\, dt \\ &\leq \int_0^1 \frac{1}{c}|\gamma'(t)|\, dt = \frac{1}{c}\ell(\gamma) \\ &= \frac{1}{c} d_M(p_n, p_m) \end{aligned}$$

But $\{p_n\}$ is Cauchy sequence in M, hence $d_M(p_n, p_m) \to 0$ so $\{f(p_n)\}$ is a Cauchy sequence. Thus there exists $q \in \overline{M}$ s.t.

$$d_{\overline{M}}(f(p_n), q) \to 0$$

Now since f is a diffeomorphism, it has a smooth inverse, hence define $p := f^{-1}(q)$ and by continuity

$$d_M(p_n, p) \to 0$$

□

Non-extendible Notice any proper open subset of a complete manifold is not complete, i.e., for open embedding

$$i : M \hookrightarrow \overline{M} \qquad i(M) \subsetneq (\overline{M}, \overline{g}) \qquad \text{open and proper}$$

The manifold $(M, i^*\overline{g})$ is not complete.

Definition 2.7.4 (Extendible)**.** *Let M, M' be connected. A Riemannian manifold (M, g) is extendible if there exists an isometric open embedding*

$$i : (M, g) \hookrightarrow (M', g') \qquad i(M) \subsetneq M'$$

Remark 2.7.1. *If M is compact then M is complete. If M is complete then M is non-extendible.*
But both converses are not true.

1. *$(\mathbb{R}^n, g_0)$ is complete but not compact*
2. *The map*

$$\begin{aligned} \exp : \mathbb{C} &\to (\mathbb{C} \setminus \{0\}, dx^2 + dy^2) \\ z &\mapsto e^z \end{aligned}$$

gives $(\mathbb{C} \setminus \{0\}, dx^2 + dy^2)$ extendible. Hence this is incomplete. But then

$$(\mathbb{C}, \exp^*(dx^2 + dy^2))$$

is incomplete and inextendible.

2.7.2.2 Applications of Hopf-Rinow

Complete Non-compact manifold contains a Ray

Definition 2.7.5 (Ray)**.** *A geodesic $\gamma : [0, \infty) \to M$ in a Riemannian manifold M is called a ray starting from $\gamma(0)$ if it minimizes the distance between $\gamma(0)$ and $\gamma(s)$ for any $s \in (0, \infty)$.*

Proposition 2.7.3 ([dC92] Exercise 7.6)**.** *Let M be complete and non-compact.*
Then for any $p \in M$, there exists a ray starting from p in M.

Proof. 1. Since M is geodesically complete, for any $p \in M$, the exponential map $\exp_p(v)$ is defined for all $v \in T_pM$. Since M is non-compact, there exists a sequence of points $q_n \in M$ s.t. $d(p, q_n) \to \infty$.

2. Using (f) in Hopf-Rinow 2.7.1, for any $q_n \in M$ one can pick a minimizing geodesic γ_n between p and q_n s.t.

$$\ell(\gamma_n) = d(p, q_n)$$

WLOG one may parametrize γ_n using arc-length, i.e.

$$\gamma_n(0) = p \qquad \gamma_n(d(p, q_n)) = q_n$$

3. Now consider the family of tangent vectors $\{\gamma_n'(0)\} \subset S_pM \subset T_pM$ where $|\gamma_n'(0)| = 1$ and S_pM denotes the unit sphere in T_pM. Since S_pM is compact, one may extract a convergnt subsequence $\gamma_{n_k}'(0) \to v \in S_pM$. Again since M is geodesically complete, the geodesic

$$\gamma : [0, \infty) \to M \qquad \gamma(0) = p, \qquad \gamma'(0) = v$$

exists.

4. We claim that γ is a ray. To see this, one needs to show γ minimizes the distance between p and $\gamma(s)$ for any $s \in (0, \infty)$. Now fix s, there exists k large enough s.t.

$$d(p, q_{n_k}) \geq d(p, \gamma(s))$$

hence

$$\ell(\gamma_{n_k}|_{[0,s]}) = d(p, \gamma_{n_k}(s)) \qquad \text{is length minimizing}$$

Push $k \to \infty$, since both

$$\gamma_{n_k}'(0) \to v \qquad \gamma_{n_k}(s) \to \gamma(s)$$

By continuous dependence on initial conditions

$$d(\gamma_{n_k}(s), \gamma(s)) \to 0$$

Hence $\gamma|_{[0,s]}$ is length minimizing.

□

Hyperbolic Plane In this paragraph we consider the upper half plane

$$H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

equipped with the Lobatchevski metric

$$g_{11} = g_{22} = \frac{1}{y^2} \qquad g_{12} = 0$$

Recall the 'Minimizing' characterisation for geodesics.

Proposition 2.7.4. *If a piecewise differentiable curve $\gamma : [a,b] \to M$ with parameter proportional to arc length has length less or equal to any other piecewise differentiable curve joining $\gamma(a)$ and $\gamma(b)$, then γ is a geodesic in M.*

Lemma 2.7.3 (Geodesics of H)**.** *Geodesics of H are either*

1. *Upper semi-circles*
2. *rays $x = x_0$ for $y > 0$*

Proof.

1. We claim the segment for $a > 0$
$$\gamma : [a,b] \to H$$
$$t \mapsto (0,t)$$
is the image of a geodesic. Indeed, for any arc $c : [a,b] \to H$ s.t.
$$c(t) = (x(t), y(t)) \qquad c(a) = (0,a) \qquad c(b) = (0,b)$$
One has
$$\ell(c) = \int_a^b |\frac{dc}{dt}|\, dt = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, \frac{1}{y(t)} dt$$
$$\geq \int_a^b |\frac{dy}{dt}| \frac{1}{y} dt \geq \int_a^b \frac{dy}{y} = \ell(\gamma)$$
Hence γ minimizes arc length for piecewise differentiable curves, and using Proposition 2.7.4, the image of γ is a geodesic.

2. The isometries of H are the Möbius Transforms
$$z \mapsto \frac{az+b}{cz+d} \qquad z = x + iy \qquad ad - bc = 1 \tag{2.149}$$
and it transforms the $0y$ axis into upper semi-circles or rays $x = x_0$ for $y > 0$. Since isometries preserve geodesics, these are geodesics. In fact they're the only geodesics. Indeed, for any $p \in H$, and any direction in T_pH, there passes either a semi-circle with center on the $0x$ axis or the circle degenerates to a ray normal to $0x$.

□

Proposition 2.7.5 ([dC92] Exercise 7.10)**.** *The Upper Half Plane $H = \mathbb{R}^2_+$ with the Lobatchevski metric g*
$$g_{11} = g_{22} = \frac{1}{y^2} \qquad g_{12} = 0$$
is complete

Proof. We want to make use of Hopf-Rinow 2.7.1 (a). We have to show the geodesic starting at the point $(0,1) \in \mathbb{R}^2_+$ is well-defined for all $v \in T_{(0,1)}\mathbb{R}^2_+$ for all time t. Since we require to exist for $t \geq 0$ it suffices to take $|v| = 1$.

1. If $v = (0,1)$ the geodesic is
$$\gamma(t) = (0, e^t)$$
since from Proposition 2.7.4
$$\ell(c) \geq \int_a^b \frac{dy}{y} = \log(b) - \log(a)$$
2. If $v = (0,-1)$ the geodesic is accordingly
$$\gamma(t) = (0, e^{-t})$$
3. If $v = (\sin(\theta), -\cos(\theta))$ we make the identification $y = iy$ in the complex field. Then we claim
$$\gamma(t) = \frac{\sin(\frac{\theta}{2}) ie^t - \cos(\frac{\theta}{2})}{\cos(\frac{\theta}{2}) ie^t + \sin(\frac{\theta}{2})}$$
is the geodesic with origin $i = (0,1)$ and initial velocity $v = e^{i\theta} = (\cos(\theta), \sin(\theta))$.
   (a) As in (2.149) $\sin^2(\frac{\theta}{2}) + \cos^2(\frac{\theta}{2}) = 1$ so image of γ is indeed a geodesic.

(b) Compute

$$\gamma(0) = \frac{\sin(\frac{\theta}{2})i - \cos(\frac{\theta}{2})}{\cos(\frac{\theta}{2})i + \sin(\frac{\theta}{2})} = i = (0,1)$$

(c) Compute

$$\begin{aligned}
\gamma'(t) &= -\frac{1}{(\cos(\frac{\theta}{2})ie^t + \sin(\frac{\theta}{2}))^2}\cos(\frac{\theta}{2})ie^t(\sin(\frac{\theta}{2})ie^t - \cos(\frac{\theta}{2})) + \frac{\sin(\frac{\theta}{2})ie^t}{\cos(\frac{\theta}{2})ie^t + \sin(\frac{\theta}{2})} \\
\gamma'(0) &= -\frac{1}{(\cos(\frac{\theta}{2})i + \sin(\frac{\theta}{2}))^2}\cos(\frac{\theta}{2})i(\sin(\frac{\theta}{2})i - \cos(\frac{\theta}{2})) + \frac{\sin(\frac{\theta}{2})i}{\cos(\frac{\theta}{2})i + \sin(\frac{\theta}{2})} \\
&= \frac{1}{(\cos(\frac{\theta}{2})i + \sin(\frac{\theta}{2}))^2}\left(\cos(\frac{\theta}{2})\sin(\frac{\theta}{2}) + i\cos(\frac{\theta}{2})^2 - \cos(\frac{\theta}{2})\sin(\frac{\theta}{2}) + i\sin(\frac{\theta}{2})^2\right) \\
&= i\frac{1}{(\cos(\frac{\theta}{2})i + \sin(\frac{\theta}{2}))^2} = \frac{i}{-\cos(\frac{\theta}{2})^2 + 2i\sin(\frac{\theta}{2})\cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2})^2} \\
&= \frac{i}{i\sin(\theta) - \cos(\theta)} = \frac{1}{\sin(\theta) + i\cos(\theta)} = \sin(\theta) - i\cos(\theta)
\end{aligned}$$

Since all above $\gamma(t)$ exists for all $t \geq 0$, $\exp_{(0,1)}(v)$ is defined for all $v \in T_{(0,1)}\mathbb{R}^2_+$. Hence $H = \mathbb{R}^2_+$ is geodesically complete, thus complete. □

Homogeneous Manifold

Definition 2.7.6. *A Riemannian manifold M is homogeneous if for any p, $q \in M$ there exists an isometry of M which takes p to q.*

Proposition 2.7.6 ([dC92] Exercise 7.12)**.** *Any homogeneous manifold M is complete.*

Proof. By Hopf-Rinow 2.7.1 it suffices to show M is geodesically complete. Suppose we have a unit speed geodesic

$$c : [a, 1) \to M \qquad \text{s.t.} \qquad \text{it is not extendiable to } t = 1$$

Now for any $p \in M$, due to local existence of geodesic, there exists another geodesic c_2 starting at p and $\alpha > 0$ small s.t.

$$\ell(c_2) \geq \alpha > 0$$

Let's denote

$$\delta := \min\{\frac{\alpha}{2}, \frac{1-a}{2}\} > 0$$

Since M is homogenegous, for points p and $c(1-\delta)$, there exists an isometry of M that takes p to $c(1-\delta)$. But isometry also preserves geodesics, hence our c_2 should be isometrically mapped to some geodesic of equal length with starting point $c(1-\delta)$. But

$$\ell(c_2) \geq \alpha \geq 2\delta$$

hence

$$c : [1-\delta, 1+\delta) \to M \qquad \text{is extended}$$

But this contradicts with our assumption. Thus M is complete. □

2.7.3 Hadamard's Theorem

Theorem 2.7.2 (Hadamard [dC92] Theorem 7.3.1)**.** *Let M be a complete Riemannian manifold with sectional curvature*

$$K(p, \sigma) \leq 0 \qquad \forall\, p \in M, \qquad \forall\, \sigma \subseteq T_pM \text{ two-dimensional subspace}$$

Then for any $p \in M$

$$\exp_p : T_pM \to M$$

is a covering map.

If in addition we assume M is simply-connected, then M is diffeomorphic to $\mathbb{R}^n$ where $n = \dim M$ with

$$\exp_p : T_pM \to M$$

a diffeomorphism.

2.7.3.1 Lemma one: Non-positive Sectional Curvature to $\exp_p$ Local Diffeomorphism

This has been essentially shown in Proposition 2.5.6.

Pole p gives $\exp_p$ local diffeomorphism Recall Conjugate Local

$$C(p) := \{\gamma(t_0) \mid \gamma(t_0) \text{ is the first conjugate point to } p \text{ along } \gamma, \text{ for all } \gamma \text{ geodesic s.t. } \gamma(0) = p\}$$

In particular by varying $\gamma'(0) = v \in T_pM$, $C(p)$ is achieved by conjugate points to p along γ emanating in any direction v.

Definition 2.7.7 (Pole)**.** *Let (M, g) be a complete Riemannian manifold. We say p is a pole if the conjugate locus $C(p) = \varnothing$ is empty.*

In particular, if p is a pole, for γ geodesic emitting in any directions $v \in T_pM$, i.e.

$$\gamma(t) = \exp_p(tv)$$

There is no $\gamma(t_0)$ conjugate point to p along γ. Now using Proposition 2.5.4

$$(d\exp_p)_{tv}$$

is surjective for all $t > 0$, for any direction $v \in T_pM$ fixed. Thus

$$\exp_p : T_pM \to M$$

has no critical points.

In particular, at any $v \in T_pM$, $\exp_p$ is a **local diffeomorphism** by the inverse function theorem. (Note this is stronger than our result in Proposition 2.3.4)

Lemma One: Complete manifold with $K(p, \sigma) \leq 0$ is pole everywhere

Lemma 2.7.4 ([dC92] Lemma 3.2)**.** *Let (M, g) be a complete Riemannian Manifold with $K(p, \sigma) \leq 0$ for any $p \in M$ and $\sigma \subset T_pM$ 2-plane.*

Then for any $p \in M$, p is a pole.

Proof. See Proposition 2.5.6. Compute $\langle J, J\rangle''$.

Let J be nontrivial Jacobi Field along a geodesic

$$\gamma : [0, \infty) \to M$$

where $\gamma(0) = p$ and $J(0) = 0$. Notice here we're using that (M, g) is complete via Theorem 2.7.1 so that the geodesic $\exp_p$ is defined for all $v \in T_pM$.

Then

$$\begin{aligned} \langle J, J\rangle'' &= 2\langle J', J'\rangle + 2\langle J, J''\rangle \\ &\overset{(2.101)}{=} 2\langle J', J'\rangle - 2\langle R(\gamma', J)\gamma', J\rangle \\ &\overset{(2.65)}{=} 2|J'|^2 - 2K(\gamma', J)|\gamma' \wedge J|^2 \geq 0 \end{aligned} \tag{2.150}$$

Therefore for any $t_2 > t_1$

$$\langle J, J\rangle'(t_2) \geq \langle J, J\rangle'(t_1)$$

Since we assumed J is nontrivial, $J'(0) \neq 0$, but $\langle J, J'\rangle(0) = 0$. Now for t sufficiently small,

$$\langle J, J\rangle(t) > \langle J, J\rangle(0)$$

It follows that for all $t > 0$

$$\langle J, J\rangle(t) = |J(t)|^2 > 0$$

thus $\gamma(t)$ is never conjugate to $\gamma(0)$ along γ. □

Remark 2.7.2. *Notice Lemma 2.7.4 does not mean if there exists $p \in M$ s.t. $K(p, \sigma) \leq 0$ for any $\sigma \subset T_pM$ then it implies p is a pole. This is because in the proof step (2.150), one really need the sign on sectional curvature along $\gamma(t)$ instead of just one point $p = \gamma(0)$.*

Example for poles in non-compact manifolds with positive sectional curvature Notice that poles can exist in non-compact manifolds which have positive sectional curvature. The point $p = (0,0,0)$ of the paraboloid

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}$$

is a pole of S. On the other hand, notice the curvature is positive. ([dC92] Exercise 7.13)

2.7.3.2 Lemma Two: Covering Map

Lemma 2.7.5 ([dC92] Lemma 7.3.3)**.** *Let (M, g) be a complete Riemannian manifold, and let (N, h) be another Riemannian manifold s.t. there exists*

$$f : M \to N$$

that is surjective and a local diffeomorphism.

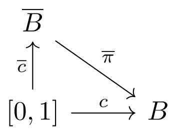
Assume for any $p \in M$, for any $v \in T_pM$, we have

$$\|df_p(v)\|_{f(p)} \geq \|v\|_p \tag{2.151}$$

Then we have f is a covering map.

Path-Lifting Property determines Covering Map

Remark 2.7.3 (Path-lifting Property)**.** *Assume we have a path $c : [0,1] \to B$. Let $\overline{\pi} : \overline{B} \to B$ be a continuous surjective map, local homeomorphism. Let $\overline{B}$ be locally path connected and B be locally simply connected. If in addition c satisfies the* ***path lifting property*** *s.t. the diagram commutes*



Then $\overline{\pi}$ is a covering map.

Proof of Lemma Two

Proof of Lemma 2.7.5. By the above fact Remark 2.7.3, we only need to check that f satisfies the path lifting property with $\overline{B} = M$ and $B = N$. Given

$$c : [0,1] \to N$$

We want to prove

(a) $\overline{c}$ can be defined. If $\overline{c}$ is defined in some small interval then one can extend it. In particular if

$$\overline{c} : [0, t_0] \to M \qquad 0 \leq t_0 < 1$$

s.t.

$$f \circ \overline{c} = c$$

Then there exists a $\delta > 0$ s.t. $\overline{c}$ is defined on $[0, t_0 + \delta]$ and also satisfies

$$f \circ \overline{c} = c$$

(b) If $\overline{c}$ is now defined in

$$\overline{c} : [0, t_0) \to M \qquad 0 < t_0 \leq 1$$

s.t.

$$f \circ \overline{c} = c$$

Then $\overline{c}$ can be extended to t_0

$$c : [0, t_0] \to M$$

with

$$f \circ \overline{c} = c$$

In particular $f(\overline{c}(t_0)) = c(t_0)$.

Proof of (a). Since f is local diffeomorphism, there exists U open neighborhood of $\bar{c}(t_0)$ s.t.

$$f|_U : U \to f(U) \qquad \text{is a diffeomorphism}$$

Then $f(U)$ is an open neighborhood of $c(t_0) = f(\bar{c}(t_0))$. Then there exists $\delta > 0$ s.t. the image of $(t_0 - \delta, t_0 + \delta)$ through c is contained in U. For $t \in (t_0 - \delta, t_0 + \delta)$ define

$$\bar{c}(t) := (f|_U)^{-1}(c(t))$$

since f is surjective.

Proof of (b)

In this why we need (2.151). Let $\{t_n\}$ be a sequence $t_{n+1} > t_n$ s.t. $t_n \to t_0$. Then for any $m < n$. Compute the distance between $\bar{c}(t_n)$ and $\bar{c}(t_m)$ because we want to show $\{\bar{c}(t_n)\}$ is Cauchy.

$$\begin{aligned} d_M(\bar{c}(t_n), \bar{c}(t_m)) \le \ell(\bar{c}|_{[t_m,t_n]}) &= \int_{t_m}^{t_n} \left\| \frac{d\bar{c}}{dt}(t) \right\|_{\bar{c}(t)} dt \\ &\overset{(2.151)}{\le} \int_{t_m}^{t_n} \left\| df_{\bar{c}(t)}(\frac{d\bar{c}}{dt}(t)) \right\|_{c(t)} dt = \int_{t_m}^{t_n} \left\| \frac{d}{dt}(f \circ \bar{c}) \right\|_{c(t)} dt \\ &= \int_{t_m}^{t_n} \left\| \frac{d}{dt} c(t) \right\|_{c(t)} dt \le C|t_n - t_m| \qquad \text{where } C := \max_{[0,1]} |\frac{dc}{dt}(t)| \end{aligned}$$

Now $\{\bar{c}(t_n)\}$ is Cauchy. Since M is complete, by Hopf-Rinow 2.7.1, $\bar{c}(t_n)$ converges, so there exists $r \in M$ s.t.

$$\bar{c}(t_n) \to r$$

We define

$$r := \bar{c}(t_0)$$

It suffices to check $f(\bar{c}(t_0)) = c(t_0)$. But using continuity of f

$$f(\bar{c}(t_0)) = f(r) = f(\lim_{n\to\infty} \bar{c}(t_n)) = \lim_{n\to\infty} (f \circ \bar{c})(t_n) = \lim_{n\to\infty} c(t_n) = c(t_0)$$

□

2.7.3.3 Proof of Hadamard's

Corollary 2.7.3 (Corollary of Lemma 2.7.5). *Let (M, g) be a complete Riemannian manifold. Suppose $p \in M$ is a pole. Then*

$$\exp_p : T_pM \to M$$

is a covering map. In particular, if M is simply connected, then $\exp_p$ is a global diffeomorphism, and hence M is diffeomorphic to $\mathbb{R}^n$.

