[2024Liu] Modern Geometry I

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Contents

1 Topological manifold and Differentiable Structure

Definition 1.1 (Topological n-manifold). *A topological manifold of dimension n is a topological space M which is locally homeomorphic to $\mathbb{R}^n$ w.r.t. the standard topology, i.e., for any $p \in M$, there exists open neighborhood $U \subset M$ of p, and there exists a local homeomorphism $\phi : U \to \phi(U) \subset \mathbb{R}^n$ (a bijective continuous map with continuous inverse).*

- *(U, ϕ) is a chart for M around p.*
- *$\phi = (x_1, \cdots, x_n) \in U$ are coordinates of U in $\mathbb{R}^n$ where $x_i : U \subset M \to \mathbb{R}$ are C^0.*

Remark 1.1. *We require in addition for the topology of M to satisfy the following*

- *M is a Hausdorff topological space, i.e., for any $p, q \in M$ distinct, there exists disjoint open neighborhoods U around p and V around q.*
- *M is second countable, i.e., M has a countable basis of open sets. So every open set of M is a union of elements in this countable collection.*

Example 1.1. *Standard example: $\mathbb{R}^n$. It is topological n-manifold that is Hausdorff and secound countable with basis $\{B_r(a) \mid a \in \mathbb{Q}^n, r \in \mathbb{Q}\}$*

Recall Quotient Topology, which is one way to construct topology on some set.

Definition 1.2 (Quotient Topology). *Let $\pi : X \to M$ be surjective map from a topological space X to some set M. One wish to use topology of the source X to equip a topology on M. $U \subset M$ is open in the quotient topology defined by the surjective map π iff the preimage $\pi^{-1}(U) \subset X$ is open. It is not hard to see that*

- *$\pi : X \to M$ is continuous for M equipped with quotient topology.*
- *Let Y be any topological space. Then $f : M \to Y$ is continuous iff $f \circ \pi : X \to Y$ is continuous*

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f\circ\pi"] & \\ M \arrow[r, "f"] & Y \end{tikzcd} \tag{1}$$

Example 1.2 (Bug-eyed line; Line with 2 origins). *Consider 2 copies of the real line.*

$$\pi : \mathbb{R} \times \{0, 1\} \to M = (\mathbb{R} \times \{0, 1\})/\{(x, 0) \sim (x, 1) \; iff \; x \neq 0\}$$

for M equipped with quotient topology. Then M is a topological 1-dim manifold, second countable, but it is not Hausdorff.

Example 1.3 (Bunching Line). *Consider 2 copies of the real line.*

$$\pi : \mathbb{R} \times \{0, 1\} \to M = (\mathbb{R} \times \{0, 1\})/\{(x, 0) \sim (x, 1) \; iff \; x < 0\}$$

for M equipped with quotient topology. Then M is a 1-manifold, second countable, but the positive part has 2 copies, so not Hausdorff.

Example 1.4 (Long Line). *The usual ray is $[0, \infty) = \bigcup_{i=1}^{\infty} [i - 1, i)$. But Long ray is countable copies of this. Imagine if put 2 rays together one gets $\mathbb{R}$, if put 2 long rays one gets the long line. It is connected, Hausdorff, 1-manifold, but not 2nd countable. (This is example 45 in "Counterexamples in topology" by Steen-Seebach).*

Definition 1.3 (Atlas). *An atlas of a topological n-manifold M is a collection of charts for M*

$$\Phi = \{(U_\alpha, \phi_\alpha) \mid \alpha \in I\} \quad s.t. \quad \bigcup_\alpha U_\alpha = M$$

along with transition functions $\phi_\beta \circ \phi_\alpha^{-1}$ that are homeomorphism

$$\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n \xrightarrow{\phi_\beta\circ\phi_\alpha^{-1}} \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$$

Definition 1.4 (Differentiable Structure & Differentiable n-manifold). *k positive integer or ∞.*

- *A C^k-atlas on a topological manifold M is an atlas $\Phi = \{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$ for M s.t. all the transition functions $\phi_\beta \circ \phi_\alpha^{-1}$ are C^k diffeomorphisms.*

- *We say two C^k-atlas $\Phi = \{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$ and $\Psi = \{(V_\beta, \psi_\beta) \mid \beta \in J\}$ are equivalent (compatible) if $\Phi \cup \Psi$ is again a C^k atlas.*
- *A C^k-differentiable structure on a topological manifold M is an equivalence class of C^k-atlases on M.*
- *A C^k-manifold is a topological manifold M equipped with a C^k-differentiable structure.*

If $k = \infty$, the above C^∞-differentiable structure is called smooth structure, C^∞ manifolds are smooth manifolds, and C^∞ maps are smooth maps.

Example 1.5. *The Bug-eyed line, the Branching Line and the Long Line are C^∞-manifolds.*

Example 1.6. *The real projective space $P_n(\mathbb{R})$ or $(\mathbb{R}P^n)$ is*

- *A set $P_n(\mathbb{R}) := \{\ell \subset \mathbb{R}^{n+1} \mid 1 - dim\ \mathbb{R} - vector\ subspace\}$*
- *One has 2 equivalent ways to define Topology on $P_n(\mathbb{R})$. First of all equip $P_n(\mathbb{R})$ with quotient topology defined by $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to P_n(\mathbb{R})$ that maps $x \mapsto \mathbb{R}x$. Notation $\pi(x_1, \cdots, x_{n+1}) = [x_1, \cdots, x_{n+1}]$.*

  (a) Let $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to (\mathbb{R}^{n+1} \setminus \{0\})/\{x \sim \lambda x\ iff\ \lambda \in \mathbb{R} \setminus \{0\}\}$ be surjective quotient map s.t.

  $$x \overset{\pi}{\sim} y \in \mathbb{R}^{n+1} \setminus \{0\} \qquad iff \qquad \exists\ \lambda \in \mathbb{R} \setminus \{0\}\ s.t.\ y = \lambda x$$

  (b) Let $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\} \subset \mathbb{R}^{n+1}$ be unit sphere in $\mathbb{R}^{n+1}$. Let $\pi : \mathbb{S}^n \to \mathbb{S}^n/\{x \sim -x\}$ be surjective quotient map s.t.

  $$x \overset{\pi}{\sim} y \in \mathbb{S}^n \qquad iff \qquad x = -y$$

  In fact,

  $$P_n(\mathbb{R}) = (\mathbb{R}^{n+1} \setminus \{0\})/\{x \sim \lambda x\ iff\ \lambda \in \mathbb{R} \setminus \{0\}\} = \mathbb{S}^n/\{x \sim -x\}$$

  Claim: $P_n(\mathbb{R})$ is compact and Hausdorff.

  Proof. $P_n(\mathbb{R})$ is equivalently equipped with quotient topology defined by $\pi|_{\mathbb{S}^n} : \mathbb{S}^n \to P_n(\mathbb{R})$. Since $\pi|_{\mathbb{S}^n}$ is continuous, and $\mathbb{S}^n$ is compact, $P_n(\mathbb{R})$ is Hausdorff and compact. □

- *$P_n(\mathbb{R})$ is a topological n-manifold with an Atlas.*

  Proof. For Altas, $1 \le i \le n+1$, define

  $$U_i := \{[x_1, \cdots, x_{n+1}] \in P_n(\mathbb{R}) \mid x_i \neq 0\} \subset P_n(\mathbb{R}) \tag{2}$$

  Then U_i is an open subset of $P_n(\mathbb{R})$ since $\pi^{-1}(U_i) = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_i \neq 0\}$ is an open subset of $\mathbb{R}^{n+1} \setminus \{0\}$. Indeed $P_n(\mathbb{R}) = \bigcup_{i=1}^{n+1} U_i$. Define $\phi_i : U_i \to \mathbb{R}^n$ that maps

  $$\phi_i([x_1, \cdots, x_{n+1}]) := \left(\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_{n+1}}{x_i}\right) \tag{3}$$

  and is bijection with inverse map $\phi_i^{-1} : \mathbb{R}^n \to U_i$

  $$\phi_i^{-1}(y_1, \cdots, y_n) := [y_1, \cdots, y_{i-1}, 1, y_i, \cdots, y_n]$$

  In fact, one has the following diagram for each $i = 1, \cdots, n+1$

  $$\begin{tikzcd} \mathbb{R}^{n+1} \setminus \{0\} \arrow[d, "\pi"] & \overset{open}{\supseteq} & \pi^{-1}(U_i) \arrow[d, "\pi_i"] & \\ P_n(\mathbb{R}) & \overset{open}{\supseteq} & U_i \arrow[r, shift left, "\phi_i"] & \mathbb{R}^n \arrow[l, shift left, "\phi_i^{-1}"] \arrow[ul, "s_i"'] \end{tikzcd}$$

  If define $s_i : \mathbb{R}^n \to \pi^{-1}(U_i) \subset \mathbb{R}^{n+1} \setminus \{0\}$ s.t. $s(y_1, \cdots, y_n) := (y_1, \cdots, y_{i-1}, 1, y_i, \cdots, y_n)$. Then $\phi_i^{-1} = \pi_i \circ s_i$ as composition of continuous function is continuous. For ϕ_i, notice

  $$\begin{array}{rcl} \phi_i \circ \pi_i : \pi^{-1}(U_i) \subset \mathbb{R}^{n+1} \setminus \{0\} & \longrightarrow & \mathbb{R}^n \\ (x_1, \cdots, x_n) & \longmapsto & \left(\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_{n+1}}{x_i}\right) \end{array}$$

  is indeed a continuous map. Hence using (1) due to quotient topology defined on U_i, one has $\phi_i : U_i \to \mathbb{R}^n$ continuous. Thus ϕ_i are homeomorphisms. One obtain $P_n(\mathbb{R})$ as a topological n-manifold with atlas $\Phi = \{(U_i, \phi_i)\}_{i=1}^{n+1}$ on $P_n(\mathbb{R})$ where open sets U_i and local homeomorphisms are given by (2) and (3).

  □

- *Transition functions $\phi_i \circ \phi_j^{-1}$ make $(P_n(\mathbb{R}), \Phi)$ a C^∞-manifold of dimension n.*

  Proof. WLOG $U_1 \cap U_2 = \{[x_1, x_2, \cdots, x_{n+1}] \mid x_1, x_2 \neq 0\}$, so

$$\begin{aligned}\phi_2 \circ \phi_1^{-1}(y_1, \cdots, y_n) &= \phi_2([1, y_1, \cdots, y_n]) \\ &= \left(\frac{1}{y_1}, \frac{y_2}{y_1}, \cdots, \frac{y_n}{y_1}\right)\end{aligned}$$

  The transition functions

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) = (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{n-1} \longrightarrow \phi_2(U_1 \cap U_2) = (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{n-1}$$

  are indeed smooth maps. Same works for general i, j. In general, for $i > j$ s.t. $U_i \cap U_j \neq \varnothing$

$$\begin{tikzcd} P_n(\mathbb{R}) & \overset{open}{\supseteq} & U_i \cap U_j \arrow[d, "\phi_i"'] \arrow[dr, "\phi_j"] & & \\ \mathbb{R}^n & \overset{open}{\supseteq} & \phi_i(U_i \cap U_j) \arrow[r, "\phi_j \circ \phi_i^{-1}"] & \phi_j(U_i \cap U_j) & \overset{open}{\subseteq} & \mathbb{R}^n \end{tikzcd}$$

  for any $(x_1, \cdots, x_n) \in \phi_i(U_i \cap U_j)$

$$\begin{aligned}\phi_j \circ \phi_i^{-1}(x_1, \cdots, x_n) &= \phi_j([x_1, \cdots, x_{i-1}, 1, x_i, x_{i+1}, \cdots, x_n]) \\ &= \left(\frac{x_1}{x_j}, \cdots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \cdots, \frac{x_{i-1}}{x_j}, \frac{1}{x_j}, \frac{x_{i+1}}{x_j}, \cdots, \frac{x_n}{x_j}\right)\end{aligned}$$

  Hence Φ is a C^∞ altas on $P_n(\mathbb{R})$. $\square$

2 Differentiable Maps

Definition 2.1 (C^k maps)**.** *Let M be C^ℓ manifold of dimension m and N a C^ℓ manifold of dimension n, where $1 \le k \le \ell \le \infty$. A continuous map $f : M \to N$ is C^k-differentiable if for any $p \in M$, there exists a C^ℓ-chart (U, ϕ) for M around p and (V, ψ) for N around $f(p)$ s.t. $f(U) \subset V$, and $g := \psi \circ f \circ \phi^{-1}$ is C^k. When $k = \infty$, C^∞ maps are smooth maps.*

$$\begin{tikzcd} M & \overset{open}{\supseteq} & p \in U \arrow[r, "f"] \arrow[d, "\phi"] & V \arrow[d, "\psi"] & \overset{open}{\subseteq} & N \\ \mathbb{R}^m & \overset{open}{\supseteq} & \phi(U) \arrow[r, "g"] & \psi(V) & \overset{open}{\subseteq} & \mathbb{R}^n \end{tikzcd}$$

Remark 2.1. *The above C^k is indeed well-defined.*

- *If $\tilde{g} := \tilde{\psi} \circ f \circ \tilde{\phi}^{-1}$ is another composition for $(\tilde{U}, \tilde{\phi})$ chart of M around p and $(\tilde{V}, \tilde{\psi})$ chart of N around $f(p)$ then $\tilde{g} = (\tilde{\psi} \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \tilde{\phi}^{-1}) = (\tilde{\psi} \circ \psi^{-1}) \circ g \circ (\phi \circ \tilde{\phi}^{-1})$ remains C^k as transition functions are C^ℓ diffeomorphisms and g is C^k. Hence Definition 2.1 works for any charts, and f C^k map is well-defined.*

Example 2.1. *Let $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to P_n(\mathbb{R})$ where $P_n(\mathbb{R})$ real projective space, which we know is C^∞-n manifold. π is continuous. In fact, projection π is a C^∞ map.*

Proof. For any $p \in \mathbb{R}^{n+1} \setminus \{0\}$, recall U_i and ϕ_i as in (2) and (3). $\pi(p) \in P_n(\mathbb{R})$, so there exists some i s.t. $\pi(p) \in U_i$. Hence $p \in \pi^{-1}(U_i)$.

$$\begin{tikzcd} \mathbb{R}^{n+1} \setminus \{0\} & \overset{open}{\supseteq} & p \in \pi^{-1}(U_i) \arrow[r, "\pi"] \arrow[d, "id"] & U_i \arrow[d, "\phi_i"] & \overset{open}{\subseteq} & P_n(\mathbb{R}) \\ \mathbb{R}^{n+1} & \overset{open}{\supseteq} & \pi^{-1}(U_i) \arrow[r, "g"] & \mathbb{R}^n & \overset{open}{\subseteq} & \mathbb{R}^n \end{tikzcd}$$

$g := \phi_i \circ \pi \circ id^{-1} : \pi^{-1}(U_i) \subset \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^n$ s.t.

$$g(x_1, \cdots, x_{n+1}) := (\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_{n+1}}{x_i})$$

is a C^∞ map. □

Definition 2.2 (Diffeomorphism)**.** *M, N C^∞ manifold. $f : M \to N$ continuous. $\dim M = m$, $\dim N = n$.*

- *f is C^∞ diffeomorphism if f is a homeomorphism, and f, f^{-1} are C^∞ maps. In particular, $m = n$.*
- *For $p \in M$, f is a local diffeomorphism(C^∞) at p if there exist a open neighborhood U of p in M and V of $f(p)$ in N s.t. $f|_U : U \to V$ is a C^∞-diffeomorphism. In particular, $m = n$.*

Remark 2.2. *For M C^k-manifold of dimension m, $U \subset M$ open. $\Phi := \{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$ some C^k-atlas of M. Then $\Phi_U := \{(U_\alpha \cap U, \phi_\alpha|_{U_\alpha \cap U}) \mid \alpha \in I, U_\alpha \cap U \neq \varnothing\}$ is C^k-atlas for U. So U is a C^k-manifold of dimension m.*

2.1 Submersion and Immersion

Definition 2.3 (Submersion/Immersion in $\mathbb{R}^m$)**.** *$f = (f_1, \cdots, f_n) : U \subset \mathbb{R}^m \to \mathbb{R}^n$ is C^k-map for $1 \le k \le \infty$ and U open. f is a submersion(immersion) at $x = (x_1, \cdots, x_m) \in U$ if*

$$df_x : \mathbb{R}^m \to \mathbb{R}^n \text{ s.t. } df_x := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_m}(x) \\ \frac{\partial f_n}{\partial x_1}(x) & \cdots & \frac{\partial f_n}{\partial x_m}(x) \end{pmatrix} \text{ is surjective (injective)}$$

under whose case $m \ge n$ ($m \le n$). f is a submersion(immersion) if f is a submersion(immersion) at every $x \in U$.

Example 2.2 (Canonical Submersion)**.** *For $m \ge n$, $\pi : \mathbb{R}^m \to \mathbb{R}^n$ s.t. $\pi(x_1, \cdots, x_m) := (x_1, \cdots, x_n)$ is projection. Here $d\pi_x = \pi : \mathbb{R}^m \to \mathbb{R}^n$ for any $x \in \mathbb{R}^m$.*

Example 2.3 (Canonical Immersion)**.** *For $m \le n$, $i : \mathbb{R}^m \to \mathbb{R}^n$ s.t. $i(x_1, \cdots, x_m) := (x_1, \cdots, x_m, 0, \cdots, 0)$ where $di_x = i : \mathbb{R}^m \to \mathbb{R}^n$ for any $x \in \mathbb{R}^m$.*

Definition 2.4 (Submersion/Immersion)**.** *Let M and N be C^∞-manifold of dimension m, n. $f : M \to N$ C^∞ map is a submersion(immersion) at $p \in M$ if there exists (U, ϕ) chart for M around p and (V, ψ) chart for N around $f(p)$ s.t.*

- *$f(U) \subset V$ and*
- *$g := \psi \circ f \circ \phi^{-1}$ the C^∞ map is a submersion(immersion) at $\phi(p)$, which implies $m \geq n$ ($m \leq n$).*

f is a submersion(immersion) if f is a submersion(immersion) at any point $p \in M$.

$$\begin{tikzcd}
M & \overset{open}{\supseteq} & p \in U \arrow[r, "f"] \arrow[d, "\phi"] & f(p) \in V \arrow[d, "\psi"] & \overset{open}{\subseteq} & N \\
\mathbb{R}^m & \overset{open}{\supseteq} & \phi(p) \in \phi(U) \arrow[r, "g"] & \psi(V) & \overset{open}{\subseteq} & \mathbb{R}^n
\end{tikzcd}$$

Remark 2.3. *This is well-defined as $\tilde{g} = (\tilde{\psi} \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \tilde{\phi}^{-1}) = (\tilde{\psi} \circ \psi^{-1}) \circ g \circ (\phi \circ \tilde{\phi}^{-1})$ and so*

$$d\tilde{g}_{\tilde{\phi}(p)} = d(\tilde{\psi} \circ \psi^{-1})_{g(\phi(p))} \circ (dg)_{\phi(p)} \circ d(\phi \circ \tilde{\phi}^{-1})_{\tilde{\phi}(p)} \text{ is surjective (injective)}$$

for $(\tilde{U}, \tilde{\phi})$ another chart of M around p and $(\tilde{V}, \tilde{\psi})$ another chart of N around $f(p)$ s.t. $f(\tilde{U}) \subset \tilde{V}$.

Proposition 2.1. *M C^∞-manifold of dimension m and N C^∞-manifold of dimension n.*

- *If f is a submersion(immersion) at $p \in M$ ($m \geq n$ ($m \leq n$)), then there exists charts (U, ϕ) for M around p and (V, ψ) for N around $f(p)$ s.t.*

  $$\phi(p) = 0 \in \mathbb{R}^m \qquad \psi(f(p)) = 0 \in \mathbb{R}^n$$

  and

  $$g = \psi \circ f \circ \phi^{-1} : \phi(U) \subset \mathbb{R}^m \to \psi(V) \subset \mathbb{R}^n \text{ is the canoncial submersion (immersion)}$$

  i.e.

  $$g(x_1, \cdots, x_m) = (x_1, \cdots, x_n) \qquad (g(x_1, \cdots, x_m) = (x_1, \cdots, x_m, 0, \cdots, 0)))$$

- *If f is both a submersion and an immersion at p, i.e., $dg_0 : \mathbb{R}^m \to \mathbb{R}^{n=m}$ is a linear isomorphism, then f is a local diffeomorphism at p.*

Proof. Follows from the Rank Theorem. □

2.2 Smooth Embedding and Submanifolds

Definition 2.5 (C^∞ Embedding & Submanifolds)**.** *$f : M \to N$ C^∞ map between C^∞-manifolds. dimension $M = m$, dimension $N = n$. We say f is a smooth embedding if*

- *f is a smooth immersion at any point $p \in M$ (implies $m \leq n$) and*
- *$f : M \to f(M) \subset N$ is a homeomorphism w.r.t. the subspace topology.*

In this case, we call $f(M)$ a C^∞ submanifold of N of dimension m.

Remark 2.4. *Embedding $\implies$ Injective + Immersion, but the converse is not true.*

Definition 2.6 (Alternative definition of submanifold)**.** *Let N be C^∞ manifold of dimension n, M subset of N. M is a C^∞ submanifold of N of dimension $m \leq n$ if*

- *for any $p \in M$, there exists chart (U, ϕ) for N around p s.t. $\phi(p) = 0 \in \mathbb{R}^n$ and*
- *$\phi(U \cap M) = \phi(U) \cap (\mathbb{R}^m \times \{0\})$.*

$$\begin{tikzcd}
M & \overset{open}{\supseteq} & p \in U \cap M \arrow[r, "id"] \arrow[d, "\phi|_{U \cap M}"] & p \in U \arrow[d, "\phi"] & \overset{open}{\subseteq} & N \\
\mathbb{R}^m & \overset{open}{\supseteq} & \phi(U) \cap (\mathbb{R}^m \times \{0\}) \arrow[r] & \phi(p) = 0 \in \phi(U) & \overset{open}{\subseteq} & \mathbb{R}^n
\end{tikzcd}$$

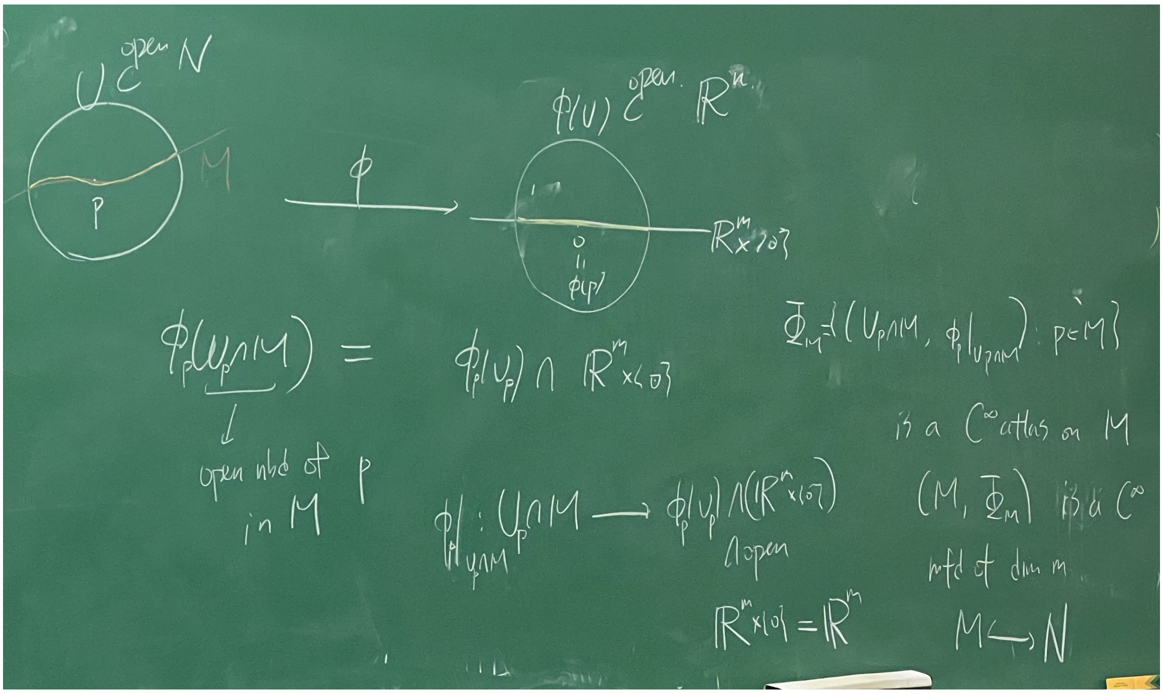

Figure 1: Chart for point on Submanifold Definiton 2.6

Proof for $M \subset N$ is smooth manifold of dimension m in Definition 2.6. For any $p \in M$, there exists local charts (U_p, ϕ_p) for N around p s.t. $\phi_p(p) = 0 \in \mathbb{R}^n$. Moreover, $\phi_p(U_p \cap M) = \phi_p(U_p) \cap (\mathbb{R}^m \times \{0\})$. One wish to define an Atlas on M. Indeed, let $\Phi_M := \left\{(U_p \cap M,\ \phi_p|_{U_p \cap M}) \mid p \in M\right\}$. Since U_p are open in N, $M \subset N$, w.r.t. the subspace topology, $U_p \cap M$ are open neighborhoods of p in M. Moreover, $\phi_p(U_p \cap M) = \phi_p(U_p) \cap (\mathbb{R}^m \times \{0\}) \subset (\mathbb{R}^m \times \{0\}) \cong \mathbb{R}^m$ are open w.r.t. subspace topology. Hence $\phi_p|_{U_p \cap M}$ are local homeomorphisms to subsets of $\mathbb{R}^m$, equipping M with topological m-manifold structure. That $M = M \cap N = \bigcup_{p \in M} M \cap U_p$ and transition functions inherits C^∞ w.r.t. subspace topology make M a m-dim C^∞ manifold. $\square$

Example 2.4. *$f : \mathbb{R} \to \mathbb{R}^2$ for $f(t) := (x(t),\ y(t))$, $f'(t) = (x'(t),\ y'(t))$, then*

$$df_t : \mathbb{R} \to \mathbb{R}^2 \qquad s.t. \qquad df_t(v) := \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} v$$

f is immersion at t iff $f'(t) \neq (0,0)$. For example

- *$f(t) = (t, t^2)$, $f'(t) = (1, 2t)$ is a immersion, and in fact, C^∞-embedding since f is a homeomorphism (in particular, bijective) from $\mathbb{R}$ onto $f(\mathbb{R})$.*
- *$f(t) = (\cos t, \sin t)$ then $f'(t) = (-\sin t, \cos t)$ so $f(\mathbb{R}) = \mathbb{S}^1$. This is immersion but not embedding because f is not injective.*
- *$f(t) = (t^3 - 4t,\ t^2 - 4)$ then $f'(t) = (3t^2 - 4, 2t)$. f is a immersion but not an embedding because f is not injective at $(0,0)$. Note both $t = -2$ and $t = 2$ correspond to $f(-2) = f(2) = (0,0)$.*
- *$f(t) = (t^3,\ t^2)$, $f'(t) = (3t^2,\ 2t)$. This is not immersion at $t = 0$. But $f(\mathbb{R})$ is homeomorphic to $\mathbb{R}$.*

Example 2.5 (counter-example for injective immersion but not embedding)**.** *$f : (-3, 0) \to \mathbb{R}^2$ smooth*

$$f(t) = \begin{cases} (0, -t-2) & -3 < t < -1 \\ \cdots & -1 < t < \frac{-1}{\pi} \\ (-t, -\sin(\frac{1}{t})) & \frac{-1}{\pi} < t < 0 \end{cases}$$

This is not an embedding because $f(-3, 0) \subset \mathbb{R}^2$ is not a topological manifold. In particular, f^{-1} is not continuous at the point $(0,0)$, hence that f needs to be homeomorphism fails.

Now we discuss tool to construct a smooth submanifold using preimage of a regular value.

Remark 2.5. *An immediate observation says preimage of singletons are closed subsets.*

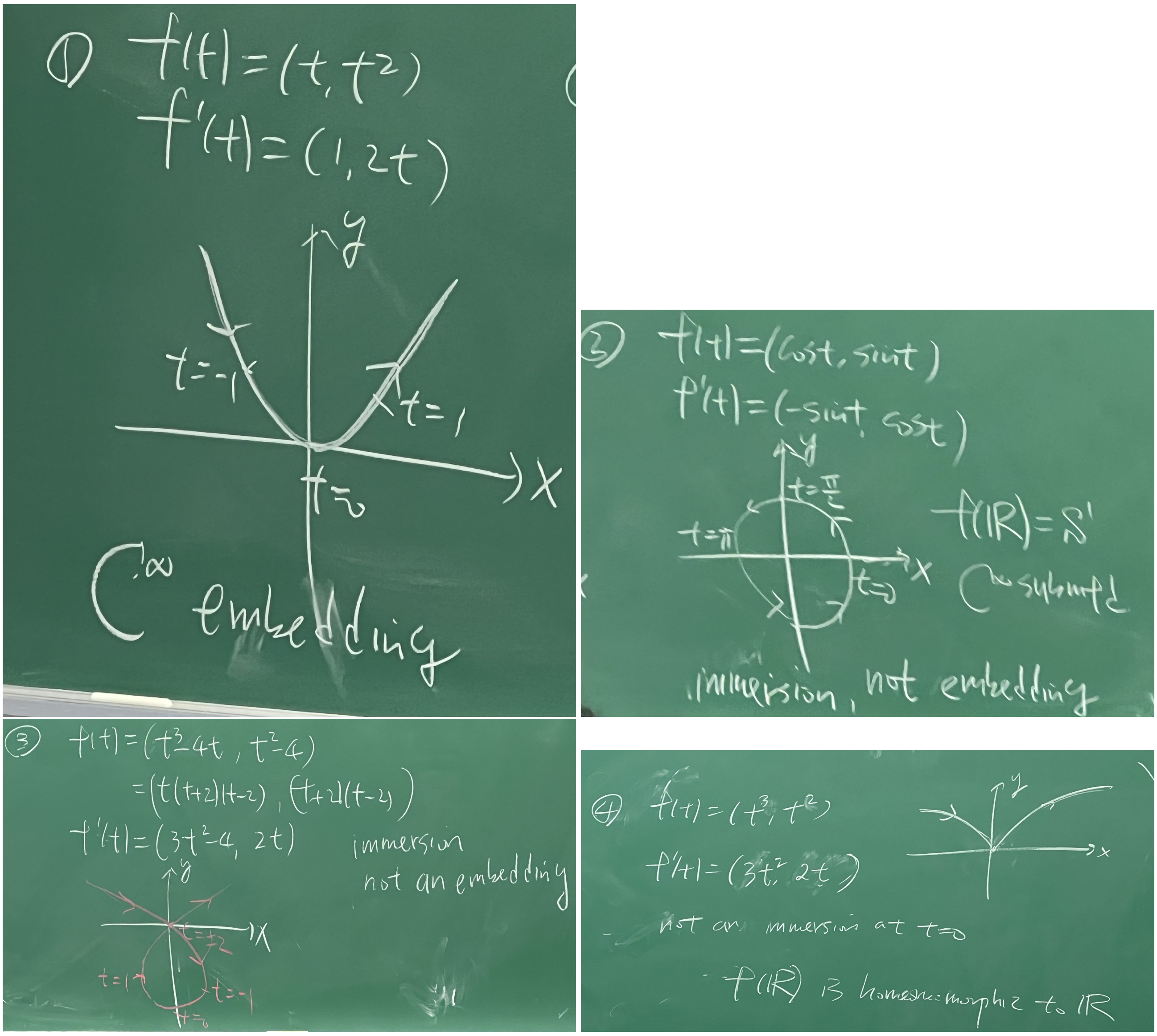

Figure 2: Examples from Example 2.4

- *A topological manifold M may not be a Hausdorff (T_2) space. But this is always a T_1 space, i.e., for any p, $q \in M$ s.t. $p \neq q$, there exists U, V open subsets of M s.t. $p \in U$ but $p \notin U$ and $q \in V$ but $p \notin V$. This is equivalent to saying for any $p \in M$, $\{p\}$ the singleton is closed in M.*

- *Hence for any $f : M \to N$ continuous map between topological manifolds, for any $q \in N$, $f^{-1}(q) \subset M$ is in fact closed.*

Definition 2.7 (Critical Value & Regular Value)**.** *M, N smooth manifolds, and $f : M \to N$ smooth map.*

- *We say $p \in M$ is a critical point of f if f is not a submersion at p.*
- *$q \in N$ is a critical value of f is there exists $p \in M$ critical point of f s.t. $p \in f^{-1}(q)$.*
- *$q \in N$ is a regular value of f if q is not a critical value of f. In other words, for any $p \in f^{-1}(q)$, f is a submersion at p.*

In particular, if $f^{-1}(q)$ is empty, then $q \in N$ is regular value of f.

Theorem 2.1 (Preimage Theorem)**.** *M, N smooth manifolds, and $f : M \to N$ smooth map. Suppose $q \in N$ is a regular value of f, and suppose $f^{-1}(q)$ is not empty (hence $dim(M) = m \geq dim(N) = n$). Then $f^{-1}(q)$ is a closed smooth submanifold of M of dimension $m - n \geq 0$.*

Example 2.6. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ s.t. $f(x_1, \cdots, x_{n+1}) = x_1^2 + \cdots x_{n+1}^2$. f is C^∞ map, and $df_x : \mathbb{R}^{n+1} \to \mathbb{R}$*

$$df_x = (2x_1, \cdots, 2x_{n+1})$$

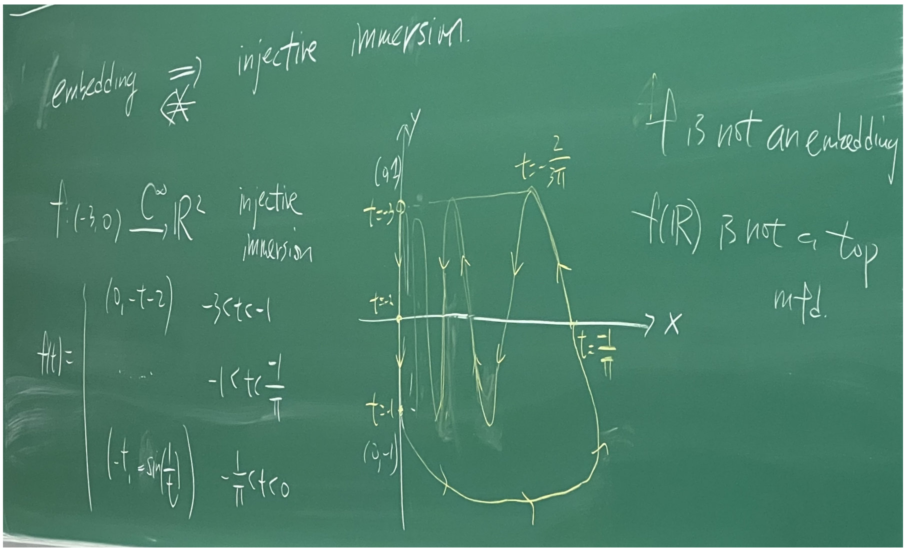

Figure 3: Counter-example for injective immersion but not embedding Example 2.5

the only critical point is $0 \in \mathbb{R}^{n+1}$ *and the only critical value is* $0 \in \mathbb{R}$. *Regular values are* $\mathbb{R} \setminus \{0\}$. *By Preimage Theorem, for any* $a > 0$

$$f^{-1}(a) = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_i x_i^2 = a\} \subset \mathbb{R}^{n+1} =: \mathbb{S}^n(\sqrt{a})$$

is a C^∞*-submanifold of dimension* n. $\mathbb{S}^n(1) = \mathbb{S}^n \subset \mathbb{R}^{n+1}$ *is a* C^∞ *submanifold of dimension* n. *If* $a = 0$, $f^{-1}(0) = 0$ *is just single point. If* $a < 0$, $f^{-1}(0) = \varnothing$.

Example 2.7 (Orthogonal Group)**.** $O(n) := \{A \in M_n(\mathbb{R}) \mid AA^T = I_n \ n \times n \ identity\} \subset M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ *where the latter is linear isomorphism. The subset* $O(n) \subset M_n(\mathbb{R})$ *is a* C^∞ *submanifold of* $M_n(\mathbb{R})$ *of dimension* $\frac{n(n-1)}{2}$.

Proof. Define $f : M_n(A) \cong \mathbb{R}^{n^2} \to S_n(\mathbb{R}) \cong \mathbb{R}^{\frac{n(n+1)}{2}}$ where $S_n(\mathbb{R})$ are real $n \times n$ symmetric matrices. Define $f(A) = AA^T - I_n$ so $O(n) = f^{-1}(0)$. Now if $B = f(A)$, $b_{ij} = \sum_{k=1}^n a_{ik}a_{kj} - \delta_{ij}$. So f is C^∞ map. It remains to show that 0 is a regular value of the map f. For any $A \in M_n(\mathbb{R})$, $df_A : \mathbb{R}^{n^2} \to \mathbb{R}^{\frac{n(n+1)}{2}}$

$$df_A(B) = \lim_{h\to 0} \frac{f(A+hB) - f(A)}{h} = \lim_{h \to 0} \frac{(A+hB)(A^T + hB^T) - I_n - (AA^T - I_n)}{h} = BA^T + AB^T \tag{4}$$

Claim: for $A \in f^{-1}(0) = O(n)$, for $C \in S_n(\mathbb{R})$, there exists $B \in M_n(\mathbb{R})$ s.t. $C = df_A(B) = BA^T + AB^T$. But

$$C = df_A(B) = BA^T + AB^T = BA^T + (BA^T)^T$$
$$\implies \text{Let } BA^T = \frac{1}{2}C \iff B = \frac{1}{2}CA$$

so $B = \frac{1}{2}CA \in M_n(\mathbb{R})$ gives $df_A(B) = \frac{1}{2}CAA^T + A\frac{1}{2}A^TC = C$. Moreover, we conclude that $O(n)$ is submanifold of $M_n(\mathbb{R}) \subset \mathbb{R}^{n^2}$ of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$. □

Example 2.8. *Similarly,* $O(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid A\overline{A}^T = I_n\} \subset M_n(\mathbb{C})$. $O(n, \mathbb{C})$ *is* C^∞ *submanifold of* $M_n(\mathbb{C})$ *of dimension* n^2. *(*$M_n(\mathbb{C}) \cong \mathbb{C}^n \cong \mathbb{R}^{2n^2}$*).*

3 Orientation

Definition 3.1 (Orientation)**.** *Let M be C^k manifold of dimension n. We say M is orientable if there exists a C^k-atlas $\Phi = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ on M s.t. for any $U_\alpha \cap U_\beta \neq \varnothing$,*

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n \to \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$$

is C^k diffeomorphism, and for any $x \in \phi_\alpha(U_\alpha \cap U_\beta)$,

$$d(\phi_\beta \circ \phi_\alpha^{-1})_x \in GL(n, \mathbb{R}) := \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\} \qquad \textit{where } \det(d(\phi_\beta \circ \phi_\alpha^{-1})_x) > 0 \tag{5}$$

Note we only require there exists one such Atlas.

- *If M is orientable, an orientation Φ on M is a choice of C^k-altas satisfying (5).*
- *if both Φ and Ψ on M satisfy (5), we say they define the same orientation if $\Phi \cup \Psi$ still satisfies (5).*

Example 3.1 ($P_n(\mathbb{C})$)**.** *$P_n(\mathbb{C})$ is orientable. One compute*

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \subset \mathbb{C}^n \to \phi_j(U_i \cap U_j) \subset \mathbb{C}^n$$

its differential

$$d(\phi_j \circ \phi_i^{-1})_{y_1, \cdots, y_n} : \mathbb{C}^n \to \mathbb{C}^n \qquad \mathbb{C} - \textit{linear map}$$

In general, for L a $\mathbb{C}$-linear map,

$$\begin{array}{ccc} x + iy \in \mathbb{C}^n & \xrightarrow{L} & L(x+iy) \in \mathbb{C}^n \\ \big\downarrow & & \big\downarrow \\ (x, y) \in \mathbb{R}^{2n} & \xrightarrow{L_{\mathbb{R}}} & L_{\mathbb{R}}(x, y) \in \mathbb{R}^{2n} \end{array}$$

there exists $C \in M_n(\mathbb{C})$ s.t.

$$x + iy \mapsto C(x + iy) \qquad \textit{for } C = A + iB \textit{ where } A,\, B \in M_n(\mathbb{R})$$

hence

$$C(x+iy) = (A+iB)(x+iy) = (Ax - By) + i(Bx + Ay) \qquad \textit{i.e. } \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $\det(\begin{bmatrix} A & -B \\ B & A \end{bmatrix}) = |\det(C)|^2$. *So L being linear isomorphism implies* $\det(\begin{bmatrix} A & -B \\ B & A \end{bmatrix}) > 0$. *Hence*

$$\det(d(\phi_j \circ \phi_i^{-1})_{y_1, \cdots, y_n}) > 0$$

More generally, if M is a complex manifold of complex dimension n, then M is an orientable C^∞ manifold of real dimension $2n$. It is indeed oriented.

Example 3.2 ($P_n(\mathbb{R})$)**.** *For real, $P_n(\mathbb{R})$ is orientable $\iff$ n is odd. Look at some examples. $P_1(\mathbb{R}) \cong \mathbb{S}^1$ so orientable, but $P_2(\mathbb{R})$ is not.*

4 Tangent Space and Tangent Bundles

Idea: first, let M be an n-dim C^∞ submanifold of $\mathbb{R}^{n+k}$. For any $p \in M$, there exists U open neighborhood of p that maps $\phi(U) \subset \mathbb{R}^n$. Now we view its inverse

$$\phi^{-1} : \phi(U) \subset \mathbb{R}^n \to M \subset \mathbb{R}^{n+k}$$

as smooth embedding so

$$d(\phi^{-1})_{\phi(p)} : \mathbb{R}^n \to \mathbb{R}^{n+k}$$

is injective linear map. We define the tangent space

$$T_pM = Im(d(\phi^{-1})_{\phi(p)}) \subset \mathbb{R}^{n+k}$$

This is well-defined as if there's another chart (V, ψ) around p s.t. $T_pM = Im(d(\psi^{-1})_{\psi(p)})$, then $d(\psi \circ \phi^{-1})_{\phi(p)}$ transits smoothly.

4.1 Tangent Space and Differential

Definition 4.1 (Tangent Space)**.** *M C^k manifold for $k \geq 1$ of dimension n. $p \in M$.*

$$T_pM := \{(U, \phi, u) \mid (U, \phi) \text{ is } C^k \text{ chart for } M \text{ around } p,\ u \in \mathbb{R}^n\} / \sim p$$

where

$$(U, \phi, u) \underset{p}{\sim} (V, \psi, v) \iff d(\psi \circ \phi^{-1})_{\phi(p)}(u) = v$$

define the map

$$\theta_{U,\phi,p} : \mathbb{R}^n \to T_pM \quad s.t.\ u \mapsto [U, \phi, u] \qquad \text{this is bijection} \tag{6}$$

Use this to equip T_pM with the structure of a vector space over $\mathbb{R}$. This structure is well-defined because diagram commutes.

$$\begin{tikzcd} \mathbb{R}^n \arrow[d, "d(\psi\circ\phi^{-1})|_{\phi(p)}"'] \arrow[dr, "\theta_{U,\phi,p}"] & \\ \mathbb{R}^n \arrow[r, "\theta_{U,\psi,p}"] & T_pM \end{tikzcd}$$

Notice the diagram is equivalent to saying

$$d(\psi \circ \phi^{-1})\big|_{\phi(p)} = \theta^{-1}_{U,\psi,p} \circ \theta_{U,\phi,p} \tag{7}$$

Call T_pM tangent space to M at p. A tangent vector to M at p is an element in T_pM.

Definition 4.2 (Differential)**.** *M, N C^k manifolds $k \geq 1$ with dimension m, n. $f : M \to N$ C^k map. The differential of f at p is a linear map*

$$df_p : T_pM \to T_{f(p)}N$$

s.t. for any (U, ϕ) C^k chart around p in M and (V, ψ) C^k chart around $f(p)$ in N, letting $g = \psi \circ f \circ \phi^{-1}$ be local representation of f, df_p denotes the composition

$$df_p := \theta_{V,\psi,f(p)} \circ dg_{\phi(p)} \circ \theta^{-1}_{U,\phi,p} \qquad so \qquad df_p([U, \phi, u \in \mathbb{R}^m]) := [V, \psi, dg_{\phi(p)}(u) \in \mathbb{R}^n]$$

Indeed the diagram for differential commutes

$$\begin{tikzcd} M & p \in U \arrow[l, phantom, "\overset{open}{\supseteq}"] \arrow[r, "f"] \arrow[d, "\phi"] & V \arrow[d, "\psi"] & N \arrow[l, phantom, "\overset{open}{\subseteq}"'] \\ \mathbb{R}^m & \phi(p) \in \phi(U) \arrow[l, phantom, "\overset{open}{\supseteq}"] \arrow[r, "g"] & \psi(V) & \mathbb{R}^n \arrow[l, phantom, "\overset{open}{\subseteq}"'] \end{tikzcd} \qquad \begin{tikzcd} T_pM \arrow[r, "df_p"] & T_{f(p)}N \\ \mathbb{R}^m \arrow[u, "\theta_{U,\phi,p}"] \arrow[r, "dg_{\phi(p)}"] & \mathbb{R}^n \arrow[u, "\theta_{V,\psi,f(p)}"'] \end{tikzcd}$$

Theorem 4.1. *f is a submersion(immersion) at p if $df_p : T_pM \to T_{f(p)}N$ is surjective (injective).*

Lemma 4.1 (Chain Rule for manifolds)**.** *If $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are C^k maps between C^k manifolds, where $k \geq 1$.*

- *$g \circ f : M_1 \to M_3$ is C^k*
- *For any $p \in M_1$, $df_p : T_pM_1 \to T_{f(p)}M_2$, $dg_{f(p)} : T_{f(p)}M_2 \to T_{g(f(p))}M_3$, then*

$$d(g \circ f)_p = dg_{f(p)} \circ df_p : T_pM_1 \to T_{g\circ f(p)}M_3$$

One has tool to construct tangent space via preimage theorem.

Theorem 4.2 (Linear Subspace and closed submanifold)**.**

- *If $M \subset N$ for C^∞ manifolds. Let $i : M \to N$ be inclusion map (hence smooth embedding, in particular, immersion at any point). For any $p \in M$,*

$$di_p : T_pM \to T_pN \qquad \textit{is an injection}$$

T_pM is a linear subspace of T_pN.

- *If $f : M \to N$ C^∞ map with $q \in N$ regular value of f s.t. $f^{-1}(q)$ is not empty. Hence $m = dimM \geq n = dimN$. By Preimage theorem, $S := f^{-1}(q) \subset M$ is a closed submanifold of M of dimension $n - m$. Now for any $p \in S$*

$$T_pS = ker(df_p : T_pM \cong \mathbb{R}^m \to T_{f(p)}N \cong \mathbb{R}^n) \tag{8}$$

In other words, there is a short exact sequence of real vector spaces

$$0 \to T_pS \to T_pM \to T_{f(p)}N \to 0$$

One make use of (8) to compute explicitly tangent space of submanifolds.