Proof. Since p is a pole, $\exp_p : T_pM \to M$ is a local diffeomorphism. Since M is complete Riemannian Manifold, by Hopf-Rinow 2.7.1 (f), there exists minimizing geodesic connecting any two points, thus

$$\exp_p : T_pM \to M \qquad \text{is surjective}$$

We define

$$\tilde{g} := \exp_p^* g$$

to be a Riemannian metric on T_pM. Then the exponential map

$$\exp_p : (T_pM, \tilde{g}) \to (M, g) \qquad \text{is a local isometry}$$

In particular

$$\left\| d\exp_p(v) \right\|_g = \|v\|_{\tilde{g}}$$

so (2.151) is satisfied.

Now we only need to check $(T_pM, \tilde{g})$ is complete to apply Lemma 2.7.5.

By Hopf-Rinow 2.7.1 (d), we want to show that the exponential map of $(T_pM, \tilde{g})$ is defined everywhere.

$$\forall\, v \in T_0(T_pM) \cong T_pM \qquad \gamma(t) := \exp_p(tv) \qquad \forall\, t \in \mathbb{R} \qquad \text{is a geodesic in } M$$

and

$$\tilde{\gamma}(t) := tv \qquad \forall\, t \in \mathbb{R} \qquad \text{is a geodesic in } T_pM$$

One define

$$\begin{aligned} \tilde{\exp}_0 : T_0(T_pM) &\to T_pM \\ tv &\mapsto \tilde{\exp}_0(tv) := \tilde{\gamma}(t) = tv \end{aligned}$$

Thus

$$\tilde{\exp}_0 : T_0(T_pM) = T_pM \to T_pM$$

is the identity.

In particular $\tilde{\exp}_0$ is defined everywhere at the point 0. By Hopf-Rinow 2.7.1 we know $(T_pM, \tilde{g})$ is complete. Hence by Lemma 2.7.5, $\exp_p$ is a covering map. □

Proof of Theorem 2.7.2. Let (M, g) be complete Riemannian manifold with $K(p, \sigma) \leq 0$. By Lemma 2.7.4 for any $p \in M$, p is a pole. By Corollary 2.7.3 we know

$$\exp_p : T_pM \to M$$

is a covering map. □

2.8 Space of Constant Curvature

So far we have encountered two examples of Riemannian manifold with constant sectional curvature K, namely Euclidean space $\mathbb{R}^n$ with $K = 0$, and the unit sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ with $K = 1$.

In this chapter we shall introduce the hyperbolic space H^n with sectional curvature -1. These are complete and simply connected. The main theorem of this chapter is that these are essentially the only complete, simply connected Riemannian manifolds with constant sectional curvature.

2.8.1 Theorem of Cartan on Determination of the Metric by Curvature

If two Riemannian manifolds have the same Riemannian curvature, then they have the same metric. How do we compare two Riemannian manifolds of the same dimension?

Setup to Compare Manifolds of same dimension Let $p \in M$ and $\tilde{p} \in \tilde{M}$ with the same dimension. In particular they both have tangent space $T_pM \cong T_{\tilde{p}}\tilde{M} = \mathbb{R}^n$. Then one can cook up a mapping i between the tangent spaces

$$\begin{aligned} i : T_pM &\to T_{\tilde{p}}\tilde{M} \\ e_i &\mapsto i(e_i) = \tilde{e}_i \end{aligned}$$

as a linear isometry, i.e., sends an orthonormal basis $\{e_i\}$ to an orthonormal basis $\{\tilde{e}_i\}$.

Using Proposition 2.3.4 there exists $r > 0$ s.t.

$$\begin{aligned} \exp_p : B_r(0) \subset T_pM &\to B_r(p) \subset M \\ v &\mapsto \exp_p(v) \\ \tilde{\exp}_{\tilde{p}} : B_r(0) \subset T_{\tilde{p}}\tilde{M} &\to B_r(\tilde{p}) \subset \tilde{M} \\ v &\mapsto \tilde{\exp}_{\tilde{p}}(v) \end{aligned}$$

are diffeomorphisms.

Now we define

$$\begin{aligned} f : B_r(p) \subset M &\to B_r(\tilde{p}) \subset \tilde{M} \\ q &\mapsto f(q) := \tilde{\exp}_{\tilde{p}} \circ i \circ (\exp_p)^{-1}(q) \end{aligned} \tag{2.152}$$

f is a diffeomorphism.

$$\begin{array}{ccccccc} T_pM & \overset{open}{\supseteq} & B_r(0) & \xrightarrow{\ i\ } & B_r(0) & \overset{open}{\subseteq} & T_{\tilde{p}}\tilde{M} \\ & & \downarrow{\scriptstyle \exp_p} & & \downarrow{\scriptstyle \tilde{\exp}_{\tilde{p}}} & & \\ M & \overset{open}{\supseteq} & B_r(p) & \xrightarrow{\ f\ } & B_r(\tilde{p}) & \overset{open}{\subseteq} & \tilde{M} \end{array}$$

For all $q \in B_r(0)$ there exists a unique initial velocity $v \in T_pM$ s.t. q can be reached via the normalized geodesic

$$\gamma(t) = \exp_p(tv)$$

s.t.

$$p = \exp_p(0) = \gamma(0), \qquad q = \exp_p(\ell v) = \gamma(\ell) \qquad \ell := d(p, q)$$

Now let

$$P_{p,q} : T_pM \to T_qM$$

be the parallel transport (1.94) along the geodesic $\gamma(t) := \exp_p(tv)$ from $\gamma(0) = p$ to $\gamma(\ell) = q$.

Similarly let

$$\tilde{P}_{\tilde{p}, f(q)} : T_{\tilde{p}}\tilde{M} \to T_{f(q)}\tilde{M}$$

be the parallel transport along the geodesic $\tilde{\gamma}(t) := \tilde{\exp}_{\tilde{p}}(t\tilde{v})$ from $\tilde{\gamma}(0) = \tilde{p}$ to

$$\tilde{\gamma}(\ell) = \tilde{\exp}_{\tilde{p}}(\ell\tilde{v}) = \tilde{\exp}_{\tilde{p}}(i \circ \exp_p^{-1}(q)) = f(q)$$

Now for any $q \in B_r(p)$, define

$$\begin{aligned} \phi_q : T_qM &\to T_{f(q)}\tilde{M} \\ v &\mapsto \phi_q(v) := \tilde{P}_{\tilde{p}, f(q)} \circ i \circ (P_{p,q})^{-1}(v) \end{aligned} \tag{2.153}$$

as an a linear isometry.

$$
\begin{array}{ccc}
T_pM & \xrightarrow{\ i\ } & T_{\tilde{p}}\tilde{M} \\
{\scriptstyle P_{p,q}}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{P}_{\tilde{p},f(q)}} \\
T_qM & \xrightarrow[\phi_q]{} & T_{f(q)}\tilde{M}
\end{array}
$$

Finally denote by R and $\tilde{R}$ the Riemannian curvature of M and $\tilde{M}$ respectively.

Cartan's Theorem

Theorem 2.8.1 (Cartan [dC92] Theorem 8.2.1)**.** *With the above notations, assume for all $q \in B_r(p)$, and for all*

$$x,\, y,\, v,\, u \in T_qM$$

the Riemannian curvature agrees

$$R(x, y, v, u) = \tilde{R}(\phi_q(x), \phi_q(y), \phi_q(v), \phi_q(u))$$

where $\phi_q =$ is as in (2.153).

Then f as in (2.152)

$$f : B_r(p) \to B_r(\tilde{p})$$

is an isometry and

$$df_p = i$$

Proof of Cartan 2.8.1. We already know that f is a diffeomorphism. We really need to show that for all $q \in B_r(p)$ and for every $w \in T_qM$, the norm is preserved

$$\|df_p(w)\|_{f(q)} = \|w\|_q$$

Observe that

$$
\begin{aligned}
df_p &= d(\tilde{\exp}_{\tilde{p}} \circ i \circ (\exp_p)^{-1}) = d\tilde{\exp}_{\tilde{p}} \circ i \circ d(\exp_p)^{-1} \\
&\overset{(2.43)}{=} \mathrm{id}_{T_{\tilde{p}}\tilde{M}} \circ i \circ (\mathrm{id}_{T_pM})^{-1} = i
\end{aligned}
$$

Remark that even though the identity is known, it doesn't mean f is an isometry. We need to show norms are the same. We do it through Jacobi fields. Those will allow us to use the hypothesis.

We may assume $p \neq q$ and $w \neq 0$. There exists unit vector $v \in T_pM$ s.t.

$$q = \exp_p(\ell v) \qquad \ell = d(p, q) > 0$$

There exists a unique $w_0 \in T_{\ell v}(T_pM) \cong T_pM$ such that

$$(d\exp_p)_{\ell v}(w_0) = \frac{w}{\ell}$$

This exists since if $\ell v < r$ one can find the preimage due to $d\exp_p$ is a linear isomorphism. Now let

$$\gamma : [0, \ell] \to M \qquad \text{be the geodesic s.t.} \qquad \gamma(0) = p \qquad \gamma(\ell) = q$$

Look at the Jacobi Field

$$J(t) := (d\exp_p)_{tv}(tw_0)$$

Then

$$
\begin{aligned}
&J(0) = 0 \\
&J(\ell) = (d\exp_p)_{\ell v}(\ell w_0) = \ell(d\exp_p)_{\ell v}(w_0) = \ell\frac{w}{\ell} = w
\end{aligned}
$$

Now we write in coordinates. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of T_pM. We let

$$e_n := v = \gamma'(0)$$

Let $\{e_1(t), \cdots, e_n(t)\}$ be the parallel transport along the geodesic γ. Then Jacobi Field has local coordinates

$$J(t) = \sum_{i=1}^{n} y_i(t)e_i(t) \qquad y_i \in C^\infty([0, \ell]; M)$$

Now here the Jacobi Equation is, upon contraction with $e_i(t)$

$$J''(t) + R(\gamma', J)\gamma' = 0 \implies \frac{d^2 y_i}{dt^2} + \sum_{j=1}^{n} R(e_n, e_j, e_n, e_i) y_j = 0$$

Now let

$$\tilde{\gamma} : [0, \ell] \to \tilde{M} \qquad \tilde{\gamma} := f \circ \gamma \qquad \text{be geodesic so that} \qquad \tilde{\gamma}(0) = f(p) = \tilde{p} \qquad \tilde{\gamma}(\ell) = f(q)$$

Let $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$ be ONB of $T_{\tilde{p}}\tilde{M}$, and $\{\tilde{e}_1(t), \cdots, \tilde{e}_n(t)\}$ be their parallel transport along $\tilde{\gamma}$. Hence

$$\tilde{e}_i(t) = \phi_q(e_i(t))$$

Now we define

$$\tilde{J}(t) = \phi_{\gamma(t)}(J(t)) = \sum_{i=1}^{n} y_i(t)\tilde{e}_i(t)$$

But this is not in principle a Jacobi Field. Here we use our assumption that two Riemannian curvatures are the same. Hence

$$\begin{aligned} &\frac{d^2 y_i}{dt^2} + \sum_{j=1}^{n} \tilde{R}(\tilde{e}_n, \tilde{e}_j, \tilde{e}_n, \tilde{e}_i) y_j \\ &= \frac{d^2 y_i}{dt^2} + \sum_{j=1}^{n} R(e_n, e_j, e_n, e_i) y_j \\ &= 0 \end{aligned}$$

Hence $\tilde{J}(t)$ is Jacobi field with $\tilde{J}(0) = 0$. What about its length?

$$\left\| \tilde{J}(t) \right\|_{\gamma(t)} = \sqrt{\sum_{i=1}^{n} y_i^2} = \|J(t)\|_{\gamma(t)} \qquad \forall\ t$$

Now $\tilde{J}(t)$ is a Jacobi Field along $\tilde{\gamma}(t)$ with

$$\begin{aligned} \tilde{J}'(0) &= \sum_{i=1}^{n} y_i'(0)\tilde{e}_i(0) \\ &= \sum_{i=1}^{n} i(y_i'(0)e_i) = i(w_0) \\ \tilde{J}(t) &= (d\tilde{\exp}_{\tilde{p}})_{ti(v)}(ti(w_0)) \\ \tilde{J}(\ell) &= (d\tilde{\exp}_{\tilde{p}})_{\ell i(v)} \circ i \circ (\ell w_0) \\ &= (d\tilde{\exp}_{\tilde{p}})_{\ell i(v)} \circ i \circ ((d\exp_p)_{\ell v})^{-1}(w) \qquad \text{using definition of } w_0 \\ &= d(\tilde{\exp}_{\tilde{p}} \circ i \circ \exp_p^{-1})_{\exp_p(\ell v)}(w) \\ &= d(f)_q(w) \qquad \text{using } q = \exp_p(\ell v) \end{aligned}$$

Thus

$$\|df_q(w)\| = \left\| \tilde{J}(\ell) \right\| = \|J(\ell)\| = \|w\|$$

□

Determination by Constant Sectional Curvature

Corollary 2.8.1. *Let (M, g) and $(\tilde{M}, \tilde{g})$ be two Riemannian manifolds of dimension n, with the same constant sectional curvature K_0. Let $p \in M$ and $\tilde{p} \in \tilde{M}$. Let $\{e_1, \cdots, e_n\}$ be ONB of T_pM and $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$ ONB of $T_{\tilde{p}}\tilde{M}$.*

Then there exists U open neighborhood of p in M and $\tilde{U}$ open neighborhood of $\tilde{p}$ in $\tilde{M}$ and an isometry

$$f : U \to \tilde{U}$$

s.t.

$$f(p) = \tilde{p} \qquad df_p(e_i) = \tilde{e}_i$$

Proof. Notice via (2.67)

$$R(x,y,u,v) = K_0(g(x,u)g(y,v) - g(x,v)g(y,u))$$

Choose

$$i : T_pM \to T_{f(p)}\tilde{M} \qquad i(e_j) = \tilde{e}_j$$

and

$$f = \exp_p \circ i \circ (\tilde{\exp}_{\tilde{p}})^{-1}$$

Apply Cartan's Theorem 2.8.1. □

2.8.2 Conformal Deformation of the Curvature

Let (M,g) be a Riemannian manifold.

Definition 2.8.1 (Conformal Deformation)**.**

$$\tilde{g} = e^{2f} g$$

for some $f \in C^\infty(M)$ smooth function on the manifold M.

This is known as a conformal change(deformation). Note we're working on the same manifold with different metrics given by the deformation.

2.8.2.1 Conformal Change in Connection

We denote ∇ as Levi-Civita connection of g and $\tilde{\nabla}$ as Levi-Civita connection of $\tilde{g}$. Using the expression for $\tilde{g}(\tilde{\nabla}_X Y, Z)$ (2.24) one may compute for $\tilde{\nabla}_X Y$.

Proposition 2.8.1. *Let g be a Riemannian metric on a manifold M and let $\tilde{g} := e^{2f} g$ where $f \in C^\infty(M)$ be its deformation.*

Let ∇ and $\tilde{\nabla}$ denote respectively the Levi-Civita connections on (M,g) and $(M,\tilde{g})$.

Then for any X, $Y \in \mathfrak{X}(M)$ one has

$$\tilde{\nabla}_X Y = \nabla_X Y + X(f)Y + Y(f)X - g(X,Y)\mathrm{grad}(f) \qquad \text{where } g(\mathrm{grad}(f), Y) := df(Y) \tag{2.154}$$

Proof. Let's prove using local coordinates. Denote

$$\tilde{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} := \sum_{\ell=1}^n \tilde{\Gamma}_{ij}^\ell \frac{\partial}{\partial x_\ell}$$

where

$$\begin{aligned}
\tilde{\Gamma}_{ij}^\ell &= \frac{1}{2}\sum_{k=1}^n \tilde{g}^{\ell k}(\tilde{g}_{ik,j} + \tilde{g}_{kj,i} - \tilde{g}_{ij,k}) \\
&= \frac{1}{2}\sum_{k=1}^n e^{-2f} g^{\ell k}\left(\frac{\partial}{\partial x_j}(e^{2f} g_{ik}) + \frac{\partial}{\partial x_i}(e^{2f} g_{kj}) - \frac{\partial}{\partial x_k}(e^{2f} g_{ij})\right) \\
&= \frac{1}{2}\sum_{k=1}^n e^{-2f} g^{\ell k}\left(2e^{2f}\frac{\partial f}{\partial x_j} g_{ik} + e^{2f} g_{ik,j} + 2e^{2f}\frac{\partial f}{\partial x_i} g_{kj} + e^{2f} g_{kj,i} - 2e^{2f}\frac{\partial f}{\partial x_k} g_{ij} - e^{2f} g_{ij,k}\right) \\
&= \sum_{k=1}^n g^{\ell k}\left(\frac{\partial f}{\partial x_j} g_{ik} + \frac{\partial f}{\partial x_i} g_{kj} - \frac{\partial f}{\partial x_k} g_{ij}\right) + \Gamma_{ij}^\ell \\
&= \Gamma_{ij}^\ell + \delta_i^\ell \frac{\partial f}{\partial x_j} + \delta_j^\ell \frac{\partial f}{\partial x_i} - \sum_{k=1}^n g^{\ell k} g_{ij} \frac{\partial f}{\partial x_k}
\end{aligned}$$

Now

$$\begin{aligned}
\tilde{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} &:= \sum_{\ell=1}^n \tilde{\Gamma}_{ij}^\ell \frac{\partial}{\partial x_\ell} \\
&= \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + \sum_{\ell=1}^n \left(\delta_i^\ell \frac{\partial f}{\partial x_j} + \delta_j^\ell \frac{\partial f}{\partial x_i} - \sum_{k=1}^n g^{\ell k} g_{ij} \frac{\partial f}{\partial x_k}\right)\frac{\partial}{\partial x_\ell} \\
&= \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + \frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i} + \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j} - g_{ij}\sum_{\ell=1}^n\sum_{k=1}^n g^{\ell k}\frac{\partial f}{\partial x_k}\frac{\partial}{\partial x_\ell} \\
&\overset{(2.89)}{=} \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + \frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i} + \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j} - g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\mathrm{grad}(f)
\end{aligned}$$

Hence (2.154) is true for coordinate basis. In general let

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \qquad a_i \in C^\infty(U)$$
$$Y = \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j} \qquad b_j \in C^\infty(U)$$

in local charts. We compute

$$\begin{aligned}
\tilde{\nabla}_X Y &= \tilde{\nabla}_{\sum_i a_i \frac{\partial}{\partial x_i}} (\sum_j b_j \frac{\partial}{\partial x_j}) \\
&= \sum_i a_i \left(\sum_j \left(\frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} + b_j \tilde{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \right) \right) \\
&= \sum_i a_i \left(\sum_j \left(\frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} + b_j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + b_j \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i} + b_j \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} - b_j g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \mathrm{grad}(f) \right) \right) \\
&= \sum_i a_i \left(\sum_j \left(\nabla_{\frac{\partial}{\partial x_i}} (b_j \frac{\partial}{\partial x_j}) + b_j \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i} + b_j \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} - b_j g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \mathrm{grad}(f) \right) \right) \\
&= \nabla_{\sum_i a_i \frac{\partial}{\partial x_i}} (\sum_j b_j \frac{\partial}{\partial x_j}) + \sum_j b_j \frac{\partial f}{\partial x_j} \sum_i a_i \frac{\partial}{\partial x_i} + \sum_i a_i \frac{\partial f}{\partial x_i} \sum_j b_j \frac{\partial}{\partial x_j} - g(\sum_i a_i \frac{\partial}{\partial x_i}, \sum_j b_j \frac{\partial}{\partial x_j}) \mathrm{grad}(f) \\
&= \nabla_X Y + Y(f) X + X(f) Y - g(X,Y) \mathrm{grad}(f)
\end{aligned}$$

□

Scaling As immediate consequence one observe scaling.