Example 4.1. *For any $p \in \mathbb{R}^n$, we have linear isomorphism $T_p\mathbb{R}^n \cong \mathbb{R}^n$ given by (6)*

$$[\mathbb{R}^n, id, u] \in T_p\mathbb{R}^n \mapsto \theta^{-1}_{\mathbb{R}^n,id,p}([\mathbb{R}^n, id, u]) = u \in \mathbb{R}^n$$

Example 4.2 ($T_x\mathbb{S}^n$)**.** *$f : \mathbb{R}^{1+n} \to \mathbb{R}$ for $f(x_1, \cdots, x_{n+1}) = \sum_{i=1}^{n+1} x_i^2$. f is C^∞ map, 1 is regular value of f. so $\mathbb{S}^n := f^{-1}(1)$ is a C^∞ submanifold of f of dimension n. For any $x \in \mathbb{R}^{1+n}$, $df_x(v) = 2x \cdot v$. And for any $x \in \mathbb{S}^n$, using (8)*

$$T_x\mathbb{S}^n := \{v \in T_x\mathbb{R}^{1+n} \mid df_x(v) = 0\} = \{v \in \mathbb{R}^{1+n} \mid x \cdot v = 0\} \subset T_x\mathbb{R}^{1+n} \cong \mathbb{R}^{1+n}$$

where the linear isomorphism is viewed via $\theta_{\mathbb{R}^{1+n},id,x}$ (6).

Example 4.3 ($T_AO(n)$)**.** *$O(n) = f^{-1}(I_n)$ for*

$$f : M_n(\mathbb{R}) \cong \mathbb{R}^{n^2} \to S_n(\mathbb{R}) \cong \mathbb{R}^{\frac{n(n+1)}{2}} \qquad s.t.\ f(A) = AA^T$$

here I_n is a regular value of f. For any $A \in O(n)$, using Remark (8)

$$T_AO(n) = \{B \in M_n(\mathbb{R}) \mid df_A(B) = 0\} \subset T_AM_n(\mathbb{R}) \cong M_n(\mathbb{R})$$

where $\cong$ is done via $\theta_{M_n(\mathbb{R}),id,A}$ (6). Then recalling $df_A(B) = BA^T + AB^T$ (4)

$$T_AO(n) = \{B \in M_n(\mathbb{R}) \mid BA^T + AB^T = 0\}$$

In particular at identity

$$T_{I_n}O(n) = \{B \in M_n(\mathbb{R}) \mid B + B^T = 0\} \quad \textit{skew symmetric matrices}$$

4.2 Tangent Bundle

Definition 4.3 (Tangent Bundle)**.** *Given C^k manifold M of dimension n where $k \in \mathbb{N}$. We will construct the tangent bundle TM of M as a C^{k-1} manifold of dimension $2n$.*

- *As a set, the tangent bundle of M is*

$$TM = \{(p, v) \mid p \in M, v \in T_pM\} = \bigsqcup_{p \in M} T_pM$$

Define $\pi : TM \to M$ as $(p, v) \mapsto p$. π is a surjective map.

- *Topology. If (U, ϕ) is a C^k chart for M, we define*

$$\tilde{\phi} : \pi^{-1}(U) \subset TM \to \phi(U) \times \mathbb{R}^n \subset \mathbb{R}^{2n} \ s.t.\ (p, v) \mapsto (\phi(p), \theta^{-1}_{(U,\phi,p)}(v))$$

where $\theta_{(U,\phi,p)}(u) = [U, \phi, u] \in T_pM$. It is bijection. Now take any C^k atlas $\Phi = \{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$ on M.

$$F : \bigsqcup_{\alpha \in I} \phi_\alpha(U_\alpha) \times \mathbb{R}^n \to TM \qquad s.t.\ (x, u) \to (\phi_\alpha^{-1}(x) \in M, \theta_{(U_\alpha,\phi_\alpha,\phi_\alpha^{-1}(x))}(u) \in T_{\phi_\alpha(x)}M)$$

We equip TM with the quotient topology determined by the surjective map F. Then TM is a topological $2n$-manifold with

1. $\tilde{\Phi} = \{(\pi^{-1}(U_\alpha), \tilde{\phi_\alpha}) \mid \alpha \in I\}$ *Atlas*
2. $\tilde{\phi_\alpha} : \pi^{-1}(U_\alpha) \subset TM \to \phi_\alpha(U_\alpha) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ *s.t.* $(p, v) \mapsto (\phi(p), \theta^{-1}_{(U,\phi,p)}(v))$

$$\begin{array}{ccccccc} TM & \overset{open}{\supseteq} & (p,v) \in \pi^{-1}(U_\alpha) & \xrightarrow{\pi} & p \in U_\alpha & \overset{open}{\subseteq} & M \\ & & \downarrow{\scriptstyle \tilde{\phi_\alpha}} & & \downarrow{\scriptstyle \phi_\alpha} & & \\ \mathbb{R}^{2n} & \overset{open}{\supseteq} & \phi_\alpha(U_\alpha) \times \mathbb{R}^n & \xrightarrow{\pi_{can}} & \phi_\alpha(U_\alpha) & \overset{open}{\subseteq} & \mathbb{R}^n \end{array}$$

where the diagram commutes and $\pi_{can} = \phi_\alpha \circ \pi \circ \tilde{\phi}_\alpha^{-1}$ *is the canonical submersion from* $\phi_\alpha(U_\alpha) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ *onto the first* n *coordinates* $\phi_\alpha(U_\alpha) \subset \mathbb{R}^n$.

- *We wish to compute transition functions. For any* U *open set of* M, *one may identify*

$$\pi^{-1}(U) = TU = \bigsqcup_{p \in U} T_p U$$

Note $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta) = \pi^{-1}(U_\alpha \cap U_\beta)$. *And given two charts* (U_α, ϕ_α), (U_β, ϕ_β) *for* M, *we have two corresponding charts* $(TU_\alpha, \tilde{\phi_\alpha})$, $(TU_\beta, \tilde{\phi_\beta})$ *for* TM. *Hence*

$$\tilde{\phi_\alpha}(\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)) = \tilde{\phi_\alpha}(\pi^{-1}(U_\alpha \cap U_\beta)) = \phi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

For any $U_\alpha \cap U_\beta \neq \varnothing$

$$\tilde{\phi_\beta} \circ \tilde{\phi_\alpha}^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n \to \phi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^n \qquad (x, u) \mapsto (\phi_\beta \circ \phi_\alpha^{-1}(x),\ \theta^{-1}_{U_\beta, \phi_\beta, \phi_\beta^{-1}(x)} \circ \theta_{U_\alpha, \phi_\alpha, \phi_\alpha^{-1}(x)}(u))$$

using diagram (7), *one may write our transition function as*

$$\tilde{\phi_\beta} \circ \tilde{\phi_\alpha}^{-1}(x, u) := \big(\phi_\beta \circ \phi_\alpha^{-1}(x),\ d(\phi_\beta \circ \phi_\alpha^{-1})_x(u)\big)$$

Since $\phi_\beta \circ \phi_\alpha^{-1}$ *is* C^k *in* $x \in \phi_\alpha(U_\alpha \cap U_\beta)$ *while* $d(\phi_\beta \circ \phi_\alpha^{-1})_x$ *in* C^{k-1} *in* $u \in \mathbb{R}^n$, *our* $\tilde{\phi_\beta} \circ \tilde{\phi_\alpha}^{-1}(x, u)$ *are* C^{k-1} *maps in* $(x, u) \in \phi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n$. *So* $\tilde{\Phi}$ *is a* C^{k-1} *atlas on* TM. $(TM, \tilde{\Phi})$ *is a* C^{k-1} *manifold of dimension* $2n$.

- *Our surjective map* $\pi : TM \to M$ *is* C^{k-1} *map due to* $\pi = \phi_\alpha^{-1} \circ \pi_{can} \circ \tilde{\phi}_\alpha$ *as composition with* C^{k-1} *charts. For* $k \geq 2$, π *is a submersion.*

- *Moreover,* TM *is orientable* C^{k-1} *manifold of dimension* $2n$, *even though* M *might not be.*

Definition 4.4. *Suppose* $f : M \to N$ C^k *map where* $k \geq 1$ *or* $k = \infty$. *Define*

$$df : TM \to TN \qquad s.t.\ (p, v) \mapsto (f(p),\ df_p(v)) \qquad for\ p \in M\ and\ v \in T_pM$$

Proposition 4.1. *If* $f : M \to N$ *is* C^k *map between* C^k *manifolds where* $k \geq 1$. *Then* $df : TM \to TN$ *is a* C^{k-1} *map between* C^{k-1} *manifolds. For* $k \geq 2$, $d(df) : T(TM) \to T(TN)$ *is defined.*

- *If* f *is a submersion(immersion), then* df *is a submersion(immersion). If* f *is submersion(immersion) at some point* $p \in M$, *then* df *is a submersion(immersion) at* (p, v) *for any* $v \in T_pM$.

- *If* N *is smooth manifold of dimension* n *and* M *smooth submanifold of dimension* $m \leq n$. *Then* $TM = \{(p, v) \mid p \in M,\ v \in T_pM\} \subset TN = \{(p, v) \mid p \in N,\ v \in T_pN\}$ C^∞ *manifold of dimension* $2n$. *Hence* TM *is* C^∞ *submanifold of dimension* $2m$.

Example 4.4. *Recall* $id : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. $T\mathbb{S}^n \subset T\mathbb{R}^{n+1} \overset{\tilde{id}}{\to} \mathbb{R}^{1+n} \times \mathbb{R}^{1+n}$. *Here*

$$\begin{aligned} T\mathbb{S}^n &= \{(x, v) \in \mathbb{R}^{1+n} \times \mathbb{R}^{1+n} \mid x \in \mathbb{S}^n\ v \in T_x\mathbb{S}^n\} \\ &= \{(x, v) \in \mathbb{R}^{1+n} \times \mathbb{R}^{1+n} \mid x \cdot x = 1,\ x \cdot v = 0\} \end{aligned}$$

and

$$TO(n) = \{(A, B) \in M_n(\mathbb{R}) \times M_n(\mathbb{R}) : AA^T = I_n,\ BA^T + AB^T = 0\} \subset TM_n(\mathbb{R}) \cong M_n(\mathbb{R}) \times M_n(\mathbb{R})$$

$TO(n)$ *is* C^∞ *submanifold of dimension* $n(n-1)$.

5 Vector Bundles

5.1 Vector Bundle and examples

Definition 5.1 (Vector Bundles). *Let M be C^k manifold with $n = dim M$. A C^k real vector bundle of rank r over M is*

- *a C^k manifold E together with*
- *a surjective C^k map*

$$\pi : E \to M$$

s.t.

1. *Local Trivialization. There exists an open over $\{U_\alpha\}_{\alpha \in I}$ of M (not necessarily the open charts) and a family of associated C^k diffeomorphisms h_α for $k \geq 1$ (or homeomorphism for $k = 0$)*

$$h_\alpha : \pi^{-1}(U_\alpha) \subset E \to U_\alpha \times \mathbb{R}^r$$

s.t. for $pr_1 : (p, v) \in U_\alpha \times \mathbb{R}^r \mapsto p \in U_\alpha$

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[d, "h_\alpha"'] \arrow[dr, "\pi_\alpha"] & \\
U_\alpha \times \mathbb{R}^n \arrow[r, "pr_1"] & U_\alpha
\end{tikzcd}$$

the diagram commutes $\pi_\alpha := \pi|_{\pi^{-1}(U_\alpha)} = pr_1 \circ h_\alpha$ (implying π is a submersion if $k \geq 1$)

2. *Transition Functions. For any U_α, U_β open subsets of M (not necessarily homeomorphic to open subsets of $\mathbb{R}^n$).*

$$h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r \qquad h_\beta : \pi^{-1}(U_\beta) \to U_\beta \times \mathbb{R}^r \qquad \textit{local trivializations}$$

Then for any $U_\alpha \cap U_\beta \neq \varnothing$

$$h_\beta \circ h_\alpha^{-1} : U_\alpha \cap U_\beta \times \mathbb{R}^r \to U_\alpha \cap U_\beta \times \mathbb{R}^r \qquad s.t.\ (p, v) \mapsto (p, g_{\beta\alpha}(p)(v)) \qquad is\ a\ C^k\ diffeomorphism$$

where

$$\mathbb{R}^r \cong \{p\} \times \mathbb{R}^r \overset{g_{\beta\alpha}(p)}{\to} \{p\} \times \mathbb{R}^r \cong \mathbb{R}^r$$

s.t. $g_{\beta\alpha}(p) \in GL(r, \mathbb{R})$ a linear isomorphism between $\mathbb{R}^r$ for any p. In other words

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(r, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\} \subset M_n(\mathbb{R}) \qquad C^k\ map$$

Here E is called total space and M is called the base of the vector bundle.

Definition 5.2 (Alternative definition of vector bundle). *Let M be a C^k manifold, $k \in \mathbb{N} \cup \{\infty\}$. We say $\pi : E \to M$ is C^k real vector bundle of rank r with total space E and base M if*

- *E is a C^k manifold*
- *π is a surjective C^k map*

and

- *For any $x \in M$, the fiber of E at x, $E_x := \pi^{-1}(x)$, is equipped with the structure of a real vector space of dimension r. π is defined by*

$$E = \bigsqcup_{x \in M} E_x \overset{\pi}{\to} M \qquad s.t. \qquad \pi(E_x) = x$$

- *Local Trivialization. For any $x \in M$, there exists open neighborhood U of x in M and a C^k diffeomorphism $h : \pi^{-1}(U) \to U \times \mathbb{R}^r$ s.t. $\pi = pr_1 \circ h$ diagram commutes and*

$$\forall\ x \in U,\ h|_{E_x} : E_x \to \{x\} \times \mathbb{R}^r \qquad \textit{is a linear isomorphism}$$

Remark 5.1. *It follows from the above definition that $\pi : E \to M$ is a C^k vector bundle of rank r with total space E and base M. Hence one may find open cover $\{U_\alpha\}_{\alpha\in I}$ of the base M where the open cover is not necessarily the local coordinate chart. And the local trivializations*

$$h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r \qquad are\ C^k\ diffeomorphisms$$

s.t. $\pi_\alpha := \pi|_{\pi^{-1}(U_\alpha)} = pr_1 \circ h_\alpha$ *diagram commutes and*

$$\forall\ x \in U_\alpha \quad h_\alpha|_{E_x} : E_x \to \{x\} \times \mathbb{R}^r \qquad is\ a\ linear\ isomorphism$$

Now one may consider transition functions

$$h_\beta \circ h_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^r \to (U_\alpha \cap U_\beta) \times \mathbb{R}^r\ s.t.\ (x, v) \mapsto (x, g_{\alpha\beta}(x)v)$$
$$where\ g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{R}) \subset M_r(\mathbb{R})\ s.t.\ x \mapsto g_{\alpha\beta}(x) = (g_{\alpha\beta}(x))_{ij}\ is\ C^k\ map$$

Example 5.1 (Product Vector Bundle). $E = M \times \mathbb{R}^r$ *where* $\pi = pr_1 : E \to M$. *This is product vector bundle of rank r over M*

Definition 5.3 (vector bundle isomorphism). *Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be 2 C^k vector bundles over the same C^k manifold M. A C^k vector bundle isomorphism from $\pi_E : E \to M$ to $\pi_F : F \to M$ is a C^k diffeomorphism h*

$$h : E \to F \quad s.t.\ \pi_E = \pi_F \circ h\ diagram\ commutes$$

in other words

$$\forall\ x \in M,\ \ h|_{E_x} : E_x \to F_x \qquad is\ a\ linear\ isomorphism$$

We say 2 C^k vector bundles are isomorphic if there exists such a C^k isomorphism.

Example 5.2 (Trivial Vector Bundle). *We say a C^k vector bundle $\pi : E \to M$ is trivial vector bundle of rank r if it is isomorphic to the product vector bundle $pr_1 : M \times \mathbb{R}^r \to M$. In other words, there exists $h : E \to M \times \mathbb{R}^r$ C^k diffeomorphism (or homeomorphism for $k = 0$) s.t.*

1. *$\pi = pr_1 \circ h$ diagram commutes.*
2. *the restriction of h to each fiber E_x is a linear isomorphism*

$$h|_{E_x} : E_x \subset E \to \{x\} \times \mathbb{R}^r$$

In a word, $\pi : E \to M$ is trivial vector bundle if there exists only one global trivialization $h : E \to M \times \mathbb{R}^r$.

Example 5.3 (Tangent Bundle). *Let M be a C^k manifold where $k \geq 1$. Then $\pi : TM \to M$ is a C^{k-1} vector bundle over M of rank $n = dimM$. Recall we've constructed*

$$TM = \bigsqcup_{p\in M} T_pM\ with\ \Phi = \{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}\ C^k\ atlas\ on\ M$$
$$a\ new\ \tilde{\Phi} = \{(\pi^{-1}(U_\alpha), \tilde{\phi_\alpha}) \mid \alpha \in I\}\ C^{k-1}\ atlas\ on\ TM$$

- *Local Trivialization of TM.*

$$h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n\ s.t.\ (p, v) \mapsto (p, \theta^{-1}_{U_\alpha,\phi_\alpha,p}(v))$$

- *Transition Functions (as C^{k-1} manifold of dimension $2n$)*

$$\tilde{\phi_\beta} \circ \tilde{\phi_\alpha}^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n \to \phi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^n\ s.t.\ (x, u) \mapsto (\phi_\beta \circ \phi_\alpha^{-1}(x),\ d(\phi_\beta \circ \phi_\alpha^{-1})_x(u)$$
$$h_\beta \circ h_\alpha^{-1} : U_\alpha \cap U_\beta \times \mathbb{R}^n \to U_\alpha \cap U_\beta \times \mathbb{R}^n\ s.t.\ (p, u) \mapsto (p, d(\phi_\beta \circ \phi_\alpha^{-1})_{\phi_\alpha(p)}(u))$$

5.2 Sections

Definition 5.4 ($C^\ell(M)$). *For M a C^k manifold, let $C^\ell(M)$ be space of C^ℓ functions for $f : M \to \mathbb{R}$ with $\ell \leq k$. One has inclusion $C^k(M) \subset C^{k-1}(M) \subset \cdots$*

Definition 5.5 (C^k section). *A C^k section of a C^k vector bundle $\pi : E \to M$ over C^k manifold M is a C^k map $s : M \to E$ s.t. $\pi \circ s : M \to M$ is the identity map, i.e.*

$$\forall\ x \in M,\ \ s(x) \in E_x = \pi^{-1}(x)$$

Define

$$C^k(M, E) = \{C^k\ sections\ s : M \to E\}$$

Indeed $C^k(M, E)$ is itself vector space

Lemma 5.1. *For any $f \in C^k(M)$ and $s \in C^k(M,E)$, one has $fs \in C^k(M,E)$ where for any $x \in M$, $fs(x) := f(x)s(x)$ where $f(x) \in \mathbb{R}$ and $s(x) \in E_x$. So $C^k(M,E)$ is a $C^k(M)$-module.*

Proposition 5.1. *Let $\pi : E \to M$ be a C^k vector bundle of rank r over a C^k manifold M of dimension n. Then it is trivial iff there exists C^k sections $\{s_1, \cdots, s_r\}$ of $\pi : E \to M$ s.t. for any $x \in M$, $\{s_1(x), \cdots, s_r(x)\} \subset E_x$ is a basis of E_x.*

Proof. $\Longrightarrow$. $\pi : E \to M$ is trivial, then there exists $h : E \to M \times \mathbb{R}^r$ C^k diffeomorphism that is global trivialization s.t. $\pi = pr_1 \circ h$ diagram commutes. For any C^k section $s : M \to E$, their composition are

$$(h \circ s)(x) = (x, f(x)) \qquad for\ f : M \to \mathbb{R}^r\ C^k\ map$$

For $\{e_1, \cdots, e_r\}$ standard basis of $\mathbb{R}^r$, one define for $1 \le i \le r$

$$s_i := h^{-1}(x, e_i)$$

Then s_i are C^k sections of $\pi : E \to M$. Now for any $x \in M$, using $h|_{E_x}$ as linear isomorphism between E_x and $\mathbb{R}^r$

$$E_x \overset{h|_{E_x}}{\to} \{x\} \times \mathbb{R}^r = \mathbb{R}^r \qquad s.t. \qquad h \circ s_i(x) = (x, e_i) \mapsto\ e_i$$

so $\{s_1(x), \cdots, s_r(x)\}$ are basis of E_x.
$\Longleftarrow$. Let $\{s_1, \cdots, s_r\}$ be C^k sections of $\pi : E \to M$ s.t. for any $x \in M$, $s_1(x), \cdots, s_r(x) \in E_x$ is a basis of $E_x \cong \mathbb{R}^r$. Define

$$\phi : M \times \mathbb{R}^r \to E\ s.t.\ \phi(x, v) := \sum_{i=1}^{r} v_i s_i(x)\ \in E_x \subset E$$

Then $pr_1 = \pi \circ \phi$ diagram commutes. For any $x \in M$, $\{x\} \times \mathbb{R}^r \overset{\phi|_{\{x\} \times \mathbb{R}^r}}{\to} E_x$ is a linear isomorphism. It remains to show that ϕ is a C^k diffeomorphism so that ϕ is a vector bundle isomorphism between the product vector bundle and $\pi : E \to M$. Since $\pi : E \to M$ is a C^k vector bundle, there exists open cover $\{U_\alpha \mid \alpha \in I\}$ of M and local trivializations s.t. $\pi = pr_1 \circ h_\alpha$ diagram commutes. One needs to check that $h_\alpha \circ \phi : U_\alpha \times \mathbb{R}^r \to U_\alpha \times \mathbb{R}^r$ is a C^k diffeomorphism. But for any $j \in \{1, \cdots, r\}$

$$h_\alpha \circ s_j : U_\alpha \to U_\alpha \times \mathbb{R}^r\ s.t.\ (x) \mapsto (x, \begin{pmatrix} s_{1j}(x) \\ \vdots \\ s_{rj}(x) \end{pmatrix})\ where\ s_{ij}(x)\ are\ C^k\ functions\ on\ U_\alpha$$

hence $A(x) = (s_{ij}(x)) \in GL(r, \mathbb{R})$. Now

$$h_\alpha \circ \phi(x, v = \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix}) = h_\alpha(\sum_{i=1}^{r} v_j s_j(x)) = (x, \begin{pmatrix} \sum_{j=1}^{r} v_j s_{1j}(x) \\ \vdots \\ \sum_{j=1}^{r} v_j s_{rj}(x) \end{pmatrix}) = (x, A(x)v)\ where\ A(x) = \begin{pmatrix} s_{11}(x) & \cdots & s_{1r}(x) \\ \vdots & \ldots & \vdots \\ s_{r1}(x) & \cdots & s_{rr}(x) \end{pmatrix}$$

here $(h_\alpha \circ \phi)(x, v) = (x, A(x)v)$ and $(h_\alpha \circ \phi)^{-1}(x, u) = (x, A(x)^{-1}u))$ so $A, A^{-1} : U_\alpha \to GL(r, \mathbb{R})$ are C^k maps. Hence $h_\alpha \circ \phi$ indeed defines C^k diffeomorphisms. $\square$

6 Derivations and Vector Fields

6.1 Local Derivations and Tangent Space Isomorphism

Definition 6.1 (Germs)**.** *Let M be C^k manifold. $k \in \mathbb{N} \cup \{\infty\}$. Given $p \in M$, we define*

$$C_p^k(M) = \{(f : U \to \mathbb{R}) \mid U \text{ open neighborhood of } p \text{ in } M,\ f \text{ is } C^k \text{ function}\}/\sim p$$

where we write the equivalence class as

$$(f : U \to \mathbb{R}) \overset{p}{\sim} (g : V \to \mathbb{R}) \iff \text{there exists open neighborhood } W \text{ of } p \text{ in } M \text{ s.t. } W \subset U \cap V \text{ and } f|_W = g|_V$$

an element $[f : U \to \mathbb{R}]$ in $C_p^k(M)$ is called a germ of C^k functions at p.

Remark 6.1. *$C^k(M) \subset C^{k-1}(M) \subset \cdots$ and $\forall\ p \in M$, $C_p^k(M) \subset C_p^{k-1}(M) \subset \cdots$. These are inclusion of subrings.*

$$[f : U \to \mathbb{R}] + [g : V \to \mathbb{R}] = [f + g : U \cap V \to \mathbb{R}]$$
$$[f : U \to \mathbb{R}][g : V \to \mathbb{R}] = [fg : U \cap V \to \mathbb{R}]$$

Remark 6.2. *One has useful ring homomorphisms that simplifies the problem.*

- *If (U, ϕ) is a C^k chart for M around p s.t. $\phi(p) = 0$*

  $$C_p^k(M) \to C_0^k(\mathbb{R}^n) \text{ s.t. } [f : V \to \mathbb{R}] = [f|_{U\cap V} : U \cap V \to \mathbb{R}] \mapsto [f \circ \phi^{-1} : \phi(U \cap V) \to \mathbb{R}]$$

  is a ring isomorphism

- $$C^k(M) \to C_p^k(M) \text{ s.t. } (f : M \to \mathbb{R}) \mapsto [f : M \to \mathbb{R}]$$

  is a surjective ring homomorphism. To see it is surjective, given $[f : V \to \mathbb{R}] \in C_p^k(M)$, there exists $\beta \in C^k(V)$ with $\mathrm{supp}(\beta) \subset V$ s.t. $(\beta : V \to \mathbb{R}) \overset{p}{\sim} (1 : M \to \mathbb{R})$. Hence

  $$[f : V \to \mathbb{R}] = [\beta f : V \to \mathbb{R}]$$

  and βf can be extended to M due to Hausdorff topology on M. But it is not injective.

- *If M is a real analytic C^w manifold and $U \subset M$ open connected, then for any $p \in U$, we may consider $C^w(U) \to C_p^w(U)$ s.t.*

  $$(f : U \to \mathbb{R}) \mapsto [f : U \to \mathbb{R}]$$

  This is injective ring homomorphism. But it is not surjective.

  $$C^w(\mathbb{R}) \subset C^w(-\varepsilon, \varepsilon) \hookrightarrow C_0^w(\mathbb{R})$$

  Look at elements of the form $\sum_{n=0}^{\infty} a_n x^n$, e.g., $\frac{1}{\frac{\varepsilon}{2} - x} = \sum_{n=0}^{\infty} (\frac{2}{\varepsilon})^{n+1} x^n \in C_0^w(\mathbb{R}) \setminus C^w(-\varepsilon, \varepsilon)$.

Definition 6.2 (Derivation)**.** *A Derivation on $C_p^k(M)$ is a $\mathbb{R}$-linear map*

$$\delta : C_p^k(M) \to \mathbb{R} \text{ s.t. Leibniz rule } \delta(fg) = \delta(f)g + f\delta(g) \text{ is satisfied}$$

If c_1, $c_2 \in \mathbb{R}$ and δ_1, δ_2 are derivations on $C_p^k(M)$, then

$$c_1\delta_1 + c_2\delta_2 : C_p^k(M) \to \mathbb{R} \text{ s.t. } (c_1\delta_1 + c_2\delta_2)(f) := c_1\delta(f) + c_2\delta(f)$$

is also a derivation. Hence the set of derivations on $C_p^k(M)$ has the structure of a vector space.

Example 6.1. *$k \geq 1$.*

- *$\frac{\partial}{\partial x_i}(0) : C_0^k(\mathbb{R}^n) \to \mathbb{R}$ s.t. $[f : U \to \mathbb{R}] \mapsto \frac{\partial}{\partial x_i} f(0) \in \mathbb{R}$ Then $\frac{\partial}{\partial x_i}(0)$ is a derivation for any $1 \leq i \leq n$.*
- *For any $a_i \in \mathbb{R}$, $\sum_i a_i \frac{\partial}{\partial x_i}(0) : C_0^k(\mathbb{R}^n) \to \mathbb{R}$ is a derivation.*

Lemma 6.1. *$k \in \mathbb{N} \cup \{\infty\}$.*

(i) If $\delta : C_0^k(\mathbb{R}) \to \mathbb{R}$ is a derivation and c is a constant, then $\delta(c) = 0$.

Proof. $\delta(c) = c\delta(1)$ by $\mathbb{R}$-linear, and

$$\delta(1) = \delta(1 \cdot 1) = \delta(1) \cdot 1 + 1 \cdot \delta(1) \implies \delta(1) = 0$$

$\square$

(ii) δ is a derivation on $C_0^0(\mathbb{R}) \iff \delta \equiv 0$.

Proof. By $\mathbb{R}$-linear and (i), $\delta(f) = \delta(f - f(0))$. May assume $f(0) = 0$. Then $f = f_+ + f_-$ with

$$f_\pm = \frac{f \pm |f|}{2} \; for \; f_\pm \in C_0^0(\mathbb{R}),\; f_+ \geq 0,\; f_- \leq 0,\; f_\pm(0) = 0$$

One may assume that $f \geq 0$ and $f(0) = 0$. Now we may do

$$g = \sqrt{f} \in C_0^0(\mathbb{R}) \; so\ that \; \delta(f) = \delta(g^2) = \delta(g)g(0) + g(0)\delta(g) = 0$$

Hence f must be 0. $\square$

(iii) δ is a derivation on $C_0^\infty(\mathbb{R})$ then $\delta = \sum_{i=1}^n \delta(x_i)\frac{\partial}{\partial x_i}(0)$

Proof. Want to show for any $f \in C_0^\infty(\mathbb{R}^n)$, $\delta(f) = \sum_{i=1}^n \delta(x_i)\frac{\partial f}{\partial x_i}(0)$. So fix $x \in \mathbb{R}^n$, define $g(t) := f(tx)$ so that $g'(t) = \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(tx)$ Then

$$f(x) - f(0) = g(1) - g(0) = \int_0^1 g'(t)\, dt = \sum_i x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx)\, dt$$

Define $h_i(x) := \int_0^1 \frac{\partial f}{\partial x_i}(tx)\, dt$ so that $h_i \in C_0^\infty(\mathbb{R}^n)$ with $h_i(0) = \int_0^1 \frac{\partial f}{\partial x_i}(0)\, dt = \frac{\partial f}{\partial x_i}(0)$

$$\delta(f) = \delta(f - f(0)) = \sum_i \delta(x_i h_i) = \sum_i \delta(x_i)h_i(0) + \sum_i x_i(0)\delta(h_i) = \sum_i \delta(x_i)\frac{\partial f}{\partial x_i}(0)$$

$\square$

Remark 6.3. *$1 \leq k < \infty$ and $n > 0$. Then the vector space of derivations on $C_0^k(\mathbb{R}^n)$ is infinite dimensional.*

From now on we discuss smooth derivations.

Definition 6.3 (D_pM)**.** *Let M be C^∞ manifold of dimension n, $p \in M$. We denote D_pM as the vector space of derivations on $C_p^\infty(M)$.*

Theorem 6.1 (Linear isomorphism between T_pM and D_pM)**.** *Let M be C^∞ manifold of dimension n, $p \in M$. Define (U, ϕ) a C^∞ chart for M around p, and we write $\phi : U \to \phi(U) \subset \mathbb{R}^n$ open with*

$$\phi(p) = 0 \in \mathbb{R}^n \qquad and \qquad \phi = (x_1, \cdots, x_n) \in C^\infty(U; \mathbb{R}^n)$$

Then there is linear isomorphism between T_pM and D_pM

$$T_pM \to D_pM = \bigoplus_{i=1}^n \mathbb{R}\frac{\partial}{\partial x_i}(p) \; s.t. \; [U, \phi, u] \mapsto \sum_{i=1}^n u_i \frac{\partial}{\partial x_i}(p)$$

with the derivation $\frac{\partial}{\partial x_i}(p) : C_p^\infty(M) \cong C_0^\infty(\mathbb{R}^n) \to \mathbb{R}$ defined as

$$\frac{\partial}{\partial x_i}(p)f := \frac{\partial}{\partial x_i}(f \circ \phi^{-1})(\phi(p)) = \frac{\partial}{\partial x_i}(f \circ \phi^{-1})(0)$$

noticing that $C_p^\infty(M) \cong C_0^\infty(\mathbb{R}^n)$ s.t. $[f : U \to \mathbb{R}] \mapsto [f \circ \phi^{-1} : \phi(U) \to \mathbb{R}]$

6.2 Global Derivations and Smooth Vector Field isomorphism

Definition 6.4 (smooth vector field)**.** *A C^∞ vector field on C^∞ manifold M is a C^∞ section of $\pi : TM \to M$, call it $X : M \to TM$. Notice this implies for any $p \in M$, $X(p) \in T_pM$. Write*

$$\mathfrak{X} = C^\infty(M, TM) = \{C^\infty \ vector \ fields \ on \ M\}$$

Theorem 6.2 (Isomorphism as $C^\infty(U)$-module)**.** *Let M be C^∞ manifold of dim n.*

- *For (U, ϕ) C^∞ chart with $\phi = (x_1, \cdots, x_n) \in \mathbb{R}^n$*

$$\frac{\partial}{\partial x_i} : U \to TU = \pi^{-1}(U) \ s.t. \ p \mapsto \frac{\partial}{\partial x_i}(p) \in D_pM = T_pM = T_pU$$

  is a C^∞ vector field on U.

- *In particular, $\frac{\partial}{\partial x_i}$ as C^∞ vector fields on U implies by definition that $\frac{\partial}{\partial x_i}$ is C^∞ section of $TU \to U$. Hence for any $p \in M$,*

$$\left\{ \frac{\partial}{\partial x_i}(p) \right\}_{i=1}^{n} \ is \ a \ basis \ of \ T_pM = T_pU$$

  Moreover

$$\mathfrak{X}(U) = \bigoplus_{i=1}^{n} C^\infty(U) \frac{\partial}{\partial x_i}$$

  is isomorphism as free $C^\infty(U)$-module.

- *In general, for $s : U \to TU$ continuous section, for any $p \in U$*

$$s(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i}(p) \qquad a_i(p) \in \mathbb{R} \quad a_i : U \to \mathbb{R}$$

  and s is a C^k vector field iff $a_i \in C^k(U)$.

Definition 6.5 (Derivation in $C^\infty(M)$)**.** *Let M be C^∞ manifold. A derivation on M is an $\mathbb{R}$-linear map*

$$\delta : C^\infty(M) \to C^\infty(M) \ s.t. \ \delta(fg) = \delta(f)g + f\delta(g) \ for \ f, \, g \in C^\infty(M)$$

Let $D(M)$ be set of all derivations $C^\infty(M) \to C^\infty(M)$. If δ_1, $\delta_2 \in D(M)$, $c_1 \, c_2 \in C^\infty(M)$, then

$$c_1\delta_1 + c_2\delta_2 : C^\infty(M) \to C^\infty(M) \ s.t. \ (c_1\delta_1 + c_2\delta_2)(f) := c_1\delta(f) + c_2\delta(f)$$

is also a derivation. $D(M)$ is a $C^\infty(M)$-module.

Remark 6.4. *For any $p \in M$, there is a localizing $\mathbb{R}$-linear map. Suppose*

$$D(M) \to D_p(M) \ s.t. \ \delta \mapsto \delta(p) \ where \ \delta(p) : C_p^\infty(M) \to \mathbb{R} \ with \ [f : M \to \mathbb{R}] \mapsto (\delta f)(p) \in \mathbb{R}$$

It is also useful to define

$$\delta_p : C_p^\infty(M) \to C_p^\infty(M) \ s.t. \ [f : M \to \mathbb{R}] \mapsto [\delta f : M \to \mathbb{R}]$$

7 Lie Derivative on smooth functions

7.1 Lie Derivative and Lie Brackets

Definition 7.1 (Lie Derivative)**.** *Define* L_X

$$\mathfrak{X}(M) \to D(M) \qquad s.t. \qquad X \mapsto L_X$$

with

$$L_X : C^\infty(M) \to C^\infty(M) \qquad s.t. \qquad f \mapsto L_X(f) := Xf$$

and

$$Xf(p) = X(p)f \qquad \forall\ X(p) \in T_pM = D_p \qquad and \qquad Xf : M \to \mathbb{R}$$

one use local coordinates to check this is C^∞ *function. On* (U, ϕ) $X = \sum_i^n a_i \frac{\partial}{\partial x_i}$ *for* $a_i \in C^\infty(U)$*. This is a morphism of* $C^\infty(M)$*-modules. Indeed this is an isomorphism.*

Proof that $D(M) \cong \mathfrak{X}(M)$*.* We have surjectivity. Given any $\delta \in D(M)$

$$X(p) := \delta(p) \in D_pM = T_pM$$

and define $X : M \to TM$. One use local coordinates to check that X is C^∞. For injectivity, if $X \neq 0$, there exists $p \in M$ s.t. $X(p) \neq 0$. Then there exists $f \in C_p^\infty(M)$ s.t. $X(p)f \neq 0$ implying $L_X f \neq 0$. We conclude $D(M) \cong \mathfrak{X}(M)$. □

Definition 7.2 (Lie Bracket)**.** *For* $X,\, Y \in \mathfrak{X}(M) = D(M)$*, define*

$$[X,Y] : C^\infty(M) \to C^\infty(M) \text{ s.t. } [X,Y]f := XYf - YXf$$

Then $[X,Y]$ *is a* $\mathbb{R}$*-linear map. Indeed it also satisfies the Liebniz rule so* $[X,Y]$ *defines a derivation.*

$$[X,Y](fg) = ([X,Y]f)g + f([X,Y]g)$$

So $[X,Y] \in D(M) = \mathfrak{X}(M)$*. More explicitly, for* (U,ϕ) C^∞ *chart on* M *with* $\phi = (x_1, \cdots, x_n)$ *local coordinates. One may write on* U

$$X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \qquad Y = \sum_{j=1}^n b_j \frac{\partial}{\partial x_j} \qquad \text{for } a_j,\, b_j \in C^\infty(U)$$

So

$$[X,Y] = \sum_j^n \left(\sum_i^n a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}$$

Proposition 7.1.

$$[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \qquad s.t. \qquad (X,\, Y) \mapsto [X,Y]$$

satisfies

(i) $\mathbb{R}$*-linear in both* X*,* Y*. (not* C^∞*-linear)*

$$[c_1X_1 + c_2X_2, Y] = c_1[X_1, Y] + c_2[X_2, Y]$$

(ii) $[X,Y] = -[Y,X]$

(iii) Jacobi Identity.

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0 \tag{9}$$

with these above, $(\mathfrak{X}(M), [\cdot,\cdot])$ *is a Lie algebra over* $\mathbb{R}$*.*

7.2 Differential as map between Derivations

Definition 7.3 (pullback of $C^\ell(N)$)**.** *Let* $F : M \to N$ *be* C^k*-map between* C^k *manifolds, and let* $\ell \leq k$ *be a positive integer. Then the map* F *induces the pullback*

$$F^* : C^\ell(N) \to C^\ell(M) \text{ s.t. } f \mapsto f \circ F$$

For a point $p \in M$*, we get a map* F_p^* *local pullback s.t.*

$$F_p^* : C^\ell_{F(p)}(N) \to C^\ell_p(M) \text{ s.t. } [(V,f)] \mapsto [F^{-1}(V),\, f \circ F]$$

Remark 7.1. *If M and N are C^k manifolds, and $F : M \to N$ is continuous map, then for each $p \in M$, there exists local pullback F_p^* s.t.*

$$F_p^* : C^0_{F(p)}(N) \to C^0_p(M)$$

here F is a C^k map iff for each $p \in M$, $F_p^*(C^k_{F(p)}(N))$ is a subring of $C^k_p(M)$. We may also use this to define C^k maps.

Lemma 7.1. *Let $F : M \to N$ be a smooth map between smooth manifolds. For each $p \in M$, the differential*

$$dF_p : T_pM = D_pM \to T_{F(p)}N = D_{F(p)}N$$

is given by the map

$$dF_p(X)f = X(F^*f) = X(f \circ F)$$

for any $X \in T_pM = D_pM$ and $f \in C^\infty_{F(p)}(N)$.

Proof. Pass to local coordinates. Assume $M \subset \mathbb{R}^m$ open subset and $N \subset \mathbb{R}^n$ open subset. $p = 0 \in \mathbb{R}^m$ and $F(p) = 0 \in \mathbb{R}^n$. Then one write

$$F(x) = (y_1(x), \cdots, y_n(x)) \ \forall\ x \in \mathbb{R}^m$$

Then for any tangent vector $X \in T_0\mathbb{R}^n$, $X = \sum_{i=1}^m a_i \frac{\partial}{\partial x_i}(0)$

$$dF_p(X) = \sum_{j=1}^n \left(\sum_{i=1}^m \frac{\partial y_j}{\partial x_i}(0) a_i \right) \frac{\partial}{\partial y_j}(0) \in T_0(N)$$

To compute explicitly

$$LHS = dF_p(X)f = \sum_{i=1}^m \sum_{j=1}^n a_i \frac{\partial y_j}{\partial x_i}(0) \frac{\partial f}{\partial y_j}(0)$$

$$RHS = \sum_{i=1}^m a_i \frac{\partial}{\partial x_i}(f \circ F)(0)$$

which is equal by chain rule. □

Remark 7.2. *We may also use $dF_p(X)f = X(F^*f)$ to define dF_p.*

7.3 Differential as map between curve velocity

Definition 7.4 (smooth curve). *Let M be smooth manifold. A smooth curve in M is a smooth map $\gamma : (a, b) \to M$ for $-\infty \le a < b \le \infty$. Notation: for any $t \in (a, b)$, let $\gamma'(t)$ or $\frac{d\gamma}{dt}(t)$ to denote the tangent vector $d\gamma_t(\frac{\partial}{\partial t}) \in T_{\gamma(t)}M$.*

Example 7.1. *If $M = \mathbb{R}^n$ then the smooth map*

$$\gamma : (a, b) \to M \text{ s.t. } \gamma(t) = (x_1(t), \cdots, x_n(t))$$

where $x_i : (a, b) \to \mathbb{R}$ are C^∞ functions on (a, b). Then

$$\gamma'(t) = (x_1'(t), \cdots, x_n'(t)) = \sum_{i=1}^n x_i'(t) \frac{\partial}{\partial x_i}(\gamma(t))$$

Lemma 7.2. *Let M be a smooth manifold and $\gamma : (-\varepsilon, \varepsilon) \to M$ be a smooth curve. Let $\gamma(0) = p$. Then $\gamma'(0)$ is a derivation at p s.t.*

$$\gamma'(0)f = \left.\frac{d}{dt}\right|_{t=0} (f \circ \gamma)$$

Proof. This is special case of $dF_p(X)f = X(F^*f)$. □

Remark 7.3. *One may alternatively define the derivation $\gamma'(0) : C^\infty_p(M) \to \mathbb{R}$ The tangent space T_pM is hence the collection of all such $\gamma'(0)$. Under this definition, $dF_p : T_pM \to T_{F(p)}N$ of a smooth map $F : M \to N$ at $p \in M$ is defined by*

$$dF_p : T_pM \to T_{F(p)}N \text{ s.t. } \gamma'(0) \mapsto (F \circ \gamma)'(0)$$

8 Integral Curves and Flows

8.1 Integral Curve Local Existence and Uniqueness

Definition 8.1 (Integral Curves)**.** *Let X be a smooth vector field on a smooth manifold M and let $\gamma : I \to M$ be a smooth curve. We say that γ is a integral curve of X if*

$$\gamma'(t) = X(\gamma(t)) \ \forall \ t \in I$$

Example 8.1. *$M = \mathbb{R}^n$ and $\gamma(t) = (x_1(t), \cdots, x_n(t))$ for $x_i : I \to \mathbb{R}$ smooth functions on I. A smooth vector field on $\mathbb{R}^n$ is of the form*

$$X(x) = (a_1(x), \cdots, a_n(x)) = \sum_i a_i(x) \frac{\partial}{\partial x_i}$$

where a_i are smooth functions s.t. $a_i : \mathbb{R}^n \to \mathbb{R}$. Therefore X can be viewed as a smooth map from $\mathbb{R}^n \to \mathbb{R}^n$. γ is an integral curve of X is equivalent to the solution to the system of ODEs

$$\frac{dx_i}{dt}(t) = a_i(x_1(t), \cdots, x_n(t)) \qquad for \ i = 1, \cdots, n$$

Theorem 8.1 (Local Existence and Uniqueness of Integral Curves)**.** *Let M be a smooth manifold and X be a smooth vector field on M.*

(i) For any $p \in M$ there is an open interval $I_p \subset \mathbb{R}$ containing 0 and an integral curve $\phi_p : I_p \to M$ of X s.t.

$$\phi_p(0) = p \quad and \quad I_p \ is \ a \ maximal \ interval \ for \ such \ \phi_p$$

(ii) Moreover, this integral curve is unique in the following sense. If $\gamma : I' \to M$ is integral curve of the vector field X on I' s.t. $\gamma(0) = p$, then the interval $I' \subset I_p$ and the curve γ is the restriction $\gamma = \phi_p|_{I'}$.

(iii) Existence of Local Flow. For any $p \in M$, there is

- *an open neighborhood U of p in M*
- *an open interval I of 0 in $\mathbb{R}$*
- *a smooth map $\phi : I \times U \to M$ (local flow)*

s.t.

$$\begin{cases} \frac{\partial}{\partial t}\phi(t, q) = X(\phi(t, q)) \\ \phi(0, q) = q \end{cases} \qquad \forall \ (t, q) \in I \times U$$

Proof. Assume $M = \mathbb{R}^n$ and $p = 0$ then the proof is a theorem in ODE. □

Example 8.2. *$M = \mathbb{R}^n$ and $p = (a_1, \cdots, a_n) \in \mathbb{R}^n$. Suppose X is the identity vector field so $X(x) = x$ for any $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. Then*

$$\begin{cases} \frac{d}{dt}x_i = x_i \\ x_i(0) = a_i \end{cases} \quad for \ i = 1, \cdots n$$

hence $x_i = a_i e^t$. We conclude that the integral curves are straight lines emanating the origin. We also calculate the local flow

$$\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \ s.t. \ \phi(t, x_1, \cdots, x_n) = (x_1 e^t, \cdots, x_n e^t)$$

or in short, $\phi(t, x) = e^t x$.

Example 8.3. *$M = \{x \in \mathbb{R}^n \mid |x| < 1\}$, and X is identity vector field. If $p = a = (a_1, \cdots, a_n)$ then*

$$\phi_p : I_p \to \mathbb{R}^n \ s.t. \ \phi_p(t) = e^t a \ for \ I_p = (-\infty, -\log|a|)$$

Example 8.4. *Given flow $\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ s.t.*

$$\phi(t, (x, y)) := \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

To find the corresponding vector field, use $\frac{\partial}{\partial t}\phi(0, q) = X(\phi(0, q)) = X(q)$. So

$$X((x, y)) = \frac{\partial}{\partial t}\phi(0, (x, y)) = \begin{pmatrix} -\sin(t) & -\cos(t) \\ \cos(t) & -\sin(t) \end{pmatrix}\Bigg|_{t=0} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

Hence $X(x, y) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$.

8.2 Integral Curves Global Existence

Definition 8.2 (Global Flow)**.** *$\phi_t : U \to M$ for $\phi_t(q) := \phi(t, q)$ This tells us where the point in M gets mapped after flowing a certain time t.*

Remark 8.1. *Let $\phi_{t_1} \circ \phi_{t_2} = \phi_{t_1+t_2}$ on the subset of M where both sides are defined.*

Lemma 8.1. *Let X be smooth vector field on a smooth manifold M s.t. the support of X is compact, where*

$$\text{supp}(X) := \overline{\{p \in M \mid X(p) \neq 0\}}$$

Then there exists a unique smooth map $\phi : \mathbb{R} \times M \to M$ where

$$\frac{\partial \phi}{\partial t}(t, q) = X(\phi(t, q)) \qquad \textit{with } \phi(0, q) = q$$

In other words, we have a global flow

$$\phi_t : M \to M$$

which exists for all times $t \in \mathbb{R}$.