Corollary 2.8.2. *If f is a constant, then $\tilde{g} = kg$ for $k > 0$ is constant times g. In this case*

$$\begin{aligned}
\tilde{\nabla}_X Y &= \nabla_X Y \\
\tilde{R}(X,Y)Z &= R(X,Y)Z \\
\tilde{R}(X,Y,Z,W) &= kR(X,Y,Z,W) \\
\tilde{K}(X,Y) &= \frac{1}{k} K(X,Y) \\
\tilde{\mathrm{Ric}}(X,Y) &= \mathrm{Ric}(X,Y) \\
\tilde{S} &= \frac{1}{k} S
\end{aligned}$$

Proof. The first two remain unchanged because Christoffel symbols are unchanged. The Riemannian curvature tenor scales by k due to the formula (2.64). For sectional curvature, note

$$K(X,Y) = \frac{1}{g(X,X)g(Y,Y) - g(X,Y)^2} R(X,Y,X,Y)$$

so that

$$\tilde{K}(X,Y) = \frac{1}{k} K(X,Y)$$

Ricci remains unchanged because $\frac{1}{k}$ pops out upon taking trace (2.79). Scalar curvature has another $\frac{1}{k}$ because we're taking trace again (2.81). □

2.8.2.2 Riemannian Curvature Deformation

Kulkarni-Nomizu Product But what happens in general?

Definition 2.8.2 (Kulkarni-Nomizu Product). *For S, $T \in C^\infty(M, \mathrm{Sym}^2 T^* M)$ symmetric $(0,2)$-tensors on M, $(S \circ T)$ gives a $(0,4)$-tensor on M. We define their Kulkarni-Nomizu Product as*

$$(S \circ T)(X,Y,Z,W) = S(X,Z)T(Y,W) + S(Y,W)T(X,Z) - S(X,W)T(Y,Z) - S(Y,Z)T(X,W) \qquad (2.155)$$

Moreover, the symmetries take the form

$$\begin{aligned}(S\circ T)(X,Y,Z,W) &= -(S\circ T)(Y,X,Z,W) && \text{anti-symmetric in first two components}\\ &= -(S\circ T)(X,Y,W,Z) && \text{anti-symmetric in second two components}\\ &= (S\circ T)(Z,W,X,Y) && \text{symmetric w.r.t. the two sets of components}\end{aligned}$$

Thus $S\circ T \in C^\infty(M, \mathrm{Sym}^2\left(\Lambda^2 T^*M\right))$.

Riemannian Curvature Deformation Let R denote curvature tensor of g and $\tilde{R}$ denote curvature tensor of $\tilde{g}$.

Lemma 2.8.1. *If (M,g) has constant sectional curvature κ, the Riemannian Curvature is determined via the product*

$$R = \frac{1}{2}\kappa g\circ g \tag{2.156}$$

Proof. Using (2.155)

$$\begin{aligned}(g\circ g)(X,Y,Z,W) &= g(X,Z)g(Y,W) + g(Y,W)g(X,Z) - g(X,W)g(Y,Z) - g(Y,Z)g(X,W)\\ &= 2g(X,Z)g(Y,W) - 2g(X,W)g(Y,Z)\end{aligned}$$

Thus recalling (2.67) gives

$$\begin{aligned}R(X,Y,Z,W) &= \kappa\left(g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\right)\\ &= \frac{1}{2}\kappa(g\circ g)(X,Y,Z,W)\end{aligned}$$

□

Proposition 2.8.2. *Under conformal deformation $\tilde{g} = e^{2f}g$, we have*

$$\tilde{R} = e^{2f}(R - (\mathrm{Hess}(f))\circ g + (df\otimes df)\circ g - \frac{1}{2}|df|^2 g\circ g) \tag{2.157}$$

where

1. *the Hessian writes (2.94), for $f_{;ij}$ the covariant derivative w.r.t. g,*

$$\mathrm{Hess}(f) = \sum_{i,j=1}^{n} f_{;ij}dx_i dx_j$$

2. *and (2.87)*

$$|df|^2 = \sum_{i,j} g^{ij} f_{;i} f_{;j}$$

In the following we demonstrate examples from Hyperbolic Space that we study using (2.157) and (2.156).

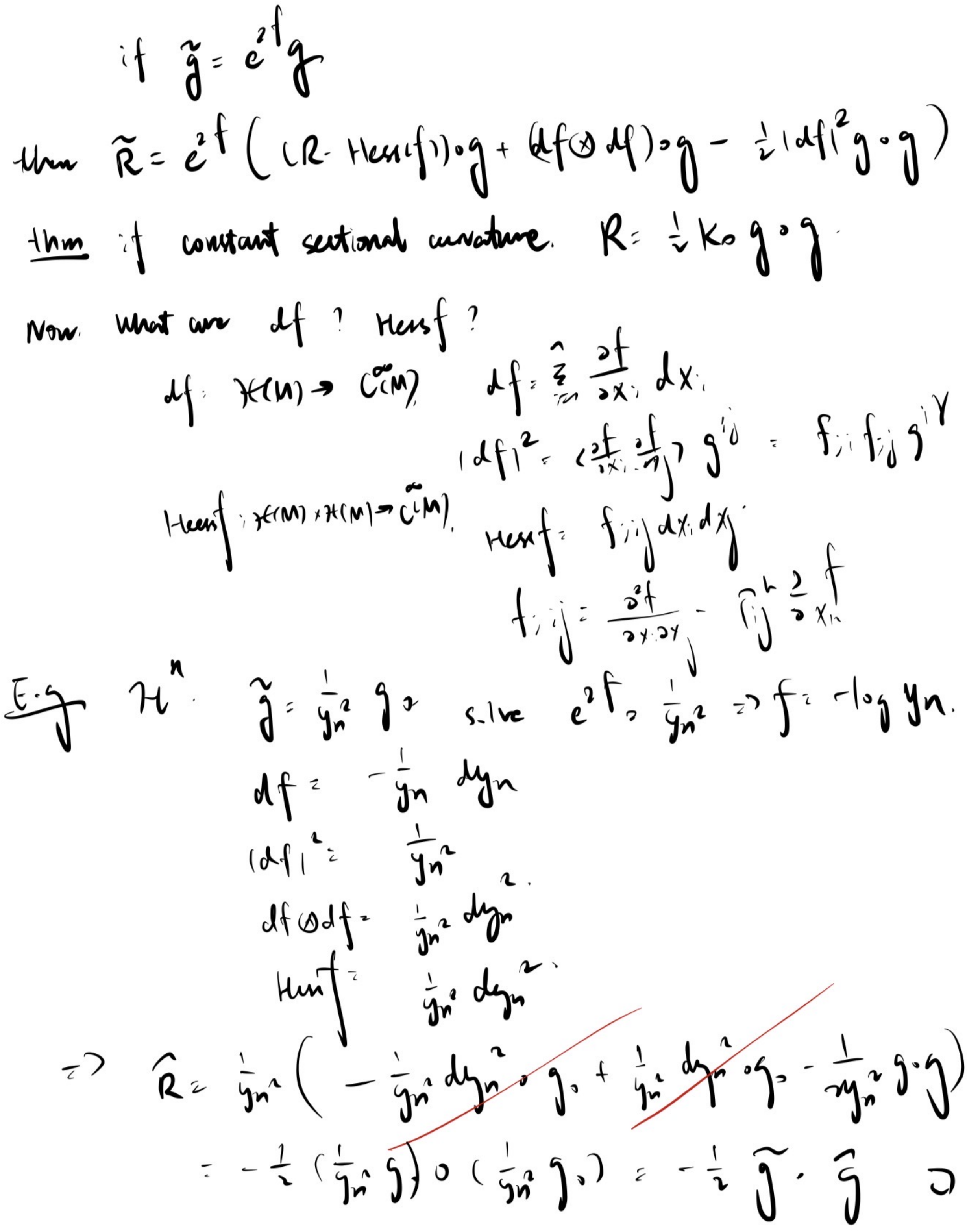


Figure 2.13: Hyperbolic Space Upper Half Space Constant Sectional Curvature -1

Example one: Hyperbolic Space Upper Half Space model

Definition 2.8.3 (Upper Half Hyperbolic Space). *We take*

$$\mathcal{H}^n := \{(y_1, \cdots, y_n) \in \mathbb{R}^n \mid y_n > 0\}$$

and metric

$$\tilde{g} := \frac{dy_1^2 + \cdots dy_n^2}{y_n^2} = e^{2f} g_0$$

where $g_0 = dy_1^2 + \cdots dy_n^2$ is the Euclidean metric.

Here one may solve for f

$$e^{2f} = \frac{1}{y_n^2} \implies f = -\log(y_n)$$

We compute

$$\begin{aligned} df &= d(-\log(y_n)) = -\frac{dy_n}{y_n} \\ df \otimes df &= \frac{dy_n \otimes dy_n}{y_n^2} = \frac{dy_n^2}{y_n^2} \\ |df|^2 &= \frac{1}{y_n^2} \\ \mathrm{Hess}(f) &= \frac{1}{y_n^2} dy_n^2 \end{aligned}$$

Then we apply the formula (2.157)

$$\begin{aligned} \tilde{R} &= \frac{1}{y_n^2}\left(R - \frac{1}{y_n^2} dy_n^2 \circ g_0 + \frac{dy_n^2}{y_n^2} \circ g_0 - \frac{1}{2}\frac{1}{y_n^2} g_0 \circ g_0\right) \\ &= -\frac{1}{2}(\frac{1}{y_n^2} g_0) \circ (\frac{1}{y_n^2} g_0) \\ &= -\frac{1}{2}\tilde{g} \circ \tilde{g} \end{aligned}$$

Hence $(\mathcal{H}^n, \tilde{g})$ has constant sectional curvature -1 using (2.156).

Example two: Hyperbolic Space Unit Disc model

Definition 2.8.4 (Unit Disk Hyperbolic Space)**.** *We take*

$$D^n := \{(u_1, \cdots, u_n) \in \mathbb{R}^n \mid |u| < 1\}$$

and metric

$$\tilde{g} := \frac{4}{(1 - |\vec{u}|^2)^2}(du_1^2 + \cdots + du_n^2) = e^{2f} g_0$$

One solve for f

$$\begin{aligned} g_0 &= du_1^2 + \cdots + du_n^2 \\ e^{2f} &= \frac{4}{(1 - |\vec{u}|^2)^2} \\ e^f &= \frac{2}{(1 - |\vec{u}|^2)} \\ f &= \log 2 - \log(1 - |\vec{u}|^2) \end{aligned}$$

We compute

$$\begin{aligned} f_i &= -\frac{-2u_i}{1 - |\vec{u}|^2} = \frac{2u_i}{1 - |\vec{u}|^2} \\ df &= \frac{\sum_{i=1}^n 2u_i du_i}{1 - |\vec{u}|^2} \\ df \otimes df &= \sum_{i,j} \frac{4u_i u_j du_i du_j}{(1 - |\vec{u}|^2)^2} \\ |df|^2 &= \frac{4|\vec{u}|^2}{(1 - |\vec{u}|^2)^2} \\ f_{ij} &= \frac{2\delta_{ij}(1 - |\vec{u}|^2) + 4u_i u_j}{(1 - |\vec{u}|^2)^2} = \frac{2\delta_{ij}}{1 - |\vec{u}|^2} + \frac{4u_i u_j}{(1 - |\vec{u}|^2)^2} \\ \mathrm{Hess}(f) &= 2\frac{\sum du_i^2}{1 - |\vec{u}|^2} + \frac{4\sum_{i,j} u_i u_j du_i du_j}{(1 - |\vec{u}|^2)^2} \end{aligned}$$

So we apply (2.157)

$$
\begin{aligned}
\tilde{R} &= e^{2f}\left(R - (2\frac{\sum du_i^2}{1-|\vec{u}|^2} + \frac{4\sum_{i,j} u_i u_j du_i du_j}{(1-|\vec{u}|^2)^2}) \circ g_0 + (\sum_{i,j}\frac{4u_iu_jdu_idu_j}{(1-|\vec{u}|^2)^2}) \circ g_0 - \frac{1}{2}\frac{4|\vec{u}|^2}{(1-|\vec{u}|^2)^2} g_0 \circ g_0\right) \\
&= e^{2f}\left(-2\frac{\sum_i du_i^2}{1-|\vec{u}|^2} \circ g_0 - \frac{2|\vec{u}|^2}{(1-|\vec{u}|^2)^2} g_0 \circ g_0\right) \\
&= -2\frac{1}{(1-|\vec{u}|^2)^2} e^{2f}\left(1-|\vec{u}|^2 + |\vec{u}|^2\right) g_0 \circ g_0 \\
&= -\frac{2}{(1-|\vec{u}|^2)^2} e^{2f} g_0 \circ g_0 \\
&= -\frac{1}{2}(e^{2f} g_0) \circ (e^{2f} g_0) \qquad \text{using } \frac{2}{(1-|\vec{u}|^2)^2} = \frac{1}{2}e^{2f} \\
&= -\frac{1}{2}\tilde{g} \circ \tilde{g}
\end{aligned}
$$

Thus $(D^n, \tilde{g})$ has constant sectional curvature -1.

Example Three One look at a non-hyperbolic example.

Given any positive constant $K > 0$, define a Riemannian metric g_K on $\mathbb{R}^n$ by

$$g_K = \frac{4\sum_{i=1}^n dx_i^2}{(1+K|x|^2)^2}$$

Then

1. $(\mathbb{R}^n, g_K)$ has constant sectional curvature K.

   Proof. Let $f \in C^\infty(M)$ s.t.

   $$g_K = \frac{4\sum_{i=1}^n dx_i^2}{(1+K|x|^2)^2} = e^{2f} g_0$$

   where g_0 denotes the flat metric. Hence

   $$
   \begin{aligned}
   g_0 &= \sum_{i=1}^n dx_i^2 \\
   e^{2f} &= \frac{4}{(1+K|x|^2)^2} \\
   e^f &= \frac{2}{1+K|x|^2} \\
   f &= \log 2 - \log(1+K|x|^2)
   \end{aligned}
   $$

   We compute

   $$
   \begin{aligned}
   f_i &= -\frac{2Kx_i}{1+K|x|^2} \\
   df &= -\frac{\sum_{i=1}^n 2Kx_i dx_i}{1+K|x|^2} \\
   df \otimes df &= \frac{\sum_{i,j=1}^n 4K^2 x_i x_j dx_i dx_j}{(1+K|x|^2)^2} \\
   |df|^2 &= \frac{4K^2|x|^2}{(1+K|x|^2)^2} \\
   f_{ij} &= \frac{-2K\delta_{ij}(1+K|x|^2) + 4K^2 x_i x_j}{(1+K|x|^2)^2} = -\frac{2K\delta_{ij}}{1+K|x|^2} + \frac{4K^2 x_i x_j}{(1+K|x|^2)^2} \\
   \text{Hess}(f) &= -\frac{2K\sum_{i=1}^n dx_i^2}{1+K|x|^2} + \frac{4K^2\sum_{i,j=1}^n x_i x_j dx_i dx_j}{(1+K|x|^2)^2}
   \end{aligned}
   $$

Now we apply (2.157) so that

$$
\begin{aligned}
R_K &= e^{2f}\left(R - (-\frac{2K\sum_{i=1}^n dx_i^2}{1+K|x|^2} + \frac{4K^2\sum_{i,j=1}^n x_i x_j dx_i dx_j}{(1+K|x|^2)^2}) \circ g_0 + (\frac{\sum_{i,j=1}^n 4K^2 x_i x_j dx_i dx_j}{(1+K|x|^2)^2}) \circ g_0 - \frac{1}{2}\frac{4K^2|x|^2}{(1+K|x|^2)^2} g_0 \circ g_0\right) \\
&= e^{2f}\left(\frac{2K\sum_{i=1}^n dx_i^2}{1+K|x|^2} \circ g_0 - \frac{2K^2|x|^2}{(1+K|x|^2)^2} g_0 \circ g_0\right) \\
&= \frac{2}{(1+K|x|^2)^2} e^{2f}(K + K^2|x|^2 - K^2|x|^2) g_0 \circ g_0 \\
&= \frac{1}{2} K \frac{4}{(1+K|x|^2)^2} e^{2f} g_0 \circ g_0 \\
&= \frac{1}{2} K (e^{2f} g_0) \circ (e^{2f} g_0) \\
&= \frac{1}{2} K g_K \circ g_K
\end{aligned}
$$

Thus constant sectional curvature equals $K > 0$ from (2.156). □

2. $(\mathbb{R}^n, g_K)$ is not complete.

Proof. Consider the radial path

$$\gamma(t) := tv \qquad \text{where } v \in \mathbb{R}^n \text{ and } |v| = 1 \qquad \forall\, t \geq 0$$

Fix any $R > 0$, we compute the length

$$
\begin{aligned}
\ell(\gamma|_{[0,R]}) &= \int_0^R \sqrt{g_{K\,\gamma(t)}(\gamma'(t), \gamma'(t))}\, dt \\
&= \int_0^R \sqrt{g_{K\,tv}(v, v)}\, dt \\
&= \int_0^R \sqrt{\frac{4\sum_{i=1}^n v_i^2}{(1+Kt^2)^2}}\, dt \\
&= \int_0^R \frac{2}{1+Kt^2}\, dt \\
&= \frac{2}{\sqrt{K}} \arctan(\sqrt{K} R)
\end{aligned}
$$

Now

$$\lim_{R\to\infty} \ell(\gamma|_{[0,R]}) = \frac{2}{\sqrt{K}}\frac{\pi}{2} = \frac{\pi}{\sqrt{K}}$$

so we conclude the radial path has finite length. Thus consider the sequence

$$x_n := nv$$

we observe

$$
\begin{aligned}
d_K(x_n, x_m) &= \int_n^m \frac{2}{1+Kt^2}\, dt \\
&= \frac{2}{\sqrt{K}} \arctan(\sqrt{K} m) - \frac{2}{\sqrt{K}} \arctan(\sqrt{K} n) \to 0
\end{aligned}
$$

as n, $m \to \infty$ hence x_n is a Cauchy sequence. However this sequence diverges in $(\mathbb{R}^n, g_0)$, and in particular, since length of the radial path is finite, the sequence x_n does not converge to a point in $(\mathbb{R}^n, g_K)$. □

2.8.2.3 Ricci Curvature Deformation

Let Ric be Ricci Curvature of g and $\tilde{\text{Ric}}$ be Ricci of $\tilde{g}$.