Proof. It suffices to prove existence. Let $K = \text{supp}(X)$. First step, look at $V = M \setminus K$ open, $X(q) = 0$ for any $q \in V$. Then define

$$\phi : \mathbb{R} \times V \to M \textit{ s.t. } \phi(t, q) = q$$

Then ϕ is smooth and

$$\frac{\partial \phi}{\partial t}(t, q) = 0 = X(q) = X(\phi(t, q)) \quad \textit{with } \phi(0, q) = q$$

Step 2, given $p \in K$, there exists open neighborhood U_p of p in M and $\varepsilon_p > 0$ s.t. there is a C^∞ map

$$\psi_p : (-\varepsilon_p, \varepsilon_p) \times U_p \to M$$

a local flow which satisfies

$$\begin{cases} \frac{\partial \psi_p}{\partial t}(t, q) = X(\psi_p(t, q)) \\ \psi_p(0, q) = q \end{cases}$$

Moreover, if $p_1, p_2 \in K$ and $U_{p_1} \cap U_{p_2} \neq \varnothing$, then

$$\psi_{p_1}|_{(-\varepsilon,\varepsilon)\times(U_{p_1}\cap U_{p_2})} = \psi_{p_2}|_{(-\varepsilon,\varepsilon)\times(U_{p_1}\cap U_{p_2})}$$

where $\varepsilon := \min\{\varepsilon_{p_1}, \varepsilon_{p_2}\} > 0$. So we obtain a smooth map $\psi(t, q)$ defined on $(-\varepsilon, \varepsilon) \times (U_{p_1} \cup U_{p_2})$ Since K is compact, $K \subset \bigcup_{p\in K} U_p$ hence there are finitely many $p_1, \cdots, p_N \in K$ s.t. $K \subset \bigcup_{i=1}^N U_{p_i}$. Let $\varepsilon := \min\{\varepsilon_{p_1}, \cdots, \varepsilon_{p_N}\} > 0$ and $U := \bigcup_{i=1}^N U_{p_i}$ we obtain a smooth map

$$\psi : (-\varepsilon, \varepsilon) \times U \to M$$

s.t.

$$\begin{cases} \frac{\partial \psi}{\partial t}(t, q) = X(\psi(t, q)) \\ \psi(0, q) = q \end{cases}$$

Step 3, again by uniqueness

$$\phi|_{(-\varepsilon,\varepsilon)\times(U\cap V)} = \phi : \mathbb{R} \times V \to M \qquad \textit{and} \qquad \psi : (-\varepsilon, \varepsilon) \times U \to M$$

We also have $U \cup V = M$ so we obtain

$$\phi : (-\varepsilon, \varepsilon) \times M \to M$$

satisfying assumptions. Step 4, for any $t \in \mathbb{R}$, there exists $n \in \mathbb{N}$ with $|t| < n\varepsilon$, we define $\phi(t, q) = \phi(\frac{t}{n}, \phi(\frac{t}{n}, \cdots, \phi(\frac{t}{n}, q)) \cdots)$ Then $\phi : \mathbb{R} \times M \to M$ satisfy the assumptions. □

8.3 Flow and Lie Derivative on Vector Fields

Now we talk about Flow and Lie derivative.

Definition 8.3 (Lie Derivative)**.** *Let M be smooth manifold, let $X \in \mathfrak{X}(M) = C^\infty(M, TM)$ space of smooth vector fields on M, which is $C^\infty(M)$-module. Recall that $L_X : C^\infty(M) \to C^\infty(M)$ s.t. $L_X f := Xf$ is a derivation. We extend this definition via*

$$L_X : \mathfrak{X}(M) \to \mathfrak{X}(M) \textit{ s.t. } Y \mapsto L_X Y := [X, Y]$$

Notice

$$L_X(fY) = (L_X f)Y + fL_X Y \qquad \text{for } f \in C^\infty(M) \text{ and } Y \in \mathfrak{X}(M)$$
$$L_{fX}(g) = fL_X(g) \qquad \text{for } f,\, g \in C^\infty(M), \text{ and } X \in \mathfrak{X}(M)$$

but in general $L_{fX}(Y) \neq fL_X Y$ *since*

$$L_{fX}(Y) = [fX, Y] = f[X,Y] - Y(f)X = fL_X Y - Y(f)X$$

Definition 8.4 (pushforward and pullback of smooth vector fields)**.** *Let* $F : M \to N$ *be* C^∞ *diffeomorphism. Define the pushforward*

$$F_* : \mathfrak{X}(M) \to \mathfrak{X}(N) \qquad s.t. \qquad X \mapsto F_* X$$
$$(F_* X)(p) := dF_{F^{-1}(p)}(X(F^{-1}(p))) \in T_p N$$

where $p \in N$*,* $F^{-1}(p) \in M$*, and* $X(F^{-1}(p)) \in T_{F^{-1}(p)}M$*. Define pullback*

$$F^* := (F^{-1})_* : \mathfrak{X}(N) \to \mathfrak{X}(M)$$

Proposition 8.1 (Lie Derivative using Flow)**.** *M smooth manifold, $X \in \mathfrak{X}(M)$, $p \in M$ and U open neighborhood of p in M. Let $\phi_t : U \to M$ smooth be flow of X at p for $t \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$. Then*

- *For* $[f : M \to \mathbb{R}] \in C_p^\infty(M)$*, pick a representative* f

$$(L_X f)(p) := X(p)f = \left.\frac{d}{dt}\right|_{t=0} (\phi_t^* f)(p)$$

- $Y \in \mathfrak{X}(V)$ *for* V *open neighborhood of* p

$$(L_X Y)(p) := [X,Y](p) = \left.\frac{d}{dt}\right|_{t=0} (\phi_t^* Y)(p) = -\left.\frac{d}{dt}\right|_{t=0} (\phi_{t*} Y)(p) = \lim_{t \to 0} \frac{Y(p) - (d\phi_t)_{\phi_{-t}(p)}(Y(\phi_{-t}(p)))}{t} \tag{10}$$

  using the fact
$$\phi_{t*} Y = -(\phi_{-t})_* Y = -\phi_t^* Y$$
  and recalling $(\phi_{t*} Y)(p) = (d\phi_t)_{\phi_{-t}(p)}(Y(\phi_{-t}(p)))$

Lemma 8.2. *If $h : (-\delta, \delta) \times U \to \mathbb{R}$ s.t. $(t, q) \mapsto h(t, q)$ is C^∞ map for $U \subset M$ open, $\delta > 0$, and suppose that $h(0, q) = 0$. Then there exists C^∞ map $g : (-\delta, \delta) \times U \to \mathbb{R}$ s.t.*

$$h(t, q) = tg(t, q)$$

Proof. Fix t, q. Let $u(s) := h(st, q)$. Then $\frac{d}{ds}u(s) = t\frac{\partial}{\partial t}h(st, q)$ with

$$h(t, q) = h(t, q) - h(0, q) = u(1) - u(0) = \int_0^1 \frac{d}{ds}u(s)\, ds = t\int_0^1 \frac{\partial}{\partial t}h(st, q)\, ds = tg(t, q)$$

where $g(t, q) = \int_0^1 \frac{\partial}{\partial t}h(st, q)\, ds$. Here g is C^∞ map. Notice $g(0, q) = \frac{\partial}{\partial t}h(0, q)\, ds = \frac{\partial}{\partial t}h(0, q)$. □

Proof of Proposition 8.1. For $f \in C_p^\infty(M)$,

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} (\phi_t^* f)(p) &= \left.\frac{d}{dt}\right|_{t=0} f(\phi_t(p)) \\
&= \left.\frac{d}{dt}\right|_{t=0} (f \circ \phi_p)(t) \\
&= \phi_p'(0)f = X(p)f
\end{aligned}$$

since $\phi_p(t) = \phi_t(p)$ for $\phi_p : (-\varepsilon, \varepsilon) \to M$ integral curves of X s.t. $\phi_p(0) = p$ and $\phi_p'(t) = X(\phi_p(t))$. Now for the second item, claim that

$$\left.\frac{d}{dt}\right|_{t=0} (\phi_{t*} Y)(p)(f) = -[X,Y](p)f \quad \forall\, f \in C_p^\infty(M)$$

To see this, let

$$h(t, q) = f \circ \phi_t(q) - f(q)$$

Here $h : (-\delta, \delta) \times V \to \mathbb{R}$ is C^∞ with $h(0, q) = 0$. By lemma 8.2, there exists C^∞ $g : (-\delta, \delta) \times V \to \mathbb{R}$ s.t. $h(t, q) = tg(t, q)$. For fixed $t \in (-\delta, \delta)$, $g_t : V \to \mathbb{R}$ smooth with $g_t(q) := g(t, q)$. So

$$f \circ \phi_t(q) = f(q) + h(t, q) = (f + tg_t)(q)$$

Also note

$$g_0(q) = \frac{\partial}{\partial t} h(0, q) = \left.\frac{d}{dt}\right|_{t=0} f \circ \phi_t(q) = X(q)f$$

from first item. Hence using Lemma 7.1

$$\begin{aligned}(\phi_{t*}Y)(p)(f) &= (d\phi_t)_{\phi_{-t}(p)}(Y(\phi_{-t}(p)))f = Y(\phi_{-t}(p))(f \circ \phi_t)\\ &= Y(\phi_{-t}(p))(f + tg_t) = Y(\phi_{-t}(p))f + Y(\phi_{-t}(p))(tg_t)\end{aligned}$$

$$\left.\frac{d}{dt}\right|_{t=0} Y(\phi_{-t}(p))(f \circ \phi_t) = \left.\frac{d}{dt}\right|_{t=0} (Yf)(\phi_{-t}(p)) + Y(p)g_0 = -X(p)Yf + Y(p)Xf = -[X, Y](p)f$$

□

9 Frobenius Theorem

9.1 Subbundle

Definition 9.1 (subbundle)**.** *Let $\pi : E \to M$ be C^∞ vector bundle of rank r over a C^∞ manifold M. $F \subset E$ is a subbundle of rank $k \leq r$ if for any $p \in M$, there exists open neighborhood U of p in M and a local trivialization*

$$h : \pi^{-1}(U) \to U \times \mathbb{R}^r \qquad C^\infty\ diffeomorphism$$

s.t. diagram $\pi = pr_1 \circ h$ commutes and

$$h(F \cap \pi^{-1}(U)) = U \times (\mathbb{R}^k \times \{0\}) \quad for\ \mathbb{R}^k \times \{0\} \subset \mathbb{R}^r$$

Remark 9.1. *Some remarks for a smooth Subbundle F of E*

- *Recall for any $x \in U$, $E_x \cong \mathbb{R}^r$*

$$E_x = \pi^{-1}(x) \to \{x\} \times \mathbb{R}^r \ is\ linear\ isomorphism$$

While in the case of F as subbundle, for any $x \in U$, $F_x := F \cap E_x$ is a subspace of dimension k in E_x.

Proposition 9.1 (Subbundle Equivalent Definition)**.** *Given $\pi : E \to M$ smooth vector bundle of rank r over a C^∞ manifold M. For any $x \in M$, $F_x \subset E_x$ is subspace of dimension $k \leq r$. Take disjoint union*

$$F := \bigsqcup_{x \in M} F_x \subset E := \bigsqcup_{x \in M} E_x$$

Then F is a C^∞ subbundle of E of rank k iff for any $p \in M$, there exists open neighborhood U of p in M and C^∞ sections $\{s_1, \cdots, s_k\} \subset C^\infty(U; \pi^{-1}(U) = E|_U)$ s.t. for any $q \in U$

$$s_1(q), \cdots s_k(q)\ is\ a\ basis\ of\ F_q$$

Example 9.1. *$E = \{(\ell, v) \mid \ell \in P_n(\mathbb{R}),\ v \in \ell\} \subset P_n(\mathbb{R}) \times \mathbb{R}^{n+1}$. E is a smooth vector bundle of rank 1 of the product vector bundle. Here $pr_1 : P_n(\mathbb{R}) \times \mathbb{R}^{n+1} \to P_n(\mathbb{R})$.*

9.2 Distribution: Involutive and Completely Integrable

Definition 9.2 (distribution)**.** *Let M be C^∞ manifold. A C^∞ distribution of dimension k for $k \leq n$ on M is a collection $\{F_p \subset T_pM \mid p \in M\}$ where F_p are k-dimensional subspaces of T_pM s.t.*

$$F = \bigsqcup_{p \in M} F_p \subset TM = \bigsqcup_{p \in M} T_pM$$

is a C^∞ subbundle of TM of rank k.

Remark 9.2. *One has an equivalent definition for smooth distribution using Prop 9.1*

- *The collection $\{F_p \subset T_pM \mid p \in M\}$ of k-dimensional subspaces of T_pM is a smooth distribution iff for any $p \in M$, there exists open neighborhood U of p in M and $X_1, \cdots, X_k \in \mathfrak{X}(U)$ s.t. for any $q \in U$*

$$F_q = \bigoplus_{i=1}^{k} \mathbb{R} X_i(q)$$

Remark 9.3. *Given a smooth subbundle $F \to M$ of $\pi : TM \to M$, and denoting $C^\infty(M, F)$ as space of smooth sections of the subbundle $F \to M$. Then*

$$C^\infty(M, F) \subset C^\infty(M, TM) = \mathfrak{X}(M)$$

is $C^\infty(M)$-submodule.

Definition 9.3 (involutive and integrable)**.** *Let F be C^∞ distribution of dimension k on a C^∞ manifold M of dimension k.*

- *We say F is involutive if $C^\infty(M, F)$ is a Lie subalgebra of $(\mathfrak{X}(M), [\cdot, \cdot])$.*

$$X, Y \in C^\infty(M, F) \implies [X, Y] \in C^\infty(M, F)$$

- *F is completely integrable if for any $p \in M$, there exists (U, ϕ) for $\phi = (x_1, \cdots, x_n)$ C^∞-chart for M around p s.t.*
$$F_q = \bigoplus_{i=1}^{k} \mathbb{R} \frac{\partial}{\partial x_i}(q) \qquad \forall\, q \in U$$
This is equivalent to saying for any $p \in M$, there is a k-dimensional submanifold $S \subset M$ s.t. $p \in S$ and for any $q \in S$, the subspace $T_qS = F_q$.

Example 9.2. *One has some examples motivating the Frobenius Theorem*

- *For $\dim F = \dim M$, then $F_p = T_pM$ for any $p \in M$, here F is involutive and completely integrable.*
- *For $\dim F = 1$, F is involutive and completely integrable.*
- *For $U \subset \mathbb{R}^3$ open, there exists $2 - \dim$ distributions not involutive and not completely integrable.*

Theorem 9.1 (Frobenius Theorem)**.** *A C^∞ distribution F on a C^∞ manifold is completely integrable if and only if it is involutive.*

Proof. Let $k := rank\, F \leq n = dim\, M = rank\, TM$. For $\implies$. If F completely integrable, for any $X, Y \in C^\infty(M, F)$, for any $p \in M$, there exists (U, ϕ) C^∞ chart for M around p s.t. for any $q \in U$
$$F_q = \bigoplus_{i=1}^{k} \mathbb{R} \frac{\partial}{\partial x_i}(q)$$
On U, $X = \sum_{i=1}^k a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_{j=1}^k b_j \frac{\partial}{\partial x_j}$ so
$$[X, Y] = \sum_j^k \left(\sum_i^k a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j} \implies [X, Y] \in C^\infty(M, F)$$
For $\impliedby$. Let F involutive. As a distribution, since F is smooth subbundle of TM, for any $p \in M$, there exists open neighborhood U of p in M and $X_1, \cdots, X_k \in \mathfrak{X}(U)$ s.t.
$$F_q = \bigoplus_{i=1}^{k} \mathbb{R} X_i(q) \qquad for\ any\ q \in U$$
For any $p \in M$, there exists (U, ϕ) $\phi = (x_1, \cdots, x_n)$ so $X_i = \sum_{j=1}^n a_{ij} \frac{\partial}{\partial x_j}$ for $a_{ij} \in C^\infty(U)$, $i = 1, \cdots, k$. For any $p \in U$, consider
$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix}(q)\ of\ rank\ k$$
by permuting $x_1, \cdots, x_n$ if necessary, we may assume the minor matrix
$$\det \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \cdots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}(p) \neq 0$$
Due to smoothness of a_{ij}, by shrinking U if necessary, we may assume
$$det \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \cdots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}(q) \neq 0 \qquad for\ any\ q \in U$$
Let $A := \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \cdots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}$ so $A = (a_{ij})_{i,j=1}^k : U \to GL(r, \mathbb{R})$ and $A^{-1} =: (a^{ij})_{i,j=1}^k : U \to GL(r, \mathbb{R})$ are smooth. Using $A^{-1}A = I_k$ we write
$$\sum_{\ell=1}^{k} a^{i\ell} a_{\ell j} = \delta_{ij}$$

For $i = 1, \cdots, k$, define

$$E^i := \sum_{j=1}^{k} a^{ij} X_j \in \mathfrak{X}(U) \qquad \textit{for any } q \in U$$

Hence for any $q \in U$, $F_q = \bigoplus_{i=1}^{k} \mathbb{R}E^i(q)$. Using $X_j = \sum_{\ell=1}^{n} a_{i\ell} \frac{\partial}{\partial x_\ell}$

$$\begin{aligned}
E^i &:= \sum_{j=1}^{k} a^{ij} \left(\sum_{\ell=1}^{n} a_{j\ell} \frac{\partial}{\partial x_\ell} \right) = \sum_{\ell=1}^{k} \delta_{i\ell} \frac{\partial}{\partial x_\ell} + \sum_{\ell=k+1}^{n} \gamma_\ell^i \frac{\partial}{\partial x_\ell} \\
&= \frac{\partial}{\partial x_i} + \sum_{\ell=k+1}^{n} \gamma_\ell^i \frac{\partial}{\partial x_\ell} \\
\implies [E^i, E^j] &= [\frac{\partial}{\partial x_i} + \sum_{\ell=k+1}^{n} \gamma_\ell^i \frac{\partial}{\partial x_\ell}, \frac{\partial}{\partial x_j} + \sum_{\ell=k+1}^{n} \gamma_\ell^j \frac{\partial}{\partial x_\ell}] \\
&= \sum_{m=k+1}^{n} c_m^{ij} \frac{\partial}{\partial x_m}
\end{aligned}$$

For any $q \in U$

$$[E^i, E^j](q) \in \bigoplus_{m=k+1}^{n} \mathbb{R} \frac{\partial}{\partial x_m}(q) =: G_q$$

where $\dim G_q = n - k$. Now G is completely integrable distribution of dimension $n-k$ on U. Since F is involutive with $E^i \in C^\infty(U, F|_U)$, for any $q \in U$

$$[E^i, E^j](q) \in F_q = \bigoplus_{i=1}^{k} \mathbb{R}E^i(q)$$

But as vector spaces $F_q \cap G_q = \{0\}$, so

$$[E^i, E^j](q) = 0$$

Conclusion: If F is an involutive C^∞ distribution of dimension k on M, then for any $p \in M$, there exists smooth chart (U, ϕ) for $\phi = (x_1, \cdots, x_n)$ of p in M and $E^1, \cdots, E^k \in \mathfrak{X}(U)$ s.t. $E^i = \frac{\partial}{\partial x_i} + \sum_{\ell=k+1}^{n} \gamma_\ell^i \frac{\partial}{\partial x_\ell}$

$$[E^i, E^j] = 0 \quad \textit{and} \quad \forall\, q \in U \quad F_q = \bigoplus_{i=1}^{k} \mathbb{R}E^i(q)$$

The strategy is to construct new coordinates $(t_1, \cdots, t_n)$ on $U' \subset U$ s.t. $E^i = \frac{\partial}{\partial t_i}$ for $i = 1, \cdots, k$ on U'. Recall Assignment 4(2): For M C^∞ manifold, $X, Y \in \mathfrak{X}(M)$ with $[X, Y] = 0$, let $p \in M$, and suppose $\phi_s^X \circ \phi_t^Y(p)$ and $\phi_t^Y \circ \phi_s^X(p)$ are defined for $(s, t) \in I \times J$ with I, J open intervals containing 0, then one has

$$\phi_s^X \circ \phi_t^Y(p) = \phi_t^Y \circ \phi_s^X(p) \qquad \forall\, (s, t) \in I \times J$$

Hence to use this, we may assume $\phi(p) = 0 \in \mathbb{R}^n$. Define for V open neighborhood of $0 \in \mathbb{R}^n$

$$\psi : V \subset \mathbb{R}^n \to M \ s.t.\ \psi(t_1, \cdots, t_n) := \phi_{t_1}^{E^1} \circ \phi_{t_2}^{E^2} \circ \cdots \circ \phi_{t_k}^{E^k} \circ \phi^{-1}(0, \cdots, 0, t_{k+1}, \cdots, t_n)$$

Then ψ is a C^∞ map. But for each $i \in \{1, \cdots, k\}$ one in fact has

$$\psi(t_1, \cdots, t_k) = \phi_{t_i}^{E^i} \left(\psi(t_1, \cdots, t_{i-1}, 0, t_{i+1}, \cdots, t_k) \right)$$

For fixed $t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_k$. Integral curve of E^i are

$$\gamma(s) := \psi(t_1, \cdots, t_{i-1}, s, t_{i+1}, \cdots, t_n) \ with\ \gamma(0) = \psi(t_1, \cdots, t_{i-1}, 0, t_{i+1}, \cdots, t_n)$$

so for $\psi : V \subset \mathbb{R}^n \to M$

$$d\psi_t(\frac{\partial}{\partial t_i}) = \frac{\partial \psi}{\partial t_i}(t_1, \cdots, t_n) = E^i(\psi(t_1, \cdots, t_n)) \qquad \forall t = (t_1, \cdots, t_n) \in V$$

At $t = 0$, $d\psi_0(\frac{\partial}{\partial t_i}) = \begin{cases} E^i(p) \ 1 \le i \le k \\ \frac{\partial}{\partial x_i}(p) \ k+1 \le i \le n \end{cases}$ Hence $d\psi_0 : T_0 V \cong \mathbb{R}^n \to T_p M$ is a linear isomorphism. There exists open neighborhood V' of 0 in $V \subset \mathbb{R}^n$, U' of p in M $U' \subset U$ s.t.

$$\psi|_{V'} : V' \to U' \qquad \textit{is a } C^\infty \textit{ diffeomorphism}$$

Then define $\phi' := (\psi|_{V'})^{-1} : U' \to V' \subset \mathbb{R}^n$ with $E^i = \frac{\partial}{\partial t_i}$ on $U' \subset U$, where $\phi' = (t_1, \cdots, t_n)$. □

Example 9.3 (1-dim distribution F)**.** *For any $p \in M$, there exists U open neighborhood of p in M, $X \in \mathfrak{X}(U)$ s.t. for any $q \in U$, $F_q = \mathbb{R}X(q)$. For k-dim distribution F, involutive iff completely integrable, this is foliation.*

10 Operation on Vector Bundles

Recall operations on vector spaces. V, W finite dimensional vector spaces of dimension r, s. Then

- V^* dual vector space is of dimension r
- $V \oplus W$ direct sum dimension $r + s$
- $V \otimes W$ tensor product dimension of rs
- $V^{\otimes k} = V \otimes \cdots \otimes V$ k-tensor product of V, dimension of r^k.
- $\Lambda^k V$ Wedge product, dimension $\binom{r}{k}$.

Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be C^∞ vector bundles of rank r, s over a C^∞ manifold M. Let the fibers be denoted as $E_p := \pi_E^{-1}(p) \cong \mathbb{R}^r$ and $F_p := \pi_F^{-1}(p) \cong \mathbb{R}^s$ for any $p \in M$, i.e.,

$$\pi_E : E = \bigsqcup_{p \in M} E_p \to M \qquad and \qquad \pi_F : F = \bigsqcup_{p \in M} F_p \to M$$

Since each E_p, F_p has structure of a vector space, one may perform the above vector space operations to fibers and define the following bundles at the set level.

- $E^* := \bigsqcup_{p \in M} E_p^*$ where $E_p^* := (E_p)^*$.
- $E \oplus F := \bigsqcup_{p \in M} (E \oplus F)_p$ where $(E \oplus F)_p := E_p \oplus F_p$.
- $E \otimes F := \bigsqcup_{p \in M} (E \otimes F)_p$ where $(E \otimes F)_p := E_p \otimes F_p$.
- $E^{\otimes k} := \bigsqcup_{p \in M} (E^{\otimes k})_p$ where $(E^{\otimes k})_p := E_p^{\otimes k}$.
- $\Lambda^k E := \bigsqcup_{p \in M} (\Lambda^k E)_p$ where $(\Lambda^k E)_p := \Lambda^k E_p$.

10.1 Dual Bundle

Let $\pi_E : E \to M$ be C^∞ vector bundles of rank r over a C^∞ manifold M.

- As a set, let $E^* := \bigsqcup_{p \in M} E_p^*$.
- As a map, let $\pi_{E^*} : E^* \to M$ s.t. $\pi_{E^*}(E_p^*) := \{p\}$.

We wish to construct $\pi_{E^*} : E^* \to M$ a smooth vector bundle of rank r. First recall the smooth structure on E.

(i) Local Trivialization and Smooth Frame. Since $\pi_E : E \to M$ is vector bundle of rank r, there exists $\{U_\alpha \mid \alpha \in I\}$ open cover of M and local trivializations

$$h_\alpha^E : \pi_E^{-1}(U_\alpha) \subset E \to U_\alpha \times \mathbb{R}^r$$

C^∞ diffeomorphisms s.t. $\pi_E = pr_1 \circ h_\alpha^E$. For any $x \in U_\alpha$, $h_\alpha^E\big|_{E_x} : E_x = \pi_E^{-1}(x) \to \{x\} \times \mathbb{R}^r$ are linear isomorphisms. One shall notice that

- h_α^E are local trivialization iff
- h_α^E are isomorphisms from $\pi_E^{-1}(U_\alpha)$ to the product vector bundle of rank r over U_α iff
- There exists C^∞ frame $e_{\alpha_1}, \cdots, e_{\alpha_r}$ where $e_{\alpha_i} \in C^\infty(U_\alpha, \pi_E^{-1}(U_\alpha))$. In particular, for any $x \in U_\alpha$, $\{e_{\alpha_i}(x)\}_{i=1}^r$ are defined as

$$e_{\alpha_i} : U_\alpha \to \pi_E^{-1}(U_\alpha) \qquad s.t. \qquad e_{\alpha_i}(x) = (h_\alpha^E)^{-1}(x, e_i)$$

where $e_i = (0, \cdots, 1, \cdots 0)$ are standard basis in $\mathbb{R}^r$. Notice

$$(h_\alpha^E)^{-1} : U_\alpha \times \mathbb{R}^r \to \pi_E^{-1}(U_\alpha) \qquad s.t. \qquad (x, v) \mapsto (x, \sum_{i=1}^r v_i e_i(x))$$

(ii) Smooth Transition Functions. On $U_\alpha \cap U_\beta$, one has smooth frames $\{e_{\alpha_i}(x)\}_{i=1}^r$ defined by h_α^E and $\{e_{\beta_i}(x)\}_{i=1}^r$ defined by h_β^E. Due to definition of vector bundle, one has the linear isomorphisms in $\mathbb{R}^r$

$$(g_{\beta\alpha}^E(x))_{i,j=1}^r \in C^\infty(U_\alpha \cap U_\beta; GL(r, \mathbb{R}))$$

s.t.

$$e_{\alpha j}(x) = \sum_{i=1}^{r} e_{\beta i}(x) g^E_{\beta\alpha}(x)_{ij}$$

or in short

$$e_\alpha = e_\beta g^E_{\beta\alpha}$$

with notation $e_\alpha = [e_{\alpha_1}, \cdots, e_{\alpha_r}]$ and $e_\beta = [e_{\beta_1}, \cdots, e_{\beta_r}]$. The $g^E_{\beta\alpha}$ corresponds to the transition functions

$$h^E_\beta \circ (h^E_\alpha)^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^r \to (U_\alpha \cap U_\beta) \times \mathbb{R}^r$$

via the following

$$\begin{aligned} h^E_\beta \circ (h^E_\alpha)^{-1}(x, v) &= h^E_\beta(x, \sum_{j=1}^{r} v_j e_{\alpha_j}(x)) \\ &= h^E_\beta(x, \sum_{j=1}^{r} v_j \sum_{i=1}^{r} e_{\beta_i}(x) g^E_{\beta\alpha}(x)_{ij}) \\ &= h^E_\beta(x, \sum_{i=1}^{r} (\sum_{j=1}^{r} v_j g^E_{\beta\alpha}(x)_{ij}) e_{\beta_i}(x)) \\ &= (x, g^E_{\beta\alpha}(x) v) \end{aligned}$$

So the transition functions $h^E_\beta \circ (h^E_\alpha)^{-1}$ are given by

$$h^E_\beta \circ (h^E_\alpha)^{-1}(x, v) = (x, g^E_{\beta\alpha}(x) v)$$

Now one wish to define the smooth structure on the set E^*.

(i) Local Trivialization and Smooth Frame. For smooth frames, define

$$e^*_{\alpha_i} : U_\alpha \to \pi^{-1}_{E^*}(U_\alpha) = \bigsqcup_{x \in U_\alpha} E^*_x \subset E^*$$

s.t. for any $x \in U_\alpha$ with $e_{\alpha_j}(x) \in E_x$, $e^*_{\alpha_i}(x) \in (E^*)_x = (E_x)^*$, we have

$$\langle e^*_{\alpha_i}(x), e_{\alpha_j}(x) \rangle = \delta_{ij} \tag{11}$$

i.e., $\{e^*_{\alpha_i}(x)\}_{i=1}^r$ is a dual basis for the dual space E^*_x w.r.t. $\{e_{\alpha_i}(x)\}_{i=1}^r$ as basis of E_x. For local trivializations, define

$$h^{E^*}_\alpha : \pi^{-1}_{E^*}(U_\alpha) \subset E^* \to U_\alpha \times \mathbb{R}^r \qquad s.t. \qquad (x, \sum_{i=1}^{r} v_i e^*_{\alpha_i}(x)) \mapsto (x, v = \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix})$$

bijection. We use this bijection to equip $\pi^{-1}_{E^*}(U_\alpha)$ with topology and a smooth structure s.t. the map $h^{E^*}_\alpha$ is C^∞ diffeomorphism. Then $\pi^{-1}_{E^*}(U_\alpha)$ is a C^∞ manifold of dimension $n + r$ where $n = dim\, M$. Indeed $\pi_{E^*} = pr_1 \circ h^{E^*}_\alpha$ for any $x \in U_\alpha$ and $E^*_x \cong \mathbb{R}^r$.

(ii) Smooth Transition Functions. On $U_\alpha \cap U_\beta \neq \varnothing$, recall

$$e_{\alpha_j}(x) = \sum_{i=1}^{r} e_{\beta_i}(x) g^E_{\beta\alpha}(x)_{ij} \in E_x$$

Then by our definition of $e^*_{\beta_k}$ (11)

$$\begin{aligned} \langle e^*_{\beta_k}(x), e_{\alpha_j}(x) \rangle &= \sum_{i=1}^{r} \delta_{ik} g^E_{\beta\alpha}(x)_{kj} = g^E_{\beta\alpha}(x)_{kj} \\ \implies e^*_{\beta_k}(x) &= \sum_{i=1}^{r} g^E_{\beta\alpha}(x)_{ki} e^*_{\alpha_i}(x) \\ &= \sum_{i=1}^{r} e^*_{\alpha_i}(x) \left(g^E_{\beta\alpha}(x)\right)^T_{ik} \\ &:= \sum_{i=1}^{r} e^*_{\alpha_i}(x) g^{E^*}_{\alpha\beta}(x)_{ik} \\ \implies (g^{E^*}_{\beta\alpha})^{-1} &= g^{E^*}_{\alpha\beta} = (g^E_{\beta\alpha})^T \end{aligned}$$

Now

$$g_{\beta\alpha}^{E^*} = ((g_{\beta\alpha}^E)^T)^{-1} : U_\alpha \cap U_\beta \to GL(r, \mathbb{R}) \qquad is\ C^\infty\ map$$

The transition map

$$h_\alpha^{E^*} \circ \left(h_\beta^{E^*}\right)^{-1} : U_\alpha \cap U_\beta \times \mathbb{R}^r \to U_\alpha \cap U_\beta \times \mathbb{R}^r$$

is given by

$$h_\alpha^{E^*} \circ \left(h_\beta^{E^*}\right)^{-1}(x, v) = \left(x, g_{\alpha\beta}^{E^*}(x)v\right) = \left(x, (g_{\beta\alpha}^E)^T(x)v\right)$$

while its inverse is given by

$$h_\beta^{E^*} \circ \left(h_\alpha^{E^*}\right)^{-1}(x, v) = \left(x, g_{\beta\alpha}^{E^*}(x)v\right) = \left(x, ((g_{\beta\alpha}^E)^T)^{-1}(x)v\right)$$

The above smooth structures gives

$$\pi_{E^*} : E^* \to M\ is\ C^\infty\ vector\ bundle\ of\ rank\ r$$

10.2 Other Operations

Similarly, for $\{e_{\alpha_i}\}_{i=1}^r$ C^∞ frame of $E|_{U_\alpha} := \pi_E^{-1}(U_\alpha)$ and $\{f_{\alpha_j}\}_{j=1}^s$ C^∞ frame of $F|_{U_\alpha} := \pi_F^{-1}(U_\alpha)$

- $\{e_{\alpha_i}\}_{i=1}^r \cup \{f_{\alpha_j}\}_{j=1}^s$ is C^∞ frame of $(E \oplus F)|_{U_\alpha}$.
- $\{e_{\alpha_i} \otimes f_{\alpha_j} \mid 1 \le i \le r,\ 1 \le j \le s\}$ is C^∞ frame of $(E \otimes F)|_{U_\alpha}$.
- $\{e_{\alpha_{i_1}} \wedge \cdots \wedge e_{\alpha_{i_k}} \mid 1 \le i_1 \le \cdots \le i_k \le r\}$ is C^∞ frame of $(\Lambda^k E)\big|_{U_\alpha}$ for $k \le r$.

11 Tensor Bundles

11.1 Tensor and Forms

Definition 11.1 (Cotangent Bundle). *Let M be C^∞ manifold with dimension n. Let $p \in M$*

- *A cotangent vector at $p \in M$ is a vector in $T_p^*M := (T_pM)^*$.*
- *T_p^*M is the cotangent vector space at p.*
- *$T^*M := (TM)^* = \bigsqcup_{p\in M} T_p^*M$ a C^∞ vector bundle of rank n is the cotangent bundle.*

Definition 11.2 ((r, s)-tensor and s-form). *Let M be C^∞ manifold with dimension n.*

- *$T_s^r(M) := (TM)^{\otimes r} \otimes (T^*M)^{\otimes s}$ is C^∞ vector bundle of rank n^{r+s}. A C^∞ (r, s)-tensor on M is a C^∞ section of $T_s^r(M)$.*
$$\text{Space of smooth (r, s)-tensors on } M := C^\infty(M, T_s^r(M))$$
- *$\Lambda^s T^*M$ is C^∞ vector bundle of rank $\binom{n}{k}$. A C^∞ s-form on M is a C^∞ section of $\Lambda^s T^*M \subset T_s^0 M = (T^*M)^{\otimes s}$.*
$$\Omega^s(M) := C^\infty(M, \Lambda^s T^*M)$$
is the space of C^∞ s-forms on M.

Remark 11.1. *Given smooth manifold M.*

- *$f \in C^\infty(M)$ is $(0, 0)$-tensor.*
- *$X \in \mathfrak{X}(M)$ is $(1, 0)$-tensor.*
- *1-form are exactly $(0, 1)$-tensors.*
- *s-forms are examples of $(0, s)$-tensors.*

Example 11.1 (Differential of smooth function). *Let M be smooth manifold of dimension n. Let $p \in M$ and (U, ϕ) a C^∞ chart around p where $\phi = (x_1, \cdots, x_n)$. Let $f \in C^\infty(U)$, then its differential df*

$$df_p : T_pU \to \mathbb{R} \ \in \ T_p^*U$$

and satisfies

$$\langle df, \frac{\partial}{\partial x_i}\rangle = \frac{\partial f}{\partial x_i} \in C^\infty(U)$$

Hence df is $(0, 1)$-tensor, or equivalently, 1-form.

Example 11.2 (dx_i, tensors and forms in local coordinates). *We pass to local coordinates. Let (U, ϕ) be C^∞ chart for M with $\phi = (x_1, \cdots, x_n)$ for $x_i \in C^\infty(U)$.*

(i) The differentials of coordinate functions $\{dx_i\}$ are smooth sections of $T^*M|_U = T^*U \to U$ s.t.

$$dx_i : U \to T^*M|_U \quad s.t.\ p \mapsto (dx_i)_p : T_pM \to T_{\phi(p)}\mathbb{R} \cong \mathbb{R}$$
$$(dx_i)_p(\frac{\partial}{\partial x_j}(p)) := \frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

where $\{\frac{\partial}{\partial x_j}\}$ is C^∞ frame of $TM|_U = TU$. Hence $\{dx_i\}$ is the C^∞ dual frame of $T^*M|_U = T^*U$.

(ii) For any $f \in C^\infty(U)$ one writes

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$$

More generally, on U, C^∞ vector fields as $(1, 0)$-tensors are

$$\sum_i^n a_i \frac{\partial}{\partial x_i}$$

where $a^i \in C^\infty(U)$, and C^∞ 1 -forms as $(0, 1)$-tensors are

$$\sum_i a_i dx_i$$

where $a^i \in C^\infty(U)$.

(iii) C^∞ (r,s)-tensors are

$$\sum_{\substack{1\le i_1,\cdots,i_r\le n\\ 1\le j_1,\cdots,j_s\le n}} a^{i_1,\cdots,i_r}_{j_1,\cdots,j_s}\frac{\partial}{\partial x_{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x_{i_r}}\otimes dx_{j_1}\otimes\cdots\otimes dx_{j_s} \tag{12}$$

for $a^{i_1,\cdots,i_r}_{j_1,\cdots,j_s}\in C^\infty(U)$. And C^∞ s-form is

$$\sum_{1\le j_1,\cdots,j_s\le n} a_{j_1,\cdots,j_s}dx_{j_1}\wedge\cdots\wedge dx_{j_s}$$

with convection $dx_1\wedge dx_2 = dx_1\otimes dx_2 - dx_2\otimes dx_1$.

11.2 Pullback and Pushforwards

Definition 11.3 (Pullback of $(0,s)$-tensor under C^∞ map)**.** *Let M, N smooth manifolds. $\phi: M\to N$ C^∞ map.*

(i) $d\phi_p: T_pM\to T_{\phi(p)}N$. One get pullback dual map $d\phi_p^*: T^*_{\phi(p)}N\to T^*_pM$ s.t.

$$d\phi_p^*(Y)(X) := Y(d\phi_p(X)) \qquad \forall\ X\in T_pM \quad \text{and} \quad Y\in T^*_{\phi(p)}N$$

which generalizes to s inputs

$$(d\phi_p^*)^{\otimes s}: (T^0_sN)_{\phi(p)} = (T^*_{\phi(p)}N)^{\otimes s}\to (T^0_sM)_p = (T^*_pM)^{\otimes s}$$

s.t.

$$\begin{aligned}(d\phi_p^*)^{\otimes s}(Y_1\otimes\cdots\otimes Y_s)(X_1,\cdots,X_s) &:= (Y_1\otimes\cdots\otimes Y_s)\left(d\phi_p^{\otimes s}(X_1,\cdots,X_s)\right)\\ &= (Y_1\otimes\cdots\otimes Y_s)\left(d\phi_p(X_1),\cdots,d\phi_p(X_s)\right)\end{aligned}$$

$\forall\ X_1\cdots X_s\in T_pM$ and $Y_1\cdots Y_s\in T^*_{\phi(p)}N$.

(ii) We define the pullback of $(0,s)$-tensor

$$\phi^*: C^\infty(N,T^0_sN)\to C^\infty(M,T^0_sM) \qquad T\mapsto\phi^*T$$

from $(0,s)$-tensor on N to $(0,s)$-tensor on M s.t. $\forall\ p\in M$

$$(\phi^*T)(p) := (d\phi_p^*)^{\otimes s}\left(T(\phi(p))\right)$$

where $T(\phi(p))\in T^0_s(N)_{\phi(p)}$ and $(d\phi_p^*)^{\otimes s}\left(T(\phi(p))\in T^0_s(N)_{\phi(p)}\right)\in T^0_s(M)_p$. In particular, for $T\in\Omega^s(N)$, for any $X_1,\cdots,X_s\in\mathfrak{X}(M)$

$$(\phi^*T)(X_1,\cdots,X_s) := T(d\phi(X_1),\cdots,d\phi(X_s))$$

One can check $\phi^*T: M\to T^r_sM$ is a C^∞ section using local coordinates.

(iii) The above definition works for pullback of s-forms, i.e. $\phi^*:\Omega^s(N)\to\Omega^s(M)$. As a particular example, consider $\Omega^1(N)$ the space of 1-forms.

(a) If $f\in C^\infty(N)=\Omega^0(N)$, so $df\in\Omega^1(N)$ as in Example 11.1. For any $q\in N$

$$df(q) = df_q: T_qN\to\mathbb{R} \qquad \text{s.t.} \qquad df = \sum_{i=1}^n\frac{\partial f}{\partial y_i}dy_i \qquad \text{on } V$$

where $(y_1,\cdots,y_n)$ is local coordinates on $V\subset N$ open. One has the following commutative lemma

Lemma 11.1. *$\phi^*df = d(\phi^*f)\in\Omega^1(M)$*

Proof. For any $p\in M$

$$(\phi^*df)(p) = d\phi_p^*(df_{\phi(p)}) = df_{\phi(p)}\circ d\phi_p = d(f\circ\phi)_p = d(\phi^*f)(p)$$

∎

(b) If more generally take any 1-form over N with smooth frame $\{dy_i\}_{i=1}^n$ in local coordinates, one has

$$\phi^* dy_i = \sum_{j=1}^n \frac{\partial y_i}{\partial x_j} dx_j \ \in \Omega^1(M)$$

so for the local coordinate representation,

$$\phi^*(\sum_{i=1}^n a_i dy_i) = \sum_{i=1}^n (a_i \circ \phi)\ \phi^* dy_i \ \in \Omega^1(M)$$

for $a_i \in C^\infty(N)$.

Example 11.3. *Let $\phi : (0,\infty) \times \mathbb{R} \to \mathbb{R}^2$ be*

$$\phi(r,\theta) := (r\cos(\theta), r\sin(\theta)) = (x,y) \in \mathbb{R}^2$$

We'd like to compute $\phi^* dx$, $\phi^* dy$ and $\phi^*(dx \wedge dy)$. Recall $\phi^*(x) = r\cos(\theta)$ and $\phi^*(y) = r\sin(\theta)$.

1. $\phi^*(dx) = d(\phi^* x) = d(r\cos(\theta)) = \cos(\theta)dr - r\sin(\theta)d\theta$.
2. $\phi^*(dy) = d(\phi^* y) = d(r\sin(\theta)) = \sin(\theta)dr + r\cos(\theta)d\theta$.
3. $\phi^*(dx \wedge dy) = d(\phi^* x) \wedge d(\phi^* y) = r\cos^2(\theta)dr \wedge d\theta + r\sin^2(\theta)dr \wedge d\theta = rdr \wedge d\theta$.

We may also compute

$$\begin{aligned}\phi^*(-ydx + xdy) &= -r\sin(\theta)(\cos(\theta)dr - r\sin(\theta)d\theta) + r\cos(\theta)(\sin(\theta)dr + r\cos(\theta)d\theta)\\ &= r^2 d\theta\end{aligned}$$

Lemma 11.2. *For $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$*

$$(g \circ f)^* = f^* g^* : C^\infty(M_3, T_s^0(M_3)) \to C^\infty(M_1, T_s^0(M_1))$$

Definition 11.4 (Pullback and Pushforward of (r,s)-tensor under C^∞ diffeomorphism)**.** *Let M, N be smooth manifolds with the same dimension. Let $F : M \to N$ be C^∞ diffeomorphism with inverse $F^{-1} : N \to M$. Note for any $p \in M$ we have $F(p) \in N$.*

(i) Define pullback $F^* : C^\infty(N, T_s^r(N)) \to C^\infty(M, T_s^r M)$ that takes (r,s)-tensor T on N to F^*T, a (r,s)-tensor on M

$$(F^*T)(p) := \left(dF_p^{-1}\right)^{\otimes r} \otimes ((dF_p)^*)^{\otimes s} (T(F(p)))$$

for $T(F(p)) \in (T_s^r N)_{F(p)} = (T_{F(p)}N)^{\otimes r} \otimes (T^*_{F(p)}N)^{\otimes s}$. One can check $F^*T : M \to T_s^r M$ is a C^∞ section using local coordinates.

(ii) Define pushforward

$$F_* := (F^{-1})^* : C^\infty(M, T_s^r M) \to C^\infty(N, T_s^r N)$$

Lemma 11.3. *For $M_1 \xrightarrow{F} M_2 \xrightarrow{G} M_3$ C^∞ diffeomorphism.*

$$(G \circ F)^* = G^* \circ F^*$$

Example 11.4. *Let $M = \{(r,\theta) \mid r > 0,\ |\theta| < \frac{\pi}{2}\}$ and $F : M \to \mathbb{R}^2$ s.t. $F(r,\theta) = (r\cos(\theta), r\sin(\theta))$. Consider the pullback of tensor field $A = \frac{1}{x^2} dy \otimes dy$ by F*

$$\begin{aligned}F^*A &= \frac{1}{r^2\cos^2(\theta)} d(r\sin(\theta)) \otimes d(r\sin(\theta))\\ &= \frac{1}{r^2\cos^2(\theta)} (\sin(\theta)dr + r\cos(\theta)d\theta) \otimes (\sin(\theta)dr + r\cos(\theta)d\theta)\\ &= \frac{\tan^2(\theta)}{r^2} dr \otimes dr + \frac{\tan(\theta)}{r}(dr \otimes d\theta + d\theta \otimes dr) + d\theta \otimes d\theta\end{aligned}$$

11.3 Lie Derivatives of Tensors

We discuss Lie Derivative L_X on $(r,s)-$tensors for $X \in \mathfrak{X}(M)$.

Definition 11.5 (Lie Derivative on Tensors)**.** *Given $X \in \mathfrak{X}(M)$ for M C^∞ manifold. We want to define $L_X : C^\infty(M, T^r_s M) \to C^\infty(M, T^r_s M)$ s.t. $T \mapsto L_X T$ extending*

$$L_X : C^\infty(M) \to C^\infty(M) \text{ s.t. } f \mapsto L_X f = Xf \qquad \text{on } (0,0) - \text{tensor}$$
$$L_X : \mathfrak{X}(M) \to \mathfrak{X}(M) \text{ s.t. } Y \mapsto L_X Y := [X,Y] \qquad \text{on } (1,0) - \text{tensor}$$

- *Approach 1. We want to define $L_X : \Omega^1(M) \to \Omega^1(M)$ $(0,1)$-tensors by requiring that it is $\mathbb{R}$-linear and satisfies the following Leibnitz rule: For any*

  $$\alpha \in \Omega^1(M) \in C^\infty(M, T^*M = T^0_1(M)) \qquad \text{and} \qquad Y \in \mathfrak{X}(M) = C^\infty(M, TM = T^1_0(M))$$

  note $\alpha(Y) \in C^\infty(M)$ s.t.

  $$\alpha(Y)(p) = \alpha(p)(Y(p)) \in \mathbb{R} \qquad for\ \alpha(p) : T_pM \to \mathbb{R}$$

  The Leibnitz rule is

  $$\begin{aligned} L_X(\alpha(Y)) &= (L_X\alpha)(Y) + \alpha(L_X Y) \\ (L_X\alpha)(Y) &= L_X(\alpha(Y)) - \alpha(L_X Y) \\ &= X(\alpha(Y)) - \alpha([X,Y]) \end{aligned}$$

  The only way to define L_X is as following

  - *Define $L_X : \Omega^1(M) \to \Omega^1(M)$ s.t. For any*

    $$\alpha \in \Omega^1(M) \in C^\infty(M, T^*M = T^0_1(M)) \qquad \text{and} \qquad Y \in \mathfrak{X}(M) = C^\infty(M, TM = T^1_0(M))$$
    $$(L_X\alpha)(Y) = X(\alpha(Y)) - \alpha([X,Y])$$

  - *tensor product*

    $$L_X(S \otimes T) = (L_X S) \otimes T + S \otimes (L_X T)$$

    this extends to tensors of any type.