Proposition 2.8.3. *Under conformal deformation $\tilde{g} = e^{2f} g$, we have*

$$\tilde{\text{Ric}} = \text{Ric} + \frac{n-2}{n-1}\left(df \otimes df - \text{Hess}(f) - |df|^2 g\right) - \frac{\Delta f}{n-1} g \tag{2.158}$$

Proof. Notice from definition of Ricci

$$(n-1)\tilde{\text{Ric}}(X,Y) \overset{(2.68)}{=} \tilde{g}^{k\ell}\tilde{R}(X,\frac{\partial}{\partial x_k},Y,\frac{\partial}{\partial x_\ell}) = e^{-2f}g^{k\ell}\tilde{R}(X,\frac{\partial}{\partial x_k},Y,\frac{\partial}{\partial x_\ell}) \tag{2.159}$$

In general if $S \in C^\infty(M, \text{Sym}^2 T^*M)$ then

$$\begin{aligned} g^{k\ell}(S\circ g)(X,\frac{\partial}{\partial x_k},Y,\frac{\partial}{\partial x_\ell}) &= S(X,Y)g^{k\ell}g(\partial_k,\partial_\ell) + g^{k\ell}S(\partial_k,\partial_\ell)g(X,Y) - g^{k\ell}S(X,\partial_k)g(Y,\partial_\ell) - g^{k\ell}S(Y,\partial_\ell)g(X,\partial_k) \\ &= nS(X,Y) + \text{tr}(S)g(X,Y) - S(X,Y) - S(X,Y) \\ &= (n-2)S(X,Y) + \text{tr}(S)g(X,Y) \qquad \text{since } g^{k\ell}g(Y,\partial_\ell) = Y^k \end{aligned}$$

Therefore writing

$$\tilde{R} = e^{2f}(R + U\circ g) \qquad \text{where } U = df\otimes df - \text{Hess}(f) - \frac{1}{2}|df|^2 g$$

One has

$$\begin{aligned} (n-1)\tilde{\text{Ric}} &= e^{-2f}g^{k\ell}e^{2f}\left(R(X,\partial_k,Y,\partial_\ell) + (U\circ g)(X,\partial_k,Y,\partial_\ell)\right) \\ &= (n-1)\text{Ric}(X,Y) + (n-2)U(X,Y) + \text{tr}(U)g(X,Y) \\ &= (n-1)\text{Ric}(X,Y) + (n-2)(df\otimes df - \text{Hess}(f) - \frac{1}{2}|df|^2 g)(X,Y) + (|df|^2 - \Delta f - \frac{n}{2}|df|^2)g(X,Y) \end{aligned}$$

notice

$$-\frac{1}{2}(n-2) + 1 - \frac{n}{2} = -n+2 = 2-n$$

Thus we have the formula (2.158). □

2.8.2.4 Scalar Curvature Deformation

Let S be Scalar Curvature of g and $\tilde{S}$ be Scalar Curvature of $\tilde{g}$.

Proposition 2.8.4. *Under conformal deformation $\tilde{g} = e^{2f}g$, we have*

$$\tilde{S} = e^{-2f}\left(S - \frac{n-2}{n}|df|^2 - \frac{2}{n}\Delta f\right) \tag{2.160}$$

Proof. We write

$$\begin{aligned} n\tilde{S} &\overset{(2.81)}{=} \tilde{g}^{k\ell}\tilde{\text{Ric}}_{k\ell} = e^{-2f}g^{k\ell}\left(\text{Ric}_{k\ell} + \frac{n-2}{n-1}(df\otimes df - \text{Hess}(f) - |df|^2 g)_{k\ell} - \frac{\Delta f}{n-1}g_{k\ell}\right) \\ &= e^{-2f}\left(nS + \frac{n-2}{n-1}(|df|^2 - \Delta f - n|df|^2) - \Delta f\frac{n}{n-1}\right) \\ &= e^{-2f}nS - (n-2)|df|^2 - 2\Delta f \qquad \text{since } -n+2-n = -2n+2 \end{aligned}$$

□

Yamabe Equation Notice this is Elliptic Problem if we want to give conditions on S. We further define for $n \geq 3$

$$\begin{aligned} u &= e^{\frac{n-2}{2}f} \\ e^{2f} &= (e^{\frac{n-2}{2}f})^{\frac{4}{n-2}} = u^{\frac{4}{n-2}} \\ \tilde{g} &= u^{\frac{4}{n-2}}g \\ \log u &= \frac{n-2}{2}f \\ f &= \frac{2}{n-2}\log(u) \\ df &= \frac{2}{n-2}\frac{du}{u} \\ |df|^2 &= \frac{4}{(n-2)^2}\frac{|du|^2}{u} \\ f_{;i} &= \frac{2}{n-2}\frac{u_{;i}}{u} \\ f_{;ij} &= \frac{2}{n-2}\left(\frac{u_{;ij}}{u} - \frac{u_{;i}u_{;j}}{u^2}\right) \\ \Delta_g f &= \frac{2}{n-2}\left(\frac{\Delta_g u}{u} - \frac{|du|^2}{u^2}\right) \end{aligned}$$

Hence

$$
\begin{aligned}
\tilde{S} &\overset{(2.160)}{=} u^{-\frac{4}{n-2}}\left(S - \frac{n-2}{n}\frac{4}{(n-2)^2}\frac{|du|^2}{u} - \frac{2}{n}\frac{2}{n-2}\left(\frac{\Delta_g u}{u} - \frac{|du|^2}{u^2}\right)\right)\\
&= u^{-\frac{4}{n-2}}\left(S - \frac{4}{n(n-2)}\frac{\Delta_g u}{u}\right)\\
&= u^{-\frac{4}{n-2}}u^{-1}\left(Su - \frac{4}{n(n-2)}\Delta_g u\right)\\
&= u^{-\frac{n+2}{n-2}}\left(Su - \frac{4}{n(n-2)}\Delta_g u\right)
\end{aligned}
$$

We derived the Yamabe Equation

$$
u^{\frac{n+2}{n-2}}\tilde{S} - Su + \frac{4}{n(n+2)}\Delta_g u = 0 \tag{2.161}
$$

Yamabe Conjecture We denote $\mathrm{scal} = n(n-1)S = g^{ik}g^{j\ell}R_{ijk\ell}$. Then

$$
u^{\frac{n+2}{n-2}}\frac{1}{n(n-1)}\tilde{\mathrm{scal}} - \frac{1}{n(n-1)}\mathrm{scal}u + \frac{4}{n(n-2)}\Delta_g u = 0
$$

$$
\frac{4(n-1)}{n-2}\Delta_g u - \mathrm{scal}u + \tilde{\mathrm{scal}}u^{\frac{n+2}{n-2}} = 0 \tag{2.162}
$$

Proposition 2.8.5 (Yamabe Conjecture)**.** *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. Then there exists a metric $\tilde{g}$ that is conformal to g, i.e.,*

$$
\tilde{g} = e^{2f}g \qquad \textit{for some } f \in C^\infty(M)
$$

s.t. it has constant scalar curvature $C = \tilde{\mathrm{scal}}$. In particular

$$
\frac{4(n-1)}{n-2}\Delta_g u - \mathrm{scal}u + Cu^{\frac{n+2}{n-2}} = 0 \qquad \textit{for } C \in \mathbb{R}
$$

Remark 2.8.1. *In $n = 2$, the uniformization theorem says that any compact manifold (M,g) is conformal to one that is constant sectional curvature.*

In the following we demonstrate a brief history of the proof for Yamabe Conjecture.
Consider the Einstein-Hilbert Action.

$$
\int_M R_g \mathrm{dvol}_g
$$

and the normalized Einstein-Hilbert Action

$$
\mathcal{E}(g) := \frac{\int_M R_g \mathrm{dvol}_g}{\mathrm{vol}(M,g)^{\frac{n-2}{2}}}
$$

so that

$$
\mathcal{E}(\lambda^2 g) = \mathcal{E}(g) \qquad \forall\, \lambda \in \mathbb{R}
$$

The critical points of Einstein-Hilbert Action are Einstein manifolds, i.e.

$$
\mathrm{Ric}(g) = \Lambda g
$$

We define the Yamabe Invariant

$$
Y(M,g) := \inf\{\mathcal{E}(\tilde{g}) \mid \tilde{g} \text{ conformal to } g\} = \inf\{\mathcal{E}(u^{\frac{4}{n-2}}g) \mid u \in C^\infty(M),\ u > 0\}
$$

Theorem 2.8.2 (Aubin)**.** *For $n = \dim M$*

$$
Y(M,g) \leq Y(\mathbb{S}^n, g_{can})
$$

One needs three theorems.

Theorem 2.8.3 (1976 Yamabe-Trudinger-Aubin)**.** *If $Y(M,g) < Y(\mathbb{S}^n, g_{can})$ for $n = \dim M$, then the Yamabe Conjecture 2.8.5 holds.*

Theorem 2.8.4 (Aubin)**.** *If (M,g) is of dimension ≥ 6 and not locally conformally flat. Then $Y(M,g) < Y(\mathbb{S}^n, g_{can})$ for $n = \dim M$.*

Theorem 2.8.5 (1984 Schoen)**.** *If (M,g) has dimension $3,4,5$ or (M,g) is locally conformally flat, then $Y(M,g) < Y(\mathbb{S}^n, g_{can})$ unless (M,g) is conformal to $(\mathbb{S}^n, g_{can})$.*

Combining above, the Yamabe Conjecture is proved.

2.8.3 Hyperbolic Space (D^n, h)

Recall the Hyperbolic Space.

Definition 2.8.5 (Unit Disc Model)**.** (D^n, h)

$$D^n := \{u \in \mathbb{R}^n \mid |u| < 1\} \qquad h := \frac{4\sum_{i=1}^n du_i^2}{(1-|u|^2)^2}$$

Definition 2.8.6 (Upper Half Space Model)**.** $(\mathcal{H}^n, g)$

$$\mathcal{H}^n := \{y \in \mathbb{R}^n \mid y_n > 0\} \qquad g := \frac{\sum_{i=1}^n dy_i^2}{y_n^2}$$

Completeness of (D^n, h) One needs the following lemma to show (D^n, h) is complete.

Lemma 2.8.2. *Let (M, g) be a Riemannian manifold and let $\sigma : M \to M$ be an isometry. Denote*

$$M^\sigma := \{x \in M \mid \sigma(x) = x\} \qquad \text{as the set of fixed points of } \sigma$$

Suppose M^σ is non-empty and is a submanifold of M. Then M^σ is a totally geodesic submanifold of M.

Proof. Since $M^\sigma \neq \varnothing$, there exists $p \in M^\sigma$. Due to local existence, for given $v \in T_pM^\sigma$, there exists $\varepsilon > 0$ s.t.

$$\gamma : [0, \varepsilon) \to M^\sigma \qquad \gamma(0) = p \qquad \gamma'(0) = v$$

is geodesic in M. Since $\sigma(x) = x$, and using that σ is an isometry

$$\sigma \circ \gamma(0) = \sigma(p) = p \qquad (\sigma \circ \gamma)'(0) = d\sigma_p \circ \gamma'(0) = d\sigma_p(v) = v$$

and thus by uniqueness of geodeiscs, σ fixes geodesics. Hence

$$\sigma \circ \gamma = \gamma \implies \gamma \subset M^\sigma$$

Thus γ is a geodesic in M^σ. Hence any geodesic in M starting in $p \in M^\sigma$, $v \in T_pM^\sigma$ remains a geodesic in M^σ. This is equivalent to say M^σ is a totally geodesic submanifold of M using Proposition 2.6.5. □

Proposition 2.8.6. *(D^n, h) is complete.*

Proof. By Hopf-Rinow 2.7.1, it suffices to show that $\exp_0$ is defined on the whole tangent space of T_0D^n. Note

$$h(0) = 4\sum_{i=1}^n du_i^2$$

Now we simplify it further by rotating. Given initial velocity at 0. For any $v \in T_0D^n$ unit length, there exists $A \in O(n)$ s.t.

$$Av = (\frac{1}{2}, 0, \cdots, 0) \qquad \text{and} \qquad A : D^n \to D^n \qquad \text{for } A \in O(n) \text{ is an isometry}$$

It suffices to show that the geodesic γ with $\gamma(0) = 0$ and $\gamma'(0) = (\frac{1}{2}, 0, \cdots, 0)$ is defined for all times. $\gamma'(0) = (\frac{1}{2}, 0, \cdots, 0)$ is to say

$$\gamma'(0) = \frac{1}{2}\left.\frac{\partial}{\partial u_1}\right|_0$$

Consider

$$\sigma : D^n \to D^n$$
$$(u_1, \cdots, u_n) \mapsto (u_1, -u_2, -u_3, \cdots, -u_n)$$

In other words $\sigma = \text{diag}(-1, 1, 1, \cdots, 1) \in O(n)$ is an isometry of D^n. Now the fixed points of σ are

$$(D^n)^\sigma = \{(u_1, 0, \cdots, 0) \in D^n \mid u_1 \in [-1, 1]\}$$

Since Lemma 2.8.2 says $(D^n)^\sigma$ is a totally geodesic submanifold. We need to prove that the geodesic lives forever. Denote

$$\beta : (-1, 1) \to D^n$$
$$t \mapsto (t, 0, \cdots, 0)$$

This is a curve β with the same image as a geodesic.

$$\beta'(t) = (1, 0, \cdots, 0) \qquad |\beta'(t)|^2 = \frac{4}{(1-t^2)^2} \qquad |\beta'(t)| = \frac{2}{1-t^2}$$

To reparametrize it in arc length

$$\begin{aligned} s(t_0) &= \int_0^{t_0} |\beta'(t)|\,dt = \int_0^{t_0} \frac{2}{1-t^2}\,dt = \int_0^{t_0} \left(\frac{1}{1+t} + \frac{1}{1-t}\right) dt = \log(1+t) - \log(1-t)|_0^{t_0} \\ &= \log(\frac{1+t_0}{1-t_0}) \\ e^{s(t_0)} &= \frac{1+t_0}{1-t_0} \\ t_0 &= \frac{e^{s(t_0)}-1}{e^{s(t_0)}+1} \\ t &= \frac{e^{s(t)}-1}{e^{s(t)}+1} = \tanh(\frac{s(t)}{2}) = \tanh(\frac{s}{2}) \end{aligned}$$

Now

$$\begin{aligned} \gamma : \mathbb{R} &\to D^n \\ s &\mapsto (t, 0, \cdots, 0) = (\tanh(\frac{s}{2}), 0, \cdots, 0) \end{aligned}$$

satisfies

$$\gamma(0) = 0 \qquad \gamma'(0) = (\frac{1}{2}, 0, \cdots, 0)$$

Hence the Disc Model is complete. □

Geodesics in (D^n, h) In general, the geodesic starting at 0 is given by

$$\begin{aligned} \exp_0 : T_0 D^n &\to D^n \\ \sum_i a_i \frac{\partial}{\partial u_i} &\mapsto \begin{cases} 0 & \vec{a} = 0 \\ \tanh(|\vec{a}|\frac{s}{2})\frac{\vec{a}}{|\vec{a}|} & \vec{a} \neq 0 \end{cases} \end{aligned}$$

Now we want to find geodesics on $\mathcal{H}^n$ where $\gamma(0) = p$ and $\gamma'(0) = v$. We need reduction to 2-dim. For $\vec{y} = (\xi, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ we want to define

$$\phi_{A,b}(\xi, t) := (A\xi + \vec{b}, t) \qquad A \in O(n-1), \qquad \vec{b} \in \mathbb{R}^{n-1} \qquad \text{is an isometry}$$

We may assume $p = (0, \cdots, 0, y)$ and $\vec{v} = (0, \cdots, 0, a, b)$. Now

$$\sigma = \begin{pmatrix} -I_{n-2} & 0 \\ 0 & 1 \end{pmatrix}$$

The fixed points are

$$(\mathcal{H}^n)^\sigma = \{(0, \cdots, 0, a, b) \in \mathcal{H}^n\} \cong (\mathcal{H}^2, \frac{dx^2 + dy^2}{y^2})$$

But for the $\mathcal{H}^2$ the isometries are given by as in Lemma 2.7.3

$$\mathrm{PSL}(2, \mathbb{R}) \cup \sigma\mathrm{PSL}(2, \mathbb{R})$$

where

$$\mathrm{PSL}(2, \mathbb{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \mid ad - bc = 1\}/\{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$$

and

$$\sigma : \mathcal{H}^2 \to \mathcal{H}^2 \qquad (x, y) \mapsto (-x, y)$$

2.8.4 Space Form

Definition 2.8.7 (Space Form)**.** *A Space form is a connected complete Riemannian Manifold with constant sectional curvature.*

Classification for Universal Cover of Complete Riemannian Manifold with Constant Sectional Curvature

Theorem 2.8.6 ([dC92] Theorem 8.4.1)**.** *Let (M^n, g) be a Space Form of Dimension n. Let $(\tilde{M}, \tilde{g})$ be its universal cover, i.e., $(\tilde{M}, \tilde{g})$ is simply connected, complete with constant sectional curvature.*
Then $(\tilde{M}, \tilde{g})$ is isometric to (up to rescaling $K_{\lambda^2 g} = \frac{1}{\lambda^2} K_g$) one of the following

1. $(\mathcal{H}^n, g)$ *with* $K = -1$
2. $(\mathbb{R}^n, g_0)$ *with* $K = 0$
3. $(\mathbb{S}^n, g_{can})$ *with* $K = 1$.

Local Isometry on connected manifold is determined by pointwise information In the proof we will need a lemma.

Lemma 2.8.3. *Given two Riemannian Manifolds (M, g), (N, h) where M is connected, and f_1, f_2*

$$f_1,\ f_2 : (M, g) \to (N, h)$$

are smooth maps, and local isometires. Also assume there exists $p \in M$ s.t.

$$f_1(p) = f_2(p)$$
$$(df_1)_p = (df_2)_p : T_pM \to T_{f_1(p)}N$$

Then $f_1 = f_2$.

Proof. Take the set

$$A := \{q \in M \mid f_1(q) = f_2(q),\ (df_1)_q = (df_2)_q\} \subset M$$

Notice

1. $A \neq \varnothing$ because $p \in A$.
2. A is closed by definition.

Now for any $q \in A$, there exists $r > 0$ s.t.

1. the exponential map
$$\exp_q : B_r(0) \subset T_qM \to B_r(q) \subset M \qquad \text{is a diffeomorphism}$$
2. f_1, f_2 maps isometrically $B_r(q)$ to $B_r(\tilde{q})$ where (this is guaranteed via f_1 and f_2 are local isometry)
$$\tilde{q} = f_1(q) = f_2(q)$$

But notice that

$$\exp_{\tilde{q}} \circ (df_1)_q = f_1 \circ \exp_q$$
$$\exp_{\tilde{q}} \circ (df_2)_q = f_2 \circ \exp_q$$

This is shown via the following: Take geodesic in M starting at q with velocity $v \in T_pM$, i.e., $\exp_q(tv)$, then since f_1 locally preserves geodesic as an isometry, $f_1 \circ \exp_q(tv)$ is a geodesic in N with

$$f_1 \circ \exp_q(0) = f_1(q) = \tilde{q}, \qquad (df_1)_q \circ d(\exp_q)_0(v) = (df_1)_q(v)$$

By uniqueness of geodesics with given initial data and velocity, one conclude

$$f_1 \circ \exp_q(tv) = \exp_{\tilde{q}} \circ (df_1)_q$$

Also since $q \in A$, assume their linear isometry takes the form

$$(df_1)_q = (df_2)_q = i : T_qM \to T_{\tilde{q}}N$$

So we have diagram that commutes

$$f_1 = \exp_{\tilde{q}} \circ i \circ (\exp_q)^{-1} = f_2 \qquad \text{on } B_r(q)$$

Thus $B_r(q) \subset A$ and so A is open. ∎

Proof of Theorem 2.8.6

Proof. 1. Case $K = -1, 0$. Let

$$\Delta_n := \begin{cases} \mathcal{H}^n & K = -1 \\ \mathbb{R}^n & K = 0 \end{cases}$$

Given two exponential maps

$$\begin{aligned} \exp_{\tilde{p}} &: T_{\tilde{p}}\tilde{M} \to \tilde{M} \\ \exp_p &: T_0\Delta_n \to \Delta_n \end{aligned}$$

Let any $\tilde{p} \in \tilde{M}$, and any point $p \in \Delta_n$, define any linear isometry

$$i : T_{\tilde{p}}\tilde{M} \to T_0\Delta$$

under the same setup as Cartan's Theorem. **Since $K \leq 0$, by Hadamard Theorem 2.7.2,** $\exp_{\tilde{p}}$ **and** $\exp_p$ **are diffeomorphisms**. Since K **is constant sectional curvature, by Cartan's Theorem 2.8.1**

$$f := \exp_p \circ i \circ \exp_{\tilde{p}}^{-1}$$

is an isometry.