- *Approach 2. Given $X \in \mathfrak{X}(M)$ we want to define $L_X T$ where T is (r,s)-tensor on M, using the local flow of X. For any $p \in M$, there exists open neighborhood U of p in M, for $\varepsilon > 0$*

  $$\phi_t : U \overset{C^\infty}{\to} M \qquad t \in (-\varepsilon, \varepsilon)$$

  Define

  $$\left(\tilde{L}_X T\right)(p) := \left.\frac{d}{dt}\right|_{t=0} (\phi_t^* T)(p)$$

  where $(-\varepsilon,\varepsilon) \overset{C^\infty}{\to} (T^r_s M)_p = (T_pM)^{\otimes r} \otimes (T^*_pM)^{\otimes s}$ maps $t \mapsto (\phi_t^*T)(p)$. We have seen that

  $$(\tilde{L}_X f)(p) = X(p)f \qquad \forall\ f \in C^\infty(M)$$
  $$(\tilde{L}_X Y)(p) = [X,Y](p) \qquad \forall\ Y \in \mathfrak{X}(M)$$

  Claim: $\tilde{L}_X T = L_X T$ for any T tensor on M of any type (r,s). It suffices to check that

  (a) $(\tilde{L}_X\alpha)(Y) = X(\alpha(Y)) - \alpha([X,Y])$ *for any*

  $$\alpha \in \Omega^1(M) \in C^\infty(M, T^*M = T^0_1(M)) \qquad \text{and} \qquad Y \in \mathfrak{X}(M) = C^\infty(M, TM = T^1_0(M))$$

  (b)

  $$\tilde{L}_X(S \otimes T) = (\tilde{L}_X S) \otimes T + S \otimes (\tilde{L}_X T)$$

  To do so, one use local flow

  $$\begin{cases} \phi_t^*(\alpha(Y)) = \phi_t^*(\alpha)\phi_t^*(Y) \\ \phi_t^*(\alpha(S \otimes T)) = \phi_t^*(S) \otimes \phi_t^*(T) \end{cases}$$

  and take derivative $\left.\frac{d}{dt}\right|_{t=0}$ to determine uniquely.

Lemma 11.4. *For $\omega \in \Omega^k(M)$, $\tau \in \Omega^\ell(M)$ and $X \in \mathfrak{X}(M)$*

$$L_X(\omega \wedge \tau) = (L_X\omega) \wedge \tau + \omega \wedge (L_X\tau)$$

Lemma 11.5. *For $\omega \in \Omega^k(M)$, $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$*

$$L_X(f\omega) = L_X(f)\omega + f(L_X\omega) = (Xf)\omega + fL_X\omega$$

Lemma 11.6 (Leibnitz Rule for Lie Derivative)**.** *For any $\omega \in \Omega^s(M)$, $X \in \mathfrak{X}(M)$ and $Y_1, \cdots, Y_s \in \mathfrak{X}(M)$*

$$L_X(\omega(Y_1, \cdots, Y_s)) = (L_X\omega)(Y_1, \cdots, Y_s) + \sum_{i=1}^{s} \omega(Y_1, \cdots, Y_{i-1}, [X, Y_i], Y_{i+1}, \cdots, Y_s)$$

Example 11.5. *Let $\omega = -ydx + xdy \in \Omega^1(\mathbb{R}^2)$, and $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{S}^1)$. We want to compute $L_X\omega$. Using that L_X is a derivation and L_X commutes with d*

$$\begin{aligned}
L_X(-ydx + xdy) &= -L_X(ydx) + L_X(xdy) \\
&= -(L_X(y)dx + yL_X(dx)) + (L_X(x)dy + xL_X(dy)) \\
&= -L_X(y)dx - yd(L_X(x)) + L_X(x)dy + xd(L_X(y))
\end{aligned}$$

it suffices to compute

$$\begin{aligned}
L_X(x) &= -yL_{\frac{\partial}{\partial x}}(x) + xL_{\frac{\partial}{\partial y}}(x) = -y \\
L_X(y) &= \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) y = x
\end{aligned}$$

so

$$L_X(-ydx + xdy) = -xdx + ydy - ydy + xdx = 0$$

Example 11.6. *Let $A \in C^\infty(M, T_2^0(M))$ be covariant 2-tensor field for M with dimension n. Let $V \in \mathfrak{X}(M)$. We wish to compute L_VA in local coordinates. First note $L_V(dx^i) = d(L_Vx^i) = d(Vx^i) = dV^i = \sum_{k=1}^n \frac{\partial V^i}{\partial x^k}dx^k$.*

$$\begin{aligned}
L_V(A_{ij}dx^i \otimes dx^j) &= L_V(A_{ij})dx^i \otimes dx^j + A_{ij}(d(Vx^i) \otimes dx^j + dx^i \otimes d(Vx^j)) \\
&= \left(VA_{ij} + A_{kj}\frac{\partial V^k}{\partial x^i} + A_{ik}\frac{\partial V^k}{\partial x^j}\right) dx^i \otimes dx^j
\end{aligned}$$

11.4 Exterior and Interior derivatives on Forms

We discuss exterior and interior derivatives on forms. Let $L_X : \Omega^s(M) \to \Omega^s(M)$ be Lie derivative on s-forms.

Definition 11.6 (Exterior Derivative on forms)**.** *$d : \Omega^s(M) \to \Omega^{s+1}(M)$ is exterior derivative if it is $\mathbb{R}$-linear and satisfies*

(a) For any $f \in C^\infty(M) = \Omega^0(M)$, $df \in \Omega^1(M)$, $df(p) = df_p : T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$ where $df(X) = X(f)$ for $X \in \mathfrak{X}(M)$, i.e., df is the differential of f.

(b) For any $f \in \Omega^0(M)$ we have $df \in \Omega^1(M)$ and $d(df) = 0$

(c) For $\alpha \in \Omega^r(M)$ and $\beta \in \Omega^s(M)$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta$$

In local coordinates (U, ϕ) C^∞ chart on M. For $\alpha \in \Omega^s(M)$, on U

$$\alpha = \sum_{1 \le j_1, \cdots, j_s \le n} a_{j_1, \cdots, j_s} dx_{j_1} \wedge \cdots \wedge dx_{j_s}$$

for $a_{j_1, \cdots, j_s} \in C^\infty(U)$. Then we compute

$$\begin{aligned}
d\alpha &= d\left(\sum_{1 \le j_1, \cdots, j_s \le n} a_{j_1, \cdots, j_s} dx_{j_1} \wedge \cdots \wedge dx_{j_s}\right) \\
&= \sum_{1 \le j_1, \cdots, j_s \le n} da_{j_1, \cdots, j_s} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_s} \\
&= \sum_{1 \le j_1, \cdots, j_s \le n} \sum_{k=1}^{n} \frac{\partial a_{j_1, \cdots, j_s}}{\partial x_k} dx_k \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_s}
\end{aligned}$$

Proposition 11.1. *Let d be the exterior derivative.*

(i) $dd\omega = 0$ *for any* $\omega \in \Omega^s(M)$.

(ii) For $F : M \to N$ C^∞ *map, for any* $\omega \in \Omega^s(N)$

$$d(F^*\omega) = F^*(d\omega) \in \Omega^{s+1}(M)$$

This is naturality of d that it commutes with pullbacks $d \circ F^* = F^* \circ d$

(iii) For $X \in \mathfrak{X}(M)$ *and* $\omega \in \Omega(M)$

$$d(L_X\omega) = L_X(d\omega) \in \Omega^{s+1}(M)$$

so d commutes with Lie derivatives $d \circ L_X = L_X \circ d$

(iv) For $\alpha \in \Omega^s(M)$ *and* $X_0 \cdots X_s \in \mathfrak{X}(M)$

$$(d\alpha)(X_0 \cdots X_s) = \sum_{i=0}^{s}(-1)^i X_i\left(\alpha(X_0, \cdots, \hat{X}_i, \cdots, X_s)\right) + \sum_{0\le i<j\le s}(-1)^{i+j}\alpha\left([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_s\right)$$

or in short, for $\alpha \in \Omega^1(M)$, X, $Y \in \mathfrak{X}(M)$

$$(d\alpha)(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]) \tag{13}$$

Proof for Prop 11.1 (iv) $\Omega^1(M)$ case. By linearity in $\mathbb{R}$, it suffices to assume $\alpha = fdg$ where f, $g \in C^\infty(U)$ for U open set on M.

$$\begin{aligned}
(d\alpha)(X, Y) &= (df \wedge dg)(X, Y) = df(X)dg(Y) - dg(X)df(Y) = (Xf)Yg - (Xg)Yf \\
X\alpha(Y) &= X((fdg)(Y)) = X(f)dg(Y) + fX(dg(Y)) = (Xf)Yg + fX(Yg) \\
Y\alpha(X) &= Y(fdg(X)) = YfXg + fY(Xg) \\
\alpha([X, Y]) &= fdg(XY - YX) = fXYg - fYXg
\end{aligned}$$

□

Example 11.7. • *Let* $f \in C^\infty(\mathbb{R}^3)$, *then*

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

• *Let* $\alpha = Adx + Bdy + Cdz$ *for* A, B, $C \in C^\infty(\mathbb{R}^3)$. *Then*

$$\begin{aligned}
d\alpha &= dA \wedge dx + dB \wedge dy + dC \wedge dz \\
&= \left(\frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial z}dz\right) \wedge dx + \left(\frac{\partial B}{\partial x}dx + \frac{\partial B}{\partial y}dy + \frac{\partial B}{\partial z}dz\right) \wedge dy + \left(\frac{\partial C}{\partial x}dx + \frac{\partial C}{\partial y}dy + \frac{\partial C}{\partial z}dz\right) \wedge dz \\
&= -\frac{\partial A}{\partial y}dx \wedge dy + \frac{\partial A}{\partial z}dz \wedge dx + \frac{\partial B}{\partial x}dx \wedge dy - \frac{\partial B}{\partial z}dy \wedge dz - \frac{\partial C}{\partial x}dz \wedge dx + \frac{\partial C}{\partial y}dy \wedge dz \\
&= \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy + \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) dy \wedge dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) dz \wedge dx
\end{aligned}$$

• *Let* $\alpha = Cdx \wedge dy + Ady \wedge dz + Bdz \wedge dx$ *for* A, B, $C \in C^\infty(\mathbb{R}^3)$

$$\begin{aligned}
d\alpha &= dC \wedge dx \wedge dy + dA \wedge dy \wedge dz + dB \wedge dz \wedge dx \\
&= \frac{\partial C}{\partial z}dz \wedge dx \wedge dy + \frac{\partial A}{\partial x}dx \wedge dy \wedge dz + \frac{\partial B}{\partial y}dy \wedge dz \wedge dx \\
&= \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) dx \wedge dy \wedge dz
\end{aligned}$$

Since $d^2 = 0$, *this is to say for any* $f \in C^\infty(M)$, $\mathrm{curl}(\nabla f) = 0$, *and for any* $X \in \mathfrak{X}(\mathbb{R}^3)$, $\mathrm{div}(\mathrm{curl}(X)) = 0$.

Definition 11.7 (Interior Derivative on forms)**.** $X \in \mathfrak{X}(M)$. *Define interior derivative*

$$i_X : \Omega^s(M) \to \Omega^{s-1}(M) \qquad s.t. \qquad \alpha \in \Omega^s(M) \mapsto i_X\alpha \in \Omega^{s-1}(M)$$

by satisfying the following

- *$i_X f = 0$ for any $f \in C^\infty(M)$.*
- *$(i_X\alpha)(Y_1, \cdots, Y_{s-1}) = \alpha(X, Y_1, \cdots, Y_{s-1})$ for $Y_1, \cdots, Y_{s-1} \in \mathfrak{X}(M)$.*

Proposition 11.2. *Let i_X denote interior derivative*

(i) *$i_X \circ i_X \omega = 0$ for any $\omega \in \Omega^s(M)$*

(ii) *$\alpha \in \Omega^r(M)$, $\beta \in \Omega^s(M)$*
$$i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^r \alpha \wedge i_X\beta$$

(iii) *Cartan's formula.*
$$d \circ i_X + i_X \circ d = L_X$$

Lemma 11.7. *For any $\omega \in \Omega^s(M)$, X, $Y \in \mathfrak{X}(M)$*

$$L_X(i_Y\omega) - i_Y(L_X\omega) = i_{[X,Y]}\omega$$

12 Riemannian Metric

Let M be C^∞ manifold.

Definition 12.1 (Riemannian Metric). *A Riemannian Metric on M is a C^∞ $(0,2)$-tensor g on M s.t. $\forall\, p \in M$, $g(p) \in T_p^*M \otimes T_p^*M$*

$$g(p) : T_pM \times T_pM \to \mathbb{R} \qquad \text{defines an inner product s.t. } (v_1,\, v_2) \mapsto g(p)(v_1,\, v_2)$$

- $g(p)(v_1, v_2) = g(p)(v_2, v_1)$
- $g(p)(v, v) > 0$ *if* $v \neq 0$

Let $n = dimM$. Then the tensor bundle $T_2^0M = T^*M \otimes T^*M = S^2T^*M \otimes \Lambda^2T^*M$ splits into product of symmetric and anti-symmetric tensor bundles, with rank $\frac{n(n+1)}{2}$ and $\frac{n(n-1)}{2}$ respectively.
For any $p \in M$,

- $(S^2T^*M)_p = \{$*symmetric bilinear forms on* $T_pM\}$
- $(\Lambda^2T^*M)_p = \{$*skew-symmetric bilinear forms on* $T_pM\}$

and $g \in C^\infty(M, S^2T^*M) = \{C^\infty$ symmetric (0, 2)-tensors$\}$.
The pair (M, g) is a Riemannian manifold.

In local coordinates, let (U, ϕ) be C^∞ chart for M with $\phi = (x_1, \cdots, x_n)$.

$$dx_i dx_j := \frac{dx_i \otimes dx_j + dx_j \otimes dx_i}{2} \in C^\infty(U, S^2\, T^*M|_U)$$

So $\{dx_i dx_j \mid 1 \leq i \leq j \leq n\}$ is C^∞ frame of $S^2\, T^*M|_U = S^2T^*U$. Recall that on the other hand

$$\{dx_i \wedge dx_j := dx_i \otimes dx_j - dx_j \otimes dx_i \mid 1 \leq i \leq j \leq n\}$$

is C^∞ frame of $\Lambda^2\, T^*M|_U$. One may write

$$dx_i^2 = dx_i dx_i = dx_i \otimes dx_i$$

And on U

$$g = \sum_{ij} g_{ij} dx_i \otimes dx_j = \sum_{ij} g_{ij} dx_i dx_j \qquad g_{ij} = g_{ji}$$

For $\dim M = 2$ with (x_1, x_2),

$$g = g_{11}dx_1^2 + 2g_{12}dx_1dx_2 + g_{22}dx_2^2$$

Example 12.1 (Euclidean and Polar coordinates). *Let $M = \mathbb{R}^n$ with Euclidean metric*

$$g_0 = \sum_{i=1}^n dx_i^2 = \sum_{ij} g_{ij} dx_i dx_j$$

so $g_{ij} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

- *For $\mathbb{R}^2$ with $(x, y) = (r\cos(\theta), r\sin(\theta))$, one may write in polar coordinates*

$$g_0 = dx^2 + dy^2 = (\cos(\theta)dr - r\sin(\theta)d\theta)^2 + (\sin(\theta)dr + r\cos(\theta)d\theta)^2 = dr^2 + r^2d\theta^2$$

- *For $\mathbb{R}^3$ with $(x, y, z) = (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))$ for $\rho > 0$, $\theta \in (0, 2\pi)$ and $\phi \in (0, \pi)$.*

$$\begin{aligned}
&g_0 = dx^2 + dy^2 + dz^2 \\
&= (\sin(\phi)\cos(\theta)d\rho - \rho\sin(\theta)\sin(\phi)d\theta + \rho\cos(\phi)\cos(\theta)d\phi)^2 + (\sin(\phi)\sin(\theta)d\rho + \rho\cos(\theta)\sin(\phi)d\theta + \rho\cos(\phi)\sin(\theta)d\phi)^2 \\
&+ (\cos(\phi)d\rho - \rho\sin(\phi)d\phi)^2 \\
&= d\rho^2 + \rho^2d\phi^2 + \rho^2\sin^2(\phi)d\theta^2
\end{aligned}$$

One may also do for smooth frames

- *On $\mathbb{R}^2$, $dx^2+dy^2 = dr^2 + r^2 d\theta^2$. We have $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ orthonormal with*

$$\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x}\rangle = 1 = \langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y}\rangle \qquad \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle = 0$$

We have

$$\frac{\partial}{\partial r} \qquad \frac{1}{r}\frac{\partial}{\partial \theta} \qquad \text{on } \mathbb{R}^2 \setminus \{0\}$$

as orthonormal basis

- *On $\mathbb{R}^3$, $dx^2+dy^2+dz^2 = d\rho^2 + \rho^2 d\phi^2 + \rho^2 \sin^2(\phi) d\theta^2$ with orthonormal frame $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$. One has*

$$\frac{\partial}{\partial \rho}, \qquad \frac{1}{\rho}\frac{\partial}{\partial \phi}, \qquad \frac{1}{\rho \sin(\phi)}\frac{\partial}{\partial \theta} \qquad \text{on open dense subset } U \subset \mathbb{R}^3$$

as orthonormal basis.

Definition 12.2 (pullback of Riemannian metric)**.** *(M, g) Riemannian manifold. If $f : M' \to M$ is C^∞ map from C^∞ manifold M' to M. Then f^*g is a C^∞ symmetric $(0, 2)$-tensor on M'. Moreover, for f^*g to define an inner product so that it equips a Riemannian metric on M', we have the following equivalent conditions: For any $p \in M'$, for any $v \neq 0 \in T_pM'$*

$$(f^*g)(v, v) := g(p)(df_p(v), df_p(v)) > 0$$

iff for any $p \in M'$,

$$df_p : T_pM' \to T_{f(p)}M \qquad \text{is injective}$$

iff f is an immersion

Remark 12.1. *If (M, g) is Riemannian manifold and $M' \subset M$ a C^∞ manifold, $i : M' \to M$ inclusion map as C^∞ embedding. Then (M', i^*g) is a Riemannian submanifold. For any $p \in M' \subset M$,*

$$(i^*g)(p) : T_pM' \times T_pM' \to \mathbb{R}$$

is the restriction of $g(p) : T_pM \times T_pM \to \mathbb{R}$.

Example 12.2 (Canonical metric on $\mathbb{S}^n(r)$)**.** *$\mathbb{S}^n(r) := \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = r^2\} \subset \mathbb{R}^{n+1}$ for $r > 0$. Define $i_r : \mathbb{S}^n(r) \to \mathbb{R}^{n+1}$ inclusion.*

$$g_{can}^{\mathbb{S}^n(r)} := i_r^* g_0 = i_r^*(dx_1^2 + \cdots dx_{n+1}^2)$$

defines canonical metric on the round sphere of radius r. For $n = 3$

$$g_0 = d\rho^2 + \rho^2 d\phi^2 + \rho^2 \sin^2(\phi) d\theta^2$$

One has

$$g_{can}^{\mathbb{S}^2(r)} = i_r^* g_0 = r^2(d\phi^2 + \sin^2(\phi) d\theta^2) \qquad \forall\ (\phi, \theta)$$

local coordinates on $U \subset \mathbb{S}^2(r)$ open.

Definition 12.3. *$f : (M_1, g_1) \to (M_2, g_2)$ is a C^∞ map between two Riemannian manifolds.*

- *We say f is an isometric immersion if f is an immersion and $f^*g_2 = g_1$.*
- *We say f is an isometric embedding if f is an embedding and $f^*g_2 = g_1$.*
- *We say f is an isometry (local isometry) if f is a diffeomorphism (local diffeomorphism) and $f^*g_2 = g_1$*

Example 12.3. *$i_r : (\mathbb{S}^n(r), g_{can}^{\mathbb{S}^n(r)}) \mapsto (\mathbb{R}^{n+1}, g_0)$ is an isometric embedding.*

Example 12.4. *$A \in GL(n, \mathbb{R})$. $L_A : \mathbb{R}^n \to \mathbb{R}^n$ linear isomorphism $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto Ax$ is C^∞ diffeomorphism. For $g_0 = \sum_{i=1}^n dx_i^2$, when is L_A an isometry between $(\mathbb{R}^n, g_0)$? i.e., when is $L_A^* g_0 = g_0$?. Note for $A = (a_{ij})$,*

$(Ax)_i = \sum_j a_{ij} x_j$

$$\begin{aligned}
L_A^* x_i &= \sum_j a_{ij} x_j \\
L_A^* dx_i &= d(L_A^* x_i) = \sum_j a_{ij} dx_j \\
L_A^* g_0 &= L_A^* (\sum_{i=1}^n dx_i^2) = \sum_{i,j,k} (a_{ij} dx_j)(a_{ik} dx_k) = \sum_{j,k=1}^n \left(\sum_{i=1}^n a_{ij} a_{ik} \right) dx_j dx_k \\
&= \sum_{j,k=1}^n \left(A^T A\right)_{jk} dx_j dx_k
\end{aligned}$$

So $L_A^* g_0 = g_0$ *iff* $A^T A = T_n$ *iff* $A \in O(n)$. *For* $b \in \mathbb{R}^n$, $T_b : \mathbb{R}^n \to \mathbb{R}^n$ *s.t.* $x \mapsto x + b$. *Here* $T_b^* x_i = x_i + b_i$, $T_b^* dx_i = dx_i$ *and* $T_b^* g_0 = g_0$.

Theorem 12.1. $f : (\mathbb{R}^n, g_0) \to (\mathbb{R}^n, g_0)$ *is an isometry iff*

$$f(x) = Ax + b \qquad \text{for } A \in O(n) \text{ and } b \in \mathbb{R}^n$$

i.e., f *is a rigid motion.*

Observe that, $A \in O(n+1)$ *and* $L_A : (\mathbb{R}^{n+1}, g_0) \to (\mathbb{R}^{n+1}, g_0)$ *is an isometry and* $L_A(\mathbb{S}^n) = \mathbb{S}^n$. *So* $L_A : (\mathbb{S}^n, g_{can}) \to (\mathbb{S}^n, g_{can})$ *is an isometry.*

$$g_{can} = i^* g_0 \qquad L_A^* g_0 = L_A^* g_0$$

Theorem 12.2. $f : (\mathbb{S}^n,\, g_{can}) \to (\mathbb{S}^n,\, g_{can})$ *is an isometry iff* $f : \mathbb{S}^n \to \mathbb{S}^n$ *is* $f(x) = Ax$ *for some* $A \in O(n+1)$.

Example 12.5. $f : \mathbb{R} \to \mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \overset{i^*}{\subset} \mathbb{R}^2$ *where* $f(t) := (\cos(t), \sin(t))$. *So*

$$f^* g_{can}^{\mathbb{S}^1} = f^* i^* (dx^2 + dy^2) = (d(\cos(t)))^2 + (d(\sin(t)))^2 = (-\sin(t) dt)^2 + (\cos(t) dt)^2 = dt^2$$

$f : (\mathbb{R}, dt^2) \to (\mathbb{S}^1, g_{can})$ *is a local isometry, and in fact a covering map.*

Definition 12.4 (Product Metric). *If* (M_1, g_1) *and* (M_2, g_2) *are Riemannian manifolds, then*

$$g_1 \times g_2 := \pi_1^* g_1 + \pi_2^* g_2$$

is a Riemannian metric on $M_1 \times M_2$. *For any* $p_i \in M_i$, $T_{(p_1,p_2)}(M_1 \times M_2) = T_{p_1} M_1 \oplus T_{p_2} M_2$ *so that*

$$g_1 \times g_2(p_1, p_2)|_{T_{(p_1 \times p_2)}(M_1 \times M_2)} = g_1(p_1)|_{T_{p_1} M_1} \oplus g_2(p_2)|_{T_{p_2} M_2}$$

i.e., the product metric writes

$$(g_1 \times g_2)_{(p_1,p_2)} : T_{(p_1,p_2)}(M_1 \times M_2) \times T_{(p_1,p_2)}(M_1 \times M_2) \to \mathbb{R} \qquad s.t. \quad \langle (u_1, u_2), (v_1, v_2) \rangle = \langle u_1, v_1 \rangle + \langle u_2, v_2 \rangle \qquad \forall\, u_i,\, v_i \in T_{p_i} M_i$$

Example 12.6. $f : (\mathbb{R}^n, g_0 = dt_1^2 + \cdots dt_n^2) \to \left(\mathbb{T}^n := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1, g_{can} \times \cdots \times g_{can}\right) \subset (\mathbb{R}^{2n}, g_0)$ *the flat* n*-torus.*

$$f(t_1, \cdots, t_n) = (\cos(t_1),\, \sin(t_1), \cdots, \cos(t_n),\, \sin(t_n))$$

f *is a local isometry.*

13 Volume, Length and Distance

13.1 Volume

Riemannian metric gives rise to volume, length and distance.

Definition 13.1 (Volume Form). *A volume form on a C^∞ manifold M of dimension n is a nowhere vanishing C^∞ n-form $\nu \in \Omega^n(M) = C^\infty(M, \Lambda^n T^*M)$*

Lemma 13.1. *Let M be C^∞ manifold. Then the following are equivalent:*

- *There exists a volume form $\nu \in \Omega^n(M)$ on M*
- *$\Lambda^n T^*M$ is trivial.*
- *M is orientable.*

Hence a volume form $\nu \in \Omega^n(M)$ determines an orientation on M. ν_1 and ν_2 volume forms determine the same orientation iff $\nu_1 = \rho\nu_2$ for some $\rho \in C^\infty(M)$ with $\rho > 0$.

Proof of Existence of Volume form implies orientable. Suppose $\nu \in \Omega^n(M)$ is a volume form on M. We may choose C^∞ atlas $\{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$ where $\phi_\alpha = (x_1^\alpha, \cdots, x_n^\alpha)$ on M s.t., on U_α

$$\nu = a_\alpha dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha \qquad a_\alpha \in C^\infty(U_\alpha) \qquad a_\alpha > 0$$

On $U_\alpha \cap U_\beta$

$$\nu = a_\beta dx_1^\beta \wedge \cdots \wedge dx_n^\beta = a_\alpha dx_1^\alpha \wedge \cdots \wedge x_n^\alpha$$

For

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \qquad (x_1^\alpha, \cdots, x_n^\alpha) \mapsto (x_1^\beta(x_1^\alpha, \cdots, x_n^\alpha), \cdots)$$

Hence

$$dx_1^\beta \wedge \cdots \wedge dx_n^\beta = \left(\sum_{j_1} \frac{\partial x_1^\beta}{\partial x_{j_1}^\alpha} dx_{j_1}^\alpha\right) \wedge \cdots \wedge \left(\sum_{j_n} \frac{\partial x_n^\beta}{\partial x_{j_n}^\alpha} dx_{j_n}^\alpha\right)$$

$$\begin{aligned}
\implies \det(d(\phi_\beta \circ \phi_\alpha^{-1})) &= \det(\frac{\partial x_i^\beta}{\partial x_j^\alpha}) \\
\implies a_\beta dx_1^\beta \wedge \cdots \wedge x_n^\beta &= a_\beta \det(d(\phi_\beta \circ \phi_\alpha^{-1})) dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha \\
&= a_\alpha dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha \\
\implies \det(d(\phi_\beta \circ \phi_\alpha^{-1})) &= \frac{a_\beta}{a_\alpha} > 0
\end{aligned}$$

□

Proposition 13.1 (Orientable implies Existence of compatible volume form). *Suppose (M, g) is an oriented Riemannian manifold. Then there exists a unique volume form $\nu \in \Omega^n(M)$ where $n = dimM$ which is compatible with g and the orientation. In fact, in local coordinates*

$$\nu_g(p) = \sqrt{\det(g_{ij})}(dx_1 \wedge \cdots \wedge dx_n)(p)$$

Remark 13.1. *For any $p \in M$, let $(e_1, \cdots, e_n)$ be an ordered orthonormal basis of $(T_pM, \langle\cdot,\cdot\rangle_p)$ where $\langle e_j, e_j\rangle_p = \delta_{ij}$ is the inner product defined by $g(p)$. Let $\{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$ be the atlas defining the given orientation. For $p \in U_\alpha$, one has coordinates $\phi_\alpha = (x_1^\alpha, \cdots, x_n^\alpha)$. $(e_1, \cdots, e_n)$ is compatible with the orientation in the sense that*

$$e_i = \sum_{j=1}^n a_{ij} \frac{\partial}{\partial x_j^\alpha}(p) \qquad A = (a_{ij}) \qquad \det(A) > 0$$

Hence

$$(dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha)_p (e_1, \cdots, e_n) > 0$$

Let $(e_1^*, \cdots, e_n^*)$ be ordered basis of T_p^*M dual to $(e_1, \cdots, e_n)$. Then

$$\nu(p) = e_1^* \wedge \cdots \wedge e_n^* \in \Lambda^n T_p^* M$$

iff $\nu(p)(e_1, \cdots, e_n) = 1$ for any ordered orthonormal basis $(e_1, \cdots, e_n)$ of $(T_pM, \langle\cdot,\cdot\rangle_p)$ compatible with the orientation.

$$\langle e_j, e_j\rangle_p = g(p)(e_i, e_j) = \delta_{ij} \qquad g(p) = \sum_{i=1}^n e_i^* \otimes e_i^*$$

Proof of 13.1. For Existence, for any $p \in M$, define $\nu(p) := e_1^* \wedge \cdots \wedge e_n^*$ as above. (U, ϕ) is C^∞ chart on M compatible with the orientation for $\phi = (x_1, \cdots, x_n)$. On U, $g_{ij} = \sum_{ij} g_{ij} dx_i dx_j$ for $g_{ij} = g_{ji} \in C^\infty(U)$. Let $p \in U$, let $(e_1, \cdots, e_n)$ be the orthonormal basis of T_pM compatible with the orientation. Then

$$\frac{\partial}{\partial x_i}(p) = \sum_{j=1}^n b_{ij} e_j \qquad B = (b_{ij}) \in GL(n, \mathbb{R}) \ \det(B) > 0$$

Then

$$\begin{aligned}
g_{ij}(p) &= \langle \frac{\partial}{\partial x_i}(p), \frac{\partial}{\partial x_j}(p) \rangle \\
&= \langle \sum_k b_{ik} e_k, \sum_\ell b_{j\ell} e_\ell \rangle \\
&= \sum_{k,\ell} b_{ik} b_{j\ell} \delta_{k\ell} \\
&= \sum_k b_{ik} b_{jk} = (BB^T)_{ij}
\end{aligned}$$

$$\begin{aligned}
\implies \nu(p)(\frac{\partial}{\partial x_1}(p), \cdots, \frac{\partial}{\partial x_n}(p)) &= \nu(p)\left(\sum_j b_{1j} e_1, \cdots, \sum_j b_{nj} e_n\right) \\
&= \det(B)\nu(p)(e_1, \cdots, e_n) = \det(B) \\
\nu(p) &= \det(B)(dx_1 \wedge \cdots \wedge dx_n) \\
&= \sqrt{\det(g_{ij})}(dx_1 \wedge \cdots \wedge dx_n)(p)
\end{aligned}$$

using $\det(g_{ij}(p)) = \det(BB^T) = (\det B)^2$. Now on U with $g = \sum_{ij} g_{ij} dx_i dx_j$, $\nu = \sqrt{\det(g_{ij})} dx_1 \wedge \cdots \wedge dx_n$. We write $\nu_g = \nu$. $\square$

Example 13.1. $\mathbb{S}^2(r) = r^2(d\phi^2 + \sin^2(\phi) d\theta^2)$ *with* $(\phi, \theta) = (x_1, x_2)$. *Here*

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \sin^2(\phi) \end{pmatrix} \implies \det(g) = r^4 \sin^2(\phi)$$

So $\nu = \sqrt{\det(g)} d\phi \wedge d\theta = r^2 \sin(\phi) d\phi \wedge d\theta$. *Hence*

$$Vol(\mathbb{S}^2(r), g_{can}^{\mathbb{S}^2(r)}) = \int_0^{2\pi} \int_0^\pi r^2 \sin\phi \, d\phi d\theta = 4\pi r^2$$

13.2 Length

Definition 13.2 (Length)**.** *For* (M, g) *Riemannian manifold,* $\gamma : [a, b] \to M$ *is a* C^∞ *curve for* $-\infty < a < b < \infty$. *For any* $t \in (a, b)$, $\gamma'(t) \in T_{\gamma(t)}M$.

$$|\gamma'(t)|_{g(\gamma(t))} = \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} = \sqrt{g(\gamma(t))(\gamma'(t), \gamma'(t))}$$

Define

$$\ell_g(r) := \int_a^b |\gamma'(t)| \, dt$$

Recall $f : (M, g) \to (N, h)$ is isometric immersion, iff for any $p \in M$,

$$\langle v_1, v_2 \rangle_p = \langle df_p(v_1), df_p(v_2) \rangle_{f(p)}$$

the former defined by $g(p)$ and the latter defined by $h(f(p))$. Then for any $\gamma : [a, b] \to M$ C^∞ curve, $f \circ \gamma : [a, b] \to N$ is also C^∞ curve. Moreover

$$\ell_g(\gamma) = \ell_h(f \circ \gamma)$$

Example 13.2. $H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. $g_0 = dx^2 + dy^2$ *Euclidean metric.* $h = \frac{dx^2 + dy^2}{y^2}$ *is hyperbolic metric. For* $\gamma_1 : [x_0, x_1] \to H$ *s.t.* $\gamma_1(t) := (t, y_0)$ *and* $\gamma_2 : [y_0, y_1] \to H$ *s.t.* $\gamma_2(t) = (x_0, t)$, *then*

$$\gamma_1'(t) = \frac{\partial}{\partial x}(\gamma(t)) \qquad \gamma_2'(t) = \frac{\partial}{\partial y}(\gamma(t))$$

Then

$$g_0(x,y)\left(a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y},c\frac{\partial}{\partial x}+d\frac{\partial}{\partial y}\right)=ac+bd$$
$$h(x,y)\left(a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y},c\frac{\partial}{\partial x}+d\frac{\partial}{\partial y}\right)=\frac{ac+bd}{y^2}$$
$$\begin{aligned}
|\gamma_1'(t)|_{g_0} &= 1 = |\gamma_2'(t)|_{g_0}\\
|\gamma_1'(t)|_h &= \sqrt{\frac{1}{y_0^2}}=\frac{1}{y_0}\\
|\gamma_2'(t)|_h &= \frac{1}{t}\\
\ell_{g_0}(\gamma_1) &= \int_{x_0}^{x_1}|\gamma_1'(t)|_{g_0}\,dt=\int_{x_0}^{x_1}dt=x_1-x_0\\
\ell_{g_0}(\gamma_2) &= \int_{y_0}^{y_1}|\gamma_2'(t)|_{g_0}\,dt=\int_{y_0}^{y_1}dt=y_1-y_0\\
\ell_h(\gamma_1) &= \int_{x_0}^{x_1}\frac{dt}{y_0}=\frac{x_1-x_0}{y_0}\\
\ell_h(\gamma_2) &= \int_{y_0}^{y_1}\frac{dt}{t}=\log(y_1)-\log(y_0)=\log(\frac{y_1}{y_0})
\end{aligned}$$

Let $\lambda>0$ $\phi_\lambda : H\to H$ s.t.

$$\phi_\lambda(x,y)=(\lambda x,\ \lambda y)$$

so

$$\begin{aligned}
\phi^*x=\lambda x &\qquad \phi^*dx=\lambda dx\\
\phi_\lambda^*g_0 &= \phi_\lambda^*(dx^2+dy^2)=\lambda^2(dx^2+dy^2)=\lambda^2 g_0\\
\ell_{g_0}(\phi_\lambda\circ\gamma) &= \lambda\ell_{g_0}(\gamma)\\
\phi_\lambda^*h &= \phi_\lambda^*\left(\frac{dx^2+dy^2}{y^2}\right)=\frac{\lambda^2dx^2+\lambda^2dy^2}{\lambda^2y^2}=h
\end{aligned}$$

Hence for any $\lambda>0$, $\phi_\lambda:(H,h)\to(H,h)$ is an isometry.

13.3 Distance

More generally if $\gamma:[a,b]\to[a,b]$ is a piecewise C^∞ curve s.t. $\gamma:[a,b]\to M$ is continuous. i.e., let $a=t_0<t_1<\cdots<t_{k-1}<t_k=b$ we have

$$\gamma|_{[t_i,t_{i+1}]}\qquad C^\infty\ i=0,\cdots,k$$

Then $\gamma'(t_i^+)$ and $\gamma'(t_i^-)$ exist. so

$$\ell_g(\gamma):=\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}|\gamma'(t)|_g\,dt$$

Definition 13.3. *Let (M,g) be a connected Riemannian manifold. Then for any $p,q\in M$, there exists $\gamma:[0,1]\to M$ piecewise C^∞ curve s.t.*

$$\gamma(0)=p\qquad\gamma(1)=q$$

We define the distance between p, q determined by g to be

$$d_g(p,q):=\inf\{\ell_g(t)\mid\gamma:[0,1]\to M\ \textit{piecewise}\ C^\infty\ \gamma(0)=p,\,\gamma(1)=q\}\in[0,\infty)$$

Then

- $d_g(p,q)=d_g(q,p)$ *and* $d_g(p,p)=0$
- $d_g(p,q)+d_g(q,r)\geq d_g(p,r)$.

In fact, if M is Hausdorff, then $d_g(p,q)=0\implies p=g$, Then (M,d_g) is a metric space.

Example 13.3 (Bugged-eyed Line). $M=(\mathbb{R}\times\{0,1\})\,/\,((x,0)\sim(x,1)$ *except for* $x=0)$. *Euclidean metric dx^2 on $\mathbb{R}$. Define $\pi:\mathbb{R}\times\{0,1\}\to M$ as the projection. There exists a unique metric g on M s.t. $\pi^*g=dx^2$. Now $[0,0]\neq[0,1]$ in M but $d_g([0,0],[0,1])=0$.*

Lemma 13.2. *If $f : (M_1, g_1) \to (M_2, g_2)$ is an isometry, then*

$$d_{g_2}(f(p),\, f(q)) = dg_1(p, q) \qquad \forall p,\, q \in M_1$$

Proposition 13.2. *For $x,\, y \in \mathbb{R}^n$ with $g_0 = dx_1^2 + \cdots + dx_n^2$*

$$d_{g_0}(x, y) = |x - y| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Proof. $d_{g_0}(x, x) = 0$. Suppose $x \neq y$, let $d = |x - y| > 0$. Then there exists $A \in O(n)$ s.t. upon rotation, $A(x - y) = (d, 0, \cdots, 0)$. Then since translation by y is an isometry and that rotation by $O(n)$ is isometry

$$\begin{aligned} d_{g_0}(x, y) &= d_{g_0}(x - y, 0) = d_{g_0}(A(x - y), 0) = d_{g_0}((d, 0, \cdots, 0), 0) \\ &= d_{g_0}((0, \cdots, 0), (d, 0, \cdots, 0)) \end{aligned}$$

It remains to show that $d_{g_0}((0, \cdots, 0), (d, 0, \cdots, 0)) = d$. Consider $\gamma : [0, 1] \overset{C^\infty}{\to} \mathbb{R}^n$ smooth curve so

$$\gamma(t) = (x_1(t), \cdots, x_n(t)) \qquad \gamma(0) = (0, \cdots, 0),\, \gamma(1) = (d, 0, \cdots, 0)$$

Then

$$\begin{aligned} \ell_{g_0}(\gamma) &= \int_0^1 |\gamma'(t)|_{g_0}\, dt = \int_0^1 \sqrt{x_1'(t)^2 + \cdots + x_n'(t)^2}\, dt \geq \int_0^1 |x_1'(t)|\, dt \\ &\geq \int_0^1 x_1'(t)\, dt = d - 0 = d \\ &= \ell_{g_0}(\gamma_0) \end{aligned}$$

where $\gamma_0(t) = (dt, 0, \cdots, 0)$ so $\gamma(0) = 0$ and $\gamma(1) = (d, 0, \cdots, 0)$. In fact if $\phi : (\mathbb{R}^n,\, g_0) \to (\mathbb{R}^n,\, g_0)$ is any isometry, then

$$|\phi(x) - \phi(y)| = |x - y|$$

$\square$

14 Discrete Group Action

Let G be a group acting on M, where M is

- a set
- a topological space
- a topological manifold
- a C^∞ manifold
- a C^∞ manifold equipped with a Riemannian metric g.

Denote M/G as set of G-orbits, where $M/\sim$ s.t.

$$x_1 \sim x_2 \qquad \text{iff} \qquad \exists\, g \in G \text{ s.t. } x_2 = gx_1$$

- For M a set, $\pi : M \to M/G$ is a surjective map.
- For M a topological space, $\pi : M \to M/G$ equips M/G with the quotient topology. Hence π is a surjective continuous map.
- For M topological manifold, when is M/G also a topological manifold?
- When does M/G admit a C^∞ structure s.t. $\pi : M \to M/G$ is C^∞ manifold?
- When does M/G admit a Riemannian metric $\hat{g}$ s.t.

  $$\pi : (M, g) \to (M/G, \hat{g})$$

  is a local isometry?

14.1 Group Action on Set

Definition 14.1 (Left/Right Group Action on Set)**.** *Let G be a group and M be a set. A left (right) action of G on M is a map*

$$\phi : G \times M \to M \qquad s.t. \qquad \phi(g, x) \equiv g \cdot x \quad (x \cdot g)$$

where for any $g \in G$, the map

$$\phi_g : M \to M \qquad s.t. \qquad \phi_g(x) := g \cdot x$$

is a bijection s.t. the following holds

- *$e \in G$ identity gives $\phi_e : M \to M$ identity map.*
- *For any $g_1, g_2 \in G$*
  1. *For left action, $\phi_{g_1} \circ \phi_{g_2} = \phi_{g_1 g_2}$. In other words*

     $$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x \qquad \forall\, x \in M$$

  2. *For right action, $\phi_{g_1} \circ \phi_{g_2} = \phi_{g_2 g_1}$. In other words*

     $$(x \cdot g_2) \cdot g_1 = x \cdot (g_2 g_1) \qquad \forall\, x \in M$$

- *In both cases, $\phi_{g^{-1}} \circ \phi_g = \phi_e = id_M \implies \phi_{g^{-1}} = \phi_g^{-1} : M \to M$. Hence ϕ_g as bijection is automatic.*

For any $g \in G$, it corresponds to bijection $\phi_g : M \to M$ s.t. $\phi_g(x) = g \cdot x$ on M. Hence

$$G \to (\boldsymbol{Perm}(M), \circ)$$

where $\boldsymbol{Perm}(M) = \{\phi : M \to M \mid \phi \text{ is bijection}\}$ and $\circ$ denotes composition. We have group homomorphism

1. *For Left group action*

   $$g \in G \mapsto \phi_g \in (\boldsymbol{Perm}(M),\ \circ) \qquad s.t. \qquad \phi_{g_1} \circ \phi_{g_2} = \phi_{g_1 g_2}$$

2. *For right group action*

   $$g \in G \mapsto \phi_{g^{-1}} \in (\boldsymbol{Perm}(M),\ \circ) \qquad s.t. \qquad \phi_{g_1^{-1}} \circ \phi_{g_2^{-1}} = \phi_{g_2^{-1} g_1^{-1}} = \phi_{(g_1 g_2)^{-1}}$$

Definition 14.2 (Free and Transitive)**.** *Let G be group and act on a set M. We assume left action.*

- *The G-action is Free if for any $p \in M$*

$$g \cdot p = p \iff g = e \text{ identity} \in G$$

- *The G-action is transitive if for any $p, q \in M$, there exists $g \in G$ s.t. $g \cdot p = q$*

Definition 14.3 (Stabilizer and Orbit)**.** *Let G be group and act on a set M. We assume left action. For any $p \in M$*

- *$G_p := \{g \in G \mid g \cdot p = p\}$ denotes the stabilizer of $p \in M$.*
- *$G \cdot p := \{g \cdot p \in M \mid g \in G\}$ denotes the orbit of $p \in M$.*

Lemma 14.1. *One has interpretations using stabilizer and orbit.*

- *G acts freely on M if $G_p = \{e\}$ for each $p \in M$.*
- *G acts transitively on M if $M = G \cdot p$ for some $p \in M$, which further implies $M = G \cdot p$ for any $p \in M$.*

14.2 Group Action on Topological Space

Definition 14.4 (Continuous Group Action on Topological Space)**.** *Suppose M is a topological space and G is a group acting on M (on the left/right). We say the action of G on M is a continuous if*

$$\forall\ g \in G \qquad \phi_g : M \to M \text{ is continuous}$$

A continuous action of a group G on a topological space M gives rise to a group homomorphism

$$G \mapsto (\textbf{\textit{Homeo}}(M),\ \circ)$$

where $\textbf{\textit{Homeo}}(M) := \{\varphi : M \to M \mid \varphi \text{ is homeomorphism}\}$.

Definition 14.5 (Properly Discontinuous Group Action)**.** *Let M be topological space and let G be a group acting continuously on M. We say the action of G on M is 'properly discontinuous' if for every $p \in M$, there exists open neighborhood U of p in M s.t.*

$$U \cap \phi_g(U) = \varnothing \qquad \forall\ g \in G \setminus \{e\}$$

where e denotes the identity.

Remark 14.1 (Properly Discontinuous Group Action $\implies$ Free Group Action)**.** *This implies*

$$\phi_{g_1}(U) \cap \phi_{g_2}(U) = \varnothing \qquad \forall\ g_1 \neq g_2 \in G$$

This further implies G acts freely on M in the sense that if $p \in M$, then $g \cdot p = p$ iff $g = e$.

Proposition 14.1. *Let G be a group and M be a topological space. If G acts continuously and properly discontinuously on M, then*

$$\pi : M \to M/G$$

with M/G equipped with quotient topology is a covering map.

Proof. Let $\overline{p} \in M/G$ and $p \in \pi^{-1}(\overline{p}) \in M$. There exists neighborhood U of p s.t. $\phi_{g_1}(U) \cap \phi_{g_2}(U) = \varnothing$ for any $g_1, g_2 \in G$ with $g_1 \neq g_2$. Let $\overline{U} = \pi(U) \subset M/G$ then $\overline{p} \in \overline{U}$ and

$$\pi^{-1}(\overline{U}) = \bigsqcup_{g \in G} \phi_g(U)$$

is disjoint union of open sets in M. Hence $\pi^{-1}(\overline{U})$ is open in M and so $\overline{U}$ is an open neighborhood of $\overline{p}$ in M/G. Moreover, for any $g \in G$

$$\pi|_{\phi_g(U)} : \phi_g(U) \to \overline{U}$$

is a homeomorphism. □

Corollary 14.1. *If M is topological manifold of dimension n and G is a group acting continuously and properly discontinuously opn M, then M/G is a topological manifold of same dimension n.*

Proposition 14.2 (M/G Hausdorff)**.** *Let M be a topological space. Suppose that a group G acts continuously and properly discontinuously on M, and if p, $q \in M$ are not in the same orbit of the group action, i.e.,*

$$\pi(p) \neq \pi(q) \in M/G$$

for quotient map $\pi : M \to M/G$, then

- *there exists an open neighborhood U of p in M and V of q in M s.t.*

$$U \cap \phi_g(V) = \varnothing \qquad \forall\, g \in G \setminus \{e\}$$

  which implies

$$\phi_{g_1}(U) \cap \phi_{g_2}(U) = \varnothing \qquad \forall\, g_1 \neq g_2 \in G$$

- *M/G with the quotient topology defined by $\pi : M \to M/G$ is Hausdorff.*

Proof. Suppose $\overline{p}$, $\overline{q} \in M/G$ s.t. $\overline{p} \neq \overline{q}$. Choose p, $q \in M$ s.t. $\pi(p) = \overline{p}$ and $\pi(q) = \overline{q}$. By assumption that G acts continuously and properly discontinuously, there exists U_1 open neighborhood of p in M s.t. $U_1 \cap \phi_g(U_1) = \varnothing$ for any $g \in G \setminus \{e\}$. Similarly there exists V_1 open neighborhood of q in M s.t. $V_1 \cap \phi_g(V_1) = \varnothing$ for any $g \in G \setminus \{e\}$. Secondly, by assumption that $\overline{p} \neq \overline{q}$, there exists U_2 open neighborhood of p in M and V_2 of q s.t. $U_2 \cap \phi_g(V_2) = \varnothing$ for any $g \in G \setminus \{e\}$. Then define

$$\overline{U} := \pi(U_1 \cap U_2) \qquad \overline{V} := \pi(V_1 \cap V_2)$$

$\overline{U}$ is open neighborhood of $\overline{p}$ in M/G and $\overline{V}$ is open neighborhood of $\overline{q}$ in M/G where $\overline{U} \cap \overline{V} = \varnothing$. Thus M/G is Hausdorff. □

14.3 Group Action on Smooth Manifold

Definition 14.6 (Smooth Group Action on Smooth Manifold)**.** *Suppose that a group G acts on a C^∞ manifold M. We say that the action is smooth if*

$$\forall\, g \in G \qquad \phi_g : M \to M \quad is \quad C^\infty$$

Hence ϕ_g is C^∞ diffeomorphism. We have a group homomorphism

$$G \to (\boldsymbol{Diff}(M),\ \circ)$$

where $\boldsymbol{Diff}(M) = \{\phi : M \to M \mid \phi \text{ is } C^\infty \text{ diffeomorphism}\}$. Note $\boldsymbol{Diff}(M) \subset \boldsymbol{Homeo}(M) \subset \boldsymbol{Perm}(M)$.

Theorem 14.1. *Let M be C^∞ manifold and let G be a group. If G acts on M smoothly and properly discontinuously, then there exists a unique C^∞ structure on M/G s.t. the covering map $\pi : M \to M/G$ is a local diffeomorphism.*

Proof. Let M be C^∞ manifold with smooth charts $\{(V_i,\, x_i)\}$ where $x_i : V_i \to M$.

- Since G acts properly discontinuously on M, for any $p \in M$, we may choose (V, x) open chart where $x(V) \subset U$ for U open neighborhood of M around p s.t.

$$U \cap \phi_g(U) = \varnothing \qquad \forall\, g \neq e \in G$$

  Thus $\pi|_U$ is injective, hence $y = \pi \circ x : V \to M/G$ is injective. The family $\{(V_i, y_i)\}$ covers M/G. It suffices to show for any $y_1 = \pi \circ x_1 : V_1 \to M/G$ and $y_2 = \pi \circ x_2 : V_2 \to M/G$ s.t. $y_1(V_1) \cap y_2(V_2) \neq \varnothing$, we have $y_1^{-1} \circ y_2$ smooth.

- Let $\pi_i := \pi|_{x_i(V_i)}$. Let $q \in y_1(V_1) \cap y_2(V_2)$ and $r = y_2^{-1}(q) = x_2^{-1} \circ \pi_2^{-1}(q)$. Let $W \subset V_2$ be a neighborhood of r s.t. $(\pi_2 \circ x_2)(W) \subset y_1(V_1) \cap y_2(V_2)$. Then the restriction of $y_1^{-1} \circ y_2$ to W is given by

$$y_1^{-1} \circ y_2\big|_W = x_1^{-1} \circ \pi_1^{-1} \circ \pi_2 \circ x_2$$

  It suffcies to show $\pi_1^{-1} \circ \pi_2$ is smooth at $p_2 = \pi_2^{-1}(q)$.

- Let $p_1 = \pi_1^{-1} \circ \pi_2(p_2)$ then p_1 and p_2 are equivalent in M, hence there exists $g \in G$ s.t. $gp_2 = p_1$. Thus the restriction $\pi_1^{-1} \circ \pi_2\big|_{x_2(W)}$ coincides with the diffeomorphism $\phi_g|_{x_2(W)}$. Since G acts smoothing on M, we know it is smooth at p_2.

□

14.4 Group Action on Riemannian Manifold

Definition 14.7 (Isometric Group Action on Riemannian Manifold)**.** *Let (M,g) be a Riemannian manifold and let G be a group acting on M smoothly. We say this G-action on (M,g) is isometric w.r.t. the given Riemannain structure if*

$$\forall\ a\in G \qquad \phi_a:(M,g)\to(M,g) \quad \text{is an isometry,} \quad \text{i.e.,} \quad \phi_a^*g=g$$

Theorem 14.2 (Existence of Riemannian Metric $\hat{g}$ on M/G)**.** *Let (M,g) be a Riemannian manifold. Let G be group. If G acts on (M,g) smoothly, properly discontinuously, and isometrically, then there exists a unique Riemannian metric $\hat{g}$ on M/G s.t.*

$$\pi:(M,g)\to(M/G,\hat{g})$$

is a local isometry, i.e., $\pi^*\hat{g}=g$.

Definition 14.8 (Metric on $(M/G,\hat{g})$)**.** *Notice for any $\overline{p}\in M/G$, for any $p\in\pi^{-1}(\overline{p})\in M$,*

$$d\pi_p:T_pM\to T_{\overline{p}}(M/G)$$

is a linear isomorphism. In particular

$$d\pi_p^{-1}:T_{\overline{p}}(M/G)\to T_pM$$

is injective. We define

$$\hat{g}(\overline{p})(v_1,v_2):=g(p)(d\pi_p^{-1}(v_1),d\pi_p^{-1}(v_2))$$

This is well-define independent of p.

Example 14.1. *$G=\{\pm1\}\cong\mathbb{Z}/2\mathbb{Z}$ acts on $(\mathbb{S}^n,g_{can})$ s.t. for any $g\in G$, $\phi_g:\mathbb{S}^n\to\mathbb{S}^n$ mapping $x\mapsto g\cdot x$. Here the only choice is $\phi_{\pm1}(p)=\pm p$ for any $p\in\mathbb{S}^n\subset\mathbb{R}^{n+1}$. Then G acts smoothly, isometrically and properly discontinuously on $(\mathbb{S}^n,g_{can})$. There exists unique Riemannian metric $\hat{g}$ on $P_n(\mathbb{R})=\mathbb{S}^n/\{\pm1\}$ s.t.*

$$\pi:(\mathbb{S}^n,g_{can})\to(P_n(\mathbb{R}),\hat{g})$$

is a local isometry $\pi^*\hat{g}=g_{can}$ and a covering map of degree 2. In particular for $n=1$,

$$\pi:(\mathbb{S}^1,g_{can})\to(P_1(\mathbb{R}),\hat{g})\cong\left(\mathbb{S}^1(\frac{1}{2}),g_{can}^{\frac{1}{2}}\right)$$

diffeomorphic to circle of radius a half. To see this, we consider

$$\begin{tikzcd}
(\mathbb{R},dt) \arrow[d, "\pi_1"'] \arrow[dr, "\pi_2" near start] & (\mathbb{R}^2,\, dx^2+dy^2) \\
(\mathbb{S}^1,\, g_{can}) \arrow[r, "\pi"] \arrow[ur, "i_1"' near start] & (\mathbb{S}^1(\frac{1}{2}),g_{can}^{\frac{1}{2}}) \arrow[u, "i_2"']
\end{tikzcd}$$

Here

$$\begin{aligned}
\pi_1(t)&=(\cos(t),\sin(t))\\
\pi_2(t)&=(\frac{1}{2}\cos(2t),\frac{1}{2}\sin(2t))\\
\pi_1^*g_{can}&=(i_1\circ\pi_1)^*(dx^2+dy^2)=(-\sin(t)dt)^2+(\cos(t)dt)^2=dt^2\\
\pi_2^*g_{can}^{\frac{1}{2}}&=(i_2\circ\pi_2)^*(dx^2+dy^2)=(-\sin(2t)dt)^2+(\cos(2t)dt)^2=dt^2
\end{aligned}$$

Example 14.2. *$G=(\mathbb{Z}^n,+)$ acts on $(\mathbb{R}^n,\, g_0=\sum_i dx_i^2)$ by*

$$\phi_m(x):=x+m$$

for any $m\in\mathbb{Z}^n$. This action is smooth and isometric and properly discontinuous. Then there exists a unique Riemannian metric $\hat{g}$ on $\mathbb{R}^n/\mathbb{Z}^n$ s.t. π is a local isometry

$$\pi:(\mathbb{R}^n,\,g_0)\to(\mathbb{R}^n/\mathbb{Z}^n,\,\hat{g})\cong\left(\left(\mathbb{S}^1(\frac{1}{2\pi})\right)^n,g_{can}^{\frac{1}{2\pi}}\times\cdots\times g_{can}^{\frac{1}{2\pi}}\right)$$

is diffeomorphic to flat torus. In particular for $n=1$, $\pi(t):=(\frac{1}{2\pi}\cos(2\pi t),\frac{1}{2\pi}\sin(2\pi t))$. Thus

$$\pi^*g_{can}^{\frac{1}{2\pi}}=(i\circ\pi)^*(dx^2+dy^2)=(-\sin(2\pi t)dt)^2+(\cos(2\pi t)dt)^2=dt^2$$

Definition 14.9 (Orientation preserving map). *Let $f : M_1 \to M_2$ be a local diffeomorphism between oriented C^∞ manifolds. We say f is orientation preserving if for any $p \in M_1$, there exists smooth chart (U, ϕ) for M_1 around p that is compatible with the orientation on M_1, then $f : U \to f(U) \subset M_2$ is a diffeomorphism*

$$
\begin{tikzcd}
M_1 \arrow[r, phantom, "\overset{open}{\supseteq}"] & U \arrow[d, "f"'] \arrow[dr, "\phi"] & & \\
M_2 \arrow[r, phantom, "\overset{open}{\supseteq}"] & f(U) \arrow[r, "\phi\circ f^{-1}"] & \phi(U) \arrow[r, phantom, "\overset{open}{\subseteq}"] & \mathbb{R}^n
\end{tikzcd}
$$

where $(f(U), \phi \circ f^{-1})$ is a C^∞ chart for M_2 around $f(p)$ compatible with the orientation on M_2.

Theorem 14.3. *Let M be an oriented C^∞ manifold and let G be a group. If G acts on M smoothly, properly discontinuously and for any $g \in G$, $\phi_g : M \to M$ is orientation preserving, then there exists a unique orientation on M/G s.t. $\pi : M \to M/G$ is orientation preserving.*

15 Lie Group

Definition 15.1 (Lie Group). *A Lie group is a group G with the structure of a C^∞ manifold s.t.*

$$\lambda : G \times G \to G \qquad s.t. \qquad (x, y) \mapsto xy^{-1}$$

is a C^∞ map.

Remark 15.1. *Given Lie Group G, its smooth structure satisfies the following*

- *Inverse. $G \to G$ s.t. $x \mapsto x^{-1}$ is a C^∞ map.*
- *Multiplication. $G \times G \to G$ s.t. $(x, y) \mapsto xy$ is a C^∞ map.*
- *Left Multiplication. For any $x \in G$, $L_x : G \to G$ s.t. $y \mapsto L_x(y) := xy$ is a C^∞ map.*
- *Right Multiplication. For any $x \in G$, $L_y : G \to G$ s.t. $y \mapsto R_x(y) := yx$ is a C^∞ map.*

Example 15.1. *We have a sequence of examples.*

- $(\mathbb{R}^n, +)$
- $(GL(n, \mathbb{R}), \circ)$ *with global coordinates (a_{ij}), and group action given by matrix multiplication.*
  - *The manifold $GL(n, \mathbb{R})$ has connected component $GL(n, \mathbb{R})_+ = \{A \in GL(n, \mathbb{R}) \mid \det(A) > 0\}$ as a connected Lie Group.*
  - *The Special Linear Group $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid \det(A) = 1\} \subset GL(n, \mathbb{R})$ is Lie subgroup of $GL(n, \mathbb{R})$.*
  - *The Orthogonal Group $O(n) = \{A \in GL(n, \mathbb{R}) \mid A^T A = I_n\}$ and the Special Orthogonal Group $SO(n) = O(n) \cap SL(n, \mathbb{R})$ are Lie Subgroups of $GL(n, \mathbb{R})$.*
- $(GL(n, \mathbb{C}), \circ)$ *with global coordinates (a_{ij}) with values in $\mathbb{C}$, and group action by matrix multiplication.*
  - *The Unitary Group $U(n) := \{A \in GL(n, \mathbb{C}) \mid A^* A = \overline{A}^T A = I_n\}$*
  - *and the Special Unitary Group $SU(n) := \{A \in U(n) \mid \det A = 1\}$*

15.1 Left/Right/Bi-invariant Tensor

Definition 15.2 (Left/Right/Bi-Invariant Tensors). *Let G be Lie group.*

- *A tensor T on G is left-invariant if*

$$L_x^* T = T \iff (L_x)_* T = T \qquad \forall\ x \in G$$

  due to $(L_x)_* = ((L_x)^{-1})^* = (L_{x^{-1}})^*$.

- *A tensor T on G is right-invariant if*

$$R_x^* T = T \iff (R_x)_* T = T \qquad \forall\ x \in G$$

- *We say T is bi-invariant if it is both left invariant and right invariant.*

Remark 15.2. *Given Lie group G. If T is either left or right invariant on G, then T is determined by the value $T(e)$, i.e., the value of T at the identity $e \in G$.*

- *A function $f \in C^\infty(G) = C^\infty(G, T_0^0 G)$ is left or right invariant iff f is constant.*
- *A vector field $X \in \mathfrak{X}(G) = C^\infty(G, T_0^1(G))$*
  1. *left invariant iff*

$$X(x) = d(L_x)_e(X(e)) \qquad \forall\ x \in G$$

  2. *right invariant iff*

$$X(x) = d(R_x)_e(X(e)) \qquad \forall\ x \in G$$

Remark 15.3 (Evaluation Map as Linear Isomorphism to $(T^r_sG)_e$)**.** *Given G Lie group. Then a tensor T on G is an element of*

$$T \in C^\infty(G, T^r_sG) = \{\textit{smooth (r, s) - tensors on } G\}$$

Write $\tilde{ev}_e$ as evaluation map of the tensor at the identity element $e \in G$

$$\tilde{ev}_e : C^\infty(G, T^r_sG) \to (T^r_sG)_e$$

and its restriction ev_e on either Left/Right/Bi-invariant Tensors as

$$ev_e : \{\textit{left/right/bi invariant (r, s)-tensors on } G\} \to (T^r_sG)_e$$

- *For left-invariant tensors, the diagram commutes*

$$\begin{tikzcd}
\{\textit{left invariant (r, s)-tensors on } G\} \arrow[d, phantom, "\mathbb{R}-\textit{Linear Subspace}\ \cap"] \arrow[dr, "ev_e"] & \\
C^\infty(G, T^r_sG) \arrow[r, "\tilde{ev}_e"] & (T^r_sG)_e \\
\uplus & \uplus \\
T \arrow[r, mapsto] & T(e)
\end{tikzcd}$$

  where

$$(T^r_sG)_e = (T_eG)^{\otimes r} \otimes (T^*_eG)^{\otimes s} \cong \mathbb{R}^{(\dim G)^{r+s}}$$

  Observation:

$$ev_e : \{\textit{left invariant (r, s)-tensors on } G\} \to (T^r_sG)_e \quad \textit{is a } \mathbb{R}-\textit{linear isomorphism} \tag{14}$$

  – *Injectivity. If T is left invariant, then for any $x \in G$,*

$$T_eG \underset{(dL_{-x})_x}{\overset{(dL_x)_e}{\rightleftarrows}} T_xG$$

$$T^*_xG \underset{(dL_{-x})^*_x}{\overset{(dL_x)^*_e}{\rightleftarrows}} T^*_eG$$

  – *Notice for any $x \in G$,*

$$T(x) = ((dL_x)_e)^{\otimes r} \otimes ((dL_x)^*_x)^{\otimes s}\,(T(e))$$

- *Similarly, for right-invariant*

$$\{\textit{right invariant (r, s)-tensors on } G\} \overset{ev_e}{\cong} (T^r_sG)_e \qquad \textit{as linear isomorphism}$$

- *However, for Bi-invariant tensors on G*

$$\{\textit{bi invariant (r, s)-tensors on } G\} \overset{ev_e}{\hookrightarrow} (T^r_sG)_e$$

  The evaluation maps is only injective linear map. The image is

$$\{\xi \in (T^r_sG)_e \mid \xi \quad \textit{is invariant under the adjoint action }\}$$

15.2 Left/Right-Invariant Vector Fields as Lie-Subalgebra

We first recall the definition for $F-$related vector fields.

Definition 15.3 (F-related smooth vector fields)**.** *Let $F : M \overset{C^\infty}{\to} N$ between smooth manifolds M and N. $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$. We say X and Y are F-related if for any $p \in M$*

$$dF_p(X(p)) = Y(F(p))$$

Lemma 15.1 (Equivalence for F-related)**.** *Given $F : M \overset{C^\infty}{\to} N$, and $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$*

- *X and Y are F-related iff*

$$X(F^*f) = F^*(Y(f)) \qquad \forall\ f \in C^\infty(N)$$

- *If F is diffeomorphism, then X and Y are F-related iff*

$$Y = F_* X$$

Lemma 15.2 (F-related preserves Lie-Bracket)**.** *For $F : M \overset{C^\infty}{\to} N$ where $X_1,\, X_2 \in \mathfrak{X}(M)$ and $Y_1,\, Y_2 \in \mathfrak{X}(N)$ and $X_i,\, Y_i$ are F-related. Then $[X_1, X_2]$ and $[Y_1, Y_2]$ are F-related.*

Proof. Let $f \in C^\infty(N)$

$$\begin{aligned}
[X_1, X_2](F^* f) &= X_1(X_2(F^* f)) - X_2(X_1(F^* f)) \\
&= X_1(F^*(Y_2(f))) - X_2(F^*(Y_1(f))) \\
&= F^*(Y_1(Y_2(f))) - F^*(Y_2(Y_1(f))) = F^*[Y_1, Y_2](f)
\end{aligned}$$

□

Corollary 15.1. *$F : M \overset{C^\infty}{\to} N$ is smooth diffeomorphism, hence pushforward under F*

$$F_* : \mathfrak{X}(M) \to \mathfrak{X}(N) \qquad X \mapsto F_* X$$

defines X and $F_* X$ as F-related vector fields. Thus

$$F_*[X_1, X_2] = [F_* X_1, F_* X_2]$$

One realize that Left/Right-invariant vector fields are automatically L_a/R_a-related to themselves for any $a \in G$.

Definition 15.4 (Left/Right Invariant Vector Field)**.** *G Lie Group.*

$$\begin{aligned}
\mathfrak{X}(G)^L &:= \{\text{Left Invariant } C^\infty \text{ vector fields on } G\} \\
\mathfrak{X}(G)^R &:= \{\text{Right Invariant } C^\infty \text{ vector fields on } G\}
\end{aligned}$$

Lemma 15.3. *Using (14) we have $\mathbb{R}$-linear isomorphism*

- *$T_e G = \mathfrak{g} \cong \mathfrak{X}(G)^L$ described by*

$$\xi \to (X_\xi^L)(x) := (dL_x)_e(\xi) \qquad \forall\; x \in G$$

where X_ξ^L is the unique left invariant vector field on G s.t. $X_\xi^L(e) = \xi$.

- *$T_e G = \mathfrak{g} \cong \mathfrak{X}(G)^R$ described by*

$$\xi \to (X_\xi^R)(x) := (dR_x)_e(\xi) \qquad \forall\; x \in G$$

where X_ξ^R is the unique right invariant vector field on G s.t. $X_\xi^R(e) = \xi$.

Lemma 15.4 ($T_e G$ as Lie-subalgebra of $\mathfrak{X}(G)$ w.r.t. Lie-Bracket)**.** *For $X,\, Y \in \mathfrak{X}(G)^L$*

- *$[X, Y] \in \mathfrak{X}(G)^L$. This is because for any $a \in G$,*

$$(L_a)_*[X, Y] = [(L_a)_* X, (L_a)_* Y] = [X, Y]$$

- *This shows that $\mathfrak{X}(G)^L \cong T_e G = \mathfrak{g} \subset \mathfrak{X}(G)$ is a Lie-subalgebra of $(\mathfrak{X}(G),\, [\cdot,\, \cdot])$ where we define*

$$[\cdot, \cdot] : T_e G \times T_e G \to T_e G \qquad (\xi, \eta) \mapsto [X_\xi^L, X_\eta^L](e)$$

Definition 15.5 ($\mathfrak{g}$)**.** *The Lie Algebra $\mathfrak{g}$ of G is defined to be $T_e G$ equipped with the above $[\cdot, \cdot]$.*

Similarly, for $X,\, Y \in \mathfrak{X}(G)^R$

- *$[X, Y] \in \mathfrak{X}(G)^R$.*
- *$\mathfrak{X}(G)^R \cong T_e G = \mathfrak{g} \subset \mathfrak{X}(G)$ with Lie Bracket forms Lie-subalgebra*

$$[\cdot, \cdot] : T_e G \times T_e G \to T_e G \qquad (\xi, \eta) \mapsto [X_\xi^R, X_\eta^R](e)$$

Proposition 15.1 (Trivial TG)**.** *The Tangent Bundle of a Lie Group G is trivial, i.e. TG has a global trivialization. In fact*

$$T_s^r G = (TG)^{\otimes r} \otimes (T^* G)^{\otimes s}$$

is a trivial vector bundle for any $r,\, s \in \mathbb{Z}_{\geq 0}$.

Proof. Let $\xi_1, \cdots, \xi_n$ be a basis of $\mathfrak{g} = T_eG$. Then $X^L_{\xi_1}, \cdots, X^L_{\xi_n}$ forms a global C^∞ frame of TG. This is because for any $x \in G$, $\mathfrak{g} \to T_xG$ s.t. $\xi \mapsto X^L_\xi(x) = (dL_x)_e(\xi)$ is a linear isomorphism. Define the map

$$\phi : G \times \mathfrak{g} \to TG \qquad s.t. \qquad (x, \xi) \mapsto (x, X^L_\xi(x)) \tag{15}$$

Notice ϕ is a C^∞ diffeomorphism. Then $\phi^{-1} : TG \to G \times \mathfrak{g}$ is a global trivialization of TG. $\square$

Example 15.2. *Let $G = (\mathbb{R}^n, +)$. For any $a_1, \cdots, a_n \in \mathbb{R}$, the vector field*

$$\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$$

is bi-invariant. We have

$$\mathfrak{X}(G)^L = \mathfrak{X}(G)^R = \{\sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \mid (a_1, \cdots, a_n) \in \mathbb{R}^n\} \cong \mathbb{R}^n$$

Then the Lie bracket $[\cdot, \cdot]$ on $T_eG = \mathfrak{g} = T_0\mathbb{R}^n = \mathbb{R}^n$ is trivial. The map (15) is given by

$$\phi : \mathbb{R}^n \times \mathbb{R}^n \to T\mathbb{R}^n \qquad (x, y) \mapsto (x, \sum_{i=1}^n y_i \frac{\partial}{\partial x_i})$$

Example 15.3. *Let $G = GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\}$. Recall $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) = M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. Then for any $A \in G$, define map*

$$L_A : G \subset M_n(\mathbb{R}) \to G \qquad s.t. \qquad B \mapsto AB$$

and consequently

$$(dL_A)_{I_n} : T_{I_n}G \cong M_n(\mathbb{R}) \to T_AG \cong M_n(\mathbb{R}) \qquad (dL_A)_{I_n}(\xi) = A\xi$$
$$(dR_A)_{I_n} : T_{I_n}G \cong M_n(\mathbb{R}) \to T_AG \cong M_n(\mathbb{R}) \qquad (dR_A)_{I_n}(\xi) = \xi A$$

We see hence, for $A = (a_{ij}) \in GL(n, \mathbb{R})$ and $\xi = (\xi_{ij}) \in \mathfrak{gl}(n, \mathbb{R}) = M_n(\mathbb{R})$, where $\frac{\partial}{\partial a_{ij}}$ are global C^∞ vector fields on $GL(n, \mathbb{R})$, we have

$$X^L_\xi(A) = A\xi = \sum_{i,j=1}^n \left(\sum_{k=1}^n a_{ik}\xi_{kj}\right) \frac{\partial}{\partial a_{ij}}$$
$$X^R_\xi(A) = \xi A = \sum_{i,j=1}^n \left(\sum_{k=1}^n \xi_{ik}a_{kj}\right) \frac{\partial}{\partial a_{ij}}$$

The map ϕ (15) is given by

$$\phi : G \times \mathfrak{g} = GL(n, \mathbb{R}) \times \mathfrak{gl}(n, \mathbb{R}) \to TG = GL(n, \mathbb{R}) \times \mathfrak{gl}(n, \mathbb{R}) \qquad (A, \xi) \mapsto (A, A\xi)$$

If moreover H is a Lie subgroup of $G = GL(n, \mathbb{R})$ and $\mathfrak{h} = T_eH$ is the Lie subalgebra, ϕ restricts to

$$\phi|_{H \times \mathfrak{h}} : H \times \mathfrak{h} \subset G \times \mathfrak{g} \to TH \subset TG$$

- *Let $H = SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid \det A = 1\}$. Then $\mathfrak{h} = \mathfrak{sl}(n, \mathbb{R}) = \{\xi \in \mathfrak{gl}(n, \mathbb{R}) \mid \mathrm{Tr}\xi = 0\}$. Note*

  $$TSL(n, \mathbb{R}) = \{(A, \xi) \in GL(n, \mathbb{R}) \times M_n(\mathbb{R}) \mid \det(A) = 1, \mathrm{Tr}(A^{-1}\xi) = 0\}$$

  and we have

  $$\phi : SL(n, \mathbb{R}) \times \mathfrak{sl}(n, \mathbb{R}) \to TSL(n, \mathbb{R}) \qquad (A, \xi) \mapsto (A, A\xi)$$

- *Let $H = O(n)$ or $H = SO(n)$. Note $I_n \in SO(n) \subset O(n)$ and*

  $$\mathfrak{h} = \mathfrak{so}(n) := \{\xi \in M_n(\mathbb{R}) \mid \xi^T + \xi = 0\} = T_{I_n}O(n) = T_{I_n}SO(n)$$

  Also note

  $$TSO(n) = \{(A, \xi) \in GL(n, \mathbb{R}) \times M_n(\mathbb{R}) \mid A^TA = I_n,\, 0 = (A^{-1}\xi) + (A^{-1}\xi)^T = A^T\xi + \xi^TA\}$$

  hence we have

  $$\phi : SO(n) \times \mathfrak{so}(n) \to TSO(n) \qquad (A, \xi) \mapsto (A, A\xi)$$

15.3 Integral Curve and Local Flow of Left/Right Invariant Vector Fields

Lemma 15.5. *For $F : M \overset{C^\infty}{\to} N$, $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ F-related. If γ is integral curve of X, $F \circ \gamma$ is integral curve of Y.*

Proof.

$$\begin{aligned}(F \circ \gamma)'(t) &= (dF)_{\gamma(t)}(\gamma'(t)) \\ &= (dF)_{\gamma(t)}(X(\gamma(t))) \\ &= Y(F(\gamma(t))) = Y(F \circ \gamma(t))\end{aligned}$$

□

Corollary 15.2. *Let G be a Lie group.*

- *If γ be integral curve of $X \in \mathfrak{X}(G)^L$. Then for any $a \in G$, $L_a \circ \gamma$ is an integral curve of $(L_a)_* X = X$.*
- *Similarly, if γ is integral curve of $X \in \mathfrak{X}(G)^R$, then $R_a \circ \gamma$ is an integral curve of $(R_a)_* X = X$.*

Definition 15.6 (Local Flow of Left/Right-Invariant Vector Field)**.** *Let G be a Lie group. $\xi \in \mathfrak{g} = T_e G$. Then*

- *let ϕ^L_ξ denote the local flow of $X^L_\xi \in \mathfrak{X}(G)^L$*
- *and ϕ^R_ξ denote the local flow of $X^R_\xi \in \mathfrak{X}(G)^R$.*

Remark 15.4. *Indeed by Local Existence Theory of integral curve 8.1, there exists $\varepsilon > 0$, an open neighborhood V of e and*

$$\phi^L_\xi : (-\varepsilon, \varepsilon) \times V \overset{C^\infty}{\to} G$$

such that

$$\begin{cases} \frac{\partial}{\partial t}\phi^L_\xi(t,x) = X^L_\xi(\phi^L_\xi(t,x)) \\ \phi^L_\xi(0,x) = x \end{cases}$$

Lemma 15.6 (Left/Right multiplication preserves left/right invariant integral curves)**.** *Let G be a Lie group. $\xi \in \mathfrak{g} = T_e G$.*

- *Let ϕ^L_ξ be local flow of X^L_ξ. For any $a \in G$*

  $$L_a \circ \phi^L_\xi(t,x) = \phi^L_\xi(t, L_a(x))$$

  i.e.

  $$a\phi^L_\xi(t,x) = \phi^L_\xi(t, ax)$$

- *Let ϕ^R_ξ be local flow of X^R_ξ. For any $a \in G$*

  $$R_a \circ \phi^R_\xi(t,x) = \phi^R_\xi(t, R_a(x))$$

  i.e.

  $$\phi^R_\xi(t,x)a = \phi^R_\xi(t, xa)$$

This is to say left(right) multiplication by 'a' carries an integral curve of left(right) invariant vector field to another integral curve of such vector field.

Proof. By uniqueness of local integral curve, it suffices to show

$$\begin{cases} (L_a \circ \phi^L_\xi)(0,x) = ax \\ \frac{d}{dt}(L_a \circ \phi^L_\xi)(t,x) = X^L_\xi((L_a \circ \phi^L_\xi)(t,x)) \end{cases}$$

The first item is true due to

$$(L_a \circ \phi^L_\xi)(0,x) = a \cdot \phi^L_\xi(0,x) = ax$$

The second is true due to

$$\begin{aligned}\frac{d}{dt}(L_a \circ \phi^L_\xi)(t,x) &= d(L_a)_{\phi^L_\xi(t,x)}(\frac{d}{dt}\phi^L_\xi(t,x)) \\ &= d(L_a)_{\phi^L_\xi(t,x)}\left(X^L_\xi(\phi^L_\xi(t,x))\right) \\ &= X^L_\xi(L_a \circ \phi^L_\xi(t,x))\end{aligned}$$

□

Proposition 15.2. *Let G be a Lie group. $\xi \in \mathfrak{g} = T_eG$. Then ϕ_ξ^L and ϕ_ξ^R are defined on $\mathbb{R} \times G$.*

Proof. We prove for ϕ_ξ^L. There exists $\varepsilon > 0$ and V open neighborhood of e in G s.t.

$$(\phi_\xi^L)_t : V \to G \qquad x \mapsto \phi_\xi^L(t,x)$$

is defined for any $t \in (-\varepsilon, \varepsilon)$. Since for any $a \in G$, from Lemma 15.6

$$(\phi_\xi^L)_t(ax) = (a\phi_\xi^L)_t(x) \iff (\phi_\xi^L)_t \circ L_a(x) = L_a \circ (\phi_\xi^L)_t(x)$$

We have

$$\phi_\xi^L : L_a(V) \to G$$

defined for any $t \in (-\varepsilon, \varepsilon)$ for any $a \in G$. Thus by arbitrariness of $a \in G$

$$(\phi_\xi^L)_t(x) = \phi_\xi^L(t,x)$$

is defined for any $t \in (-\varepsilon, \varepsilon)$ for any $x \in G$. Thus

$$(\phi_\xi^L)_{nt}(x) = (\phi_\xi^L)_t \circ \cdots \circ (\phi_\xi^L)_t(x)$$

is defined for any $t \in (-\varepsilon, \varepsilon)$, for any $n \in \mathbb{Z}_{>0}$ and for any $x \in G$. Thus

$$(\phi_\xi^L)_t(x)$$

is defined for any $t \in \mathbb{R}$ and for any $x \in G$. $\square$

Example 15.4. *Take $G = GL(n, \mathbb{R})$ or any Lie subgroup of $GL(n, \mathbb{R})$ (e.g. $SL(n, \mathbb{R})$, $O(n)$, $SO(n)$), for any $\xi \in \mathfrak{g}$*

$$X_\xi^L(A) = A\xi \qquad X_\xi^R(A) = \xi A$$

and moreover

$$\phi_\xi^L(t, A) = A\exp(t\xi) \qquad \phi_\xi^R(t, A) = \exp(t\xi)A$$

where $\exp(B) = \sum_{n=0}^{\infty} \frac{B^n}{n!}$ for any $B \in M_n(\mathbb{R})$. We use such observation to extend notion of exponential to any Lie Group.

Definition 15.7 (Exponential Map)**.** *For G Lie group and $\mathfrak{g} = T_eG$ Lie algebra of G. Define*

$$\exp : \mathfrak{g} \to G \qquad s.t. \qquad \xi \mapsto \phi_\xi^L(1, e)$$

where e is the identity for G.

Remark 15.5. *Note for any $t \in \mathbb{R}$ and $\xi \in \mathfrak{g}$*

$$\exp(t\xi) = \phi_{t\xi}^L(1, e) = \phi_\xi^L(t, e)$$

and for any $x \in G$

$$\phi_\xi^L(t, x) = x\phi_\xi^L(t, e) = x\exp(t\xi)$$

Thus

$$(\phi_\xi^L)_t = R_{\exp(t\xi)} : G \to G$$

15.4 Left/Right/Bi-Invariant Riemannian Metric

Definition 15.8 (Left/Right-invariant Riemannian Metric)**.** *As special case to Definition 15.2, let G be Lie group and $g \in C^\infty(G, S^2T^*G)$ be Riemannian metric on G. We say*

- *g is Left-invariant if*

  $$(L_x)^*g = g \iff (L_x)_*g = g \qquad \forall\ x \in G$$

  iff

  $$L_x : (G, g) \to (G, g) \qquad \textit{is an isometry} \ \ \forall\ x \in G$$

- *g is right-invariant if*

  $$(R_x)^*g = g \iff (R_x)_*g = g \qquad \forall\ x \in G$$

  iff

  $$R_x : (G, g) \to (G, g) \qquad \textit{is an isometry} \ \ \forall\ x \in G$$

Remark 15.6. *Let G be Lie group and g be Riemannian metric on G. We have one-to-one correspondence between*

$$\{\text{left-invariant metrics on } G\} \iff \{\text{Inner-products on } T_eG\}$$

1. *g is left-invariant iff*

$$g(x)(U,V) = g(e)\left(d(L_{x^{-1}})_xU, d(L_{x^{-1}})_xV\right) \qquad \forall\, x \in G,\ U,\, V \in T_xG$$

2. *g is right-invariant iff*

$$g(x)(U,V) = g(e)\left(d(R_{x^{-1}})_xU, d(R_{x^{-1}})_xV\right) \qquad \forall\, x \in G,\ U,\, V \in T_xG$$

We shall illustrate not every Lie group G admits a bi-invariant metric.

Example 15.5. *Let*

$$G = \{g : \mathbb{R} \to \mathbb{R} \mid g(t) = yt + x \qquad x \in \mathbb{R},\ y \in (0,\infty)\}$$

be the group of proper affine linear transformations of $\mathbb{R}$ s.t. multiplication is defined by composition. For $g_1(t) = y_1x + x_1,\ g_2(t) = y_2t + x_2$

$$g_1 \circ g_2(t) := g_1(y_2t + x_2) + x_1 = y_1y_2t + (y_1x_2 + x_1)$$

We may thus identity $(G, \circ)$ with the Half plane $(H, \cdot)$ where the set

$$H = \{(x,y) \in \mathbb{R}^2 \mid y > 0\} \subset \mathbb{R}^2$$

is equipped with multiplication given by

$$(x_1, y_1) \cdot (x_2, y_2) := (y_1x_2 + x_1, y_1y_2)$$

The multiplication defines a smooth map $G \times G \to G$ whose identity element is $e = (0,1)$ and inverse is given by $(x,y)^{-1} = (-\frac{x}{y}, \frac{1}{y})$. Hence G defined a Lie group. We note that the Left group action takes the form

$$L_{a,b}(x,y) = (bx + a, by) = b(x,y) + a$$

Hence

$$(dL_{a,b})_{(x,y)} : T_{(x,y)}H = \mathbb{R}^2 \to T_{(x,y)}H = \mathbb{R}^2 \qquad s.t. \qquad v \mapsto bv$$

where the left-invariant vector fields on G takes the form

$$\mathfrak{X}^L(G) = \mathbb{R}y\frac{\partial}{\partial x} \oplus \mathbb{R}y\frac{\partial}{\partial y} = \{y(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}) \mid a,\, b \in \mathbb{R}\}$$

and the left-invariant 1-forms on $(G, \circ)$ takes the form

$$\mathbb{R}\frac{1}{y}dx \oplus \mathbb{R}\frac{1}{y}dy = \{\frac{1}{y}(adx + bdy) \mid a,\, b \in \mathbb{R}\}$$

One may also observe a left-invariant Riemannian metric on $(H, \cdot) \cong (G, \circ)$

$$h = \frac{dx^2 + dy^2}{y^2} = (\frac{dx}{y})^{\otimes 2} + (\frac{dy}{y})^{\otimes 2}$$

h is in fact the unique left-invariant Riemannian metric on $(H, \cdot) \cong (G, \circ)$ s.t.

$$h(0,1) = dx^2 + dy^2$$

It is easy to check that h is not right-invariant metric since

$$R_{a,b}(x,y) = (ay + x, by) \neq (bx + a, by)$$

Indeed there is no bi-invariant Riemannian metric on $(H, \cdot) \cong (G, \circ)$.

Example 15.6. *Bi-invariant Riemannian metrics on $(\mathbb{R}^n, +)$ takes the form*

$$\sum_{i,j=1}^{n} a_{ij}dx_idx_j$$

for $a_{i,j} \in \mathbb{R}$ where (a_{ij}) is symmetric positive definite matrix. In particular, $g_0 = \sum_{i=1}^{n} dx_i^2$ is a bi-invariant Riemannian metric.

Lemma 15.7. *If G is compact Lie group, then there exists a bi-invariant Riemannian metric on G.*

Example 15.7 (Bi-invariant metric on $SO(n)$). *Let $a_{ij} : GL(n, \mathbb{R}) \to \mathbb{R}$ be entries of the matrix, hence a_{ij} are global coordinates on $GL(n, \mathbb{R})$. Let $\tilde{g}_n$ be Riemannian metric on $GL(n, \mathbb{R})$ defined by*

$$\tilde{g}_n := \sum_{i,j=1}^{n} da_{ij}^2$$

Let

$$i : SO(n) \to GL(n, \mathbb{R})$$

be the inclusion map, which is smooth embedding. Then

$$g_n = i^* \tilde{g}_n \tag{16}$$

is a bi-invariant Riemannian metric on $SO(n)$.

Proof. Recall

$$SO(n) = \{(a_{ij}) \in GL(n, \mathbb{R}) \mid A^T A = I_n \quad \det(A) = 1\}$$

Given $g_n := i^* \tilde{g}_n$ where $\tilde{g}_n := \sum_{i,j=1}^{n} da_{i,j}^2$ is Riemannian metric defined on $GL(n, \mathbb{R})$, we want to show g_n is both left and right invariant, i.e. for any $B = (b_{i,j}) \in SO(n)$, and for any $A = (a_{i,j}) \in SO(n)$

$$(L_B)^* \left(\sum_{i,j=1}^{n} da_{i,j}^2 \right) = \sum_{i,j=1}^{n} da_{i,j}^2 \qquad (R_B)^* \left(\sum_{i,j=1}^{n} da_{i,j}^2 \right) = \sum_{i,j=1}^{n} da_{i,j}^2$$

Indeed, since

$$L_B : SO(n) \to SO(n) \qquad (a_{ij}) \mapsto (\sum_{k=1}^{n} b_{ik} a_{kj})_{i,j}$$

We may calculate explicityly

$$\begin{aligned}
(L_B)^*(\tilde{g}_n) &= \sum_{i,j} d(\sum_{k=1}^{n} b_{ik} a_{kj})^2 \\
&= \sum_{i,j} \left(\sum_{k=1}^{n} b_{ik} da_{kj} \right)^2 \\
&= \sum_{i,j} \left(\sum_{k=1}^{n} b_{ik} da_{kj} \right) \left(\sum_{m=1}^{n} b_{im} da_{mj} \right) \\
&= \sum_{i,j} \sum_{k,m=1}^{n} b_{ik} b_{im} da_{kj} da_{mj} \\
&= \sum_{k,m=1}^{n} \sum_{i,j} b_{ki}^T b_{im} da_{kj} da_{mj} \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} da_{kj} da_{kj} = \sum_{j,k=1}^{n} da_{kj}^2 = \tilde{g}_n
\end{aligned}$$

Similarly, since

$$R_B : SO(n) \to SO(n) \qquad (a_{ij}) \mapsto (\sum_{k=1}^{n} a_{ik} b_{kj})_{i,j}$$

We do same calculations

$$\begin{aligned}
(R_B)^*(\tilde{g}_n) &= \sum_{i,j} d(\sum_{k=1}^{n} a_{ik}b_{kj})^2 \\
&= \sum_{i,j}\left(\sum_{k=1}^{n} b_{kj}da_{ik}\right)\left(\sum_{m=1}^{n} b_{mj}da_{im}\right) \\
&= \sum_{i,j}\sum_{k,m=1}^{n} b_{kj}b_{mj}da_{ik}da_{im} \\
&= \sum_{k,m=1}^{n}\sum_{i,j} b^T_{jk}b_{mj}da_{ik}da_{im} \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} da_{jk}da_{jk} = \sum_{j,k=1}^{n} da^2_{jk} = \tilde{g}_n
\end{aligned}$$

□

Theorem 15.1 (John Miler). *A connected Lie Group admits a bi-invariant Riemannian metric iff it is isomorphic to $G \times \mathbb{R}^n$ where G is a compact Lie Group and $(\mathbb{R}^n, +)$ is additive group.*

15.5 Adjoint Representation

Definition 15.9 (Adjoint Representation Ad of Lie Group G). *Let G be a Lie group. For any $a \in G$,*

$$R_{a^{-1}} \circ L_a : G \to G \qquad s.t. \qquad x \mapsto axa^{-1}$$

is a diffeomorphism. For $\mathfrak{g} = T_eG$ the Lie Sub-algebra

1. *$R_{a^{-1}} \circ L_a(e) = e$ sends e to the identity e.*
2. *Hence we get $Ad(a) := d(R_{a^{-1}} \circ L_a)_e : T_eG \to T_eG$ a linear isomorphism.*
3. *Furthermore we have a group homomorphism*

$$Ad : G \to GL(\mathfrak{g}) \qquad s.t. \qquad a \mapsto Ad(a) := d(R_{a^{-1}} \circ L_a)_e \tag{17}$$

where $GL(\mathfrak{g}) = \{\mathbb{R} - \text{linear isomorphisms from } \mathfrak{g} \to \mathfrak{g}\}$. One may in fact generalize this to

$$G \to GL(\mathfrak{g}^{\otimes r} \otimes (\mathfrak{g}^*)^{\otimes s}) = GL((T^r_sG)_e)$$

'Ad' the representation of G is called the adjoint representation.

Remark 15.7. *In particular, if G is abelian, then the adjoint representation is trivial*

$$R_{a^{-1}} \circ L_a = Id_G : G \to G \qquad \text{is the identity } \forall\, a \in G$$
$$Ad(a) = Id_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g} \qquad \forall\, a \in G$$

In this case, left invariant iff right invariant iff bi-invariant.

Example 15.8. *$(\mathbb{R}^n, +)$ is abelian. For any $a \in \mathbb{R}^n$*

$$L_a = R_a : \mathbb{R}^n \to \mathbb{R}^n \qquad x \mapsto x + a$$

with

- *$\frac{\partial}{\partial x_i} \in \mathfrak{X}(G)$ bi-invariant vector fields.*
- *$dx_i \in \Omega^1(G)$ bi-invariant 1-forms.*
- *$\sum_{\substack{i_1,\cdots,i_r \\ j_1,\cdots,j_s}} a^{i_1,\cdots,i_r}_{j_1,\cdots,j_s} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s}$ are bi-invariant $(r,s)-$tensors if $a^{i_1,\cdots,i_r}_{j_1,\cdots,j_s}$ are constants.*

Proposition 15.3 (Adjoint Representation ad of Lie Algebra $\mathfrak{g} = T_eG$). *Let G be a Lie group and Ad be its adjoint representation (17). For any $\xi, \eta \in \mathfrak{g}$*

$$ad(\xi)(\eta) := \left.\frac{d}{dt}\right|_{t=0} Ad(\exp(t\xi))\eta = [\xi, \eta]$$

The map

$$ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

is the Adjoint representation of the Lie Algebra $\mathfrak{g}$.

Proof. Let X_ξ^L be the unique left invariant vector field on G s.t. $X_\xi^L(e) = \xi \iff X_\xi^L(x) = (dL_x)_e(\xi)$. Similarly, define X_η^L. Then

$$[\xi, \eta] = [X_\xi^L, X_\eta^L](e) \in \mathfrak{g} = T_eG$$

Let $(\phi_\xi^L)_t = R_{\exp(t\xi)} : G \to G$ be the local flow of X_ξ^L. Using (10) and then using X_η^L is left-invariant

$$\begin{aligned}
[X_\xi^L, X_\eta^L](e) &= \lim_{t\to 0} \frac{X_\eta^L(e) - ((\phi_\xi^L)_t)_* X_\eta^L(e)}{t} \\
&= \lim_{t\to 0} \frac{X_\eta^L(e) - (R_{\exp(t\xi)})_* X_\eta^L(e)}{t} \\
&= \left.\frac{d}{dt}\right|_{t=0} (R_{\exp(-t\xi)})_* X_\eta^L(e) \\
&= \left.\frac{d}{dt}\right|_{t=0} \big((R_{\exp(-t\xi)})_* (L_{\exp(t\xi)})_* X_\eta^L\big)(e) \\
&= \left.\frac{d}{dt}\right|_{t=0} \big((R_{\exp(-t\xi)} \circ L_{\exp(t\xi)})_* X_\eta^L\big)(e) \\
&= \left.\frac{d}{dt}\right|_{t=0} d(R_{\exp(-t\xi)} \circ L_{\exp(t\xi)})_e(X_\eta^L(e)) \\
&= \left.\frac{d}{dt}\right|_{t=0} Ad(\exp(t\xi))\eta
\end{aligned}$$

□

Example 15.9 (Adjoint Representation for General Linear Group)**.** *Let $G = GL(n, \mathbb{R})$ or its subgroups. For any $A \in G$,*

$$R_A^{-1} \circ L_A : G = GL(n, \mathbb{R}) \subset M_n(\mathbb{R}) \cong \mathbb{R}^{n^2} \to G \qquad B \mapsto ABA^{-1}$$

is linear in B, so

$$Ad(A) = d(R_A^{-1} \circ L_A)_{I_n} : M_n(\mathbb{R}) \to M_n(\mathbb{R}) \qquad \eta \mapsto A\eta A^{-1}$$

Thus

$$Ad(\exp(t\xi))\eta = e^{t\xi}\eta e^{-t\xi}$$

and

$$ad(\xi)(\eta) = [\xi, \eta] = \left.\frac{d}{dt}\right|_{t=0} e^{t\xi}\eta e^{-t\xi} = \xi\eta - \eta\xi$$

16 Continuous Group Action

Recall we have defined smooth group action. Let G be in particular, a Lie Group.