2. Case $K = 1$. Again take any $\tilde{p} \in \tilde{M}$, $p \in \mathbb{S}^n$, construct a linear isometry

$$i : T_p\mathbb{S}^n \to T_{\tilde{p}}\tilde{M}$$

One has similar diagram for

$$\begin{aligned} \exp_p &: T_p\mathbb{S}^n \to \mathbb{S}^n \\ \exp_{\tilde{p}} &: T_{\tilde{p}}\tilde{M} \to \tilde{M} \end{aligned}$$

$$\begin{array}{ccccccc} T_p\mathbb{S}^n & \overset{open}{\supseteq} & B_r(0) & \xrightarrow{i} & B_r(0) & \overset{open}{\subseteq} & T_{\tilde{p}}\tilde{M} \\ & & \big\downarrow \exp_p & & \big\downarrow \exp_{\tilde{p}} & & \\ \mathbb{S}^n \setminus \{-p\} & \overset{open}{\supseteq} & B_r(p) & \xrightarrow{f} & B_r(\tilde{p}) & \overset{open}{\subseteq} & \tilde{M} \end{array}$$

Then we define

$$f = (\exp_{\tilde{p}}) \circ i \circ (\exp_p)^{-1} : \mathbb{S}^n \setminus \{-p\} \to \tilde{M}$$

Since K is constant, f is a local isometry.
Let's take another $p' \in \mathbb{S}^n \setminus \{\pm p\}$. Then define

$$i' \equiv df_{p'} : T_{p'}\mathbb{S}^n \to T_{f(p')}\tilde{M}$$

which is another linear isometry. Denote $\tilde{p'} := f(p')$. Let's define another f' s.t.

$$f' := (\exp_{\tilde{p'}}) \circ i' \circ (\exp_{p'})^{-1} : \mathbb{S}^n \setminus \{-p'\} \to \tilde{M}$$

This is another local isometry.

$$\begin{array}{ccccccc} T_{p'}\mathbb{S}^n & \overset{open}{\supseteq} & B_r(0) & \xrightarrow{i'} & B_r(0) & \overset{open}{\subseteq} & T_{\tilde{p}'}\tilde{M} \\ & & \big\downarrow \exp_{p'} & & \big\downarrow \exp_{\tilde{p}'} & & \\ \mathbb{S}^n \setminus \{-p'\} & \overset{open}{\supseteq} & B_r(p') & \xrightarrow{f'} & B_r(\tilde{p}') & \overset{open}{\subseteq} & \tilde{M} \end{array}$$

Notice

$$\begin{aligned} f(p') = f'(p') = \tilde{p'} \\ df_{p'} = df'_{p'} = i' \end{aligned}$$

Let's see this. Notice by definition $f(p') = \tilde{p}'$. On the other hand by construction

$$f'(p') = (\exp_{\tilde{p'}}) \circ i'(0) = \tilde{p}'$$

For differentials, again $i' = df_{p'}$ is by definition. And

$$df'_{p'} = d\left((\exp_{\tilde{p'}}) \circ i' \circ (\exp_{p'})^{-1}\right)_{p'} = \mathrm{Id} \circ i' \circ \mathrm{Id} = i'$$

Via Lemma 2.8.3

$$f = f' : \mathbb{S}^n \setminus \{-p, -p'\} \to \tilde{M}$$

How do we put the points back? Define

$$h(x) := \begin{cases} f(x) & x \in \mathbb{S}^n \setminus \{-p\} \\ f'(x) & x \in \mathbb{S}^n \setminus \{-p'\} \end{cases}$$

Then clearly h is a local isometry. Now h is surjective, $h(\mathbb{S}^n)$ is closed, nonempty, and also open by completeness of $\tilde{M}$. Thus $h(\mathbb{S}^n) = \tilde{M}$. Hence h is an isometry using Lemma 2.7.5.

□

Classification of Space Form via Group Theory Now by the Theorem 2.8.6, if we're given (M^n, g) complete with constant sectional curvature either $0, \pm 1$, (M, g) is isometric to $(\tilde{M}/\Gamma, \hat{g})$ where $\tilde{M}$ is either $\mathcal{H}^n$, $\mathbb{R}^n$ or $\mathbb{S}^n$, and Γ is a subgroup of discrete isometries acting in a fully discontinuous way.

We know that M/Γ has a smooth structure in which the projection $\pi : M \to M/\Gamma$ is a local diffeomorphism.

One can put a Riemannian metric $\hat{g}$ on M/Γ s.t. it is the only metric s.t.

$$(M, \tilde{g}) \to (M/\Gamma, \hat{g}) \qquad \text{is a local isometry}$$

Indeed, for any $p \in M/\Gamma$, choose $\tilde{p} \in \pi^{-1}(p)$. For every pair of $u, v \in T_p(M/\Gamma)$ define

$$\langle u, v \rangle_p := \langle d\pi^{-1}(u), d\pi^{-1}(v) \rangle_{\tilde{p}}$$

This is *metric on M/Γ induced by the covering π*.

Observe that M/Γ is complete iff M is complete. M/Γ has constant curvature iff M has constant curvature.

Taking $M = \mathbb{S}^n$ or $\mathbb{R}^n$ or $\mathcal{H}^n$ we conclude that M/Γ is a complete manifold of constant sectional curvature 1 (if $M = \mathbb{S}^n$), 0 (if $M = \mathbb{R}^n$) or -1 (if $M = \mathcal{H}^n$).

Proposition 2.8.7 ([dC92] Proposition 8.4.3)**.** *Let M be a complete Riemannian manifold with constant section curvature $K = 0, \pm 1$. Then M is isometric to $\tilde{M}/\Gamma$, where*

1. *For $K = 1$, $\tilde{M} = \mathbb{S}^n$*
2. *For $K = 0$, $\tilde{M} = \mathbb{R}^n$*
3. *For $K = -1$, $\tilde{M} = \mathcal{H}^n$*

and Γ is a subgroup of the group of isometries of $\tilde{M}$ which acts in a totally discontinuous manner on $\tilde{M}$. The metric on $\tilde{M}/\Gamma$ is induced from the covering $\pi : \tilde{M} \to \tilde{M}/\Gamma$.

We remark

$$\mathrm{Isom}(\mathcal{H}^n, g) \cong O(n, 1)$$

since one can realize $(\mathcal{H}^n, g)$ as a submanifold of $(\mathbb{R}^{n,1}, -dx_0^2 + dx_1^2 + \cdots dx_n^2)$, where

$$O(n, 1) = \{A \in \mathrm{GL}(n+1, \mathbb{R}) \mid A^T \eta A = \eta, \quad \eta = \mathrm{diag}(-1, 1, \cdots, 1)\}$$

And

$$\mathrm{Isom}(\mathbb{R}^n, g_0) \cong O(n) \rtimes \mathbb{R}^n \qquad x \mapsto Ax + \vec{b}$$
$$\mathrm{Isom}(\mathbb{S}^n, g_{\mathrm{can}}) \cong O(n+1) = \{A \in \mathrm{GL}(n+1, \mathbb{R}) \mid A^T A = I\}$$

In the following, we reduce the problem of finding all space forms to the problem of determining all subgroups of the group of isometries that act in a totally discontinuous manner on each of the simply connected models.

We demonstrate facts about the Sphere Problem.

Proposition 2.8.8 ([dC92] Proposition 8.4.4)**.** *M^n complete Riemannian manifold with $K = 1$, $n = 2m$. Then M^n is isometric to $\mathbb{S}^n$ or $\mathbb{R}P^n = \mathbb{S}^n/\{\pm 1\}$.*

Proof. $M \cong \tilde{M}/\Gamma = \mathbb{S}^n/\Gamma$ for $\Gamma \subset O(n+1) = O(2m+1)$ discrete subgroup. Let $\gamma \in \Gamma$ be its eigenvalue

$$\{e^{-i\theta_1}, e^{i\theta_1}, \cdots, e^{-i\theta_k}, e^{i\theta_k}, 1, 1, \cdots, 1, -1, \cdots, -1\} \qquad 2k + r + s = 2m$$

and that $\det \gamma = (-1)^s$.

1. If $r > 0$ then there exists $x \in \mathbb{S}^n$ s.t. $\gamma(x) = x$ so upon free group action, $\gamma = \mathrm{id}$. Then $M \cong \mathbb{S}^n$.
2. If $r = 0$, γ^2 has eigenvalues

$$\{e^{-2i\theta_1}, e^{2i\theta_1}, \cdots, e^{-2i\theta_k}, e^{2i\theta_k}, 1, 1, \cdots, 1\}$$

   Thus by first case, $\gamma^2 = \mathrm{id}$ so eigenvalues $\{-1, \cdots, -1\}$ and $\gamma = -\mathrm{id}$. Thus $M \cong \mathbb{S}^n \setminus \{\pm 1\}$.

□

Remark 2.8.2. *If n is odd, there are some more possibilities, For example $\mathbb{S}^3 \setminus \mathbb{Z}_q$ lens space has $K = 1$.*

2.8.5 Conformal Mappings and Liouville Theorem

2.8.5.1 Conformal Mapping

Conformal Mapping between Inner Product Spaces

Definition 2.8.8. *Let V, W be finite dimensional vector spaces equipped with an inner product. We say that a linear map $L : V \to W$ is a linear conformal map if*

1. *L is a linear isomorphism*
2. *and the angles are preserved, i.e.*

$$\frac{\langle L(v_1), L(v_2)\rangle_W}{|L(v_1)|_W |L(v_2)|_W} = \frac{\langle v_1, v_2\rangle_V}{|v_1|_V |v_2|_V} \qquad \forall\, v_1,\, v_2 \in V \setminus \{0\}$$

   i.e., $\cos($*angle between* $L(v_1)$ *and* $L(v_2)) = \cos($*angle between* v_1 *and* $v_2)$.

Lemma 2.8.4. *Let $L : V \to W$ be a linear isomorphism. Then the followings are equivalent*

1. *L is a conformal map.*
2. *There exists $\lambda \in \mathbb{R}_+$ s.t. $|L(v)|_W = \lambda |v|_V$ for any $v \in V$.*
3. *There exists $\lambda \neq 0$ s.t.*

$$\langle L(v), L(w)\rangle_W = \lambda^2 \langle v, w\rangle_V \qquad \forall\, v,\, w \in V \tag{2.163}$$

Conformal Mapping between Riemannian Manifold

Definition 2.8.9 (Conformal Map)**.** *Let (M, g), (N, h) be two Riemannian manifolds. A C^∞ function $f : M \to N$ map is conformal w.r.t. g and h if for any $p \in M$*

$$df_p : T_pM \to T_{f(p)}N$$

is a linear conformal map.

By Lemma 2.8.4, f is a conformal map iff

$$\begin{cases} f \text{ is a local diffeomorphism} \\ f^*h = \lambda^2 g \end{cases} \tag{2.164}$$

Here the non-vanishing function

$$\lambda : M \to (0, \infty)$$

C^∞ is called the conformal factor.

Remark 2.8.3. *A local Isometry is a conformal map with $\lambda = 1$. In fact*

$$\textit{local isometry } f^*h = g \implies \textit{conformal map } f^*h = \lambda^2 g \implies \textit{local diffeomorphism}$$

The converse is not true unless $n = 1$.

2.8.5.2 Liouville Theorem

Let's begin by observing examples of conformal Mappings.

Dilations For $\lambda > 0$

$$\begin{aligned} f : \mathbb{R}^n &\to \mathbb{R}^n \\ x &\mapsto \lambda x \end{aligned}$$

Then

$$f^* g_0 = f^*(dx_1^2 + \cdots dx_n^2) = \lambda^2(dx_1^2 + \cdots + dx_n^2) = \lambda^2 g_0$$

and for any $x \in \mathbb{R}^n$

$$df_x : \mathbb{R}^n \to \mathbb{R}^n$$

is represented by λI_d. f is an orientation-preserving conformal map of $\mathbb{R}^n$.

Inversion For any fixed $x_0 \in \mathbb{R}^n$ consider

$$\begin{aligned} f : \mathbb{R}^n \setminus \{x_0\} &\to \mathbb{R}^n \setminus \{x_0\} \\ x &\mapsto \frac{x - x_0}{|x - x_0|^2} + x_0 \end{aligned}$$

s.t.

$$|f(x) - x_0| \cdot |x - x_0| = 1$$

Here $\frac{x-x_0}{|x-x_0|}$ gives the direction of $\vec{x} - \vec{x_0}$ and $\frac{1}{|x-x_0|}$ gives the length.
Now for any $v \in T_x(\mathbb{R}^n \setminus \{0\}) = T_x\mathbb{R}^n \cong \mathbb{R}^n$

$$\begin{aligned} (df_x)(v) &= \frac{v|x - x_0|^2 - (x - x_0)2\langle v, x - x_0\rangle}{|x - x_0|^4} \\ &= \frac{1}{|x - x_0|^2}\left(v - \frac{2\langle v, x - x_0\rangle}{|x - x_0|^2}(x - x_0)\right) \end{aligned} \tag{2.165}$$

Taking the square

$$\begin{aligned} |df_x(v)|^2 &= \frac{1}{|x - x_0|^4}\left(|v|^2 - \frac{4\langle v, x - x_0\rangle}{|x - x_0|^2}\langle v, x - x_0\rangle + \frac{4\langle v, x - x_0\rangle^2}{|x - x_0|^4}|x - x_0|^2\right) \\ &= \frac{1}{|x - x_0|^4}|v|^2 \end{aligned}$$

Hence using (2.164), f is a conformal map with

$$f^* g_0 = \frac{1}{|x - x_0|^4} g_0$$

From the formula of inversion (2.165),

1. if $\langle v, x - x_0\rangle = 0$ then
$$df_x(v) = \frac{1}{|x - x_0|^2} v$$
2. If $v \in \mathbb{R}(x - x_0)$ in the span, say $v = \xi(x - x_0)$ then
$$df_x(v) = \frac{1}{|x - x_0|^2}\left(\xi(x - x_0) - \frac{2\langle \xi(x - x_0), x - x_0\rangle(x - x_0)}{|x - x_0|^2}\right) = -\xi\frac{x - x_0}{|x - x_0|^2} = -\frac{1}{|x - x_0|^2} v$$

So $df_x : \mathbb{R}^n \to \mathbb{R}^n$ is represented by

$$\frac{1}{|x - x_0|^2}\begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Hence f is an orientation reversing conformal map.

Liouville Theorem

Theorem 2.8.7 (Liouville)**.** *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a conformal map with respect to g_0, $n \geq 3$. Let U be connected.*

Then f is the restriction to U of F where F is a composition of isometries, dilation and inversion, at most one of each.

Isometries of $\mathcal{H}^n$

Theorem 2.8.8. *The isometries of $\mathcal{H}^n$ are restrictions to $\mathcal{H}^n \subset \mathbb{R}^n$ of the conformal transformations of $\mathbb{R}^n$ that take $\mathcal{H}^n$ into itself for $n \geq 2$.*

2.8.5.3 Counter-examples of Liouville Theorem in Dimensions $n = 1, 2$

Dimension $n = 1$ Consider a conformal mapping in $n = 1$

$$\begin{aligned} f : (a, b) &\to (\mathbb{R}, dx^2) \\ x &\mapsto f(x) \end{aligned}$$

It is a diffeomorphism, hence $f'(x) \neq 0$. Then

$$f^* g_0 = f^*(dx^2) = (f'(x))^2 dx^2$$

is a conformal map with conformal factor $f'(x)$. In particular, a local diffeomorphism is always a conformal map in dimension 1.

Dimension $n = 2$ Consider a conformal mapping in $n = 2$

$$\begin{aligned} f : U \subset \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x, y) &\mapsto f(x, y) = (u(x, y), v(x, y)) \end{aligned}$$

Now the differential

$$df_{(x,y)} : \mathbb{R}^2 \to \mathbb{R}^2$$

is represented by metric

$$df_{(x,y)} = \begin{pmatrix} \frac{\partial u}{\partial x}(x, y) & \frac{\partial u}{\partial y}(x, y) \\ \frac{\partial v}{\partial x}(x, y) & \frac{\partial v}{\partial y}(x, y) \end{pmatrix}$$

If f is conformal, then necessarily $\det(df_{(x,y)}) \neq 0$. We have two cases

1. If $\det(df_{(x,y)}) > 0$ we have model $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Then f is orientation preserving.

$$df_x = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

If we satisfy the Cauchy-Riemann Equations

$$\begin{aligned} u_x &= v_y \\ v_x &= -u_y \end{aligned}$$

Then $\det(df_{(x,y)}) > 0$. This means if we construct $w(z) = w(x + iy) = u(x, y) + iv(x, y)$, then

$$\frac{\partial}{\partial \bar{z}} w = 0 \qquad \text{where} \qquad \frac{\partial}{\partial \bar{z}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

This corresponds to f being holomorphic. It doesn't have to be composition of isometries, dilations or inversions.

2. If $\det(df_{(x,y)}) < 0$ we have model $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$. Then f is orientation reversing. We want

$$\begin{aligned} u_x &= -v_y \\ u_y &= v_x \end{aligned}$$

This corresponds to

$$\frac{\partial f}{\partial z} = 0 \qquad \text{where} \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

hence f is anti-holomorphic.

However the group generated by isometries, dilations and inversions in $\mathbb{R}^2$ is given by

$$\mathrm{PSL}(2,\mathbb{C}) \cup \sigma\mathrm{PSL}(2,\mathbb{C})$$

where

$$\mathrm{PSL}(2,\mathbb{C}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}) \mid ad - bc = 1\}/\{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\} \qquad z \mapsto \frac{az+b}{cz+d}$$

and

$$\sigma(z) := -\overline{z}$$

In complex coordinates,

1. Isometries of $\mathbb{R}^2$ are

$$\mathbb{R} \ltimes O(2) = \mathbb{R} \ltimes SO(2) \bigsqcup \mathbb{R} \ltimes O(2) \qquad \{z \mapsto e^{i\theta} z + z_0\} \cup \{z \mapsto e^{i\theta}\overline{z} + \overline{z_0}\}$$

2. Dilations are of the form $z \mapsto \lambda z$ for $\lambda > 0$

3. Inversion w.r.t. $z_0 \in \mathbb{C}$ are of the form

$$z \mapsto z_0 + \frac{z - z_0}{|z - z_0|^2} = z_0 + \frac{1}{\overline{z} - \overline{z_0}}$$

2.9 Variations of Energy

Variation of a curve Consider Riemannian manifold (M, g).

Definition 2.9.1. *Let*

$$c : [0, a] \to M$$

be a piecewise smooth curve. A variation of c is a continuous map

$$\begin{aligned} f : (-\varepsilon, \varepsilon) \times [0, a] &\to M \\ (s, t) &\mapsto f(s, t) \end{aligned}$$

s.t.

1. *$f(0, t) = c(t)$ for any $t \in [0, a]$.*
2. *There exists a subdivision of $[0, a]$ by points*

$$0 = t_0 < t_1 < \cdots < t_{k+1} = a$$

s.t. the restriction of f to each $(-\varepsilon, \varepsilon) \times [t_i, t_{i+1}]$ for $i = 0, \cdots, k$ is smooth.

A variation f is said to be proper if both endpoints are held fixed

$$f(s, 0) = c(0), \qquad f(s, a) = c(a) \qquad \forall\ s \in (-\varepsilon, \varepsilon)$$

For each $s \in (-\varepsilon, \varepsilon)$, the parametrized curve

$$\begin{aligned} f_s : [0, a] &\to M \\ t &\mapsto f_s(t) := f(s, t) \end{aligned}$$

is called a curve in the variation f. Thus a variation determines a family $f_s(t)$ of neighboring curves of $f_0(t) = c(t)$. A variation is proper iff the curves of this family have same initial point $c(0)$ and same endpoint $c(a)$.