Definition 16.1 (Smooth Lie Group Action on smooth Manifold)**.** *Let G be Lie group and let M be a smooth manifold. Let $\phi : G \times M \to M$ be a left action of G on M*

$$\phi : G \times M \to M \qquad \phi(g, x) := g \cdot x$$

The action is C^∞ if ϕ is C^∞ map, i.e.

$$\forall\ g \in G \qquad \phi_g : M \to M \qquad s.t. \qquad x \mapsto g \cdot x$$

is C^∞ diffeomorphism.

16.1 Continuous Action of Topological Group

We want sufficient condition on $\phi : G \times M \to M$ s.t. M/G equipped with the quotient topology is 'nice'. To do so, we discuss bit of point set topology.

Definition 16.2 (Topological Group)**.** *A topological group G is a group equipped with a topology (hence a topological space) s.t.*

$$G \times G \to G \qquad (x,\, y) \mapsto xy^{-1}$$

is continuous.

Remark 16.1. *That G is a topological group indeed implies both group multiplication and inversion are continuous*

$$\begin{aligned} G \to G \qquad & x \mapsto x^{-1} \\ G \times G \to G \qquad & (x,\, y) \mapsto x \cdot y \end{aligned}$$

Definition 16.3 (Continuous Group Action on Topological Space)**.** *Let G be a topological group and let M be a topological space. Let*

$$\phi : G \times M \to M \qquad (g,\, x) \mapsto g \cdot x$$

be a Left G-action on M. We say this action is continuous if ϕ is a continuous map, i.e.

$$\forall\ g \in G \qquad \phi_g : M \to M \qquad s.t. \qquad x \mapsto g \cdot x$$

is homeomorphism. Here $\phi_{g^{-1}} = (\phi_g)^{-1}$.

Lemma 16.1. *Let G be a group equipped with the discrete topology. Then $\phi : G \times M \to M$ is continuous iff*

$$\forall\ g \in G \qquad \phi_g : M \to M \qquad s.t. \qquad x \mapsto g \cdot x$$

is continuous.

Proof. $\Longrightarrow$. If ϕ is continuous, then

$$i_g : M \to G \times M \qquad s.t. \qquad x \mapsto (g, x)$$

is continuous due to discrete topology on G. As composition, $\phi_g = \phi \circ i_g$ is continuous.
$\Longleftarrow$. Suppose each ϕ_g is continuous. Given $U \subset M$ open subset, note

$$\phi^{-1}(U) = \bigcup_{g \in G} \left(\{g\} \times \phi_g^{-1}(U)\right)$$

Since G itself is open as topological space and all $\phi_g^{-1}(U)$ are open, $\phi^{-1}(U)$ is open. $\square$

Recall the definition of 'proper'.

Definition 16.4 (Proper Continuous Map)**.** *Let X, Y be topological spaces and $f : X \to Y$ be a continuous map. We say f is proper if for any $K \subset Y$ compact subset of Y, we have $f^{-1}(K) \subset X$ as compact subset of X.*

Definition 16.5 (Proper Group Action)**.** *Let G be a topological group and M be a topological space. Let $\phi : G \times M \to M$ be a continuous left G-action on M. The action is proper if*

$$\theta : G \times M \to M \times M \qquad s.t. \qquad \theta(g,\, x) = (g \cdot x,\, x)$$

is proper, i.e., for any $K \subset M \times M$ compact, the preimage $\theta^{-1}(K)$ is compact.

Proposition 16.1 (Equivalence for 'Proper Group Action')**.** *If G is a topological group and M is a Hausdorff topological space, then the following conditions on a continuous group action $\phi : G \times M \to M$ are equivalent*

(i) The action is proper.

(ii) For any compact set $K \subset M$
$$G_k := \{g \in G \mid \phi_g(K) \cap K \neq \varnothing\}$$
is compact.

Definition 16.6 (Locally Compact)**.** *Recall M topological space is locally compact implies for any $p \in M$, there exists open neighborhood U in M and a compact subset K in M s.t. $U \subset K$.*

Given topological group G acting continuously and properly on a locally compact Hausdorff topological space M, the quotient remains Hausdorff.

Theorem 16.1. *If G is a topological group, M is a locally compact Hausdorff topological space, and G acts continuously and properly on M, then M/G equipped with the quotient topology is Hausdorff.*

16.2 Smooth Lie Group Action and Smooth Fiber Bundle

Definition 16.7 (Smooth Fiber Bundle)**.** *$\pi : E \to B$ is a C^∞ fiber bundle with total space E, base B and fiber F if*

- *E, B, F are C^∞ manifolds.*
- *π is a surjective C^∞ map.*
- *Local Trivializations. There exists $\{U_\alpha \mid \alpha \in I\}$ open cover of B and C^∞ diffeomorphisms*
$$h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$$
s.t. the diagram commutes $\pi|_{\pi^{-1}(U_\alpha)} = pr_1 \circ h_\alpha$
$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[d, "h_\alpha"'] \arrow[dr, "\pi|_{\pi^{-1}(U_\alpha)}"] & \\ U_\alpha \times F \arrow[r, "pr_1"] & U_\alpha \end{tikzcd}$$
Hence π is a C^∞ submersion.

Example 16.1 (C^∞ fiber bundles)**.** *One has some examples for fiber bundle.*

- *$pr_1 : E = B \times F \to B$ product fiber bundle.*
- *$\pi : E \to B$ C^∞ vector bundle of rank r is indeed a C^∞ fiber bundle with total space E, base B and fiber $\mathbb{R}^r$. But the converse is not true. This is because that π is a fiber bundle only implies the transition functions take the form*
$$h_\beta \circ h_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^r \to (U_\alpha \cap U_\beta) \times \mathbb{R}^r \qquad (x, v) \mapsto (x, \phi_x(v))$$
for some $\phi_x : \mathbb{R}^r \to \mathbb{R}^r$ C^∞ diffeomorphism, but not necessarily $GL(r, \mathbb{R})$.
- *A covering space is a C^∞ fibration with discrete fiber.*

Theorem 16.2 (Quotient Manifold Theorem)**.** *Let G be a Lie Group and M be a C^∞ manifold that is Hausdorff and second countable. If G acts on M smoothly, freely and properly, then M/G equipped with quotient topology is a topological manifold (hence $dim\, M/G = dim\, M - dim\, G$), and there exists a unique C^∞ structure on M/G s.t. the quotient map*
$$\pi : M \to M/G$$
is a C^∞ fiber bundle with fiber G (hence π is a smooth submersion).

Example 16.2 (Hopf Fibration)**.**
$$\mathbb{S}^1 := \{z \in \mathbb{C} \mid |z| = 1\} = U(1)$$
is a Lie group. Let
$$\phi : \mathbb{S}^1 \times \mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1} := \left\{(z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1} \mid \sum_{i=1}^{n+1} |z_i|^2 = 1\right\} \qquad \phi(\lambda, (z_1, \cdots, z_{n+1})) := (\lambda z_1, \cdots, \lambda z_{n+1})$$

Then $\mathbb{S}^1$ acts on $\mathbb{S}^{2n+1}$ smoothly, freely and properly. The quotient map

$$\pi : \mathbb{S}^{2n+1} \to P_n(\mathbb{C}) := \mathbb{S}^{2n+1}/\mathbb{S}^1 = \left(\mathbb{C}^{n+1} \setminus \{0\}\right)/\mathbb{C}^*$$

is a C^∞ fiber bundle w.r.t. the C^∞ structure on $\mathbb{S}^{2n+1}$ (which agrees with the C^∞ structure on $\mathbb{S}^{2n+1}$ as a $(2n+1)$-dim submanifold of $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$) and the C^∞ structure on $P_n(\mathbb{C})$. Therefore the C^∞ structure on $P_n(\mathbb{C})$ agrees with the C^∞ structure on $\mathbb{S}^{2n+1}/\mathbb{S}^1$ given by the Quotient Manifold Theorem. Here π is a circle bundle (fiber bundle with fiber $\mathbb{S}^1$) known as the Hopf Fibration.

16.3 Riemannian Submersion

Let $f : (M, g) \to N$ be a C^∞ submersion (hence $m = \dim M \geq n = \dim N$) from a Riemannian manifold (M, g) to a C^∞ manifold N.

Definition 16.8 (Horizontal Distribution)**.** *We define a horizontal distribution $H := \{H_p \subset T_pM \mid p \in M\}$ (defined by f and g) which is a C^∞ distribution of dimension $n = \dim N$ as follows.*

- *For any $p \in M$, let $q = f(p) \in N$. By Preimage Theorem, $F := f^{-1}(q)$ is a C^∞ submanifold of dimension $m - n$ where $m = \dim M$. We have a short exact sequence of vector spaces*

$$0 \to T_pF \to T_pM \overset{df_p}{\to} T_qN \to 0$$

- *Define H_p to be the orthogonal complement of T_pF in T_pM, i.e.*

$$H_p := \{v \in T_pM \mid \langle u, v\rangle_p = 0 \quad \forall\ u \in T_pF\}$$

  Hence $\dim H_p = n$*. In fact we have orthogonal decomposition w.r.t. $\langle \cdot, \cdot\rangle_p$*

$$T_pM = T_pF \oplus H_p$$

- *We check $H := \{H_p \subset T_pM \mid p \in M\}$ is C^∞ distribution of dimension n. Indeed, for any $p \in M$*

$$df_p|_{H_p} : H_p \overset{\cong}{\to} T_{f(p)}N$$

  is a linear isomorphism.

Definition 16.9 (Riemannian Submersion)**.** *Let $f : (M, g) \to (N, h)$ be a C^∞ submersion between Riemannian manifolds, and let $\{H_p \mid p \in M\}$ be the horizontal distribution defined by f and g. We say f is a Riemannian submersion if for any $u, v \in H_p$*

$$\langle u, v\rangle_p = \langle df_p(u), df_p(v)\rangle_{f(p)} \tag{18}$$

where $\langle \cdot, \cdot\rangle_p$ is inner product defined by $g(p)$ and $\langle \cdot, \cdot\rangle_{f(p)}$ is inner product defined by $h(f(p))$. This is equivalent to saying

$$df_p|_{H_p} : H_p \to T_{f(p)}N$$

is a linear isometry (isomorphism of inner product spaces).

Theorem 16.3 (Metric on M/G for Riemannian Submersion)**.** *Suppose that a Lie group G acts on a Riemannian manifold (M, g) (where M is Hausdorff and 2nd countable) smoothly, freely, properly and isometrically, i.e.*

$$\forall\ a \in G \qquad \phi_a : M \to M \qquad \phi_a^* g = g$$

Then there exists a unique Riemannian metric $\hat{g}$ on M/G s.t.

$$\pi : (M, g) \to (M/G, \hat{g})$$

is a Riemannian Submersion, i.e.,

$$d\pi|_{H_p} : H_p \to T_{\pi(p)}(M/G)$$

is a linear isometry.

Proof. To define

$$\hat{g}(q) : T_q(M/G) \times T_q(M/G) \to \mathbb{R}$$

pick any $p \in \pi^{-1}(q)$ so that

$$H_p \overset{d\pi_p|_{H_p}}{\cong} T_q(M/G)$$

as linear isomorphism. Then we may write for any $u,\, v \in T_q(M/G)$

$$\hat{g}(q)(u,v) := g(p)\left(\left(d\pi_p|_{H_p}\right)^{-1}(u),\, \left(d\pi_p|_{H_p}\right)^{-1}(v)\right) \tag{19}$$

Note this is well-defined because the RHS is independent of the choice of $p \in \pi^{-1}(q)$, since any other $p' \in \pi^{-1}(q)$ is of the form $p' = a \cdot p$ for some $a \in G$, and $\phi_a^* g = g$, i.e., $(d\phi_a)_p : H_p \to H_{\phi_a(p)}$ is linear isometry. The diagram commutes

$$\begin{tikzcd} H_p \arrow[d, "(d\phi_a)_p"'] \arrow[dr, "d\pi_p"] & \\ H_{a\cdot p} \arrow[r, "d\pi_{a\cdot p}"] & T_q(M/G) \end{tikzcd}$$

□

Example 16.3. $\mathbb{S}^1$ *acts on* $(\mathbb{S}^{2n+1}, g_{can})$ *smoothly, freely, properly and isometrically. There exists a unique Riemannian metric* $\hat{g}_{can}$ *on* $P_n(\mathbb{C})$ *s.t.*

$$\pi : (\mathbb{S}^{2n+1},\, g_{can}) \to (P_n(\mathbb{C}), \hat{g}_{can})$$

is a Riemannian Submersion. In particular, for $n = 1$,

$$\pi : (\mathbb{S}^3, g_{can}) \to P_1(\mathbb{C}) \cong \mathbb{S}^2$$

and moreover

$$(P_1(\mathbb{C}), \hat{g}_{can}) \cong (\mathbb{S}^2,\, \frac{1}{4} g_{can})$$

Hence

$$\pi : \mathbb{S}^3(1) \to \mathbb{S}^2(\frac{1}{2})$$

is a Riemannian Submersion.

Proof for $(P_1(\mathbb{C}), \hat{g}_{can}) \cong (\mathbb{S}^2,\, \frac{1}{4} g_{can})$. One look at commutative diagram

$$\begin{tikzcd} \mathbb{S}^3 \arrow[d, "f"'] \arrow[dr, "\pi"] & \\ \mathbb{S}^2 \arrow[r, "j"] & P_1(\mathbb{C}) \end{tikzcd}$$

with diffeomorphism

$$j^{-1} : P_1(\mathbb{C}) \to \mathbb{S}^2 \qquad s.t. \qquad [z_1, z_2] \mapsto \left(\frac{2 z_1 \overline{z}_2}{|z_1|^2 + |z_2|^2}, \frac{|z_2|^2 - |z_1|^2}{|z_1|^2 + |z_2|^2}\right)$$

and

$$f : \mathbb{S}^3 = \{(z_1,\, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\} \to \mathbb{S}^2 = \{(\omega, z) \in \mathbb{C}\times\mathbb{R} \mid |\omega|^2 = z^2 = 1\} \qquad s.t. \qquad (z_1, z_2) \mapsto (2 z_1 \overline{z}_2, |z_2|^2 - |z_1|^2)$$

We've defined $\hat{g}_{can}$ as the unique metric on $P_1(\mathbb{C})$ s.t. $\pi = j \circ f : (\mathbb{S}^3, g_{can}) \to (P_1(\mathbb{C}), \hat{g}_{can})$ is a Riemannian submersion. To show that $(P_1(\mathbb{C}), \hat{g}_{can})$ is isometric to $(\mathbb{S}^2, \frac{1}{4} g_{can})$, it suffices to compute $j^* \hat{g}_{can}$ and verify that

$$j^* \hat{g}_{can} = \frac{1}{4} g_{can}^{\mathbb{S}^2(1)}$$

To do so, write coordinates on $\mathbb{S}^3$ as

$$\begin{cases} z_1 = \sin(\lambda) e^{i\theta_1} \\ z_2 = \cos(\lambda) e^{i\theta_2} \end{cases}$$

and if we write $z_j = x_j + \sqrt{-1} y_j$ we have

$$\begin{cases} x_1 = \sin(\lambda)\cos(\theta_1) \\ y_1 = \sin(\lambda)\sin(\theta_1) \\ x_2 = \cos(\lambda)\cos(\theta_2) \\ y_2 = \cos(\lambda)\sin(\theta_2) \end{cases}$$

as coordinates on $\mathbb{S}^3$. We compute metric $g_{can}^{\mathbb{S}^3(1)}$ so that

$$g_{can}^{\mathbb{S}^3(1)} = d\lambda^2 + \sin^2(\lambda) d\theta_1^2 + \cos^2(\lambda) d\theta_2^2$$

We use spherical metric on $\mathbb{S}^2$ as

$$\begin{cases} x = \sin(\phi)\cos(\theta) \\ y = \sin(\phi)\sin(\theta) \\ z = \cos(\phi) \end{cases}$$

and recall that

$$g_{can}^{\mathbb{S}^2(1)} = d\phi^2 + (\sin^2(\phi))d\theta^2$$

Now we look at

$f : (z_1, z_2) = (\sin(\lambda)e^{i\theta_1}, \cos(\lambda)e^{i\theta_2}) \mapsto (2\sin(\lambda)e^{i\theta_1}\cos(\lambda)e^{-i\theta_2}, \cos^2(\lambda) - \sin^2(\lambda)) = (\sin(2\lambda)e^{i(\theta_1-\theta_2)}, \cos^2(\lambda) - \sin^2(\lambda))$

But $\sin(2\lambda)e^{i(\theta_1-\theta_2)} = \sin(\phi)e^{i\theta}$ in $\mathbb{S}^2(1)$, so $\phi = 2\lambda$ and $\theta = \theta_1 - \theta_2$

$$df(\frac{\partial}{\partial\lambda}) = 2\frac{\partial}{\partial\phi} \qquad df(\frac{\partial}{\partial\theta_1}) = \frac{\partial}{\partial\theta} \qquad df(\frac{\partial}{\partial\theta_2}) = -\frac{\partial}{\partial\theta}$$

Thus

$$ker(df) = \mathbb{R}\left(\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}\right)$$

and as its orthogonal complement, the horizontal subspace H writes

$$H = (ker(df))^{\perp} = \mathbb{R}\frac{\partial}{\partial\lambda} \oplus \mathbb{R}\left(\cos^2(\lambda)\frac{\partial}{\partial\theta_1} - \sin^2(\lambda)\frac{\partial}{\partial\theta_2}\right)$$

Hence

$$\begin{aligned}
j^*\hat{g}_{can}\left(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\phi}\right) &= g_{can}^{\mathbb{S}^3(1)}\left(\frac{1}{2}\frac{\partial}{\partial\lambda}, \frac{1}{2}\frac{\partial}{\partial\lambda}\right) = \frac{1}{4} \\
j^*\hat{g}_{can}\left(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\theta}\right) &= g_{can}^{\mathbb{S}^3(1)}\left(\frac{1}{2}\frac{\partial}{\partial\lambda}, \cos^2(\lambda)\frac{\partial}{\partial\theta_1} - \sin^2(\lambda)\frac{\partial}{\partial\theta_2}\right) = 0 \\
j^*\hat{g}_{can}\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right) &= g_{can}^{\mathbb{S}^3(1)}\left(\cos^2(\lambda)\frac{\partial}{\partial\theta_1} - \sin^2(\lambda)\frac{\partial}{\partial\theta_2}, \cos^2(\lambda)\frac{\partial}{\partial\theta_1} - \sin^2(\lambda)\frac{\partial}{\partial\theta_2}\right) \\
&= \sin^2(\lambda)\cos^4(\lambda) + \cos^2(\lambda)\sin^4(\lambda) = \sin^2(\lambda)\cos^2(\lambda) = \frac{1}{4}\sin^2(2\lambda) = \frac{1}{4}\sin^2(2\phi)
\end{aligned}$$

Thus

$$j^*\hat{g}_{can} = \frac{1}{4}d\phi^2 + \frac{1}{4}\sin^2(2\phi)d\theta^2 = \frac{1}{4}g_{can}^{\mathbb{S}^2(1)}$$

□

16.4 Homogeneous Spaces

Theorem 16.4 (Cartan-Von Neumann)**.** *Let G be a Lie Group, and let H be a closed subgroup of G. Then H is a C^∞ submanifold of G. Therefore H is a Lie subgroup of G, i.e., H is both a subgroup and a C^∞ submanifold of G.*

Theorem 16.5. *Let G be a Lie group and let H be a closed subgroup of G. From Cartan-Von Neumann, we know H is a closed Lie subgroup of G.*

(i) Then we consider the action H on G by right multiplication. This action is free, proper and smooth. The Quotient

$$G/H = \{aH \mid a \in G\}$$

is the set of left cosets of H. There is a unique structure of smooth manifold on G/H s.t. the projection

$$\pi : G \to G/H$$

is a smooth fiber bundle with fiber H (hence π defines smooth submersion), using the Quotient Manifold Theorem 16.2.

(ii) Let G act on G/H on the left by

$$G \times G/H \to G/H \qquad s.t. \qquad (a, bH) \mapsto abH \tag{20}$$

left multiplication. Note

$$\begin{array}{ccc}
(a,b) \in G \times G & \xrightarrow{\ m\ } & ab \in G \\
\Big\downarrow{\scriptstyle id_G \times \pi} & & \Big\downarrow{\scriptstyle \pi} \\
(a,bH) \in G \times G/H & \longrightarrow & abH \in G/H
\end{array}$$

Then $G \times G/H \to G/H$ as in (20) is a C^∞ G-action on G/H.

Definition 16.10 (G-homogeneous Space)**.** *Let M be a C^∞ manifold. Let G be a Lie Group. M is a G-homogeneous space if G acts smoothly and transitively on M.*

In fact any G-homogeneous space is the form of (20) if we consider left action.

Lemma 16.2 (Stabilizer of G-homogeneous Space)**.** *For any $x \in M$, recall*

$$G_x := \{a \in G \mid a \cdot x = x\}$$

is the isotropy group (stabilizer) of x. Assume G Lie group and M is a G-homogeneous space.

- *Using Carton-Von Neumann G_x is a closed subgroup of G, hence G_x is a Lie subgroup.*
- *Using G is transitive action, for any $y \in M$, $y = bx$ for some $b \in G$. So*

$$\forall\ a \in G_y = G_{b\cdot x} \iff a \cdot (b \cdot x) = b \cdot x \iff (b^{-1}ab) \cdot x = x \iff b^{-1}ab \in G_x$$

  Then $G_{b\cdot x} = bG_x b^{-1}$.

Theorem 16.6 (Characterisation of G-homogeneous Space)**.** *Let M be a G-homogeneous space. Let $x \in M$ and let $H = G_x$ be the stabilizer of the G-action at x. Then the bijection*

$$G/H \to M \qquad s.t. \qquad aH \mapsto a \cdot x \tag{21}$$

is a C^∞ diffeomorphism.

Remark 16.2. *Now for some M just a set, we identify it as transient action of some Lie Group G.*

Example 16.4 ($SO(n+1)/SO(n) \cong \mathbb{S}^n$)**.** *We run through the construction as in Theorem 16.5 with $G = SO(n+1)$ and $H = SO(n)$. Then let $SO(n+1)$ act smoothly and transitively on*

$$\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} \mid |x| = 1\} \qquad where \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix}$$

via

$$SO(n+1) \times \mathbb{S}^n \to \mathbb{S}^n \qquad s.t. \qquad (A, x) \mapsto Ax$$

Hence by definition, $\mathbb{S}^n$ is $SO(n+1)$-homogeneous Space. Using Theorem 16.6, we expect

(i) $H = SO(n) \cong SO(n+1)_{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}$ stabilizer of column vector in $\mathbb{R}^{n+1}$ with all 0 but 1 at the bottom, under group action $SO(n+1)$. Indeed, the stabilizer of $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ is

$$\left\{ \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \mid B \in SO(n) \right\} \cong SO(n)$$

(ii) As a consequence, $\mathbb{S}^n$ is diffeomorphic to $SO(n+1)/SO(n)$ via (21)

$$SO(n+1)/SO(n) \overset{\cong}{\to} \mathbb{S}^n \qquad ASO(n) \to A \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

For simplicity, denote

$$f : \mathbb{S}^n \to SO(n+1)/SO(n)$$

as the diffeomorphism.

Example 16.5 ($(SO(n+1)/SO(n), \hat{g}) \cong (\mathbb{S}^n, 2g_{can})$)**.** *In fact, $(SO(n+1)/SO(n), \hat{g})$ is isometric to $(\mathbb{S}^n, \lambda g_{can})$ for some $\lambda > 0$ constant. On one hand, equipped with Riemannian Metric, it is easy to check $SO(n+1)$ acts isometrically on $(\mathbb{S}^n, g_{can})$. On the other hand*

(i) *Recall*

$$i : SO(n) \to M_n(\mathbb{R}) \cong \left(\mathbb{R}^{n^2}, \sum_{i,j=1}^{n} da_{i,j}^2\right)$$

Then as in (16)

$$g_n := i^* \left(\sum_{i,j=1}^{n} da_{i,j}^2\right)$$

is a bi-invariant Riemannian metric on $SO(n)$*.*

(ii) *Since* $SO(n) \subset SO(n+1)$ *is closed subgroup, as in Theorem 16.5,* $(SO(n), g_n)$ *acts on* $(SO(n+1), g_{n+1})$ *smoothly, freely, properly by right multiplication.*

(iii) *In fact* $SO(n)$ *also acts on* $SO(n+1)$ *isometrically. Then using Theorem 16.3, there exists a unique Riemannian metric* $\hat{g}$ *on the quotient* $SO(n+1)/SO(n)$ *s.t.*

$$\pi : (SO(n+1), g_{n+1}) \to (SO(n+1)/SO(n), \hat{g})$$

is a Riemannian submersion. We can indeed check that $SO(n+1)$ *acts smoothly, transitively, and isometrically on* $(SO(n+1)/SO(n), \hat{g})$ *on the left.*

Since $SO(n+1)$ *acts transitively and isometrically on both* $(SO(n+1)/SO(n), \hat{g})$ *and* $(\mathbb{S}^n, g_{can})$*, it suffices to show that*

$$f^*\hat{g} = \lambda g_{can} \qquad at \qquad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{S}^n$$

which implies $(SO(n+1)/SO(n), \hat{g})$ *is isometric to* $\mathbb{S}^n(\sqrt{\lambda})$*.*

Proof. We want to show

$$f^*\hat{g} = \lambda g_{can}$$

for some $\lambda > 0$. Recall that

$$f^{-1} : SO(n+1)/SO(n) \to \mathbb{S}^n \qquad s.t. \qquad ASO(n) \mapsto A \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

is a diffeomorphism. Also recall that

$$\pi : (SO(n+1), g_{n+1}) \to (SO(n+1)/SO(n), \hat{g}) \qquad s.t. \qquad A \mapsto ASO(n)$$

hence

$$f^{-1} \circ \pi : (SO(n+1), g_{n+1}) \to (\mathbb{S}^n, g_{can}) \qquad s.t. \qquad A \mapsto A \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Also notice

$$T_{I_{n+1}} SO(n+1) = \{A \in GL(n+1, \mathbb{R}) \mid A + A^T = 0\}$$

and

$$T_{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}} \mathbb{S}^n = \{v \in \mathbb{R}^{n+1} \mid v \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = 0\} = \{v \in \mathbb{R}^{n+1} \mid v_{n+1} = 0\}$$

So the differential of $f^{-1}\circ\pi$ at I_{n+1} writes

$$d(f^{-1}\circ\pi)_{I_{n+1}}: T_{I_{n+1}}SO(n+1)\to T_{\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}}\mathbb{S}^n \qquad s.t. \qquad B\mapsto B\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}$$

and the kernel writes

$$Ker(d(f^{-1}\circ\pi)_{I_{n+1}}) = \{\begin{pmatrix}B & 0\\ 0 & 0\end{pmatrix} \mid B\in T_{I_n}SO(n)\}\subset T_{I_{n+1}}SO(n+1)$$

We would love to determine the Horizontal Distribution. Indeed,

$$H_{I_{n+1}} := Ker(d(f^{-1}\circ\pi)_{I_{n+1}})^{\perp} = \{\begin{pmatrix}0 & v\\ -v^T & 0\end{pmatrix} \mid v\in\mathbb{R}^n\}$$

so that $H_{I_{n+1}}\oplus \mathrm{Ker}(d(f^{-1}\circ\pi)_{I_{n+1}}) = T_{I_{n+1}}SO(n+1)$. To compute $f^*\hat{g}$, we need to recall

$$g_{n+1} := i^*\left(\sum_{i,j=1}^{n+1} da_{ij}^2\right) \qquad \text{where} \qquad i: SO(n+1)\hookrightarrow GL(n+1,\mathbb{R})$$

We compute for any $v\in\mathbb{R}^{n+1}$ s.t. $v_{n+1}=0$. We denote $\hat{v} := (v_1,\cdots,v_n)^T$. Using (19)

$$\begin{aligned}
f^*\hat{g}_{SO(n)}(v,v) &= (f)^*\hat{g}_{SO(n)}(v,v)\\
&= (f)^*(g_{n+1})_{I_{n+1}}(\,d\pi_{I_{n+1}}|^{-1}_{H_{I_{n+1}}}(v),\ d\pi_{I_{n+1}}|^{-1}_{H_{I_{n+1}}}(v))\\
&= (g_{n+1})_{I_{n+1}}(\,d(f^{-1}\circ\pi)_{I_{n+1}}|^{-1}_{H_{I_{n+1}}}(v),\ d(f^{-1}\circ\pi)_{I_{n+1}}|^{-1}_{H_{I_{n+1}}}(v))\\
&= (g_{n+1})_{I_{n+1}}\left(\begin{pmatrix}0 & \hat{v}\\ -\hat{v}^T & 0\end{pmatrix}, \begin{pmatrix}0 & \hat{v}\\ -\hat{v}^T & 0\end{pmatrix}\right)\\
&= 2\sum_{i=1}^{n}(dv_i)^2 = 2g_{can}(v,v)
\end{aligned}$$

Hence $f^*\hat{g} = 2g_{can}$ and so $\lambda = 2$. □

Example 16.6 (Real/Complex Grassmannian $G_{k,n}(\mathbb{R})$ or $G_{k,n}(\mathbb{C})$)**.** *As a set*

$$G_{k,n}(\mathbb{R}) := \{V\subset\mathbb{R}^n \mid V \text{ } k\text{-dimensional subspace of } \mathbb{R}^n\}$$

In particular, $G_{1,n}(\mathbb{R}) = P_{n-1}(\mathbb{R})$. Aiming for Theorem 16.6, let $G = O(n)$ and $M = G_{k,n}(\mathbb{R})$, here $O(n)$ acts transitively on $G_{k,n}(\mathbb{R})$. For the first k coordinates $\mathbb{R}^k\times\{(0,\cdots,0)\}\subset\mathbb{R}^n$, the stabilizer is

$$O(k)\times O(n-k) = \left\{\begin{pmatrix}B & 0\\ 0 & C\end{pmatrix} \mid B,\, C\in O(n)\right\}$$

As a set,

$$G_{k,n}(\mathbb{R})\cong O(n)/O(k)\times O(n-k)$$

the RHS is a C^∞ manifold. Since

$$O(n)\xrightarrow{i} M_n(\mathbb{R}) \qquad g_n = i^*(\sum_{i,j=1}^{n} da_{i,j}^2)$$

is a bi-invariant Riemannian metric on $O(n)$. $O(k)\times O(n-k)$ acts smoothly, freely, properly and isometrically on $(O(n),g_n)$. There is a unique Riemannian metric $\hat{g}$ on $G_{k,n}(\mathbb{R}) = O(n)/O(k)\times O(n-k)$ s.t.

$$(O(n),g_n)\to(G_{k,n}(\mathbb{R}) = O(n)/O(k)\times O(n-k),\hat{g})$$

is a Riemannian submersion. In particular take $k=1$ and $n+1$

$$P_n(\mathbb{R}) = G_{1,n+1}(\mathbb{R}) = \frac{O(n+1)}{O(1)\times O(n)}$$

Notice $O(n+1)/O(n) = SO(n+1)/SO(n)$ hence

$$P_n(\mathbb{R}) = \frac{O(n+1)}{O(1)\times O(n)} = \frac{1}{\{\pm 1\}}\frac{O(n+1)}{O(n)} = \frac{1}{\{\pm 1\}}\frac{SO(n+1)}{SO(n)} = \frac{\mathbb{S}^n(\sqrt{\lambda})}{\{\pm 1\}}$$

How about Complex Grassmannian? For $G_{k,n}(\mathbb{C})$, we replace $O(n)$ with $U(n)$ where

$$U(n) := \{A \in GL(n,\mathbb{C}) \mid A^*A = \overline{A}^T A = I_n\}$$

and identify

$$U(n) \xrightarrow{i} M_n(\mathbb{C}) \cong \mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$$

so that for $a_{i,j} = b_{i,j} + \sqrt{-1}c_{i,j}$

$$g_n = i^*\left(\sum_{i,j=1}^{n} db_{i,j}^2 + dc_{i,j}^2\right)$$

Then there is unique Riemannian metric $\hat{g}$ on

$$G_{k,n}(\mathbb{C}) = U(n)/U(k)\times U(n-k)$$

and

$$(U(n), g_n) \to (G_{k,n}(\mathbb{C}), \hat{g})$$

is Riemannian submersion.

$$P_n(\mathbb{C}) = \frac{U(n+1)}{U(1)\times U(n)} = \frac{\mathbb{S}^{2n+1}(\sqrt{\lambda})}{U(1)}$$

Example 16.7. *Recall*

$$\pi : \mathbb{C}^n \setminus \{0\} \to P_{n-1}(\mathbb{C}) \qquad s.t. \qquad z = (z_1, \cdots, z_n) \mapsto [z_1, \cdots, z_n] = Span\{z_1, \cdots, z_n\}$$

for $\Phi = \{(U_i, \phi_i) \mid i = 1, \cdots, n\}$ and

$$U_i = \{[z_1, \cdots, z_n] \mid z_i \neq 0\} \xrightarrow{\phi_i} \mathbb{C}^{n-1} \qquad s.t. \qquad [z_1, \cdots, z_n] \mapsto \left(\frac{z_1}{z_i}, \frac{z_2}{z_i}, \cdots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \cdots, \frac{z_n}{z_i}\right)$$

Then

$$\Pi : \{A = \begin{pmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \cdots & \vdots \\ z_{k1} & \cdots & z_{kn} \end{pmatrix} \mid Rank(A) = k\} \to G_{k,n}(\mathbb{C}) \qquad s.t. \qquad A \mapsto Span \text{ of row vectors of } A$$

Here

$$\Phi = \{(U_I, \phi_I) \mid I = \{i_1, \cdots, i_k\} \subset \{1, \cdots, n\},\ 1 \leq i_1 < \cdots < i_k \leq n,\ |I| = k\}$$

and

$$U_I = \Pi\left(\begin{pmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \cdots & \vdots \\ z_{k1} & \cdots & z_{kn} \end{pmatrix} \mid \det\begin{pmatrix} z_{1i_1} & \cdots & z_{1i_k} \\ \vdots & \cdots & \vdots \\ z_{ki_1} & \cdots & z_{ki_k} \end{pmatrix} \neq 0\right)$$

For $A \in U_I$,

$$\phi_I : [A = \big((A_I)_{k\times k} \mid (A_{I'})_{k\times(n-k)}\big)] = [(I_k \mid A_I^{-1}A_{I'})] \mapsto A_I^{-1}A_{I'} \in M_{k\times(n-k)}(\mathbb{C})$$

17 Connections on Vector Bundles

17.1 Connections on a C^∞ Vector Bundle

Definition 17.1 (Connection on C^∞ vector bundle)**.** *Let M be C^∞ manifold and fix $\pi : E \to M$ a C^∞ vector bundle over M of rank r. A connection on E is a $\mathbb{R}$-linear map*

$$\nabla : \mathfrak{X}(M) \times C^\infty(M,E) := \{C^\infty \text{ sections of } \pi : E \to M\} \to C^\infty(M,E) \qquad s.t. \qquad (X,s) \mapsto \nabla_X s$$

s.t. for any $X \in \mathfrak{X}(M)$, for any $s \in C^\infty(M,E)$, and for any $f \in C^\infty(M)$

(i) $\nabla_{fX} s = f\nabla_X s$, i.e., $C^\infty(M)$-linear in X.

(ii) For fixed $X \in \mathfrak{X}(M)$, the map $\nabla_X : C^\infty(M,E) \to C^\infty(M,E)$ satisfies Leibniz Rule, i.e.,

$$\nabla_X(fs) = X(f)s + f\nabla_X s$$

Here $\mathfrak{X}(M)$ and $C^\infty(M,E)$ are $C^\infty(M)$-modules.

Remark 17.1. (i) *implies given $p \in M$, for any $v \in T_pM$ and $s \in C^\infty(M,E)$, we may define*

$$\nabla_v s \in E_p = \pi^{-1}(p) \subset E$$

Definition 17.2 (Affine Connection on smooth manifold)**.** *An affine connection on a C^∞ manifold M is a connection on the tangent bundle $\pi : TM \to M$, i.e., a $\mathbb{R}$-linear map*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \qquad s.t. \qquad (X,Y) \mapsto \nabla_X Y$$

s.t. for any X, Y, $Z \in \mathfrak{X}(M)$ and f, $g \in C^\infty(M)$

(i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$, $C^\infty(M)$-linear.

(ii) Leibniz Rule, for fixed $X \in \mathfrak{X}(M)$

$$\nabla_X(fY) = X(f)Y + f\nabla_X Y \tag{22}$$

Lemma 17.1. *If E and F are C^∞ vector bundles on a C^∞ manifold M and $\phi : C^\infty(M,E) \to C^\infty(M,F)$ is $C^\infty(M)$-linear, i.e. for $f \in C^\infty(M)$ and $s \in C^\infty(M,E)$*

$$\phi(fs) = f\phi(s)$$

Then $\phi \in C^\infty(M, E^* \otimes F)$.

Proof. On $U \subset M$ open, let $\{e_1, \cdots, e_r\}$, $\{f_1, \cdots, f_s\}$ be C^∞ frame of $E|_U$ and $F|_U$ respectively. Then in local coordinates

$$\phi(e_i) = \sum_{j=1}^{s} a_{ij} f_j \qquad for \quad a_{ij} \in C^\infty(U)$$

we have

$$\phi = \sum_{i=1}^{r}\sum_{j=1}^{s} a_{ij} e_i^* \otimes f_j$$

for $\{e_1^*, \cdots, e_r^*\}$ C^∞ frame of $E^*|_U$ dual to $(e_1, \cdots, e_r)$. $\square$

We introduce the following notation.

Definition 17.3 (E-valued p-forms)**.** *Space of E-valued p-forms*

$$\Omega^p(M,E) := C^\infty(M, \Lambda^p T^*M \otimes E)$$

In particular

1. *$\Omega^0(M,E) = C^\infty(M,E) \to \Omega^1(M,E) = C^\infty(M, T^*M \otimes E)$.*
2. *$\Omega^0(M,TM) = C^\infty(M,TM) = \mathfrak{X}(M) \to \Omega^1(M,TM) = C^\infty(M, T^*M \otimes TM)$.*

Remark 17.2 (∇s)**.** *For a fixed $s \in C^\infty(M,E) = \Omega^0(M,E)$, let*

$$\nabla s : \mathfrak{X}(M) = C^\infty(M,TM) \ \to C^\infty(M,E) \qquad s.t. \qquad X \mapsto \nabla_X s$$

then ∇s is $C^\infty(M)$-linear by (i). We may view ∇s as a smooth section of $T^*M \otimes E$, i.e.

$$\nabla s \in C^\infty(M, T^*M \otimes E) = \Omega^1(M,E) \tag{23}$$

Definition 17.4 (Connection on C^∞ vector bundle (Alternative Formulation))**.** *Let $\pi : E \to M$ be a C^∞ vector bundle over a C^∞ manifold M. A connection on E is a $\mathbb{R}$-linear map*

$$\nabla : \Omega^0(M,E) = C^\infty(M,E) \to \Omega^1(M,E) \qquad s.t. \qquad s \mapsto \nabla s$$

such that for any $f \in C^\infty(M)$, and for any $s \in \Omega^0(M,E) = C^\infty(M,E)$

$$\nabla(fs) = df \otimes s + f\nabla s \tag{24}$$

where ∇s is as in (23).

Well-definedness. Recall in general, for any $\alpha \in \Omega^p(M) = C^\infty(M, \Lambda^p T^*M)$ and $s \in C^\infty(M,E)$

$$\alpha \otimes s \in \Omega^p(M,E) = C^\infty(M, \Lambda^p T^*M \otimes E)$$

Hence for $f \in C^\infty(M)$, $df \in \Omega^1(M) = C^\infty(M, T^*M)$, and so

$$df \otimes s \in C^\infty(M, T^*M \otimes E) = \Omega^1(M,E)$$

$\square$

Lemma 17.2 ($\Omega^1(M, End(E))$)**.** *Given E as C^∞ vector bundle over M. Let $F = T^*M \otimes E$. Then any $C^\infty(M)$-linear map*

$$\phi : C^\infty(M,E) = \Omega^0(M,E) \to C^\infty(M, T^*M \otimes E) = \Omega^1(M,E)$$

can be viewed as $\phi \in C^\infty(M, E^* \otimes T^*M \otimes E) = C^\infty(M, T^*M \otimes End(E)) = \Omega^1(M, End(E))$ via Lemma 17.1.

Lemma 17.3. *If ∇_0 and ∇_1 are two connections on the same vector bundle $\pi : E \to M$, then*

$$\nabla_1 - \nabla_0 : \Omega^0(M,E) = C^\infty(M,E) \to \Omega^1(M,E) = C^\infty(M, T^*M \otimes E) \qquad s.t. \qquad s \mapsto \nabla_1 s - \nabla_0 s$$

is $C^\infty(M)$-linear. This corresponds to a section of

$$E^* \otimes T^*M \otimes E = T^*M \otimes End(E)$$

according to Lemma 17.1, i.e., $\nabla_1 - \nabla_0$ can be viewed as an element in

$$C^\infty(M, T^*M \otimes End(E)) = \Omega^1(M, End(E))$$

Proof. For any $f \in C^\infty(M)$ and $s \in C^\infty(M,E)$

$$\begin{aligned}(\nabla_1 - \nabla_0)(fs) &= \nabla_1(fs) - \nabla_0(fs) \\ &= (df \otimes s + f\nabla_1 s) - (df \otimes s + f\nabla_0 s) \\ &= f(\nabla_1 s - \nabla_0 s) = f(\nabla_1 - \nabla_0)s\end{aligned}$$

$\square$

Definition 17.5 ($A(E)$ Space of Connections on Vector Bundle)**.** *Let $A(E)$ be the space of connections on E. Then $A(E)$ is an affine space associated to the vector space $\Omega^1(M, End(E))$. Indeed, for any $\nabla_0 \in A(E)$, $\phi \in \Omega^1(M, End(E))$*

$$(\nabla_0 + \phi) : \Omega^0(M,E) \to \Omega^1(M,E)$$

so $\nabla_0 + \phi \in A(E)$. Note $\Omega^1(M, End(E))$ is ∞-dimensional if $\dim M > 0$ and $rankE > 0$.

Remark 17.3 (Connection on C^∞ Vector Bundle in Local Coordinates)**.** *Let $\pi : E \to M$ be C^∞ vector bundle of rank r over C^∞ manifold of dimension n. We write our connection on E*

$$\nabla : \Omega^0(M,E) \to \Omega^1(M,E) \qquad s \mapsto \nabla s$$

in local coordinates.

(i) Suppose (U, ϕ) for $\phi = (x_1, \cdots, x_n)$ is a C^∞ chart for M where $n = dimM$ such that $E|_U := \pi^{-1}(U)$ is trivial. So

$$h : \pi^{-1}(U) = E|_U \subset E \to U \times \mathbb{R}^r \subset M \times \mathbb{R}^r$$

is local trivialization. Then we have $\{e_1, \cdots, e_r\} \subset C^\infty(U, E|_U)$ as a C^∞ frame of $E|_U \to U$

$$e_j : U \to \pi^{-1}(U) \qquad s.t. \qquad e_j(x) := h^{-1}(x, \hat{e}_j) \qquad where \qquad \hat{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

and $r = rankE$. For any $s \in C^\infty(U, E|_U)$, we write smooth section

$$s = \sum_{k=1}^{r} a^k e_k \in C^\infty(U, E|_U)$$

in local coordinates for $a^k \in C^\infty(U)$.

(ii) *We have $\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\}$ as C^∞ frame of $TM|_U = TU$. To let ∇ act on s, we first discuss what ∇ is acting on e_j. In fact, on U we define the Christoffel Symbols $\Gamma^k_{i,j} \in C^\infty(U)$ s.t.*

$$\nabla_{\frac{\partial}{\partial x_i}} e_j := \sum_{k=1}^{r} \Gamma^k_{i,j} e_k \in C^\infty(U, E|_U) \tag{25}$$

We further define connection 1-form $\omega^k_j \in \Omega^1(U)$ s.t.

$$\nabla e_j = \sum_{k=1}^{r} \omega^k_j \otimes e_k \tag{26}$$

holds. This uses only trivialization of $E|_U$ (but not trivialization of $T^*M|_U$). This also used the observation that the element ∇e_j is an E-valued one-form on U, i.e.

$$\nabla e_j \in \Omega^1(U, E|_U) = C^\infty(U, T^*U \otimes E|_U)$$

Plugging (25) into above (26) we may identify

$$\sum_{k=1}^{r} \Gamma^k_{i,j} e_k = \nabla_{\frac{\partial}{\partial x_i}} e_j = \sum_{k=1}^{r} \omega^k_j(\frac{\partial}{\partial x_i}) e_k \implies \omega^k_j(\frac{\partial}{\partial x_i}) = \Gamma^k_{i,j}$$

Thus obtaining

$$\omega^k_j = \sum_{i=1}^{n} \Gamma^k_{i,j} dx_i \ \in \Omega^1(U) = C^\infty(U, T^*U) \tag{27}$$

Plugging back into (26) we have explicit form in both Christoffel Symbols and connection 1-forms.

$$\nabla e_j = \sum_{k=1}^{r} \omega^k_j \otimes e_k = \sum_{k=1}^{r}\sum_{i=1}^{n} \Gamma^k_{ij} dx_i \otimes e_k$$

Now we discuss how ∇ transits between two intersecting coordinate charts.

(i) *Now take open cover $\{U_\alpha \mid \alpha \in I\}$ of the base M and*

$$h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$$

local trivializations. Let

$$e_{\alpha_j} : U_\alpha \to \pi^{-1}(U_\alpha) \qquad s.t. \qquad e_{\alpha_j}(x) := h^{-1}_\alpha(x, \hat{e}_j)$$

for $j = 1, \cdots, r$, i.e., $e_{\alpha_1}, \cdots, e_{\alpha_r}$ are C^∞ frames of $E|_{U_\alpha}$. For any $U_\alpha \cap U_\beta \neq \varnothing$,

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \overset{C^\infty}{\to} GL(r, \mathbb{R}) \qquad s.t. \qquad e_{\alpha_j}(x) = e_{\beta_i}(x) g_{\beta\alpha}(x)_{i,j}$$

and we have transition functions

$$h_\beta \circ h^{-1}_\alpha : (U_\alpha \cap U_\beta) \cap \mathbb{R}^r \to (U_\alpha \cap U_\beta) \times \mathbb{R}^r \qquad s.t. \qquad (x, v) \mapsto (x, g_{\beta\alpha}(x)v)$$

for $v \in \mathbb{R}^r$. Since $s \in C^\infty(M, E)$ is a section, on U_α we have

$$s = \sum_{j=1}^{r} s^j_\alpha e_{\alpha_j} = e_\alpha s_\alpha \qquad for \quad s^j_\alpha \in C^\infty(U_\alpha), \qquad e_\alpha = [e_{\alpha_1}, \cdots, e_{\alpha_r}], \qquad s_\alpha := \begin{pmatrix} s^1_\alpha \\ \vdots \\ s^r_\alpha \end{pmatrix} \in C^\infty(U_\alpha, \mathbb{R}^r) \tag{28}$$

Now $s \in C^\infty(M, E)$ is a C^∞ section iff $s \in C^\infty(U_\alpha, \mathbb{R}^r)$ and $s_\beta = g_{\beta\alpha} s_\alpha$ on $U_\alpha \cap U_\beta$. Indeed, on $U_\alpha \cap U_\beta$

$$s = e_\alpha s_\alpha = e_\beta g_{\beta\alpha} s_\alpha = e_\beta s_\beta$$

(ii) Now suppose that we're given a connection ∇ on E. On U_α we define connection 1-form $(\omega_\alpha)_j^k \in \Omega^1(U_\alpha)$ for $j,k = 1,\cdots,r$ as in (26) by

$$\nabla e_{\alpha_j} = \sum_{k=1}^r (\omega_\alpha)_j^k \otimes e_{\alpha_k} \qquad (\omega_\alpha)_j^k \in \Omega^1(U_\alpha)$$

So

$$\nabla e_\alpha = [\nabla e_{\alpha_1},\cdots,\nabla e_{\alpha_r}] = e_\alpha \omega_\alpha \qquad s.t. \qquad \omega_\alpha := \begin{pmatrix} (\omega_\alpha)_1^1 & \cdots & (\omega_\alpha)_r^1 \\ \vdots & \cdots & \vdots \\ (\omega_\alpha)_1^r & \cdots & (\omega_\alpha)_r^r \end{pmatrix} \in \Omega^1(U_\alpha, \mathfrak{gl}(r,\mathbb{R}) = M_r(\mathbb{R}))$$

where $\mathfrak{gl}(r,\mathbb{R})$ is the Lie algebra of $GL(r,\mathbb{R})$.