Transversal Curve and Variational Field. On the other hand, if hold t fixed but vary in $s \in (-\varepsilon, \varepsilon)$

$$\begin{aligned} f_t : (-\varepsilon, \varepsilon) &\to M \\ s &\mapsto f_t(s) := f(s, t) \end{aligned}$$

is called a transversal curve of the variation.

Definition 2.9.2. *The piecewise smooth velocity vector field of a transversal curve at $s = 0$ along $c(t)$*

$$\begin{aligned} V : [0, a] &\to M \\ t &\mapsto \left.\frac{d}{ds}\right|_{s=0} f_t(s) = f_t'(0) = \frac{\partial f}{\partial s}(0, t) \end{aligned} \tag{2.166}$$

is called the variational field of f.

If f is proper, then

$$V(0) = V(a) = 0$$

Construction of Variation In fact, given V a piecewise smooth vector field along a curve c, one may recover a variation whose variational field coincides with V.

Proposition 2.9.1 ([dC92] Proposition 9.2.2)**.** *Let*

$$c : [0, a] \to M$$

be piecewise C^∞ curve, and $V(t) \not\equiv 0$ be piecewise C^∞ vector field along c.

Then there exists variation

$$f : (-\varepsilon, \varepsilon) \times [0, a] \to M$$

of c s.t.

$$V(t) = \frac{\partial f}{\partial s}(0, t)$$

If in addition that $V(0) = V(a) = 0$, it is possible to choose f as proper variation.

Proof. By compactness of $[0, a]$, there exists universal $\delta > 0$ s.t.

$$\exp_{c(t)}(v)$$

is defined for $|v| < \delta$, $v \in T_{c(t)}M$, for any $t \in [0, a]$. Now we want to consider

$$\varepsilon := \frac{\delta}{\max_{t\in[0,a]} |V(t)|}$$

Define

$$\begin{aligned} f : (-\varepsilon, \varepsilon) \times [0, a] &\to M \\ (s, t) &\mapsto \exp_{c(t)}(sV(t)) \end{aligned} \tag{2.167}$$

This is well-defined because $|sv(t)| < \delta$ for $|s| < \varepsilon$. Now verify

$$f(0, t) = \exp_{c(t)}(0) = c(t)$$

and

$$\frac{\partial f}{\partial s}(0, t) = (d \exp_{c(t)})_0(V(t)) \overset{(2.43)}{=} V(t)$$

In addition, if $V(0) = V(a) = 0$, then

$$f(s, 0) = \exp_{c(0)}(sV(0)) = \exp_{c(0)}(0) = c(0)$$

and

$$f(s, a) = \exp_{c(a)}(sV(a)) = \exp_{c(a)}(0) = c(a)$$

Thus f is proper. □

Notice that, in the construction of (2.167), one did not specify any s^2 order information of V. Thus the construction is not unique. One may alter any second order information in the definition of f to recover the same variational field.

Arc Length and Energy Functional We want to compare the arc length of c with the length of neighboring curves in a variation of c

$$f : (-\varepsilon, \varepsilon) \times [0, a] \to M$$

To do so, we define

$$\begin{aligned} L : (-\varepsilon, \varepsilon) &\to \mathbb{R} \\ s &\mapsto \int_0^a |\frac{\partial f}{\partial t}(s, t)|dt \end{aligned}$$

$L(s)$ is the length of the curve $f_s(t)$.

One prefer to work with the energy function of f given by

$$\begin{aligned} E : (-\varepsilon, \varepsilon) &\to \mathbb{R} \\ s &\mapsto \int_0^a |\frac{\partial f}{\partial t}(s, t)|^2 dt \end{aligned} \tag{2.168}$$

Let's justify why E is better to work with. Consider

$$c : [0, a] \to M$$

a curve and denote

$$L(c) = \int_0^a |c'(t)|dt$$
$$E(c) = \int_0^a |c'(t)|^2 dt$$

Now by Cauchy-Schwarz

$$L(c) = \int_0^a |c'(t)|dt \le \sqrt{a} \left(\int_0^a |c'(t)|^2 dt \right)^{\frac{1}{2}}$$
$$L(c)^2 \le a \cdot E(c)$$

Now the equality takes place iff $|c'(t)|$ is constant, i.e., t is proportional to arc length.

In fact, curves which minimize energy are automatically parametrized by parameter proportional to arc length. This is one of the advantages of working with the energy function rather than arc length function.

Lemma 2.9.1 ([dC92] Lemma 9.2.3)**.** *Let* p, $q \in M$*. Let*

$$\gamma : [0, a] \to M$$

be a minimizing geodesic joining p *to* q*.*

Then for all curves

$$c : [0, a] \to M$$

joining p *to* q*, one has*

$$E(\gamma) \le E(c)$$

with equality iff c *is a minimizing geodesic.*

Proof. For a minimizing geodesic, curve velocity is constant, thus

$$a^2 E(\gamma) = L(\gamma)^2 \le L(c)^2 \le a^2 E(c)$$

If equality holds, necessarily the parametrization of c is proportional to arc length. But also $L(\gamma) = L(c)$, which implies c is itself a minimizing geodesic. □

2.9.1 Formula for First and Second Variations

2.9.1.1 First Variation

Proposition 2.9.2 (Formula for First Variation of Energy; [dC92] Proposition 9.2.4)**.** *Let*

$$c : [0, a] \to M$$

be piecewise smooth curve and let

$$f : (-\varepsilon, \varepsilon) \times [0, a] \to M$$

be variation of c*. Let* E (2.168) *be energy of* f*.*

$$\frac{1}{2} \frac{d}{ds}\bigg|_{s=0} E(s) = -\int_0^a \langle V(t), \frac{D}{dt}\frac{dc}{dt}\rangle dt + \sum_{i=1}^{k} \langle V(t_i), \frac{dc}{dt}(t_i^-) - \frac{dc}{dt}(t_i^+)\rangle + \langle V(a), \frac{dc}{dt}(a)\rangle - \langle V(0), \frac{dc}{dt}(0)\rangle \quad (2.169)$$

where $V(t)$ (2.166) *is the variational field of* f*, and*

$$\frac{dc}{dt}(t_i^+) = \lim_{t \searrow t_i} \frac{dc}{dt}(t), \qquad \frac{dc}{dt}(t_i^-) = \lim_{t \nearrow t_i^-} \frac{dc}{dt}(t)$$

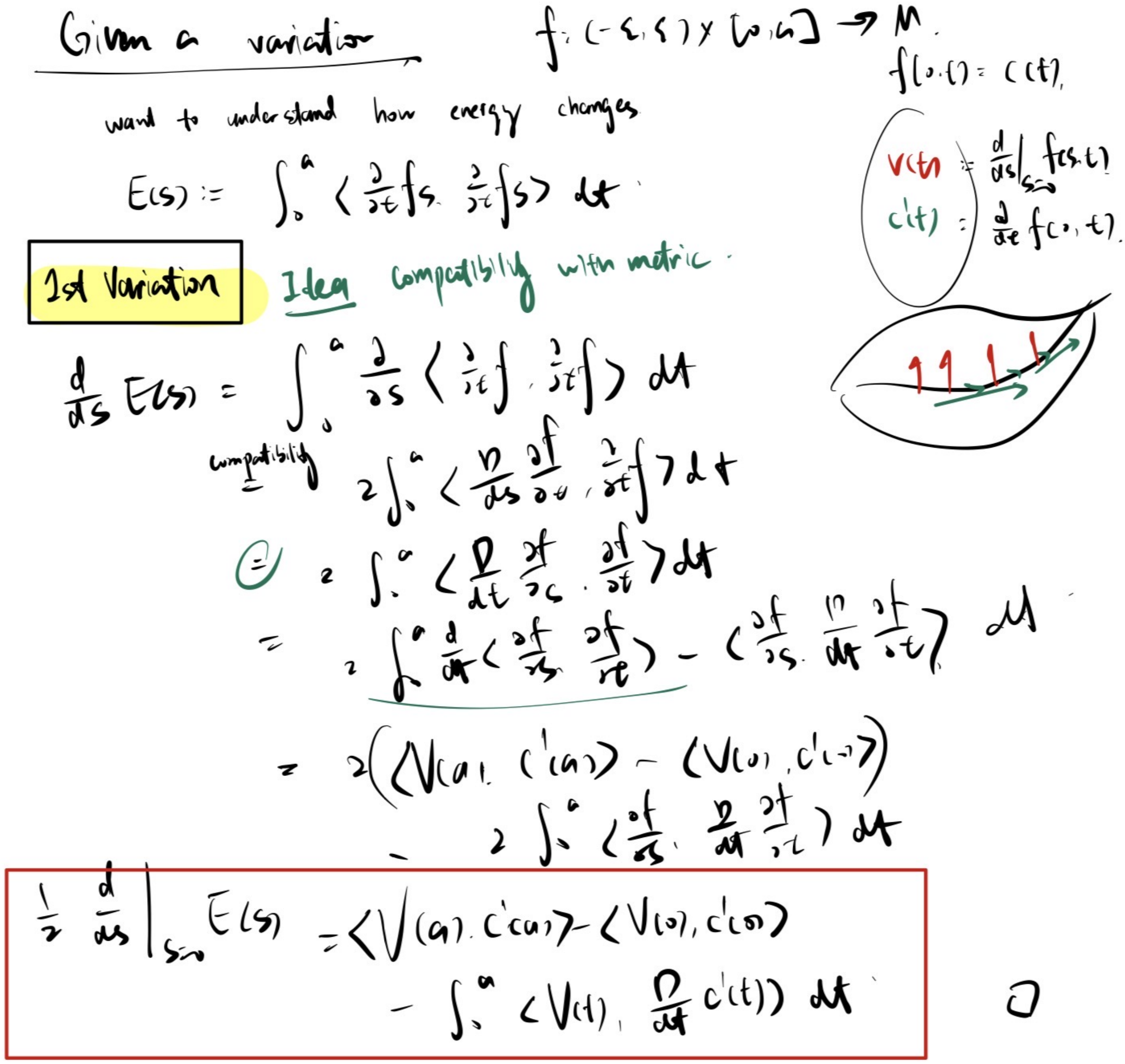


Figure 2.14: First Variation for smooth curves

Proof. Compute

$$E(s) = \int_0^a \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \rangle dt = \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \rangle dt$$

So

$$\begin{aligned}
\frac{d}{ds}\left(\int_{t_i}^{t_{i+1}} \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \rangle dt \right) &= 2\int_{t_i}^{t_{i+1}} \langle \frac{D}{ds}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \rangle dt \overset{(2.44)}{=} 2\int_{t_i}^{t_{i+1}} \langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle dt \\
&= 2\int_{t_i}^{t_{i+1}} \left(\frac{d}{dt}\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle - \langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t} \rangle \right) dt \\
&= 2 \left. \langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle \right|_{t_i^+}^{t_{i+1}^-} - 2\int_{t_i}^{t_{i+1}} \langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t} \rangle dt
\end{aligned}$$

Thus

$$\frac{1}{2}E'(s) = \sum_{i=0}^{k} \left. \langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle \right|_{t_i^+}^{t_{i+1}^-} - \int_0^a \langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t} \rangle dt \tag{2.170}$$

Now we evaluate at $s = 0$. In particular

$$\left.\frac{\partial f}{\partial t}\right|_{s=0} = c'(t)$$
$$\left.\frac{\partial f}{\partial s}\right|_{s=0} = V(t)$$

So

$$\frac{1}{2}E'(0) = \sum_{i=0}^{k} \left(\langle V(t_{i+1}^-), c'(t_{i+1}^-)\rangle - \langle V(t_i^+), c'(t_i^+)\rangle\right) - \int_0^a \langle V(t), \frac{D}{dt}\frac{dc}{dt}\rangle dt$$

Rearranging and using $f(s,t)$ is smooth in s, continuous in t to yield (2.169). □

Geodesics as critical points of E In fact critical points of E are geodesics.

Proposition 2.9.3 ([dC92] Proposition 9.2.5)**.** *A piecewise smooth curve*

$$c : [0,a] \to M$$

is a geodesic iff for every proper variation f of c, we have

$$\left.\frac{d}{ds}\right|_{s=0} E(s) = 0$$

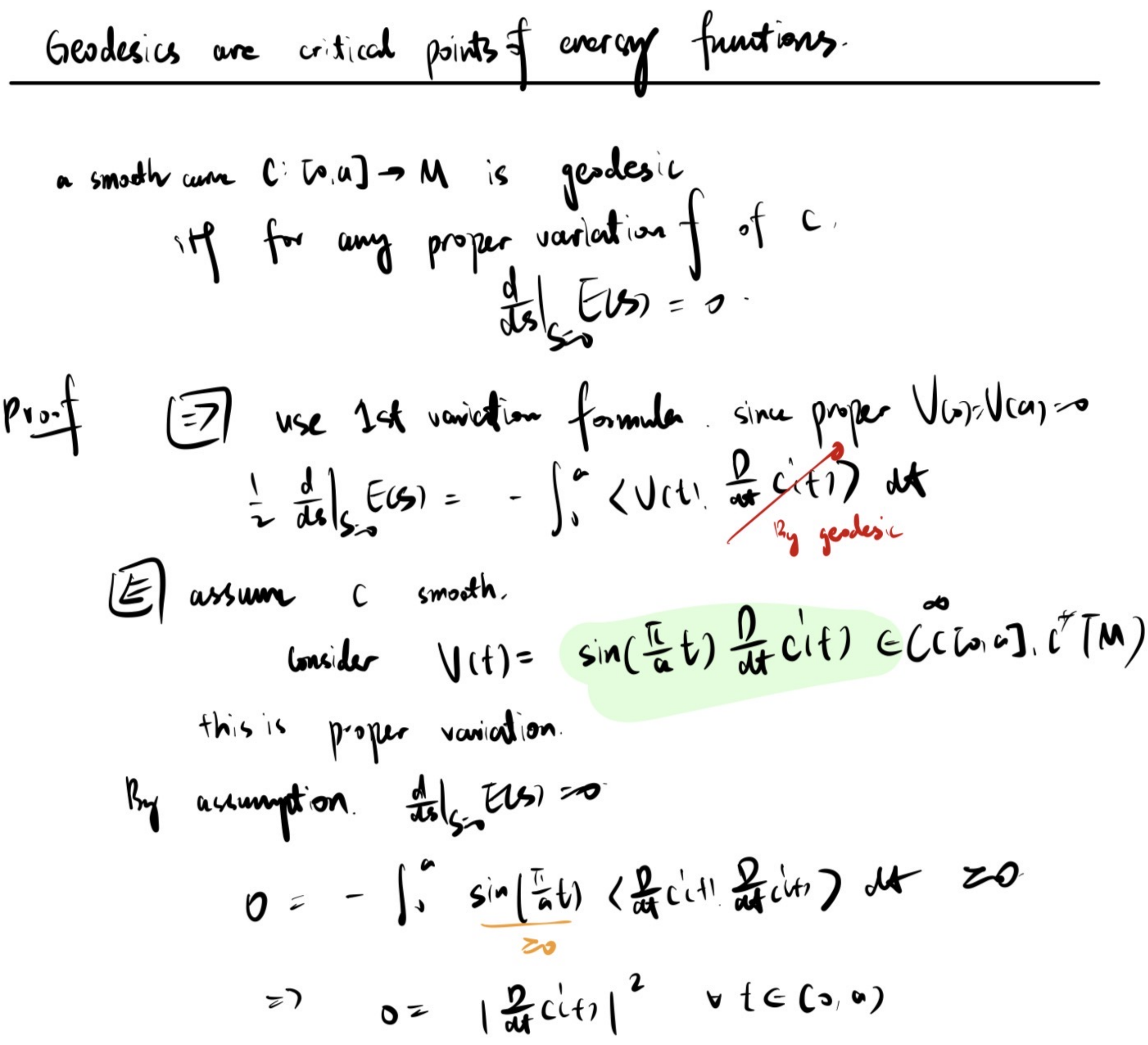


Figure 2.15: Geodesics as Critical Points to Energy

Proof. ($\Longrightarrow$) For any proper variation f of c,

$$V(0) = V(a) = 0$$

If c is a geodesic, then

$$\begin{aligned}
&\frac{1}{2}E'(0) \\
&\overset{(2.169)}{=} - \underbrace{\cancel{\int_0^a \langle V(t), \frac{D}{dt}\frac{dc}{dt}\rangle dt}}_{c \text{ is geodesic so } \frac{D}{dt}\frac{dc}{dt}=0} + \sum_{i=1}^{k} \langle V(t_i), \underbrace{\cancel{\frac{dc}{dt}(t_i^-) - \frac{dc}{dt}(t_i^+)}}_{\text{geodesics are smooth}} \rangle + \underbrace{\cancel{\langle V(a), \frac{dc}{dt}(a)\rangle - \langle V(0), \frac{dc}{dt}(0)\rangle}}_{f \text{ is proper so } V(0)=V(a)=0} \\
&= 0
\end{aligned}$$

($\Longleftarrow$) We consider the following

1. Let

$$V(t) = g(t)\frac{D}{dt}\frac{dc}{dt}$$

be C^∞ smooth curve with

$$g(t) = \sin\left(\frac{\pi(t - t_i)}{t_{i+1} - t_i}\right) \qquad \forall\, t \in [t_i, t_{i+1}]$$

In particular g vanishes at each t_i. Hence

$$V(0) = V(a) = V(t_i) = 0$$

Then V is the variational field of a proper variation of c. By our assumption

$$E'(0) = 0$$

By our first variation formula (2.169)

$$\begin{aligned}
0 &= \frac{1}{2}E'(0) = -\int_0^a \langle V(t), \frac{D}{dt}\frac{dc}{dt}\rangle dt = -\int_0^a g(t)|\frac{D}{dt}\frac{dc}{dt}|^2 dt \\
&= -\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}} g(t)|\frac{D}{dt}\frac{dc}{dt}|^2 dt \geq 0 \\
0 &= \int_{t_i}^{t_{i+1}} g(t)|\frac{D}{dt}\frac{dc}{dt}|^2 dt \qquad \forall\, i \\
0 &= g(t)|\frac{D}{dt}\frac{dc}{dt}|^2 \qquad \forall\, t \in [t_i, t_{i+1}] \\
0 &= \frac{D}{dt}\frac{dc}{dt} \qquad \forall\, t \in (t_i, t_{i+1})
\end{aligned}$$

In particular, c must be a piecewise geodesic. So in particular, for any V s.t. $V(0) = V(a) = 0$ we have

$$\frac{1}{2}E'(0) = \sum_{i=1}^{k} \langle V(t_i), \frac{dc}{dt}(t_i^-) - \frac{dc}{dt}(t_i^+)\rangle = 0$$

2. We choose $\overline{V}(t)$ such that

$$\overline{V}(0) = \overline{V}(a) = 0$$

and $\overline{V}(t_i) = \frac{dc}{dt}(t_i^-) - \frac{dc}{dt}(t_i^+)$. It has an associated proper variation satisfying

$$\begin{aligned}
0 &= \frac{1}{2}E'(0) = \sum_{i=1}^{k} |\frac{dc}{dt}(t_i^-) - \frac{dc}{dt}(t_i^+)|^2 \\
\frac{dc}{dt}(t_i^-) &= \frac{dc}{dt}(t_i^+)
\end{aligned}$$

Hence c is smooth. Because c is smooth and $\frac{D}{dt}\frac{dc}{dt} = 0$ we know c is a geodesic.

□

$\Omega_{p,q}$ **Interpretation** This is [dC92] Remark 9.2.7.

Now one has characterization of geodesics as critical points of the energy for all proper variations. It is in this sense that geodesics can be thought of as solutions to a variational problem.

Let (M, g) be a Riemannian manifold, $p, q \in M$. Denote

$$\Omega_{p,q} := \{c : [0,1] \to M \mid \text{piecewise } C^\infty,\ c(0) = p \text{ and } c(1) = q\}$$

Now points in the manifold $\Omega_{p,q}$ are piecewise smooth curves with endpoints p and q.

Pick any $c \in \Omega_{p,q}$. f as proper variation of c is thus viewed as

$$\begin{aligned} f : (-\varepsilon, \varepsilon) &\to \Omega_{p,q} \\ s &\mapsto f_s \end{aligned}$$

s.t.

$$f_s(0) = c$$

The tangent space at a point c consists of piecewise smooth vector fields V along c which vanishes at both endpoints p and q

$$T_c\Omega_{p,q} = \{V : [0,1] \to T_{c(t)}M \mid V \text{ piecewise smooth vector field along } c \text{ s.t. } V(0) = V(1) = 0\}$$

In particular for any $c \in \Omega_{p,q}$ fixed, and f as proper variation of c chosen, we define the variational field $V \in T_c\Omega_{p,q}$ as

$$\begin{aligned} V : [0,1] &\to T_{c(t)}M \\ t &\mapsto \left.\frac{d}{ds}\right|_{s=0} f_s(t) \end{aligned}$$

As in Proposition 2.9.1, for any $V \in T_c\Omega_{p,q}$, one can construct a proper variation $f : (-\varepsilon, \varepsilon) \to \Omega_{p,q}$ whose variational field recovers V.

The energy E is thus a smooth function on $\Omega_{p,q}$ and $E'(0)$ is the derivative of E in the direction of V

$$\begin{aligned} E : \Omega_{p,q} &\to \mathbb{R} \\ c &\mapsto E(c) := \int_0^1 |\frac{dc}{dt}|^2 dt \\ dE_c : T_c\Omega_{p,q} &\to \mathbb{R} \\ V &\mapsto dE_c(V) := \left.\frac{d}{ds}\right|_{s=0} E(f_s) \end{aligned}$$

where f is the proper variation associated to V.

Now if $\gamma \in \Omega_{p,q}$ is a geodesic, from Proposition 2.9.3, we know

$$dE_\gamma(V) = \left.\frac{d}{ds}\right|_{s=0} E(f_s) = 0 \qquad \forall\, V \in T_\gamma\Omega_{p,q}$$

Thus $\gamma \in \Omega_{p,q}$ is a critical point for the energy E.

2.9.1.2 Second Variation

Because $\frac{d}{ds}\big|_{s=0} E(s) = 0$ for every proper variation of a geodesic, our next information on the energy of neighboring curves is given by the second order derivatives at 0.

We first derive the second variation for general variations of a geodesic γ.

Proposition 2.9.4 (Formula for Second Variation of Energy; [dC92] Proposition 9.2.8, Remark 9.2.9)**.** *Let*

$$\gamma : [0, a] \to M$$

be a geodesic, and let

$$f : (-\varepsilon, \varepsilon) \times [0, a] \to M$$

be a variation of γ*. Let* E (2.168) *be the energy of the variation.*

$$\begin{aligned} \frac{1}{2}\left.\frac{d^2}{ds^2}\right|_{s=0} E(s) = &\langle \frac{D}{ds}\frac{\partial f}{\partial s}(0,a), \gamma'(a)\rangle - \langle \frac{D}{ds}\frac{\partial f}{\partial s}(0,0), \gamma'(0)\rangle + \langle V(a), \frac{D}{dt}V(a)\rangle - \langle V(0), \frac{D}{dt}V(0)\rangle \\ &+ \sum_{i=1}^{k} \langle V(t_i), \frac{D}{dt}V(t_i^-) - \frac{D}{dt}V(t_i^+)\rangle - \int_0^a \langle V(t), \frac{D^2}{dt^2}V(t) + R(\gamma', V)\gamma'\rangle dt \end{aligned} \tag{2.171}$$

where V *is the variational field of* f*,* R *is the Riemannian curvature of* M*, and*

$$\frac{DV}{dt}(t_i^+) = \lim_{t \searrow t_i} \frac{DV}{dt}(t), \qquad \frac{DV}{dt}(t_i^-) = \lim_{t \nearrow t_i} \frac{DV}{dt}(t)$$

Proof. Recall from the proof of first variation, we have (2.170)

$$\frac{1}{2}E'(s) = \sum_{i=0}^{k}\langle\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\rangle\Big|_{t=t_i^+}^{t=t_{i+1}^-} - \int_0^a\langle\frac{\partial f}{\partial s},\frac{D}{dt}\frac{\partial f}{\partial t}\rangle dt$$

We take another derivative

$$\begin{aligned}\frac{1}{2}E''(s) &= \sum_{i=0}^{k}\langle\frac{D}{ds}\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\rangle\Big|_{t=t_i^+}^{t=t_{i+1}^-} + \sum_{i=0}^{k}\langle\frac{\partial f}{\partial s},\frac{D}{ds}\frac{\partial f}{\partial t}\rangle\Big|_{t=t_i^+}^{t=t_{i+1}^-}\\ &\quad - \int_0^a\langle\frac{D}{ds}\frac{\partial f}{\partial s},\frac{D}{dt}\frac{\partial f}{\partial t}\rangle dt - \int_0^a\langle\frac{\partial f}{\partial s},\frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t}\rangle dt\end{aligned}$$

Notice

$$\begin{aligned}\frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} &\overset{(2.99)}{=} \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial t} + R(\frac{\partial f}{\partial t},\frac{\partial f}{\partial s})\frac{\partial f}{\partial t}\\ &= \frac{D}{dt}\underbrace{\frac{D}{dt}\frac{\partial f}{\partial s}}_{(2.44)} + R(\frac{\partial f}{\partial t},\frac{\partial f}{\partial s})\frac{\partial f}{\partial t}\end{aligned}$$

So

$$\begin{aligned}\frac{1}{2}E''(s) &= \sum_{i=0}^{k}\langle\frac{D}{ds}\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\rangle\Big|_{t=t_i^+}^{t=t_{i+1}^-} + \sum_{i=0}^{k}\langle\frac{\partial f}{\partial s},\frac{D}{dt}\frac{\partial f}{\partial s}\rangle\Big|_{t=t_i^+}^{t=t_{i+1}^-}\\ &\quad - \int_0^a\langle\frac{D}{ds}\frac{\partial f}{\partial s},\frac{D}{dt}\frac{\partial f}{\partial t}\rangle dt - \int_0^a\langle\frac{\partial f}{\partial s},\frac{D}{dt}\frac{D}{dt}\frac{\partial f}{\partial s} + R(\frac{\partial f}{\partial t},\frac{\partial f}{\partial s})\frac{\partial f}{\partial t}\rangle dt\end{aligned}$$

We assumed that $f(0,t) = \gamma(t)$ is a geodesic. Then

$$\begin{aligned}\frac{D}{dt}\frac{\partial f}{\partial t}(0,t) &= \frac{D}{dt}\gamma'(t) = 0\\ \frac{\partial f}{\partial t}(0,t) &= \gamma'(t)\end{aligned}$$

Also by assumption, V is variational field of f

$$\frac{\partial f}{\partial s}(0,t) = V(t)$$

Then plug $s = 0$ in

$$\begin{aligned}\frac{1}{2}E''(0) &= \sum_{i=0}^{k}\langle\frac{D}{ds}\frac{\partial f}{\partial s}(0,t),\gamma'(t)\rangle\Big|_{t=t_i^+}^{t=t_{i+1}^-} + \sum_{i=0}^{k}\langle V(t),\frac{D}{dt}V(t)\rangle\Big|_{t=t_i^+}^{t=t_{i+1}^-}\\ &\quad - \underbrace{\cancel{\int_0^a\langle\frac{D}{ds}\frac{\partial f}{\partial s}(0,t),\frac{D}{dt}\gamma'(t)\rangle dt}}_{\gamma\text{ is a geodesic}} - \int_0^a\langle V(t),\frac{D^2}{dt^2}V(t) + R(\gamma',V)\gamma'\rangle dt\\ &\overset{\gamma\text{ smooth}}{=} \underbrace{\langle\frac{D}{ds}\frac{\partial f}{\partial s}(0,a),\gamma'(a)\rangle - \langle\frac{D}{ds}\frac{\partial f}{\partial s}(0,0),\gamma'(0)\rangle}_{\text{differentiating only in } s \text{ preserves smoothness, so middle terms vanish}} + \langle V(a),\frac{D}{dt}V(a)\rangle - \langle V(0),\frac{D}{dt}V(0)\rangle\\ &\quad + \sum_{i=1}^{k}\langle V(t_i),\frac{D}{dt}V(t_i^-) - \frac{D}{dt}V(t_i^+)\rangle - \int_0^a\langle V(t),\frac{D^2}{dt^2}V(t) + R(\gamma',V)\gamma'\rangle dt\end{aligned}$$

Now □

Proposition 2.9.5 (Formula for Second Proper Variation of Energy; [dC92] Proposition 9.28)**.** *In addition to assumption as in Proposition 2.9.4, take f to be a proper variation.*

Then

$$\frac{1}{2}E''(0) = \int_0^a\left(\langle\frac{D}{dt}V(t),\frac{D}{dt}V(t)\rangle - R(\gamma',V,\gamma',V)\right)dt \tag{2.172}$$

Proof. If f is proper, then for any $s \in (-\varepsilon, \varepsilon)$,

$$f(s,0) = \gamma(0)$$
$$f(s,a) = \gamma(a)$$

But

$$V(0) = \frac{\partial f}{\partial s}(0,0) = V(a) = \frac{\partial f}{\partial s}(0,a) = 0$$
$$\frac{D}{ds}\frac{\partial}{\partial s}f(0,0) = \frac{D}{ds}\frac{\partial}{\partial s}f(0,a) = 0$$

Thus

$$\begin{aligned}
\frac{1}{2}E''(0) &\overset{(2.171)}{=} \cancel{\langle \frac{D}{ds}\frac{\partial f}{\partial s}(0,a), \gamma'(a)\rangle - \langle \frac{D}{ds}\frac{\partial f}{\partial s}(0,0), \gamma'(0)\rangle} + \cancel{\langle V(a), \frac{D}{dt}V(a)\rangle - \langle V(0), \frac{D}{dt}V(0)\rangle} \\
&+ \sum_{i=1}^{k} \langle V(t_i), \frac{D}{dt}V(t_i^-) - \frac{D}{dt}V(t_i^+)\rangle - \int_0^a \langle V(t), \frac{D^2}{dt^2}V(t) + R(\gamma', V)\gamma'\rangle dt \\
&= -\int_0^a \langle V(t), \frac{D^2}{dt^2}V(t) + R(\gamma', V)\gamma'\rangle dt + \sum_{i=1}^{k} \langle V(t_i), \frac{D}{dt}V(t_i^-) - \frac{D}{dt}V(t_i^+)\rangle
\end{aligned}$$

Now applying Integration by Parts,

$$\begin{aligned}
\langle \frac{DV}{dt}, \frac{DV}{dt}\rangle &= \frac{d}{dt}\left(\langle V(t), \frac{D}{dt}V(t)\rangle\right) - \langle V(t), \frac{D^2}{dt^2}V(t)\rangle \\
\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}} \langle \frac{DV}{dt}, \frac{DV}{dt}\rangle dt &= \sum_{i=0}^{k}\int_{t_i}^{t_{i+1}} \frac{d}{dt}\left(\langle V(t), \frac{D}{dt}V(t)\rangle\right) - \int_0^a \langle V(t), \frac{D^2}{dt^2}V(t)\rangle dt \\
&= \sum_{i=0}^{k} \langle V(t_i), \frac{D}{dt}V(t_i)\rangle \Big|_{t=t_i^+}^{t=t_{i+1}^-} - \int_0^a \langle V(t), \frac{D^2}{dt^2}V(t)\rangle dt
\end{aligned}$$

So we recover (2.172). □

$\Omega_{p,q}$ **Interpretation** This is [dC92] Remark 9.2.12.

Recall our energy function

$$E : \Omega_{p,q} \to \mathbb{R}$$

Take any $c \in \Omega_{p,q}$. This time we consider a two-parameter proper variation

$$\begin{aligned}
f : (-\delta, \delta) \times (-\varepsilon, \varepsilon) &\to \Omega_{p,q} \\
(r,s) &\mapsto f_{r,s}
\end{aligned}$$

s.t.

$$f_{0,0} = c$$

For a given c and f two-parameter variation, one define two variational fields V, $W \in T_c\Omega_{p,q}$

$$\begin{aligned}
V : [0,1] &\to T_{c(t)}M \\
t &\mapsto \frac{\partial}{\partial r}\Big|_{r=0} f_{r,0}(t) \\
W : [0,1] &\to T_{c(t)}M \\
t &\mapsto \frac{\partial}{\partial s}\Big|_{s=0} f_{0,s}(t)
\end{aligned}$$

As in Proposition 2.9.1, for any $V, W \in T_c\Omega_{p,q}$, one can construct a two-parameter variation $f : (-\delta, \delta) \times (-\varepsilon, \varepsilon) \to \Omega_{p,q}$ whose variational fields are V and W.

Now define for any $c \in \Omega_{p,q}$ the second variation as the Hessian of the energy function E at c

$$\begin{aligned}
d^2E_c : T_c\Omega_{p,q} \times T_c\Omega_{p,q} &\to \mathbb{R} \\
(V, W) &\mapsto \text{Hess}(E)(c)(V,W) := \frac{\partial^2}{\partial r \partial s}\Big|_{(r,s)=(0,0)} E(f_{r,s})
\end{aligned}$$

where f is the proper two-parameter variation associated to V and W.

Now take $\gamma \in \Omega_{p,q}$ to be the geodesic connecting p, q, i.e., γ is the critical point for E. Then for any V, $W \in T_\gamma\Omega_{p,q}$

$$\begin{aligned} d^2E_\gamma(V,W) &= \frac{\partial^2}{\partial r \partial s}\Big|_{(r,s)=(0,0)} E(f_{r,s}) \\ &= 2\int_0^1 \left(\langle \frac{DV}{dt}, \frac{DW}{dt}\rangle - R(\gamma', V, \gamma', W)\right) dt \end{aligned}$$

In particular

$$\begin{aligned} d^2E_\gamma(V,V) &= \frac{d^2}{dr^2}\Big|_{r=0} E(f_{r,0}) \\ &\overset{(2.172)}{=} 2\int_0^1 \left(\langle \frac{DV}{dt}, \frac{DV}{dt}\rangle - R(\gamma', V, \gamma', V)\right) dt \end{aligned}$$

2.9.2 Bonnet-Myers Theorem

Theorem 2.9.1 ([dC92] Theorem 9.3.1)**.** *Let (M^n, g) be a complete Riemannian manifold. Suppose there exists $r > 0$ s.t. either*

1. *For any $p \in M$, and any $v \in T_pM$ of unit length*

$$\mathrm{Ric}_p(v) \geq \frac{1}{r^2} > 0 \tag{2.173}$$

2. *Or for any $p \in M$, and any $\sigma \subseteq T_pM$ 2-plane*

$$K(p, \sigma) \geq \frac{1}{r^2} > 0 \tag{2.174}$$

Then M is compact, and

$$\mathrm{diam}(M, g) := \sup_{p,\, q \in M} d_g(p, q) \leq \pi r \tag{2.175}$$

Remarks on Bonnet-Myers

1. We remark that it suffices to check under the assumption of Myers (2.173), since the assumption of Bonnet (2.174) implies the former ([dC92] Corollary 9.3.3). Indeed, for $\{e_i\}_{1\leq i\leq n-1} \cup \{v\}$ an orthonormal basis of T_pM

$$\begin{aligned} \mathrm{Ric}_p(v) &= \frac{1}{n-1}\sum_{i=1}^{n-1} \langle R(v, e_i)v, e_i\rangle \\ &\overset{(2.66)}{=} \frac{1}{n-1}\sum_{i=1}^{n-1} K(p, \mathrm{Span}\{v, e_i\}) \end{aligned}$$

2. Also, note the inequalities are sharp ([dC92] Remark 9.3.6). Consider the example

$$M = \mathbb{S}^n(r)$$

Then for any $p \in \mathbb{S}^n(r)$ and $v \in T_p\mathbb{S}^n(r)$

$$\mathrm{Ric}_p(v) \overset{(2.80)}{=} K = \frac{1}{r^2} > 0$$

But

$$\mathrm{diam}(\mathbb{S}^n(r)) = \pi r^2$$

In fact there is result stating uniqueness.

Theorem 2.9.2 (Cheng-Shiohama)**.** *Let (M^n, g) be complete with*

$$\mathrm{Ric}_p(v, v) \geq \frac{1}{r^2} \qquad \forall\ p \in M,\ \forall\ v \in T_pM$$

If

$$\mathrm{diam}(M, g) = \pi r$$

Then the manifold M^n is isometric to the sphere $\mathbb{S}^n(r)$ of curvature $\frac{1}{r^2}$

$$(M^n, g) \cong (\mathbb{S}^n(r), g_{\mathrm{can}}^{\mathbb{S}^n(r)})$$

3. It is necessary that K, Ric are bounded away from 0 ([dC92] Remark 9.3.4). For example consider paraboloid
$$S = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}$$
This is complete, and has curvature
$$K = \frac{4}{(1 + x^2 + y^2)^2} > 0, \qquad \inf_{p \in S} K(p) = 0$$
but S is indeed not compact.

Proof of Bonnet-Myers Theorem 2.9.1 It suffices to assume for (2.173). This is beautiful proof leveraging the second variation formula.

Proof. 1. By contradiction, suppose that $\mathrm{diam}(M, g) > \pi r$. Then there exists two points p, $q \in M$ s.t.
$$\ell := d(p, q) > \pi r$$
Since the manifold is complete, using Hopf-Rinow Theorem 2.7.1 (f) there exists also a normalized geodesic that connects these two points p, q. Take γ
$$\gamma : [0, \ell] \to M \qquad \gamma(0) = p \qquad \gamma(\ell) = q$$
We want to apply the second variation formula to some variation that gives us a contradiction.

2. We construct a variation by imposing a vector field along this curve. We define it by using the orthonormal basis of the tangent space. Let $\{e_1, \cdots, e_n\}$ be ONB of T_pM where
$$e_n := \gamma'(0)$$
Then parallel transport them. Let $e_i(t)$ be the parallel transport of e_i along γ. We define our variational field (the one that saturates the sphere)
$$\begin{aligned} &V_i(t) := \sin(\frac{\pi t}{\ell}) e_i(t) \qquad i = 1, \cdots, n-1 \\ &V_i(0) = V_i(\ell) = 0 \end{aligned}$$
Given our Vector fields along γ, from Proposition 2.9.1, we have a family of proper variations $\{f_i\}$ associated to the Variational Field, i.e., V_i are the variational field of f_i of γ.

3. Now let $E_i(s)$ be the energy of $f_i(s, t)$, the proper variation associated to V_i. Let's compute
$$\begin{aligned} E_i(s) &:= \int_0^\ell |\frac{\partial f_i}{\partial t}(s, t)|^2 \geq \frac{1}{\ell} L(f_i)^2 \qquad \text{Cauchy Schwarz} \\ &\geq \frac{1}{\ell} L(\gamma)^2 \qquad \text{since } \gamma \text{ is geodesic} \\ &= E(\gamma) = E_i(0) \end{aligned}$$
Now that γ is geodesic and V proper, we know $E_i'(0) = 0$. We also know $E_i''(0) \geq 0$ because we've shown that 0 achieves minimum.