(iii) On U_α we defined

$$(\nabla s)_\alpha := \begin{pmatrix} (\nabla s)_\alpha^1 \\ \vdots \\ (\nabla s)_\alpha^r \end{pmatrix} \in \Omega^1(U_\alpha, \mathbb{R}^r)$$

by

$$\nabla s = \sum_{j=1}^r (\nabla s)_\alpha^j \otimes e_{\alpha_j} \in \Omega^1(U_\alpha, E|_{U_\alpha}) = C^\infty(U_\alpha, T^*U_\alpha \otimes E|_{U_\alpha})$$

where $(\nabla s)_\alpha^j \in \Omega^1(U_\alpha) = C^\infty(U_\alpha, T^*U_\alpha)$. So

$$\nabla s = e_\alpha (\nabla s)_\alpha$$

But on the other hand, by Leibniz Rule, we may unpack the definition

$$\begin{aligned} \nabla s = \nabla\left(\sum_{j=1}^r s_\alpha^j e_{\alpha_j}\right) &= \sum_{j=1}^r ds_\alpha^j \otimes e_{\alpha_j} + \sum_{j=1}^r s_\alpha^j \nabla e_{\alpha_j} \\ &= \sum_{j=1}^r ds_\alpha^j \otimes e_{\alpha_j} + \sum_{j=1}^r\sum_{k=1}^r s_\alpha^j (\omega_\alpha)_j^k \otimes e_{\alpha_k} \\ &= \sum_{j=1}^r \left(ds_\alpha^j + \sum_{k=1}^r (\omega_\alpha)_k^j s_\alpha^k\right) \otimes e_{\alpha_j} = \sum_{j=1}^r (\nabla s)_\alpha^j \otimes e_{\alpha_j} \end{aligned}$$

Hence

$$(\nabla s)_\alpha = \begin{pmatrix} (\nabla s)_\alpha^1 \\ \vdots \\ (\nabla s)_\alpha^r \end{pmatrix} = d\begin{pmatrix} s_\alpha^1 \\ \vdots \\ s_\alpha^r \end{pmatrix} + \begin{pmatrix} (\omega_\alpha)_1^1 & \cdots & (\omega_\alpha)_r^1 \\ \vdots & \cdots & \vdots \\ (\omega_\alpha)_1^r & \cdots & (\omega_\alpha)_r^r \end{pmatrix}\begin{pmatrix} s_\alpha^1 \\ \vdots \\ s_\alpha^r \end{pmatrix} = ds_\alpha + \omega_\alpha s_\alpha$$

Or in short hand notation

$$\nabla s = \nabla(e_\alpha s_\alpha) = \nabla e_\alpha s_\alpha + e_\alpha ds_\alpha = e_\alpha \omega_\alpha s_\alpha + e_\alpha ds_\alpha = e_\alpha\left(ds_\alpha + \omega_\alpha s_\alpha\right)$$

Combining with $\nabla s = e_\alpha(\nabla s)_\alpha$ we obtain

$$(\nabla s)_\alpha = ds_\alpha + \omega_\alpha s_\alpha \tag{29}$$

(iv) One may ask: On $U_\alpha \cap U_\beta$, how are ω_α and ω_β related? On $U_\alpha \cap U_\beta$, we align both representations, and using (28)

$$\begin{aligned} \nabla e_\beta &= e_\beta \omega_\beta = e_\alpha g_{\alpha\beta}\omega_\beta \\ \nabla e_\beta &= \nabla\left(e_\alpha g_{\alpha\beta}\right) = \nabla e_\alpha g_{\alpha\beta} + e_\alpha dg_{\alpha\beta} = e_\alpha\omega_\alpha g_{\alpha\beta} + e_\alpha dg_{\alpha\beta} \end{aligned}$$

for $g_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta, \mathfrak{gl}(r))$, $dg_{\alpha\beta} \in \Omega^1(U_\alpha \cap U_\beta, \mathfrak{gl}(r))$ and $\omega_\beta \in \Omega^1(U_\beta, \mathfrak{gl}(r))$. Hence

$$g_{\alpha\beta}\omega_\beta = \omega_\alpha g_{\alpha\beta} + dg_{\alpha\beta} \in \Omega^1(U_\alpha, \mathfrak{gl}(r))$$

Rewriting yields

$$\omega_\beta = g_{\alpha\beta}^{-1}\omega_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1}dg_{\alpha\beta} \tag{30}$$

Hence that

$$\nabla : \Omega^0(M,E) \to \Omega^1(M,E)$$

is connection on E iff for any $\omega \in \Omega^1(U_\alpha, \mathfrak{gl}(r))$ it satisfies (30)

$$\omega_\beta = g_{\alpha\beta}^{-1}\omega_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1}dg_{\alpha\beta} \qquad on \qquad U_\alpha \cap U_\beta$$

Remark 17.4. *Let E be C^∞ vector bundle of rank r. Let $P : GL(E) \to M$ be the frame bundle over M, i.e.*

$$GL(E)_x = \{(e_1, \cdots, e_r) \mid \text{ordered basis of } E_x \cong \mathbb{R}^r\}$$

This is fiber bundle with fiber $GL(r, \mathbb{R})$, so-called principal $GL(r, \mathbb{R})$-bundle. $M = GL(E)/GL(r, \mathbb{R})$. Our previous example $G \to G/H$ is principal H-bundle. There is notation of connection on $GL(E)$ iff $GL(r, \mathbb{R})$-valued 1-form $\omega \in \Omega^1(GL(E), \mathfrak{gl}(r))$ with some properties. Then

$$e_\alpha = [e_{1\alpha}, \cdots, e_{r\alpha}] : U_\alpha \to P^{-1}(U_\alpha)$$

with $\omega_\alpha = e_\alpha^* \omega \in \Omega^1(U_\alpha, \mathfrak{gl}(r))$.

17.2 Pullback Section and Pullback Vector Bundle

Definition 17.6 (Pullback Vector Bundles)**.** *Let $F : M \to N$ be a C^∞ map between C^∞ manifolds. Let*

$$\pi : E \to N$$

be C^∞ vector bundle on N of rank r. Define

$$\tilde{\pi} : F^*E \to M$$

the pullback vector bundle as C^∞ vector bundle on M of rank r s.t.

(i) As a set,

$$F^*E := \bigsqcup_{p \in M} E_{F(p)}$$

where $E_{F(p)} \cong \mathbb{R}^r$.

$$\begin{array}{ccc} F^*E & \longrightarrow & E \\ \big\downarrow{\scriptstyle \tilde{\pi}} & & \big\downarrow{\scriptstyle \pi} \\ M & \xrightarrow{F} & N \end{array}$$

In other words

$$F^*E := \{(x, (y, v)) \in M \times E \mid F(x) = y = \pi(y, v)\} \subset M \times E$$

s.t. $x \in M$, $y \in N$ and $v \in E_y$.

(ii) F^*E is a C^∞ submanifold of $M \times E$. Let $\{U_\alpha \mid \alpha \in I\}$ be open cover of N with

$$h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$$

as local trivializations. Then using $F : M \to N$ is C^∞ map

$$\{F^{-1}(U_\alpha) \mid \alpha \in I\}$$

is open cover of M. We want to define

$$\tilde{h}_\alpha : \tilde{\pi}^{-1}(F^{-1}(U_\alpha)) \to F^{-1}(U_\alpha) \times \mathbb{R}^r$$

as local trivialization of the vector bundles $\tilde{\pi} : F^*E \to M$.

Definition 17.7 (Pullback Sections)**.** *Let $\pi : E \to N$ be C^∞ vector bundle of rank r over a C^∞ manifold N. Let $F : M \to N$ be smooth map. For*

$$s : N \to E$$

C^∞ section of N. We define $F^*s \in C^\infty(M, F^*E)$

$$F^*s : M \to F^*E \qquad \text{s.t.} \qquad (F^*s)(p) := s(F(p)) \in E_{F(p)} = (F^*E)_p \qquad \forall\, p \in M$$

as smooth section of F^*E s.t. the diagram commutes

$$\begin{array}{ccc} F^*E & \longrightarrow & E \\ {\scriptstyle F_s^*}\big\uparrow & & {\scriptstyle s}\big\uparrow \\ M & \xrightarrow{F} & N \end{array}$$

One hence view

$$F^* : C^\infty(N, E) = \Omega^0(N, E) \to C^\infty(M, F^*E) \qquad s \mapsto F^*s$$

Now, to define the local trivialization for F^*E, given

$$h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$$

local trivializations of $\pi : E|_{U_\alpha} \to U_\alpha$ *and* $\{e_{\alpha_1}, \cdots e_{\alpha_r}\}$ *as* C^∞ *frame of* $E|_{U_\alpha}$*, recall*

$$e_{\alpha_j} : U_\alpha \to \pi^{-1}(U_\alpha) = E|_{U_\alpha} \qquad s.t. \qquad e_{\alpha_j}(y) := h_\alpha^{-1}(y, \hat{e}_j) \qquad for \quad \hat{e}_j := \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

We have pullback sections $\{F^*e_{\alpha_1}, \cdots, F^*e_{\alpha_r}\}$ *as* C^∞ *frame for* $F^*E|_{F^{-1}(U_\alpha)}$ *and we define*

$$\tilde{h}_\alpha : \tilde{\pi}^{-1}(F^{-1}(U_\alpha)) \to F^{-1}(U_\alpha) \times \mathbb{R}^r \qquad s.t. \qquad \tilde{h}_\alpha^{-1}(x, \hat{e}_j) := (F^*e_{\alpha_j})(x) = e_{\alpha_j}(F(x))$$

We define our surjective map as

$$\tilde{\pi} : F^*E \to M \qquad s.t. \qquad (p, v) \in M \times \big((F^*E)_p = E_{F(p)}\big) \mapsto p$$

(iii) *Transition Functions. On* $U_\alpha \cap U_\beta$*, for* $e_\alpha = e_\beta g^E_{\beta\alpha}$ *where* $e_\alpha = [e_{\alpha_1}, \cdots, e_{\alpha_r}]$

$$g^E_{\beta\alpha} : U_\alpha \cap U_\beta \overset{C^\infty}{\to} GL(r, \mathbb{R})$$

Note for $F^{-1}(U_\alpha) \cap F^{-1}(U_\beta) = F^{-1}(U_\alpha \cap U_\beta)$*, the diagram commutes*

$$\begin{tikzcd} M & \overset{open}{\supseteq} & F^{-1}(U_\alpha \cap U_\beta) \arrow[d, "F"'] \arrow[dr, "F^*g^E_{\beta\alpha} = g^E_{\beta\alpha}\circ F"] & \\ N & \overset{open}{\supseteq} & U_\alpha \cap U_\beta \arrow[r, "g^E_{\beta\alpha}"] & GL(r, \mathbb{R}) \end{tikzcd}$$

Then

$$F^*e_\alpha = [F^*e_{\alpha_1}, \cdots, F^*e_{\alpha_r}] = F^*e_\beta F^*g^E_{\beta\alpha}$$

and hence

$$g^{F^*E}_{\beta\alpha} := F^*g^E_{\beta\alpha}$$

Notice $s \in C^\infty(N, E)$ *iff*

$$s_\alpha = \begin{pmatrix} s^1_\alpha \\ \vdots \\ s^r_\alpha \end{pmatrix} \in C^\infty(U_\alpha, \mathbb{R}^r)$$

and $s_\beta = g^E_{\beta\alpha} s_\alpha$ *on* $U_\alpha \cap U_\beta$ *upon writing* $s = e_\alpha s_\alpha$*. Hence we have* $F^*s \in C^\infty(M, F^*E)$ *s.t.*

$$(F^*s)_\alpha = F^*s_\alpha = \begin{pmatrix} F^*s^1_\alpha \\ \vdots \\ F^*s^r_\alpha \end{pmatrix} \in C^\infty(F^{-1}(U_\alpha), \mathbb{R}^r)$$

Now we consider the special case $E = TN$. Then the pullback tangent bundle writes

$$\tilde{\pi} : F^*TN \to M$$

We consider the space of connections on the C^∞ vector bundle F^*TN, i.e. $C^\infty(M, F^*TN)$

Definition 17.8 (Pushforward and Pullback of Vector Field into Section of Pullback Tangent Bundle)**.** *Let* $F : M \to N$ *smooth map. Define*

$$F_* : \mathfrak{X}(M) = C^\infty(M, TM) \to C^\infty(M, F^*TN) \qquad s.t. \qquad X \mapsto (F_*X)(p) := dF_p(X(p)) \in T_{F(p)}N = (F^*TN)_p \tag{31}$$

This is smooth section of pushforward bundle. Also, we have pull-back as particular example of Definition 17.7

$$F^* : \mathfrak{X}(N) = C^\infty(N, TN) \to C^\infty(M, F^*TN) \qquad s.t. \qquad Y \mapsto (F^*Y)(p) := Y(F(p)) \in T_{F(p)}N = (F^*TN)_p \tag{32}$$

If moreover $X \in \mathfrak{X}(M)$ *and* $Y \in \mathfrak{X}(N)$ *are F-related as in Definition 15.3 then*

$$F_*X = F^*Y \in C^\infty(M, F^*TN)$$

In particular, we study elements in $C^\infty(M, F^*TN)$, i.e., sections of pullback Tangent Bundle.

Definition 17.9 (C^∞ vector field along F)**.** *For $F : M \to N$ smooth map between C^∞ manifold. A C^∞ vector field along F is a C^∞ map*

$$V : M \to TN \qquad s.t. \qquad \forall\, p \in M, \quad V(p) \in TN_{F(p)} = (F^*TN)_p$$

We may view V as a C^∞ section of F^*TN, i.e., $V \in C^\infty(M, F^*TN)$.

$$\begin{tikzcd} M \arrow[d, "F"'] \arrow[dr, "V"] & \\ N & TN \arrow[l, "\pi"] \end{tikzcd}$$

More generally, for smooth vector bundle $\pi : E \to N$, we study elements in $C^\infty(M, F^*E)$.

Definition 17.10 (C^∞ section along F)**.** *For $F : M \to N$ smooth map between C^∞ manifold. Let*

$$\pi : E \to N$$

be C^∞ vector bundle of rank r on N. A C^∞ section of $\pi : E \to N$ along F is a C^∞ map

$$V : M \to E \qquad s.t. \qquad \forall\, p \in M, \quad V(p) \in E_{F(p)} = (F^*E)_p$$

We may view V as a C^∞ section of $F^*E \to M$, i.e., $V \in C^\infty(M, F^*E)$.

$$\begin{tikzcd} M \arrow[d, "F"'] \arrow[dr, "V"] & \\ N & E \arrow[l, "\pi"] \end{tikzcd}$$

17.3 Pullback Connection

Definition 17.11 (Pullback Connection)**.** *Let $F : M \to N$ be C^∞ map between C^∞ manifolds. Let*

$$\pi : E \to N$$

be C^∞ vector bundle, and on it a connection

$$\nabla : \Omega^0(N, E) \to \Omega^1(N, E)$$

Then

1. *there exists a unique connection on $\tilde{\pi} : F^*E \mapsto M$ called the pullback connection s.t. symbolically*
$$F^*\nabla : \Omega^0(M, F^*E) \to \Omega^1(M, F^*E) \qquad F^*s \mapsto (F^*\nabla)(F^*s) := F^*(\nabla s) \qquad \forall\, s \in \Omega^0(N, E), \quad F^*s \in \Omega^0(M, F^*E) \tag{33}$$

2. *Equivalently using $(F^*\nabla)(F^*s) \in \Omega^1(M, F^*E) = C^\infty(M, T^*M \otimes F^*E)$ so*
$$(F^*\nabla)_X(F^*s) \in C^\infty(M, F^*E)$$
One can write explicitly as in Definition 17.1
$$(F^*\nabla)_X(F^*s) := \nabla_{F_*X}s \qquad \forall\, s \in \Omega^0(N, E) = C^\infty(N, E), \quad \forall\, X \in \mathfrak{X}(M)$$

3. *In particular, pointwise*
$$\forall\, p \in M, \quad \forall\, v \in T_pM, \qquad (F^*\nabla)_v(F^*s) := \left(\nabla_{dF_p(v)}s\right)(F(p)) \in E_{F(p)} = (F^*E)_p \tag{34}$$

Remark 17.5. *We make sense of the definition (33). We've defined pullback as in Definition 17.7*

$$F^* : \Omega^0(N, E) = C^\infty(N, E) \to \Omega^0(M, F^*E) = C^\infty(M, F^*E)$$

We may extend

$$F^* : \Omega^p(N, E) \to \Omega^p(M, F^*E)$$

as $\mathbb{R}$-linear map s.t. for any $\alpha \in \Omega^p(N)$ and $s \in C^\infty(N, E)$

$$F^*(\alpha \otimes s) \mapsto F^*\alpha \otimes F^*s \tag{35}$$

where $F^*\alpha \in \Omega^p(M)$ and $F^*s \in C^\infty(M, F^*E)$. Thus for any $s \in \Omega^0(N, E)$ and $\nabla s \in \Omega^1(N, E)$, (34) can be rewritten as the following

$$F^*(\nabla s) = (F^*\nabla)(F^*s) \in \Omega^1(M, F^*E)$$

using

$$(F^*\alpha)(p)(v) := \alpha(dF_p(v)) \qquad \forall\, p \in M, \;\; v \in T_pM, \;\; \alpha \in \Omega^1(N)$$

Pullback Connection in Local Coordinates. Let $r = \text{rank } E$.

(i) 1. For $\{U_\alpha \mid \alpha \in I\}$ as open cover of N, the local trivializations write

$$h^E_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r \iff e_{\alpha_1}, \cdots, e_{\alpha_r}\ C^\infty \text{ frame of } \ E|_{U_\alpha}$$

On U_α

$$\nabla e_{\alpha_j} = \sum_{k=1}^{r} (\omega^{E,\nabla}_\alpha)^k_j \otimes e_{\alpha_k} \qquad \forall\ (\omega^{E,\nabla}_\alpha)^k_j \in \Omega^1(U_\alpha) \qquad U_\alpha \subset N \text{ open}$$

and $\omega^{E,\nabla}_\alpha \in \Omega^1(U_\alpha, \mathfrak{gl}(r, \mathbb{R}))$ are connection 1-forms associated with ∇ on U_α.

2. On $U_\alpha \cap U_\beta$, recall (30)

$$\omega^{E,\nabla}_\beta = (g^E_{\alpha\beta})^{-1} \omega^{E,\nabla}_\alpha g^E_{\alpha\beta} + (g^E_{\alpha\beta})^{-1} dg^E_{\alpha\beta} \tag{36}$$

for transition functions $g^E_{\alpha\beta}$ on $\pi : E \to N$

$$g^E_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{R})$$

(ii) 1. For $\{F^{-1}(U_\alpha) \mid \alpha \in I\}$ open cover of M, we have $F^* e_{\alpha_1}, \cdots, F^* e_{\alpha_r}$ C^∞ frame of $F^*E|_{F^{-1}(U_\alpha)}$. Using (35)

$$(F^*\nabla)(F^* e_{\alpha_j}) = F^*(\nabla e_{\alpha_j}) = F^*(\sum_{k=1}^{r} (\omega^{E,\nabla}_\alpha)^k_j \otimes e_{\alpha_k}) = \sum_{k=1}^{r} (F^* \omega^{E,\nabla}_\alpha)^k_j \otimes F^* e_{\alpha_k}$$

Now

$$\omega^{F^*E, F^*\nabla}_\alpha := F^* \omega^{E,\nabla}_\alpha \in \Omega^1(F^{-1}(U_\alpha), \mathfrak{gl}(r, \mathbb{R}))$$

2. On $F^{-1}(U_\alpha) \cap F^{-1}(U_\beta)$, F^* acting on (36) yields

$$\omega^{F^*E, F^*\nabla}_\beta = (g^{F^*E}_{\alpha\beta})^{-1} \omega^{F^*E, F^*\nabla}_\alpha g^{F^*E}_{\alpha\beta} + (g^{F^*E}_{\alpha\beta})^{-1} dg^{F^*E}_{\alpha\beta}$$

Hence

$$\{\omega^{F^*E, F^*\nabla}_\alpha\} \subset \Omega^1(F^{-1}(U_\alpha), \mathfrak{gl}(r, \mathbb{R}))$$

defines a connection $F^*\nabla$ on $\tilde{\pi} : F^*E \to M$.

□

17.4 Covariant Derivative

Definition 17.12 (Covariant Derivative). *Let $\pi : E \to M$ be a C^∞ vector bundle over a C^∞ manifold M together with a connection*

$$\nabla : \Omega^0(M, E) \to \Omega^1(M, E) \qquad s.t. \qquad s \mapsto \nabla s$$

or equivalently

$$\nabla : \mathfrak{X}(M) \times C^\infty(M, E) \to C^\infty(M, E) \qquad (X, s) \mapsto \nabla_X s$$

For any C^∞ curve

$$c : I \subset \mathbb{R} \to M \qquad s.t. \qquad t \mapsto c(t)$$

(i) Define the covariant derivative along c as the pullback connection under c evaluated at $\frac{\partial}{\partial t} \in \mathfrak{X}(I)$. Recall (34)

$$\frac{D}{dt} : C^\infty(I, c^*E) = \{C^\infty \textit{sections of } E \textit{ along } c : I \to M\} \to C^\infty(I, c^*E) \qquad s.t. \qquad s \mapsto \frac{Ds}{dt} := (c^*\nabla)_{\frac{\partial}{\partial t}} s$$

(ii) In particular if pick $E = TM$ tangent bundle so that $C^\infty(M, E) = C^\infty(M, TM) = \mathfrak{X}(M)$

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \qquad (X, Y) \mapsto \nabla_X Y$$

is an affine connection as in Definition 17.2, then

$$\frac{D}{dt} : C^\infty(I, c^*TM) \to C^\infty(I, c^*TM) \qquad s.t. \qquad V \mapsto \frac{DV}{dt}$$

(iii) Leibniz rule holds

$$\frac{D}{dt}(fs) = \frac{df}{dt}s + f\frac{Ds}{dt} \qquad \forall\ f \in C^\infty(I), \quad s(t) \in C^\infty(I, c^*E) \tag{37}$$

Covariant Derivative in Local Coordinates. In local coordinates, for (U, ϕ) C^∞ chart with $\phi = (x_1, \cdots, x_n)$. We have

$$\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$$

smooth frame of $TM|_U = TU$ where $n = \dim M$, and

$$e_1, \cdots e_r$$

C^∞ frame of $E|_U$ where $r = \text{rank } E$. Then

$$\nabla e_j = \sum_{k=1}^r \omega_j^k \otimes e_k = \sum_{i=1}^n \sum_{k=1}^r \Gamma_{ij}^k dx_i \otimes e_k$$

$$\nabla_{\frac{\partial}{\partial x_i}} e_j = \sum_{k=1}^r \Gamma_{ij}^k e_k \qquad \text{for} \quad \Gamma_{ij}^k \in C^\infty(U)$$

If $E = TM$ and $r = n$, so $e_j = \frac{\partial}{\partial x_j}$ we have

$$\phi \circ c(t) = (x_1(t), \cdots, x_n(t))$$

and the diagram commutes

$$\begin{tikzcd} I \arrow[r, "c"] & M \\ I' \arrow[u, phantom, sloped, "\subseteq_{open}"] \arrow[r, "c"] \arrow[dr, "\phi\circ c"'] & U \arrow[u, phantom, sloped, "\subseteq_{open}"] \arrow[d, "\phi"] \\ & \mathbb{R}^n \end{tikzcd}$$

The curve velocity writes

$$c'(t) = \sum_{i=1}^n \frac{dx_i}{dt}(t)\frac{\partial}{\partial x_i}(c(t)) \in C^\infty(I', c^*TM)$$

for $s \in C^\infty(I, c^*E)$ we have

$$s(t) = \sum_{j=1}^r s^j(t)e_j(c(t)) = \sum_{j=1}^r s^j(t)(c^*e_j)(t)$$

Now we write, using Leibniz Rule (37)

$$\begin{aligned}\frac{Ds}{dt}(t) &= (c^*\nabla)_{\frac{\partial}{\partial t}} s = (c^*\nabla)_{\frac{\partial}{\partial t}}\left(\sum_{j=1}^r s^j c^* e_j\right)\\ &= \sum_{j=1}^r \frac{ds^j}{dt}(t)e_j(c(t)) + \sum_{j=1}^r s^j (c^*\nabla)_{\frac{\partial}{\partial t}}(c^*e_j)\end{aligned}$$

Here

$$\begin{aligned}(c^*\nabla)_{\frac{\partial}{\partial t}}(c^*e_j) &= \nabla_{(dc_t)(\frac{\partial}{\partial t})} e_j(c(t)) = \nabla_{c'(t)} e_j(c(t))\\ &= \nabla_{\sum_{i=1}^n \frac{dx_i}{dt}\frac{\partial}{\partial x_i}(c(t))} e_j(c(t)) = \sum_{i=1}^n \frac{dx_i}{dt}(t)\left(\nabla_{\frac{\partial}{\partial x_i}(c(t))} e_j(c(t))\right)\\ &= \sum_{i=1}^n \sum_{k=1}^r \frac{dx_i}{dt}(t)\Gamma_{ij}^k(c(t))e_k(c(t))\end{aligned}$$

Notice

$$(dc_t)(\frac{\partial}{\partial t}) = \frac{dc}{dt}(t) = \sum_{i=1}^n \frac{dx_i}{dt}(t)\frac{\partial}{\partial x_i}(c(t)) \in T_{c(t)}M$$

Hence for

$$s = \sum_{j=1}^r s^j(t)e_j(c(t))$$

we have

$$\frac{Ds}{dt}(t) = \sum_{k=1}^{r}\left(\frac{ds^k}{dt}(t) + \sum_{i=1}^{n}\sum_{j=1}^{r}\Gamma_{ij}^k(c(t))\frac{dx_i}{dt}(t)s^j(t)\right)e_k(c(t)) \tag{38}$$

In particular, if we have affine connection ∇, then $V(t) = \sum_{j=1}^{n} V^j(t)\frac{\partial}{\partial x_j}(c(t))$ is a C^∞ vector field along $c : I \to M$, and we have expression

$$\frac{DV}{dt} = \sum_{k=1}^{n}\left(\frac{dV^k}{dt} + \sum_{i,j=1}^{n}(\Gamma_{ij}^k \circ c)\frac{dx_i}{dt}V^j\right)\frac{\partial}{\partial x_k}(c(t)) \tag{39}$$

□

17.5 Parallel Transport

Definition 17.13 (Parallel Section)**.** *Let $V \in C^\infty(I, c^*E)$, i.e. a C^∞ section of E along c. We say V is parallel w.r.t. ∇ if*

$$\frac{DV}{dt} = 0 \qquad \forall\ t \in I$$

Proposition 17.1. *Let $c : I \overset{C^\infty}{\to} M$ be C^∞ curve. Given any $t_0 \in I$ and any $v \in E_{c(t_0)} \cong \mathbb{R}^r$ fiber of E over $c(t_0)$ where $r =$ rank E. Then there exists a unique parallel section V of E along c s.t. $V(t_0) = v$.*

Proof. WLOG assume $c : I \to U \subset M$ open with $\phi = (x_1, \cdots, x_n)$ and $\phi(U) \subset \mathbb{R}^n$ open, i.e., (U, ϕ) is C^∞ chart for M. Let $n = \dim M$. $E|_U$ is trivialized iff there exists $e_1, \cdots, e_r$ C^∞ frame of $E|_U$. We thus have on U

$$\nabla_{\frac{\partial}{\partial x_i}} e_j = \sum_{k=1}^{r}\Gamma_{ij}^k e_k$$

For $(\phi \circ c)(t) = (x_1(t), \cdots, x_n(t))$ and $c'(t) = \sum_{i=1}^{n}\frac{dx_i}{dt}(t)\frac{\partial}{\partial x_i}(c(t))$ and hence

$$V(t) = \sum_{j=1}^{r} V^j(t)e_j(c(t))$$

Using (38), the condition $\frac{DV}{dt} = 0$ holds iff

$$\frac{dV^k}{dt} + \sum_{i=1}^{n}\sum_{j=1}^{r}\left(\Gamma_{ij}^k \circ c\right)\frac{dx_i}{dt}V^j = 0 \qquad k = 1, \cdots, r$$

For $v = \sum_{j=1}^{r} v^j e_j(c(t_0)) \in E_{c(t_0)}$ we have initial conditions $V(t_0) = v$ iff

$$V^k(t_0) = v^k \qquad k = 1, \cdots, r$$

Thus we have 1st order ODE. Directly Apply Existence and Uniqueness theorem. □

Definition 17.14 (Parallel Transport)**.** *Define for any $t \in I$*

$$P_{c,t_0,t} : E_{c(t_0)} \to E_{c(t)} \qquad s.t. \qquad v = V(t_0) \mapsto V(t)$$

where $V \in C^\infty(I, c^*E)$ is the unique C^∞ section of E along c s.t.

$$\frac{DV}{dt} = 0$$

and $V(t_0) = v$. $P_{c,t_0,t}$ is parallel transport along c (defined by (E, v)).

Example 17.1. *In particular, let $E = TM$, ∇ is affine connection on M (which is a connection on TM). Then we define parallel transport along $c : I \to M$ C^∞ curve, for any $t_0, t_1 \in I$,*

$$P_{c,t_0,t_1} : T_{c(t_0)}M \to T_{c(t_1)}M$$

This is a linear isomorphism.

18 Riemannian Connection

Recall Affine Connection as in Definition 17.2.

Definition 18.1 (Symmetric affine connection)**.** *An affine connection ∇ on a smooth manifold M is symmetric if for any X, $Y \in \mathfrak{X}(M)$*

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

In Local Coordinates. Recall as in (25) with $e_j = \frac{\partial}{\partial x_j}$

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} - \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} = [\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$$

$$\sum_k \left(\Gamma_{ij}^k - \Gamma_{ji}^k\right) \frac{\partial}{\partial x_k} = 0$$

Hence $\Gamma_{ij}^k = \Gamma_{ji}^k$. $\square$

Definition 18.2 (Compatible with metric)**.** *An affine connection ∇ on a Riemannian manifold (M, g) is compatible with the Riemannian metric g if for any X, Y, $Z \in \mathfrak{X}(M)$ we have*

$$Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

where $g(X, Y) \in C^\infty(M)$. In fact, compatibility with the metric in equivalent to

$$\nabla_Z g = 0 \qquad \forall\ Z \in \mathfrak{X}(M) \tag{40}$$

Proposition 18.1 (Equivalence with Compatibility with Metric)**.** *Let $\frac{D}{dt}$ be defined along $c : I \to M$ smooth curve by an affine connection ∇ on M which is compatible with a Riemannian metric g on M. For V, W smooth vector fields along $c : I \to M$, i.e., V, $W \in C^\infty(I, c^*TM)$, the metric inner product writes*

$$\langle V, W\rangle(t) = (g(c(t)) \, (V(t), W(t))$$

where $\langle V, W\rangle \in C^\infty(I)$. Then we have

$$\frac{d}{dt}\langle V, W\rangle(t) = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle \tag{41}$$

(i) In fact, ∇ is compatible with g iff (41) holds.

(ii) In particular, ∇ is compatible with g implies whenever V, W are parallel, we have

$$\langle V, W\rangle = constant$$

In fact the converse holds as well.

In the following we note the more general relationship between ∇ and pullback connection.

Proposition 18.2. *Suppose $F : M \overset{\infty}{\to} (N, h)$ from smooth manifold M to Riemannian manifold (N, h). Let*

$$F_* : \mathfrak{X}(M) \to C^\infty(M, F^*TN) \qquad s.t. \qquad X \mapsto (F_*X)(p) := dF_p(X(p)) \in T_{F(p)}N = (F^*TN)_p$$

be pushforward as in (31). Let ∇ be affine connection on N and $D := F^*\nabla$ be the pullback connection on M in F^*TN as in (33).

(i) If ∇ is symmetric , then

$$D_X(F_*Y) - D_Y(F_*X) = F^*\nabla_X(F_*Y) - F^*\nabla_Y(F_*X) = F_*([X, Y]) \qquad \forall\ X, Y \in \mathfrak{X}(M) \tag{42}$$

(ii) If ∇ is compatible with the Riemannian metric h then

$$X\langle V, W\rangle = \langle D_X V, W\rangle + \langle V, D_X W\rangle \qquad \forall\ X \in \mathfrak{X}(M), \quad \forall\ W, V \in C^\infty(M, F^*TN) \tag{43}$$

Theorem 18.1 (Levi-Civita)**.** *Let (M, g) be a Riemannian manifold. Then there exists a unique affine connection ∇ on M which is symmetric and compatible with the metric g. Such connection is called the Levi-Civita Connection.*

Proof of Uniqueness. Take any X, Y, $Z \in \mathfrak{X}(M)$, if we have compatibility with the metric g, then

$$\begin{aligned}X(g(Y,Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\\ Y(g(Z,X)) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X)\\ Z(g(X,Y)) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y)\end{aligned}$$

Now add up first two and subtract the third, using g is symmetric tensor, and then using ∇ is symmetric affine connection

$$\begin{aligned}X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) &= g(\nabla_X Y + \nabla_Y X, Z) + g(Y, \nabla_X Z - \nabla_Z X) + g(X, \nabla_Y Z - \nabla_Z Y)\\ &= 2g(\nabla_Y X, Z) + g(Z, \nabla_X Y - \nabla_Y X) + g(Y, \nabla_X Z - \nabla_Z X) + g(X, \nabla_Y Z - \nabla_Z Y)\\ &= 2g(\nabla_Y X, Z) + g(Z, [X,Y]) + g(Y, [X,Z]) + g(X, [Y,Z])\end{aligned}$$

Then

$$g(\nabla_Y X, Z) = \frac{1}{2}\left(X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) - g(Y,[X,Z]) - g(X,[Y,Z]) - g(Z,[X,Y])\right) \tag{44}$$

This uniquely determines $\nabla_Y X$ for any X, $Y \in \mathfrak{X}(M)$. $\square$

Proof of Existence. We define $\nabla_Y X$ as above and check that ∇ is symmetric and compatible with the Riemannian metric g. $\square$

Local Coordinates. Let $Y = \frac{\partial}{\partial x_i}$, $X = \frac{\partial}{\partial x_j}$ and $Z = \frac{\partial}{\partial x_k}$ as in (44). Then making use of (25) with $e_j = \frac{\partial}{\partial x_j}$ so that

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x_k} \tag{45}$$

Then

$$\begin{aligned}LHS &= g(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}) = g(\sum_{\ell=1}^n \Gamma_{ij}^\ell \frac{\partial}{\partial x_\ell}, \frac{\partial}{\partial x_k}) = \sum_{\ell=1}^n \Gamma_{ij}^\ell g_{\ell k}\\ RHS &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}) + \frac{\partial}{\partial x_i} g(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j}) - \frac{\partial}{\partial x_k} g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) - g(\frac{\partial}{\partial x_i}, 0) - g(\frac{\partial}{\partial x_j}, 0) - g(\frac{\partial}{\partial x_k}, 0)\right)\\ &= \frac{1}{2}\left(g_{ik,j} + g_{kj,i} - g_{ij,k}\right)\end{aligned}$$

where $g_{ij,k} := \frac{\partial g_{ij}}{\partial x_k}$. Hence $LHS = RHS$ gives

$$\Gamma_{ij}^\ell = \frac{1}{2} \sum_{k=1}^n g^{\ell k}\left(g_{ik,j} + g_{kj,i} - g_{ij,k}\right) \tag{46}$$

$\square$

Example 18.1. *Consider* $(\mathbb{R}^n, g = dx_1^2 + \cdots dx_n^2)$ *where* $g_{ij} = \delta_{ij}$. *Then* $g_{ij,k} = 0$ *with*

$$\Gamma_{ij}^\ell = 0 \qquad \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0 \qquad \nabla_{\frac{\partial}{\partial x_j}} = 0$$

Then for $c : I \to \mathbb{R}^n$ *smooth curve with* $c(t) = (x_1(t), \cdots, x_n(t))$

$$V(t) = \sum_{j=1}^n V^j(t) \frac{\partial}{\partial x_j}(c(t))$$

C^∞ *vector field. Then plugging in* (38) *we see*

$$\frac{DV}{dt}(t) = \sum_{j=1}^n \frac{dV^j}{dt}(t) \frac{\partial}{\partial x_j}(c(t))$$

and $\frac{DV}{dt} = 0$ *iff* $\frac{dV^j}{dt}(t) = 0$.

Example 18.2. *Consider* $(\mathbb{S}^2, g_{can} = d\phi^2 + \sin^2(\phi)d\theta^2)$*. For spherical coordinates* $\theta \in (0, 2\pi)$ *and* $\phi \in (0, \pi)$*.*

$$(x, y, z) = (\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$$

And $(x_1, x_2) = (\phi, \theta)$*. We have*

$$\begin{aligned}
g_{11} &= 1 \\
g_{12} &= g_{21} = 1 \\
g_{22} &= \sin^2(\phi) \\
g^{11} &= 1 \\
g^{12} &= g^{21} = 0 \\
g^{22} &= \frac{1}{\sin^2(\phi)}
\end{aligned}$$

Thus $g_{ij} = 0$ *for any* $i \neq j$ *and* $g^{kk} = \frac{1}{g_{kk}}$*. Using* (45) *we derive relations*

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial\phi}} \frac{\partial}{\partial\phi} &= \Gamma^1_{11}\frac{\partial}{\partial\phi} + \Gamma^2_{11}\frac{\partial}{\partial\theta} \\
\nabla_{\frac{\partial}{\partial\phi}} \frac{\partial}{\partial\theta} &= \nabla_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\phi} = \Gamma^1_{12}\frac{\partial}{\partial\phi} + \Gamma^2_{12}\frac{\partial}{\partial\theta} \\
\nabla_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\theta} &= \Gamma^1_{22}\frac{\partial}{\partial\phi} + \Gamma^2_{22}\frac{\partial}{\partial\theta}
\end{aligned}$$

Since $g_{22,1} = 2\sin(\phi)\cos(\phi)$ *and* $g_{ij,k} = 0$ *otherwise, So using* (46) *we compute*

$$\begin{aligned}
\Gamma^1_{11} &= \Gamma^2_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = 0 \\
\Gamma^2_{12} &= \frac{1}{2}\sum_{k=1}^{2}\left(g^{2k}(g_{1k,2} + g_{k2,1} - g_{12,k})\right) = \frac{1}{2g_{22}}\frac{\partial}{\partial\phi}g_{22} \\
&= \frac{1}{2}\frac{\partial}{\partial\phi}\log(\sin^2(\phi)) = \frac{\cos(\phi)}{\sin(\phi)} = \cot(\phi) = \Gamma^2_{21} \\
\Gamma^1_{22} &= \frac{1}{2}g^{11}(0 + 0 - g_{22,1}) = -\frac{1}{2}\frac{\partial}{\partial\phi}(\sin^2(\phi)) = -\sin(\phi)\cos(\phi)
\end{aligned}$$

Thus

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial\phi}} \frac{\partial}{\partial\phi} &= \Gamma^1_{11}\frac{\partial}{\partial\phi} + \Gamma^2_{11}\frac{\partial}{\partial\theta} = 0 \\
\nabla_{\frac{\partial}{\partial\phi}} \frac{\partial}{\partial\theta} &= \nabla_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\phi} = \Gamma^1_{12}\frac{\partial}{\partial\phi} + \Gamma^2_{12}\frac{\partial}{\partial\theta} = \cot(\phi)\frac{\partial}{\partial\theta} \\
\nabla_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\theta} &= \Gamma^1_{22}\frac{\partial}{\partial\phi} + \Gamma^2_{22}\frac{\partial}{\partial\theta} = -\sin(\phi)\cos(\phi)\frac{\partial}{\partial\phi}
\end{aligned}$$

Hence for (26) *with* $e_j = \frac{\partial}{\partial x_j}$

$$\nabla\frac{\partial}{\partial x_j} = \sum_{k=1}^{2}\omega^k_j \otimes \frac{\partial}{\partial x_k}$$

we have

$$\begin{aligned}
\nabla\frac{\partial}{\partial\phi} &= d\phi \otimes \nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\phi} + d\theta \otimes \nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\phi} = (\cot(\phi)d\theta) \otimes \frac{\partial}{\partial\theta} \\
\nabla\frac{\partial}{\partial\theta} &= d\phi \otimes \nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\theta} + d\theta \otimes \nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\theta} = (\cot(\phi)d\phi) \otimes \frac{\partial}{\partial\theta} - \sin(\theta)\cos(\theta)d\theta \otimes \frac{\partial}{\partial\phi}
\end{aligned}$$

Hence $\omega^1_1 = 0$*,* $\omega^2_1 = \cot(\phi)d\theta$*,* $\omega^1_2 = -\sin(\phi)\cos(\phi)d\theta$ *and* $\omega^2_2 = \cot(\phi)d\phi$*. The connection* 1*-form writes*

$$\begin{pmatrix} \omega^1_1 & \omega^1_2 \\ \omega^2_1 & \omega^2_2 \end{pmatrix} = \begin{pmatrix} 0 & -\sin(\phi)\cos(\phi)d\theta \\ \cot(\phi)d\theta & \cot(\phi)d\phi \end{pmatrix} \in \Omega^1(U, \mathfrak{gl}(2, \mathbb{R}))$$

Alternatively, we can choose a different frame. Using Leibniz rule (22)

$$\begin{aligned}
\nabla_1 &= \nabla_{\frac{\partial}{\partial x_1}} = \nabla_{\frac{\partial}{\partial \phi}} \\
\nabla_2 &= \nabla_{\frac{\partial}{\partial x_2}} = \nabla_{\frac{\partial}{\partial \theta}} \\
e_1 &:= \frac{\partial}{\partial \phi} \\
e_2 &:= \frac{1}{\sin(\phi)}\frac{\partial}{\partial \theta} \\
\nabla_1 e_1 &= \nabla_{\frac{\partial}{\partial \phi}} \frac{\partial}{\partial \phi} = 0 \\
\nabla_1 e_2 &= \nabla_{\frac{\partial}{\partial \phi}} \left(\frac{1}{\sin(\phi)}\frac{\partial}{\partial \theta}\right) = -\frac{\cos(\phi)}{\sin^2(\phi)}\frac{\partial}{\partial \theta} + \frac{1}{\sin(\phi)}\nabla_{\frac{\partial}{\partial \phi}}\frac{\partial}{\partial \theta} = 0 \\
\nabla_2 e_1 &= \nabla_{\frac{\partial}{\partial \phi}} \frac{\partial}{\partial \phi} = \cot(\phi)\frac{\partial}{\partial \theta} = \cos(\phi) e_2 \\
\nabla_2 e_2 &= \nabla_{\frac{\partial}{\partial \theta}} \left(\frac{1}{\sin(\phi)}\frac{\partial}{\partial \theta}\right) = \frac{1}{\sin(\phi)}\nabla_{\frac{\partial}{\partial \theta}}\frac{\partial}{\partial \theta} = \frac{1}{\sin(\phi)}(-\sin(\phi)\cos(\phi)\frac{\partial}{\partial \phi}) = -\cos(\phi) e_1
\end{aligned}$$

Hence for $\nabla e_j = \sum_{k=1}^{2} \tilde{\omega}_j^k \otimes e_k$, since

$$\begin{aligned}
\nabla e_1 &= d\phi \otimes \nabla_{\frac{\partial}{\partial \phi}} e_1 + d\theta \otimes \nabla_{\frac{\partial}{\partial \theta}} e_1 = d\theta \otimes \nabla_2 e_1 = \cos(\phi) d\theta \otimes e_2 \\
\nabla e_2 &= d\phi \otimes \nabla_1 e_2 + d\theta \otimes \nabla_2 e_2 = -\cos(\phi) d\theta \otimes e_1
\end{aligned}$$

hence

$$[\nabla e_1, \nabla e_2] = [e_1, e_2] \begin{pmatrix} 0 & -\cos(\phi) \\ \cos(\phi) & 0 \end{pmatrix} d\theta$$

and so our $\tilde{\omega}$ writes

$$\begin{pmatrix} \tilde{\omega}_1^1 & \tilde{\omega}_2^1 \\ \tilde{\omega}_1^2 & \tilde{\omega}_2^2 \end{pmatrix} = \begin{pmatrix} 0 & -\cos(\phi) d\theta \\ \cos(\phi) d\theta & 0 \end{pmatrix} \in \Omega^1(U, \mathfrak{so}(2))$$

Remark 18.1. *In general if $e_1, \cdots, e_n$ are local **orthonormal frame** of $TM|_U = TU$, and ∇ is an affine connection compatible with the Riemannian metric, then*

$$\begin{aligned}
d\langle e_i, e_j\rangle &= \langle \nabla e_i, e_j\rangle + \langle e_i, \nabla e_j\rangle \\
\nabla e_j &= \sum_{k=1}^{n} \omega_j^k \otimes e_k \\
\omega_j^k &= -\omega_k^j \implies \omega \in \Omega^1(U, \mathfrak{so}(n))
\end{aligned}$$

Lemma 18.1. *Let $F : (M, g) \to (N, h)$ be an isometric immersion. For any $p \in M$, let π_p be the orthogonal projection from $T_{F(p)}N$ to the image of*

$$dF_p : T_pM \to T_{F(p)}N$$

Let X, $Y \in \mathfrak{X}(M)$ F-related to $\tilde{X}$, $\tilde{Y} \in \mathfrak{X}(N)$, and let ∇, $\tilde{\nabla}$ be Levi-Civita connections respectively on (M, g) and (N, h). Then for any $p \in M$

$$dF_p((\nabla_X Y)(p)) = \pi_p((\tilde{\nabla}_{\tilde{X}} \tilde{Y})(F(p)))$$

19 Geodesic

Definition 19.1. *Let (M, g) be a Riemannian manifold. Let $\gamma : I \subset \mathbb{R} \to M$ be C^∞ curve. We say γ is geodesic at $t_0 \in I$ if*

$$\frac{D}{dt}\frac{d\gamma}{dt}(t_0) = 0 \in T_{\gamma(t_0)}M$$

where $\frac{D}{dt}$ is the covariant derivative defined by the Levi-civita connection on (M, g). We say γ is geodesic if

$$\frac{D}{dt}(\frac{d\gamma}{dt}) \equiv 0$$

Lemma 19.1. *If $\gamma : I \to M$ is a geodesic in a Riemannian manifold (M, g) then*

$$|\gamma'| := |\frac{d\gamma}{dt}| = \sqrt{g(t)(\frac{d\gamma}{dt}(t), \frac{d\gamma}{dt}(t))} = constant$$

Proof. Using $\frac{D}{dt}$ defined by Levi-civita connection, which is compatible with the metric, (41)

$$\frac{d}{dt}\langle\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle = \langle\frac{D}{dt}\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle + \langle\frac{d\gamma}{dt}, \frac{D}{dt}\frac{d\gamma}{dt}\rangle = 0$$

□

Local Coordinates. Let (U, ϕ) for $\phi = (x_1, \cdots, x_n)$ be C^∞ chart on M where $n = \dim M$. On U we have

$$\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} = \sum_k \Gamma^k_{ij}\frac{\partial}{\partial x_k}$$

where

$$\Gamma^\ell_{ij} = \frac{1}{2}\sum_k g^{\ell k}\left(g_{ik,j} + g_{kj,i} - g_{ij,k}\right)$$

WLOG assume

$$\gamma : I \to U \overset{\phi}{\to} \mathbb{R}^n$$

then

$$\begin{aligned}
\phi \circ \gamma(t) &= (x_1(t), \cdots, x_n(t)) \\
\gamma'(t) &= \sum_k \frac{dx_k}{dt}(t)\frac{\partial}{\partial x_k}(\gamma(t)) \\
V(t) &= \sum_{k=1}^n V^k(t)\frac{\partial}{\partial x_k}(t) \\
\frac{DV}{dt}(t) &= \sum_{k=1}^n \left(\frac{dV^k}{dt}(t) + \sum_{i,j=1}^n \Gamma^k_{ij}(\gamma(t))\frac{dx_i}{dt}(t)V^j(t)\right)\frac{\partial}{\partial x_k}(\gamma(t))
\end{aligned}$$

Now take the curve velocity $V(t) = \gamma'(t) \equiv \frac{d\gamma}{dt}$ to be the C^∞ vector field along γ. By matching coefficients we have $V^k(t) = \frac{dx_k}{dt}(t)$. so

$$\frac{D}{dt}\frac{d\gamma}{dt} = 0 \iff \frac{d^2x_k}{dt^2} + \sum_{i,j=1}^n \Gamma^k_{ij} \circ \gamma\frac{dx_i}{dt}\frac{dx_j}{dt} = 0 \qquad \forall\ k = 1\cdots n \tag{47}$$

This is a system of 2nd order ODEs in $x_1(t), \cdots, x_n(t)$. Denote

$$y_i(t) := \frac{dx_i}{dt}(t)$$

Then they satisfy

$$\begin{cases} \frac{dx_k}{dt} = y_k \\ \frac{dy_k}{dt} = -\sum_{i,j=1}^n \Gamma^k_{ij} \circ \gamma y_i y_j \end{cases}$$

This is a system of 1st order ODE in $x_1(t), \cdots, x_n(t)$ and $y_1(t), \cdots, y_n(t)$. Hence there exists unique solution if given initial data $a_i, b_i \in \mathbb{R}$

$$\begin{aligned}
x_i(t_0) &= a_i \\
y_i(t_0) &= b_i = \frac{dx_i}{dt}(t_0)
\end{aligned}$$

or in other words

$$\gamma(t_0) = \phi^{-1}(a_1, \cdots, a_n) =: p$$
$$\gamma'(t_0) = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}(p)$$

$\square$

Theorem 19.1 (Existence and Uniqueness Theory for Geodesic)**.** *Let (M, g) be a Riemannian manifold. Given any $p \in M$ and $v \in T_pM$*

- *There exists a geodesic $\gamma : I \to M$ s.t. $0 \in I$, $\gamma(0) = p$ and $\gamma'(0) = v$.*
- *If $\beta : I' \to M$ is a geodesic s.t. $\beta(0) = p$, $\beta'(0) = v$ then we must have*
$$I' \subset I \qquad \beta = \gamma|_{I'}$$

Example 19.1. *Let $(\mathbb{R}^n, g_0 = dx_1^2 + \cdots dx_n^2)$ then*

$$g_{ij} = \delta_{ij} \qquad \Gamma_{ij}^k = 0$$

Hence using (47)

$$\frac{D}{dt}\gamma'(t) = 0 \iff \frac{d^2 x_k}{dt^2} = 0$$

so for

$$\gamma : I \to \mathbb{R}^n \qquad s.t. \qquad t \mapsto (x_1(t), \cdots, x_n(t))$$

Given any $a \in \mathbb{R}^n$ and $b \in T_a\mathbb{R}^n \cong \mathbb{R}^n$ the unique geodesic $\gamma(t)$ with $\gamma(0) = a$ and $\gamma'(0) = b$ writes

$$\gamma(t) = a + bt \qquad t \in \mathbb{R}$$

Example 19.2. *Let $(\mathbb{S}^n, g_{can})$. Given $p \in \mathbb{S}^n$ and $v \in T_p\mathbb{S}^n$. Recall*

$$(p, v) \in T\mathbb{S}^n \subset T\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$$

for $|p| = 1$ and $\langle p, v\rangle = 0$. The unique geodesic $\gamma(t)$ in $(\mathbb{S}^n, g_{can})$ is given by

$$\gamma(t) = \begin{cases} p & \text{if } v = 0 \\ \cos(|v|t)p + \sin(|v|t)\frac{v}{|v|} & \text{if } v \neq 0 \end{cases}$$

19.1 Geodesic Field and Geodesic Flow

For $\gamma : I \to M$ smooth curve in M and V a C^∞ vector field along γ, the tuple

$$\tilde{\gamma}(t) = (\gamma(t), V(t))$$

defines a smooth curve in TM s.t. the diagram commutes

$$\begin{tikzcd} I \arrow[d, "\tilde{\gamma}"'] \arrow[dr, "\gamma"] & \\ TM \arrow[r, "\pi"] & M \end{tikzcd}$$

In particular we prescribe initial data $\gamma(0) = p$ and $\gamma'(0) = v$ for $(p, v) \in TM$. Notice γ is a geodesic in (M, g), i.e., $\frac{D}{dt}\frac{d}{dt}\gamma = 0$ iff $\gamma(t)$ and $V(t)$ satisfy

$$\gamma'(t) = V(t)$$
$$\frac{DV}{dt}(t) = 0$$
$$\tilde{\gamma}(0) = (p, v)$$

Here we send γ to (γ, γ') and $\tilde{\gamma}$ to $\pi \circ \tilde{\gamma}$. Now for any $(p, v) \in TM$, define $G(p, v) \in T_{(p,v)}(TM)$ as follows.