By the second variation formula (2.172)
$$\begin{aligned} \frac{1}{2} E_i''(0) &= -\int_0^\ell \left(\langle \frac{D^2 V_i}{dt^2}, V_i \rangle + R(\gamma', V_i, \gamma', V_i) \right) dt \qquad \text{no boundary terms because all are piecewise smooth} \\ &= \int_0^\ell \left(\frac{\pi^2}{\ell^2} \sin^2(\frac{\pi t}{\ell}) - \sin^2(\frac{\pi t}{\ell}) R(e_n, e_i, e_n, e_i) \right) dt \qquad \forall\, i = 1, \cdots, n-1 \\ \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{1}{2} E_i''(0) &= \int_0^\ell \left(\frac{\pi^2}{\ell^2} \sin^2(\frac{\pi t}{\ell}) - \sin^2(\frac{\pi t}{\ell}) \mathrm{Ric}_p(e_n, e_n) \right) dt \\ &\overset{(2.173)}{\leq} \int_0^\ell \left(\frac{\pi^2}{\ell^2} - \frac{1}{r^2} \right) \sin^2(\frac{\pi t}{\ell}) dt < 0 \qquad \text{since } \pi r < \ell \text{ by our contradictory assumption} \end{aligned}$$
Now this contradicts $E_i''(0) \geq 0$.

Since M is totally bounded and complete, M is compact. ∎

Corollaries and Examples for Bonnet-Myers 2.9.1

Corollary 2.9.1 ([dC92] Corollary 9.3.2). *If (M^n, g) is a complete Riemannian manifold with*

$$\mathrm{Ric}_p(v, v) \geq \delta > 0 \qquad \forall\, p \in M,\ \forall\, v \in T_pM$$

Then the universal cover of M is compact. In particular the first fundamental group $\pi_1(M)$ is finite.

Proof. Let $(\tilde{M}, \tilde{g})$ be the universal cover equipped with the covering metric, i.e. the pullback metric under the covering map (so that π is local isometry)

$$\pi : \tilde{M} \to M$$

Then $\tilde{M}$ is complete and $\tilde{\mathrm{Ric}}_p \geq \delta > 0$.

By Myer's Theorem 2.9.1, $\tilde{M}$ is compact. Hence the number of sheets of the covering is finite. Since this is a number of elements in the fundamental group $\pi_1(M)$ of M, we conclude that $\pi_1(M)$ is finite.

Or to be precise, for any $p \in M$, $\pi^{-1}(p)$ is a discrete set in a compact manifold, so that its finite.

$$|\pi_1(M)| = \#\pi^{-1}(p) < \infty$$

□

Example 2.9.1 ([dC92] Exercise 9.2). *Introduce a complete Riemannian metric on $\mathbb{R}^2$. Prove that*

$$\lim_{r\to\infty}\left(\inf_{x^2+y^2\geq r^2} K(x, y)\right) \leq 0$$

where $(x, y) \in \mathbb{R}^2$ and $K(x, y)$ is the Gaussian curvature of the given metric at (x, y).

Proof. Assume for contradiction that

$$\lim_{r\to\infty}\left(\inf_{x^2+y^2\geq r^2} K(x, y)\right) > 0$$

Then, there exists $\varepsilon > 0$ and $R > 0$ such that

$$K(x, y) \geq \varepsilon \quad \text{for all} \quad x^2 + y^2 \geq R^2$$

Consider the closed subset $M = \{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 \geq R^2\}$ with the induced metric. Since $(\mathbb{R}^2, g)$ is complete, M as a closed submanifold is also a complete Riemannian manifold by Cororllary of Hopf-Rinow 2.7.2.

By construction, $K \geq \varepsilon > 0$ on M. Now the Bonnet-Myers Theorem 2.9.1 states that a complete connected Riemannian manifold with Ricci curvature bounded below by some strictly positive constant is compact and has finite diameter.

In dimension 2, the Ricci curvature coincides with the Gaussian curvature by Example 2.6.4. Thus, Bonnect-Myers says M must be compact.

However, M is homeomorphic to $\mathbb{R}^2 \setminus B_R(0)$, which is non-compact as it contains unbounded sequences. We reach a contradiction.

□

2.9.3 Synge-Weinstein Theorem

Theorem 2.9.3 (Weinstein; [dC92] Theorem 9.3.7). *Let (M^n, g) be a compact oriented Riemannian manifold. Suppose M has positive sectional curvature. Suppose*

$$f : (M, g) \to (M, g)$$

is an isometry s.t. f preserves (resp. reverses) the orientation if $n = \dim M$ is even (resp. odd).

Then f has a fixed point, i.e., there exists $p \in M$ s.t. $f(p) = p$.

Remark 2.9.1. *This assumption in fact excludes the case of the sphere. Consider*

$$\begin{aligned} A : \mathbb{S}^n &\to \mathbb{S}^n \\ p &\mapsto -p \end{aligned}$$

Then this is the opposite of the orientation requirement on the isometry, i.e., A is orientation preserving if n is odd and orientation reversing if n is even.

Fact from orthogonal linear transformation over $\mathbb{R}$

Lemma 2.9.2 ([dC92] Lemma 9.3.8)**.** *Let A be an orthogonal linear transformation of $\mathbb{R}^{n-1}$, i.e., $A \in O(n-1)$ over the reals. Suppose that*

$$\det(A) = (-1)^n$$

Then A leaves invariant some non-zero vector of $\mathbb{R}^{n-1}$.

Proof. If n even, then $\det(A - \lambda I)$ is a real polynomial in λ of odd degree $n-1$. Then A has a real eigenvalue. Since $A \in O(n-1)$ orthogonal, such real eigenvalue is of the form ± 1. But the product of the complex eigenvalues of A is non-negative (this is because for matrix A over the reals, complex eigenvalues come in pairs. To see this, assume $Av = \lambda v$ then taking complex conjugate $\overline{Av} = \overline{\lambda}\overline{v}$. But A is real so $A = \overline{A}$ hence $\overline{\lambda}$ is an eigenvalue), and $\det(A) = 1$, so at least one of the eigenvalues of A equals 1.

If n is odd, $\det(A) = -1$. But the product of complex eigenvalues is non-negative, thus there is at least one pair of real eigenvalues, only one of which is positive, hence equal to 1. □

Proof of Weinstein Theorem 2.9.3

Proof. 1. Suppose that f has no fixed points, i.e.

$$f(q) \neq q \qquad \forall\, q \in M$$

Consider the continuous function on M

$$\begin{aligned} h : M &\to \mathbb{R} \\ q &\mapsto d(q, f(q)) \end{aligned}$$

Since M is compact, there exists $p \in M$ s.t.

$$h(p) = \min_{q \in M} h(q)$$

i.e., using contradictory assumption

$$\ell := d(p, f(p)) = \inf_{q \in M} d(q, f(q)) > 0 \tag{2.176}$$

But by Corollary 2.7.1, M is complete due to compactness, hence Hopf-Rinow Theorem 2.7.1 (f) says there exists a normalized geodesic γ connecting the two points q and $f(q)$

$$\gamma(0) = p, \qquad \gamma(\ell) = f(p)$$

2. Now consider the two velocity vectors $\gamma'(0)$ and $\gamma'(\ell)$. We need two claims that gives a contradiction.

Claim 1. For f as in our assumption

$$df_p : T_pM \to T_{f(p)}M$$

sends $\gamma'(0) \mapsto \gamma'(\ell)$.

Indeed, let $p' := \gamma(t')$ for arbitrary $0 < t' < \ell$. So $t' = d(p, p')$. We look at the distances between p' and $f(p')$.

$$\begin{aligned} d(p', f(p')) &\leq d(p', f(p)) + d(f(p), f(p')) \\ &= d(p', f(p)) + d(p, p') \qquad \text{using } f \text{ is an isometry} \\ &= \ell - t' + t' = \ell = d(p, f(p)) \end{aligned}$$

where the last line uses our unit-speed parametrization.

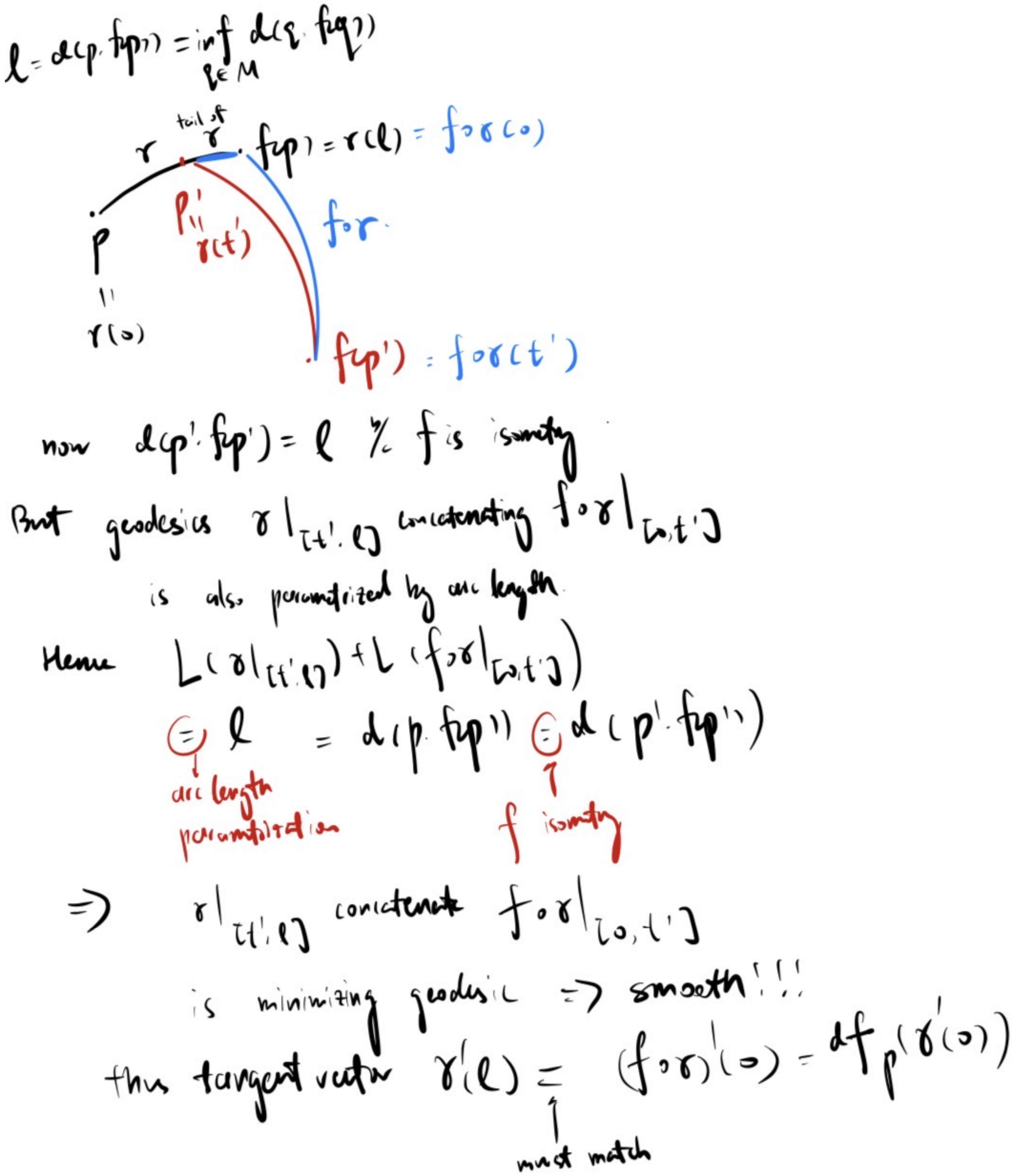


Figure 2.16: Claim 1 for Proof of Theorem 2.9.3

But on the other hand, $d(p, f(p))$ is the minimum. Thus we have

$$d(p', f(p')) = d(p, f(p)) = d(p', f(p)) + d(f(p), f(p')) = L(\gamma|_{[t',\ell]}) + L((f \circ \gamma)|_{[0,t']})$$

Hence γ and $f \circ \gamma$ are normalized geodesics. Thus

$$\gamma'(\ell) = (f \circ \gamma)'(0) = df_p(\gamma'(0)) \tag{2.177}$$

3. **Claim 2. There exists a parallel vector field $V(t)$ along $\gamma(t)$ s.t. $|V(t)| = 1$ and $\langle V(t), \gamma'(t)\rangle = 0$.**

   (a) We setup. Let $P : T_pM \to T_{f(p)}M$ be the parallel transport along γ (P is orientation preserving). Notice both maps

$$T_pM \underset{df_p}{\overset{P}{\to}} T_{f(p)}M$$

Note $P(\gamma'(0)) = \gamma'(\ell)$ since $\gamma'(t)$ is parallel to $\gamma(t)$ (because γ is geodesic). Define

$$A := P^{-1} \circ df_p : T_pM \to T_pM$$

Then $A \in O(n)$ is an isometry and $\det(A) = (-1)^n$ (using assumption f preserves orientation if n even anf reverses orientation if n odd). In particular -1 if n odd and 1 if n even. Note

$$\begin{aligned} A(\gamma'(0)) &= P^{-1}(df_p(\gamma'(0))) \\ &\overset{(2.177)}{=} P^{-1}(\gamma'(\ell)) \qquad \text{use Claim 1} \\ &= \gamma'(0) \end{aligned}$$

so $\gamma'(0)$ is an eigenvector with eigenvalue 1.

$$A(\gamma'(0)) = \gamma'(0) \tag{2.178}$$

(b) Let W be the orthogonal complement of $\mathbb{R}\gamma'(0)$ in T_pM, i.e.

$$T_pM = \mathbb{R}\gamma'(0) \oplus W$$

Consider

$$B := A|_W : W \to W \cong \mathbb{R}^{n-1} \qquad B \in O(n-1) \qquad \det(B) = (-1)^n$$

Recall that if $C \in O(m)$ and 1 is not an eigenvalue, then $\det(C) = (-1)^m$. So if $C \in O(m)$ s.t. $\det(C) = (-1)^{m+1}$, then 1 is an eigenvalue (this is essentially applying Lemma 2.9.2).

Thus our B has to have 1 as an eigenvalue. Now let $v \in W$ be the associated eigenvector and take $|v| = 1$

$$Av = Bv = v \tag{2.179}$$

(c) Now Let $V(t)$ be the parallel transport of v along $\gamma(t)$. Then since the parallel transport doesn't change the norms, and since $v \in W$ (the orthogonal complement of $\mathbb{R}\gamma'(0)$)

$$\langle v, \gamma'(0)\rangle = 0 \qquad \langle v, v\rangle = 1$$

Then using parallel transport

$$\langle V(t), \gamma'(t)\rangle = 0 \qquad \langle V(t), V(t)\rangle = 1$$

Thus

$$\begin{aligned} (P^{-1} \circ df_p)(V(0)) &= (P^{-1} \circ df_p)(v) \\ &= Av \\ &\overset{(2.179)}{=} v = V(0) \end{aligned}$$

In particular

$$df_p(v) = df_p(V(0)) = P(V(0)) = V(\ell) \tag{2.180}$$

4. We build a variational field

$$\begin{aligned} h : (-\varepsilon, \varepsilon) \times [0, \ell] &\to M \\ (s, t) &\mapsto h(s, t) := \exp_{\gamma(t)}(sV(t)) \end{aligned}$$

In particular each

$$s \mapsto h(s, t)$$

is a geodesic. Let

$$\begin{aligned} \alpha(s) &:= h(s, 0) = \exp_p(sv) \\ \beta(s) &:= h(s, \ell) \overset{(2.180)}{=} \exp_{f(p)}(s\; df_p(v)) \end{aligned}$$

These are themselves geodesics. As

$$\beta(0) = f(p) \qquad \beta'(0) \overset{(2.180)}{=} V(\ell) = df_p(V(0))$$

In fact we conclude here that

$$\beta = f \circ \alpha \tag{2.181}$$

Now we consider a curve for fixed s. For fixed $s \in (-\varepsilon, \varepsilon)$, consider

$$h_s : [0, \ell] \to M$$

a smooth curve from $\alpha(s)$ to $\beta(s)$. Now the energy for h_s writes

$$E(s) = \int_0^\ell |\frac{\partial h}{\partial t}(s,t)|^2 dt \overset{\text{Cauchy Schwarz}}{\geq} \frac{1}{\ell}\left(\int_0^\ell |\frac{\partial h}{\partial t}(s,t)| dt\right)^2 = \frac{1}{\ell} L(h_s)^2$$
$$\overset{(2.181)}{\geq} \frac{1}{\ell} d(\alpha(s), f \circ \alpha(s))^2 \overset{(2.176)}{\geq} \frac{1}{\ell} d(p, f(p))^2 = \ell = E(0)$$

5. Since the original p and $f(p)$ have shortest distance, all variations get longer. Here is where we'll get our contradiction. We use positive curvature to push off the curve along a parallel field, then necessarily it has decreasing energy. For negative curvature, pushing off the geodesic increases energy.

   Now to make this rigorous. Back to our setup: Since we have γ as a geodesic,

$$\langle V(t), \gamma'(t)\rangle = 0 \qquad \forall\ t$$

   Thus $E'(0) = 0$ as in Proposition 2.9.3. And because

$$E(s) \geq E(0) \qquad \forall\ s$$

   $E(0)$ must be a local minimum, so $E''(0) \geq 0$.

   But we have the second variation formula (2.171)

$$\frac{1}{2}E''(0) = -\int_0^\ell \langle \underbrace{\cancel{\frac{D^2 V}{dt^2}}}_{V \text{ parallel}} + R(\gamma', V)\gamma', V\rangle dt + \underbrace{\cancel{\langle \frac{D}{ds}\frac{\partial h}{\partial s}(0,\ell), \gamma'(\ell)\rangle - \langle \frac{D}{ds}\frac{\partial h}{\partial s}(0,0), \gamma'(0)\rangle}}_{\text{because } s \mapsto h(s,t) \text{ is geodesic}}$$
$$+ \underbrace{\cancel{\langle \frac{DV}{dt}(\ell), V(\ell)\rangle - \langle \frac{DV}{dt}(0), V(0)\rangle}}_{\frac{DV}{dt} \equiv 0 \text{ since } V \text{ is parallel}}$$
$$= -\int_0^\ell \langle R(\gamma', V)\gamma', V\rangle dt$$

   Since γ', V has length 1, $\gamma' \perp V$, we know $\{V(t), \gamma'(t)\}$ span the $\pi(s)$ 2-plane, whose sectional curvature is strictly negative

$$\frac{1}{2}E''(0) = -\int_0^\ell \langle R(\gamma', V)\gamma', V\rangle dt = -\int_0^\ell K(\pi(t))dt < 0$$

   Thus we have a contradiction.

□

Synge Theorem

Corollary 2.9.2 (Synge [dC92] Corollary 9.3.10)**.** *Let (M^n, g) be compact with positive sectional curvature*

1. *if M orientable, n even, then $\pi_1(M) = 1$, i.e. M is simply connected*
2. *if n odd, then M is orientable.*

Proof. 1. For universal cover

$$\pi : \tilde{M} \to M$$

   $\tilde{M}$ is complete with $K \geq c > 0$ (because M is compact). Then by Myers Theorem 2.9.1 $\tilde{M}$ is compact. We want to show $M = \tilde{M}$. If not, choose $\varphi \neq \text{id}$,

$$\varphi : \tilde{M} \to \tilde{M}$$

   a covering transformation, i.e., $\pi \circ \varphi = \pi$ (so φ has no fixed points). Then φ is an isomertry of $\tilde{M}$. By M orientable, φ preserves orientation. But this contradicts to Weinstein Theorem 2.9.3.

2. If M is not orientable, there exists orientation double cover

$$\tilde{M} \to M$$

   now $\phi : \tilde{M} \to \tilde{M}$ is orientation reversing without fixed points, so this is contradiction to Weinstein 2.9.3.

□

Examples for Weinstein Theorem 2.9.3 $\mathbb{R}P^n = \mathbb{S}^n/\{\text{antipodal}\}$ is orientable iff n is odd.

Bibliography

[dC92] M.P. do Carmo. *Riemannian Geometry*. Mathematics (Boston, Mass.). Birkhäuser, 1992.

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