Definition 19.2 (Geodesic Field)**.** *Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be the unique geodesic in (M, g) s.t. $\gamma(0) = p$, $\gamma'(0) = v$. Let*

$$\tilde{\gamma} : (-\varepsilon, \varepsilon) \to TM \qquad s.t. \qquad \tilde{\gamma}(t) = (\gamma(t), \gamma'(t))$$

Define

$$G(p, v) := \tilde{\gamma}'(0) \in T_{\tilde{\gamma}(0)}(TM) = T_{(p,v)}(TM)$$

Claim that $G \in \mathfrak{X}(TM)$.

Local Coordinates. For (U, ϕ) where $\phi = (x_1, \cdots, x_n)$ is C^∞ chart for M. We have $(\pi^{-1}(U), \tilde{\phi})$

$$\tilde{\phi} : \pi^{-1}(U) \to \phi(U) \times \mathbb{R}^n \subset \mathbb{R}^{2n} \qquad s.t. \qquad \tilde{\phi} = (x_1, \cdots, x_n,\, y_1, \cdots, y_n)$$

Now for any $(p, v) \in \pi^{-1}(U)$, $p \in U$ and $v = \sum_{i=1}^n y_i \frac{\partial}{\partial x_i}(p) \in T_pM$,

$$\tilde{\phi}(p, v) = (\phi(p), y_1, \cdots, y_n)$$

note

$$\phi \circ \gamma(t) = (x_1(t), \cdots, x_n(t))$$

implies

$$\tilde{\phi} \circ \tilde{\gamma}(t) = (x_1(t), \cdots, x_n(t), y_1(t), \cdots, y_n(t))$$

Hence writing into equations

$$\begin{aligned} G(\tilde{\gamma}(t)) := \frac{d\tilde{\gamma}}{dt}(t) &= \sum_{i=1}^n \frac{dx_i}{dt}(t) \frac{\partial}{\partial x_i}(\tilde{\gamma}(t)) + \sum_{k=1}^n \frac{dy_k}{dt}(t) \frac{\partial}{\partial y_k}(\tilde{\gamma}(t)) \\ &= \sum_{i=1}^n \frac{dx_i}{dt}(t) \frac{\partial}{\partial x_i}(\tilde{\gamma}(t)) - \sum_{i,j,k=1}^n (\Gamma_{ij}^k \circ \gamma)(t) y_i(t) y_j(t) \frac{\partial}{\partial y_k}(\tilde{\gamma}(t)) \end{aligned}$$

On $\pi^{-1}(U)$ we have

$$\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_n}$$

as C^∞ frame of $T(TM)|_{\pi^{-1}(U)}$. Hence

$$G = \sum_{k=1}^n y_k \frac{\partial}{\partial x_k} - \sum_{i,j,k=1}^n (\Gamma_{ij}^k \circ \phi^{-1}(x_1, \cdots, x_n)) y_i y_j \frac{\partial}{\partial y_k} \tag{48}$$

G is a C^∞ vector field on TM known as the geodesic field. The flow of G is called the geodesic flow. For any $(p, v) \in TM$, using Theorem 8.1, there exists $\delta > 0$ and an open neighborhood U of (p, v) in TM s.t. geodesic flow ϕ exists

$$\phi : (-\delta, \delta) \times U \overset{C^\infty}{\to} TM \qquad s.t. \qquad (t, q, w) \mapsto \phi(t, q, w)$$

for any $t \in (-\delta, \delta)$, $q \in M$ and $w \in T_pM$. (From here on we abuse of notation to denote ϕ as flow instead of coordinates) Then they solve

$$\begin{cases} \frac{\partial \phi}{\partial t}(t, q, w) = G(\phi(t, q, w)) \\ \phi(0, q, w) = (q, w) \end{cases}$$

Using the geodesic flow, one may construct geodesics in M using any initial data in the neighborhood $\mathcal{U}$ of (p, v)

$$\gamma := \pi \circ \phi : (-\delta, \delta) \times \mathcal{U} \to M \qquad (t, q, w) \mapsto \gamma(t, q, w)$$

For fixed $(q, w) \in \mathcal{U} \subset TM$ s.t. $q \in M$ and $w \in T_qM$, we have

$$\gamma_{q,w} : (-\delta, \delta) \to M \qquad s.t. \qquad t \mapsto \gamma(t, q, w) =: \gamma_{q,w}(t)$$

as a geodesic with $\gamma_{q,w}(0) = q$ and $\gamma'_{q,w}(0) = w$. □

Example 19.3. *For* $(\mathbb{R}^n, g = dx_1^2 + \cdots dx_n^2)$*, we know* $\Gamma_{ij}^k = 0$*. One identify* $T\mathbb{R}^n \cong \mathbb{R}^{2n}$ *so geodesic field writes*

$$G : T\mathbb{R}^n = \mathbb{R}^{2n} \to T(T\mathbb{R}^n) \qquad s.t. \qquad (x, y) \mapsto \sum_{k=1}^n y_k \frac{\partial}{\partial x_k}$$

and solving ODEs give the geodesic flow

$$\phi : \mathbb{R} \times T\mathbb{R}^n \to T\mathbb{R}^n \qquad s.t. \qquad (t, x, y) \mapsto (x + ty, y)$$

along with nearby geodesics in $\mathbb{R}^n$

$$\gamma : \mathbb{R} \times T\mathbb{R}^n \to \mathbb{R}^n \qquad s.t. \qquad (t, x, y) \mapsto x + ty$$

Example 19.4. *For* $(\mathbb{S}^n, g_{can})$ *we have geodesics in* $\mathbb{S}^n$

$$\gamma : \mathbb{R} \times T\mathbb{S}^n \to \mathbb{S}^n \qquad s.t. \qquad \gamma(t,x,y) = \begin{cases} x & \text{if } y = 0 \\ \cos(|y|t)x + \sin(|y|t)\frac{y}{|y|} & \text{if } y \neq 0 \end{cases}$$

For geodesic flows, we either have

$$\phi : \mathbb{R} \times T\mathbb{S}^n \to T\mathbb{S}^n \qquad s.t. \qquad (t,x,y) \mapsto (x,0)$$

or

$$\phi(t,x,y) = (\cos(|y|t)x + \sin(|y|t)\frac{y}{|y|}, -\sin(|y|t)|y|x + \cos(|y|t)y)$$

making use of

$$\phi(t,q,w) = (\gamma(t,q,w), \frac{\partial\gamma}{\partial t}(t,q,w))$$

so $|\frac{\partial\gamma}{\partial t}(t,q,w)| = |w|$*. Geodesic Flow preserves the sphere bundle, for*

$$S_{|v|}(TM) = \{(p,v) \in TM \mid |v| = r\}$$

with $r > 0$*. The geodesic field* $G(p,v)$ *is tangent to* $S_{|v|}(TM)$*.*

Proposition 19.1. *If* (M,g) *is compact Riemannin manifold. Then the geodesic flow is defined on* $\mathbb{R} \times TM$*.*

$$\phi : \mathbb{R} \times TM \to TM$$
$$\gamma : \mathbb{R} \times TM \to M$$

19.2 Exponential Map

Now we study homogeneity of geodesics. Let $\phi : (-\delta,\delta) \times \mathcal{U} \to TM$ be geodesic flow with $\mathcal{U} \subset TM$. Let $\gamma : (-\delta,\delta) \times \mathcal{U} \to M$ s.t. $\gamma := \pi \circ \phi$ and so

$$\phi(t,p,v) = (\gamma(t,p,v), \frac{\partial}{\partial t}\gamma(t,p,v)) \qquad \forall\ (t,p,v) \in (-\delta,\delta) \times \mathcal{U}$$

Lemma 19.2 (Homogeneity of geodesics)**.** *For* $\gamma(t,p,v)$ *flow defined for* $t \in (-\delta,\delta)$ *as above, then for any* $a > 0$*, the flow* $\gamma(t,p,av)$ *is defined for* $t \in (-\frac{\delta}{a}, \frac{\delta}{a})$ *and*

$$\gamma(t,p,av) = \gamma(at,p,v)$$

Proof. Fix $(p,v) \in \mathcal{U}$ and consider $\gamma = \gamma_{p,v} : (-\delta,\delta) \to M$ as geodesic on M. For another curve β, observe

$$\beta : (-\frac{\delta}{a}, \frac{\delta}{a}) \to M \qquad s.t. \qquad \beta(t) = \gamma(at) \qquad \beta'(t) = a\gamma'(at)$$

also satisfies the geodesic equation $\frac{D\beta'}{dt} = 0$ but with $\beta(0) = p$ and $\beta'(0) = av$. By uniqueness Theorem 8.1

$$\gamma(t,p,av) = \beta(t) = \gamma(at) = \gamma(at,p,v)$$

□

Now consider $(p,0) \in TM$. For any $p \in M$, there exists open neighborhood $\mathcal{U} \subset TM$ of $(p,0)$, and there exists $\delta > 0$ s.t.

$$\gamma : (-\delta,\delta) \times \mathcal{U} \to M \qquad s.t. \qquad t \mapsto \gamma(t,q,v)$$

is the unique trajectory of geodesic field $G \in \mathfrak{X}(TM)$ which satisfies initial conditions

$$\gamma(0,q,v) = (q,v) \qquad \forall\ (q,v) \in \mathcal{U}$$

In particular, it is possible to choose $\mathcal{U}$ with parameter $\varepsilon > 0$ controlling the size of tangent vectors. There exists V open neighborhood of p in M, $\varepsilon > 0$ and

$$\mathcal{U}_{V,\varepsilon} := \{(q,w) \mid q \in V,\ w \in T_qM,\ |w| < \varepsilon\}$$

this is to say $\gamma(t,q,w)$ is defined for $t \in (-\delta,\delta)$, $q \in V$, $|w| < \varepsilon$. But then by homogeneity 19.2, choose $a = \frac{\delta}{2}$ $\gamma(t,q,w)$ is defined for $t \in (-2,2)$, $q \in V$, $|w| < \frac{\varepsilon\delta}{2}$.

Lemma 19.3 (Interval of Existence for geodesic uniformly large in Neighborhood of p)**.** *For any $p \in M$, there exists open neighborhood V of p and there exists $\varepsilon > 0$ s.t. $\gamma(t, q, w)$ is defined for $t \in (-2, 2)$, $q \in V$, $w \in T_qM$ and $|w| < \varepsilon$, i.e., on*

$$\gamma : (-2, 2) \times \mathcal{U}_{V,\varepsilon} \subset \mathbb{R} \times TM \to M \qquad s.t. \qquad (t, q, w) \mapsto \gamma(t, q, w)$$

as the unique geodesic with $\gamma(0, q, w) = q$, $\frac{\partial}{\partial t}\gamma(0, q, w) = w$ for any $q \in V$ and $|w| < \varepsilon$.

Definition 19.3 (Exponential Map)**.** *For any $p \in M$, there exists $\mathcal{U}_{V,\varepsilon}$ as in Lemma 19.3. Define*

$$\exp : \mathcal{U}_{V,\varepsilon} \subset TM \to M \qquad s.t. \qquad \exp(q, w) = \gamma(1, q, w) = \gamma(|w|, q, \frac{w}{|w|}) \qquad \forall\ q \in V, \quad |w| < \varepsilon$$

on $\mathcal{U}_{v,\varepsilon} \subset TM$ open. Also define its restriction to the tangent space T_qM for any $q \in V$

$$\exp_q : B_\varepsilon(0) \subset T_qM \to M \qquad s.t. \qquad \exp_q(v) := \exp(q, v) \qquad \forall\ q \in V, \quad |v| < \varepsilon$$

Remark 19.1. *Why is this called an exponential map? If given G Lie group and g bi-invariant Riemannian metric.*

$$\exp = \exp_e : T_eG = \mathfrak{g} \to G$$

is defined for the whole Lie algebra and coincides with the previous definition 15.7.

Proposition 19.2 (Exponential Map as Diffeomorphism)**.** *For any $p \in M$, there exists $\varepsilon > 0$ s.t.*

$$\exp_p : B_\varepsilon(0) \subset T_pM \to M \qquad \exp_p(v) := \exp(p, v) \qquad \forall\ |v| < \varepsilon$$

is a diffeomorphism of $B_\varepsilon(0)$ onto an open subset of M.

Proof. By Inverse Function Theorem, it suffices to prove that

$$(d\exp_p)_0 : T_0(T_pM) \cong T_pM \to T_pM$$

is the identity.

$$(d\exp_p)_0(v) = \frac{d}{dt}\bigg|_{t=0} \exp_p(tv) = \frac{\partial}{\partial t}\gamma(t, p, v)\bigg|_{t=0} = v$$

Hence $\exp_p : B_\varepsilon(0) \to M$ is a local diffeomorphism at the origin $0 \in B_\varepsilon(0)$, i.e., there exists $\varepsilon > 0$ s.t.

$$\exp_p : B_\varepsilon(0) \subset T_pM \to \exp_p(B_\varepsilon(0)) \subset M$$

is a diffeomorphism.

$$B_\varepsilon(p) := \exp_p(B_\varepsilon(0))$$

is the geodesic ball of radius $\varepsilon > 0$ centered at p. □

Example 19.5. *For $M = \mathbb{R}^n$,*

$$\exp_p : T_p\mathbb{R}^n \to \mathbb{R}^n \qquad s.t. \qquad v \mapsto p + v$$

Example 19.6. *For $M = \mathbb{S}^n$*

$$\exp_p(v) = \begin{cases} p & v = 0 \\ \cos(|v|)p + \sin(|v|)\frac{v}{|v|} & v \neq 0 \end{cases}$$

This is diffeomorphism of $B_\pi(0)$ onto $\mathbb{S}^n \setminus \{-p\}$.

Lemma 19.4 (Geodesic Frame)**.** *Let (M, g) be Riemannian manifold of dimension n and let $p \in M$. There exists an open neighborhood $U \subset M$ of p and n vector fields $E_1, \cdots, E_n \in \mathfrak{X}(U)$ s.t.*

(i) For any $q \in U$, $\{E_1(q), \cdots, E_n(q)\}$ is an ONB of T_qM.

(ii) $(\nabla_{E_i}E_j)(p) = 0$.

Proof. Choose a normal neighborhood U of p, i.e., there exists a neighborhood $0 \in V \subset T_pM$ s.t. $\exp_p : V \to U$ is a diffeomorphism. Consider an orthonormal frame $\{E_1(p), \cdots, E_n(p)\}$ of T_pM. For any $q \in U$, there is a unique geodesic γ in U s.t. $\gamma(0) = p$ and $\gamma(1) = q$. Define

$$\{E_1(q), \cdots, E_n(q)\} \subset T_qM$$

to be the parallel transport of $\{E_1(p), \cdots, E_n(p)\}$ along γ to q. Since parallel transport is linear isometry, $\{E_1(q), \cdots, E_n(q)\} \subset T_qM$ remain orthonormal frame. Suppose γ is geodesic with $\gamma(0) = p$ and $\gamma'(0) = E_i(p)$. Since E_j is parallel vector field along γ, we have

$$\nabla_{\gamma'(0)}E_j = \nabla_{E_i}E_j(p) = 0$$

□

19.3 Minimizing Properties of Geodesics

Some notations.

- Let $s : A \subset \mathbb{R}^2 \overset{C^\infty}{\to} M$ be a parametrized surface in a smooth manifold M. Let (u, v) be global coordinates on $\mathbb{R}^2$, then
$$\frac{\partial}{\partial u}\ \frac{\partial}{\partial v}\ \in \mathfrak{X}(A) \qquad \frac{\partial s}{\partial u}(u,v)\ \frac{\partial s}{\partial v}(u,v)\ \in T_{s(u,v)}M \equiv (s^*TM)_{(u,v)}$$
- We used $s_*\frac{\partial}{\partial u}$ and $s_*\frac{\partial}{\partial v}$ in place of Do Carmo's notation $\frac{\partial s}{\partial u}$ and $\frac{\partial s}{\partial v} \in C^\infty(A, s^*TM)$, i.e., $\frac{\partial s}{\partial u}$ and $\frac{\partial s}{\partial v}$ are vector fields along the parametrized surface $s : A \to M$.
- If ∇ is an affine connection on M, then let $D = s^*\nabla$, we denote
$$\frac{D}{du} := D_{\frac{\partial}{\partial u}}, \quad \frac{D}{dv} := D_{\frac{\partial}{\partial v}} : C^\infty(A, s^*TM) \to C^\infty(A, s^*TM)$$

Lemma 19.5 (Symmetry)**.** *If ∇ is a symmetric affine connection on M, then*
$$\frac{D}{dv}\frac{\partial s}{\partial u} = \frac{D}{du}\frac{\partial s}{\partial v} \tag{49}$$

Proof. Using (42)
$$\begin{aligned}\frac{D}{dv}\frac{\partial s}{\partial u} - \frac{D}{du}\frac{\partial s}{\partial v} &= D_{\frac{\partial}{\partial v}} s_*\frac{\partial}{\partial u} - D_{\frac{\partial}{\partial u}} s_*\frac{\partial}{\partial v} \\ &= s^*\nabla_{\frac{\partial}{\partial v}} s_*\frac{\partial}{\partial u} - s^*\nabla_{\frac{\partial}{\partial u}} s_*\frac{\partial}{\partial v} \\ &= s_*\left([\frac{\partial}{\partial v}, \frac{\partial}{\partial u}]\right) = 0\end{aligned}$$
□

Lemma 19.6 (Gauss Lemma)**.** *Let (M, g) be a Riemannian Manifold. $p \in M$ and $v \in T_pM$ such that $\exp_p(v)$ is defined (i.e., defined on line segment connecting 0 and v as in Definition 33). For any $w \in T_pM = T_v(T_pM)$*
$$\langle (d\exp_p)_v(v), (d\exp_p)_v(w)\rangle = \langle v, w\rangle \qquad \forall\ v,\ w \in T_pM \tag{50}$$
notice $(d\exp_p)_v(v), (d\exp_p)_v(w) \in T_{\exp_p(v)}M$.

Proof. Define
$$f : (-\varepsilon, \varepsilon) \times (-\delta, 1+\delta) \to M \qquad s.t. \qquad f(s,t) := \exp_p(t(v+sw))$$
for δ, $\varepsilon > 0$ sufficiently small. For any $s \in (-\varepsilon, \varepsilon)$ define f_s
$$f_s : (-\delta, 1+\delta) \to M \qquad s.t. \qquad f_s(t) := f(s,t) = \exp_p(t(v+sw))$$
Here f_s is geodesic with initial position $f_s(0) = p$ and initila velocity $f_s'(0) = v + sw$. Now using f_s is geodesic
$$\frac{D}{dt}\frac{\partial f}{\partial t}(s,t) = \frac{D}{dt}f_s'(t) = 0$$
Also
$$\begin{aligned}\left\|\frac{\partial f}{\partial t}(s,t)\right\|^2 = \langle\frac{\partial f}{\partial t}(s,t), \frac{\partial f}{\partial t}(s,t)\rangle &= \langle f_s'(t), f_s'(t)\rangle = \langle f_s'(0), f_s'(0)\rangle \\ &= \langle v+sw, v+sw\rangle \\ &= \langle v,v\rangle + 2s\langle v,w\rangle + s^2\langle w,w\rangle\end{aligned}$$

Now we differentiate
$$\begin{aligned}f(t,s) &= \exp_p(t(v+sw)) \\ \frac{\partial f}{\partial t}(t,s) &= (d\exp_p)_{t(v+sw)}(v+sw) \\ \frac{\partial f}{\partial s}(t,s) &= (d\exp_p)_{t(v+sw)}(tw) \\ \frac{\partial f}{\partial t}(t,0) &= (d\exp_p)_{tv}(v) \\ \frac{\partial f}{\partial s}(t,0) &= (d\exp_p)_{tv}(tw)\end{aligned}$$

Now the LHS is equal to

$$\langle \frac{\partial f}{\partial t}(1,0), \frac{\partial f}{\partial s}(1,0)\rangle$$

We differentiate using compatibility with the Riemannian metric g (41), and that metric is symmetric (49)

$$\begin{aligned}\frac{\partial}{\partial t}\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\rangle &= \langle \frac{D}{dt}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\rangle + \langle \frac{\partial f}{\partial t}, \frac{D}{dt}\frac{\partial f}{\partial s}\rangle = \langle \frac{\partial f}{\partial t}, \frac{D}{ds}\frac{\partial f}{\partial t}\rangle \\ &= \frac{1}{2}\frac{\partial}{\partial s}\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle = \frac{1}{2}\frac{\partial}{\partial s}\left(\langle v,v\rangle + 2s\langle v,w\rangle + s^2\langle w,w\rangle\right) \\ &= \langle v,w\rangle + s|w|^2\end{aligned}$$

Thus we compute

$$\begin{aligned}\langle \frac{\partial f}{\partial t}(1,0), \frac{\partial f}{\partial s}(1,0)\rangle - \langle \frac{\partial f}{\partial t}(0,0), \frac{\partial f}{\partial s}(0,0)\rangle &= \int_0^1 \frac{\partial}{\partial t}\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\rangle(t,0)\,dt = \int_0^1 \langle v,w\rangle\,dt = \langle v,w\rangle \\ \langle \frac{\partial f}{\partial t}(1,0), \frac{\partial f}{\partial s}(1,0)\rangle &= \langle (d\exp_p)_v(v), (d\exp_p)_v(w)\rangle \\ \langle \frac{\partial f}{\partial t}(0,0), \frac{\partial f}{\partial s}(0,0)\rangle &= 0\end{aligned}$$

□

Proposition 19.3 (Geodesic Locally Minimize length)**.** *Let (M,g) be a Riemannian manifold. $p \in M$. Let U be a normal neighborhood of p in M, i.e., there exists U' open neighborhood of 0 in T_pM s.t. $\exp_p$ is defined on U' and maps U' diffeomorphically to $U = \exp_p(U')$. Let $B = B_\delta(p) \subset U$ be a geodesic ball of radius $\delta > 0$ centered at p. Let $\gamma : [0,1] \to B$ be the geodesic segment s.t.*

$$\gamma(0) = p \qquad \gamma(1) = q \neq p \qquad \gamma'(0) =: v_0 \in T_pM$$

i.e.

$$\gamma(t) = \exp_p(tv_0), \qquad q = \gamma(1) = \exp_p(v_0), \qquad \ell(\gamma) = |v_0|$$

Now for any $c : [0,1] \to M$ piecewise C^∞ curve in M s.t. $c(0) = c(1) = q$. We have

$$\ell(c) \geq \ell(\gamma)$$

Moreover, $\ell(c) = \ell(\gamma)$ implies

$$\gamma([0,1]) = c([0,1])$$

Proof. WLOG

- Assume $c([0,1]) \subset B$ otherwise consider the smallest $t_1 \in [0,1]$ s.t. $c(t_1) \in \partial B$ and show that $\ell(c) \geq \ell(c|_{[0,t_0]}) \geq \delta > \ell(\gamma)$.
- Assume $c(t) \neq p$ for $t > 0$. Otherwise consider the largest $t_2 \in (0,1)$ s.t. $c(t_2) = p$. Consider $c|_{[t_2,1]}$ and show $\ell(c) \geq \ell(c|_{[t_2,1]}) \geq \ell(\gamma)$.

Define $b : [0,1] \to B_\delta(0) \subset T_pM$ s.t.

$$b(t) = \exp_p^{-1}(c(t)) \iff c(t) = \exp_p(b(t))$$

so $b(t)$ is piecewise smooth curve in T_pM. By our assumption, $b(t) \neq 0$ for $t > 0$. Let $r(t) = |b(t)|$ so

$$r : [0,1] \to \mathbb{R}_{\geq 0}$$

is piecewise C^∞. We have $r(t) > 0$ for any $t > 0$. For $t > 0$

$$v(t) := \frac{b(t)}{|b(t)|}$$

so $v : (0,1] \to T_pM$ is piecewise C^∞. Hence using Compatibility with the metric

$$\langle v(t), v(t)\rangle = 1 \implies \langle v(t), v'(t)\rangle = 0$$

Then for $0 < t \leq 1$

$$
\begin{aligned}
c(t) &= \exp_p(b(t)) = \exp_p(r(t)v(t)) \\
\frac{d}{dt}c(t) &= (d\exp_p)_{b(t)}(r'(t)v(t) + r(t)v'(t)) \\
|\frac{d}{dt}c(t)|^2 &= \langle (d\exp_p)_{r(t)v(t)}(r'(t)v(t) + r(t)v'(t)), (d\exp_p)_{r(t)v(t)}(r'(t)v(t) + r(t)v'(t)) \rangle \\
&= (r'(t))^2 \langle (d\exp_p)_{r(t)v(t)}(v(t)), (d\exp_p)_{r(t)v(t)}(v(t)) \rangle \\
&+ 2r(t)r'(t) \langle (d\exp_p)_{r(t)v(t)}(v(t)), (d\exp_p)_{r(t)v(t)}(v'(t)) \rangle \\
&+ (r(t))^2 \langle (d\exp_p)_{r(t)v(t)}(v'(t)), (d\exp_p)_{r(t)v(t)}(v'(t)) \rangle \\
&= r'(t)^2 \langle v(t), v(t) \rangle + 2r(t)r'(t) \langle v(t), v'(t) \rangle + (r(t))^2 |(d\exp_p)_{r(t)v(t)}(v'(t))|^2 \\
&= r'(t)^2 + (r(t))^2 |(d\exp_p)_{r(t)v(t)}(v'(t))|^2
\end{aligned}
$$

where the last step uses Gauss Lemma (50). Hence

$$
|\frac{dc(t)}{dt}| = \sqrt{r'(t)^2 + (r(t))^2 |(d\exp_p)_{r(t)v(t)}(v'(t))|^2} \geq |r'(t)| \geq r'(t)
$$

so

$$
\ell(c) \geq \int_0^1 |\frac{dc(t)}{dt}| dt \geq \int_\varepsilon^1 r'(t) dt = r(1) - r(\varepsilon)
$$

for any $\varepsilon > 0$. Note $\lim_{\varepsilon \to 0} r(\varepsilon) = 0$ so using $r(1) = |v_0| = \ell(\gamma)$ yields

$$
\ell(c) \geq \ell(\gamma)
$$

Furthermore $\ell(c) = \ell(\gamma) \iff v'(t) = 0$ and $r'(t) \geq 0$. Then

$$
v(t) = \frac{v_0}{|v_0|}
$$

is constant unit vector. Now

$$
c(t) = \exp_p(r(t)\frac{v_0}{|v_0|}) \qquad r'(t) \geq 0 \quad r(0) = 0 \quad r(1) = 0
$$

and

$$
\gamma(t) = \exp_p(tv_0) \qquad c(0) = \gamma(0) = p \quad c(1) = \gamma(1) = \exp_p(v_0) = q
$$

hence

$$
c([0,1]) = \gamma([0,1])
$$

□

19.4 Killing Vector Fields

Let (M, g) be a Riemannian manifold with metric g. Let $X \in \mathfrak{X}(M)$. Let $p \in M$ and $U \subset M$ be open neighborhood of p. Let

$$
\varphi : (-\varepsilon, \varepsilon) \times U \to M \qquad s.t. \qquad t \mapsto \varphi(t, q) \text{ is trajectory of } X \text{ passing through } q \text{ at } t = 0 \qquad \forall\, q \in U \tag{51}
$$

Definition 19.4 (Killing Vector Field)**.** *X if called a Killing Vector Field if for each $t_0 \in (\varepsilon, \varepsilon)$, the mapping*

$$
\varphi(t_0, \cdot) : U \subset M \to M \qquad \textit{is an isometry, i.e.,} \qquad \varphi(t_0, \cdot)^* g = g \qquad \forall\, t_0 \in (-\varepsilon, \varepsilon)
$$

Proposition 19.4 (Killing Equation)**.** *$X \in \mathfrak{X}(M)$ is a Killing vector field iff*

$$
\langle \nabla_Y X, Z \rangle + \langle \nabla_Z X, Y \rangle = 0 \qquad \forall\, Y,\, Z \in \mathfrak{X}(M) \tag{52}
$$

Hence alternatively one has definition

Definition 19.5 (Killing Vector Field Equivalent Definition)**.** *Given Riemannian manifold (M, g). $X \in \mathfrak{X}(M)$ is Killing Field if the Lie-Derivative of the metric g w.r.t. X vanishes*

$$
L_X g = 0
$$

Proof. Let $L_X g = 0$. Then

$$\begin{aligned} 0 = L_X g(Y, Z) &= X(g(Y, Z)) - g(L_X Y, Z) - g(Y, L_X Z) \\ &= X(g(Y, Z)) - g([X, Y], Z) - g(Y, [X, Z]) \end{aligned}$$

Note for ∇ Levi-Civita connection that is compatible with the metric

$$0 = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = \nabla_X g(Y, Z)$$

and substitute using 'symmetric'

$$\nabla_Y Z - \nabla_Z Y = [Y, Z]$$

we conclude

$$0 = L_X g(Y, Z) = \langle \nabla_Y X, Z \rangle + \langle \nabla_Z X, Y \rangle$$

$\square$

Proposition 19.5. *Let X be a Killing vector field on a connected Riemannian Manifold M. If there exists point $q \in M$ s.t.*

$$X(q) = 0 \qquad \textit{and} \qquad \nabla_Y X(q) = 0 \qquad \forall\ Y(q) \in T_q M$$

Then $X \equiv 0$ identically vanishes.

20 Curvature

20.1 Curvature on Smooth Vector Bundle

Let $\pi : E \to M$ be C^∞ vector bundle over a C^∞ manifold M. Let $r = \text{rank } E$ and $n = \dim M$. Let

$$\nabla : \Omega^0(M,E) \to \Omega^1(M,E) \qquad s \mapsto \nabla s$$

be smooth connection on E. For any $X \in \mathfrak{X}(M)$ we know $\nabla_X s \in C^\infty(M,E)$

Definition 20.1 (Curvature F_∇). *For any X, $Y \in \mathfrak{X}(M)$ define $\mathbb{R}$-linear map*

$$F_\nabla(X,Y) : C^\infty(M,E) \to C^\infty(M,E) \qquad s \mapsto \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s =: F_\nabla(X,Y)s$$

Then

- *F_∇ is anti-symmetric $F_\nabla(X,Y) = -F_\nabla(Y,X)$ and*
- *$(X,Y,s) \mapsto F_\nabla(X,Y)s$ is $C^\infty(M)$-linear in X, Y, s.*

Linearity. Since $F_\nabla(X,Y) = -F_\nabla(Y,X)$ it suffices to show that for any X, $Y \in \mathfrak{X}(M)$, for any $s \in C^\infty(M,E)$ for any $f \in C^\infty(M)$

(i) $F_\nabla(fX,Y)(s) = fF_\nabla(X,Y)s$

(ii) $F_\nabla(X,Y)(fs) = fF_\nabla(X,Y)$.

We check (i).

$$\begin{aligned}
F_\nabla(fX,Y)(s) &= \nabla_{fX}\nabla_Y s - \nabla_Y\nabla_{fX}s - \nabla_{[fX,Y]}s \\
&= f\nabla_X\nabla_Y s - \nabla_Y(f\nabla_X s) - \nabla_{f[X,Y]-Y(f)X}s \\
&= f\nabla_X\nabla_Y s - Y(f)\nabla_X s - f\nabla_Y\nabla_X s - f\nabla_{[X,Y]}s + Y(f)\nabla_X s \\
&= f(\nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s) = fF_\nabla(X,Y)s
\end{aligned}$$

□

Remark 20.1. *Since $E^* \otimes E = End(E)$, for any X, $Y \in \mathfrak{X}(M)$*

$$F_\nabla(X,Y) \in C^\infty(M, End(E))$$

On the other hand we write

$$F_\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \times C^\infty(M,E) \to C^\infty(M,E) \qquad (X,Y,s) \mapsto F_\nabla(X,Y)s$$

is $C^\infty(M)$-linear. Hence

$$F_\nabla \in C^\infty(M, T^*M \otimes T^*M \otimes E^* \otimes E)$$

Since $F_\nabla(X,Y) = -F_\nabla(X,Y)$ we in fact have

$$F_\nabla \in C^\infty(M, (\Lambda^2 T^*M) \otimes End(E)) = \Omega^2(M, End(E))$$

Definition 20.2 (Metric h on Smooth Vector Bundle). *Let $\pi : E \to M$ be a C^∞ vector bundle of rank r on a C^∞ manifold M.*

(i) A metric on E is a C^∞ section $h \in C^\infty(M, Sym^2 E^*)$ such that for any $p \in M$

$$h(p) : E_p \times E_p \to \mathbb{R}$$

is an inner product on E_p.

(ii) We say a connection ∇ on E is compatible with h if for any $X \in \mathfrak{X}(M)$ for any s, $t \in C^\infty(M,E)$

$$Xh(s,t) = h(\nabla_X s, t) + h(s, \nabla_X t)$$

for $h(s,t) \in C^\infty(M)$.

Proposition 20.1 (Anti-Self adjoint). *If ∇ is a connection on $E \to M$ compatible with a metric h. Then for any X, $Y \in \mathfrak{X}(M)$, the curvature $F_\nabla(X,Y) \in C^\infty(M, End(E))$ is anti-self adjoint.*

$$h(F_\nabla(X,Y)s, t) = -h(F_\nabla(X,Y)t, s) = -h(s, F_\nabla(X,Y)t) \qquad \forall\ s,\ t \in C^\infty(M,E)$$

Proof.

$$h(F_\nabla(X,Y)s,t) + h(F_\nabla(X,Y)t,s) = h(F_\nabla(X,Y)(s+t),(s+t)) - h(F_\nabla(X,Y)s,s) - h(F_\nabla(X,Y)t,t)$$

It suffices to show that

$$h(F_\nabla(X,Y)s,s) = 0 \qquad \forall\ X,\,Y \in \mathfrak{X}(M) \qquad \forall\ s \in C^\infty(M,E)$$

so the RHS vanishes. But

$$\begin{aligned} h(F_\nabla(X,Y)s,s) &= h(\nabla_X\nabla_Y s,s) - h(\nabla_Y\nabla_X s,s) - h(\nabla_{[X,Y]}s,s) \\ &= Xh(\nabla_Y s,s) - h(\nabla_Y s,\nabla_X s) - Yh(\nabla_X s,s) + h(\nabla_X s,\nabla_Y s) - \frac{1}{2}[X,Y]h(s,s) \\ &= \frac{1}{2}XYh(s,s) - \frac{1}{2}YXh(s,s) - \frac{1}{2}[X,Y]h(s,s) = 0 \end{aligned}$$

□

Now let ∇ be an affine connection on a C^∞ manifold M, i.e., ∇ is a connection on TM.

20.2 Riemannian Curvature and Riemannian Curvature Tensor

In the Riemannian setting, first consider F_∇ curvature over $E = TM$ over tangent bundle.

Definition 20.3 (Riemannian Curvature)**.** *For any $X,\,Y \in \mathfrak{X}(M)$, define*

$$R_\nabla(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M) \qquad s.t. \qquad R_\nabla(X,Y)Z := -F_\nabla(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z - \nabla_{[Y,X]}Z \quad (53)$$

Lemma 20.1. *We have for $X(M) = C^\infty(M,TM)$*

$$R_\nabla : \mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M) \qquad s.t. \qquad (X,Y,Z) \mapsto R_\nabla(X,Y)Z$$

is $C^\infty(M)$-linear in X, Y, Z.

$$R_\nabla \in \Omega^2(M,End(TM)) = C^\infty(M,\Lambda^2T^*M\otimes T^*M\otimes TM) \subset C^\infty(M,TM\otimes(T^*M)^{\otimes 3})$$

where $TM\otimes(T^*M)^{\otimes 3} = T^1_3M$. Hence R_∇ is $(1,3)$-tensor on M.

Proposition 20.2 (First Bianchi Identity)**.** *If ∇ is a symmetric affine connection on M, i.e.,*

$$\nabla_X Y - \nabla_Y X = [X,Y] \qquad \forall\ X,\,Y \in \mathfrak{X}(M)$$

Then

$$R_\nabla(X,Y)Z + R_\nabla(Y,Z)X + R_\nabla(Z,X)Y = 0$$

Proof.

$$\begin{aligned} R_\nabla(X,Y)Z + R_\nabla(Y,Z)X + R_\nabla(Z,X)Y = &\ \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z - \nabla_{[Y,X]}Z \\ &+ \nabla_Z\nabla_Y X - \nabla_Y\nabla_Z X - \nabla_{[Z,Y]}X \\ &+ \nabla_X\nabla_Z Y - \nabla_Z\nabla_X Y - \nabla_{[X,Z]}Y \end{aligned}$$

Now using that the connection is symmetric we reduce to

$$\begin{aligned} R_\nabla(X,Y)Z + R_\nabla(Y,Z)X + R_\nabla(Z,X)Y &= \nabla_Y[X,Z] + \nabla_Z[Y,X] + \nabla_X[Z,Y] - \nabla_{[X,Z]}Y - \nabla_{[Y,X]}Z - \nabla_{[Z,Y]}X \\ &= [Y,[X,Z]] + [Z,[Y,X]] + [X,[Z,Y]] = 0 \end{aligned}$$

where we used Jacobi Indentity (9). □

Now we define Riemannian Curvature Tensor using Riemannian Curvature.

Proposition 20.3 (Riemannian Curvature Tensor)**.** *Let (M,g) be a Riemannian manifold and let ∇ be the Levi-Civita connection determined by g. Define*

$$R : \mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M) \to C^\infty(M) \qquad s.t. \qquad R(X,Y,Z,T) := g(R_\nabla(X,Y)Z,T) \qquad (54)$$

Then R is a $(0,4)$-tensor, i.e. $R(X,Y,Z,T)$ is $C^\infty(M)$-linear in X, Y, Z, T. Moreover

(a) *First Bianchi Identity holds*

$$R(X,Y,Z,T)+R(Y,Z,X,T)+R(Z,X,Y,T)=0 \tag{55}$$

(b) *$R\in C^\infty(M, Sym^2(\Lambda^2T^*M))$, i.e., for any X, Y, $Z\in\mathfrak{X}(M)$*

(b1) *$R(X,Y,Z,T)=-R(Y,X,Z,T)$ anti-symmetric in first 2 coordinates.*

(b2) *$R(X,Y,Z,T)=-R(X,Y,T,Z)$ anti-symmetric in 2 coordinates.*

(b3) *$R(X,Y,Z,T)=R(Z,T,X,Y)$ symmetric w.r.t. the 2 sets of coordinates.*

(b1) *and* (b2) *together gives $R\in C^\infty(M,\Lambda^2T^*M\otimes\Lambda^2T^*M)$. With* (b3), *$R\in C^\infty\left(M, Sym^2\left(\Lambda^2T^*M\right)\right)$.*

R is called the Riemannain Curvature Tensor of (M,g).

Proof. (b1) is clear from definition. That ∇ is compatible with g implies (b2). Assume (b1) and (b2) we derive (b3) using elementary algebra. $\square$

Local Coordinates of Riemannian Curvature. Let (U,ϕ) be C^∞ chart on M. Let $(x_1,\cdots,x_n)$ be local coordinates on U. Let T be any (r,s)-tensor on M. Then locally on U, T takes the form (12)

$$T=\sum_{\substack{1\le i_1,\cdots,i_r\le n\\ 1\le j_1,\cdots,j_s\le n}} T^{i_1,\cdots,i_r}_{j_1,\cdots,j_s}\frac{\partial}{\partial x_{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x_{i_r}}\otimes dx_{j_1}\otimes\cdots\otimes dx_{j_s}\qquad \text{for } T^{i_1,\cdots,i_r}_{j_1,\cdots,j_s}\in C^\infty(U)$$

For ∇ Levi-Civita connection. Write

$$g=\sum_{i,j} g_{i,j}dx_idx_j$$

where $g_{ij:=g(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j})}\in C^\infty(U)$. Recall we have Levi-Civita connection s.t.

$$\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}=\sum_k\Gamma^k_{ij}\frac{\partial}{\partial x_k}$$

where

$$\Gamma^\ell_{ij}=\frac{1}{2}\sum_k g^{\ell k}\left(g_{ik,j}+g_{kj,i}-g_{ij,k}\right)\qquad g_{\ell j,i}:=\frac{\partial}{\partial x_i}g_{\ell j}$$

Define $R^m_{ijk}\in C^\infty(U)$ by

$$R_\nabla(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k}=\sum_m R^m_{ijk}\frac{\partial}{\partial x_m} \tag{56}$$

On U, recall $R_\nabla\in C^\infty(M,T^1_3M)$

$$R_\nabla=\sum_{i,j,k,m}R^m_{ijk}dx_i\otimes dx_j\otimes dx_k\otimes\frac{\partial}{\partial x_m}$$

as $(1,3)$-tensor. Using definition (53)

$$R_\nabla(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k}=\nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_k}-\nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k}-\nabla_{[\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}]}\frac{\partial}{\partial x_k}$$

where by computations

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_k}&=\nabla_{\frac{\partial}{\partial x_j}}(\sum_\ell\Gamma^\ell_{ik}\frac{\partial}{\partial x_\ell})\\
&=\sum_\ell\frac{\partial}{\partial x_j}\Gamma^\ell_{ik}\frac{\partial}{\partial x_\ell}+\sum_\ell\Gamma^\ell_{ik}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_\ell}\\
&=\sum_m\left(\frac{\partial}{\partial x_j}\Gamma^m_{ik}+\sum_\ell\Gamma^\ell_{ik}\Gamma^m_{j\ell}\right)\frac{\partial}{\partial x_m}\\
\nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k}&=\nabla_{\frac{\partial}{\partial x_i}}\left(\sum_\ell\Gamma^\ell_{jk}\frac{\partial}{\partial x_\ell}\right)\\
&=\sum_\ell\frac{\partial}{\partial x_i}\Gamma^\ell_{jk}\frac{\partial}{\partial x_\ell}+\sum_\ell\Gamma^\ell_{jk}\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_\ell}\\
&=\sum_m\left(\frac{\partial}{\partial x_i}\Gamma^m_{jk}+\sum_\ell\Gamma^\ell_{jk}\Gamma^m_{i\ell}\right)\frac{\partial}{\partial x_m}\\
\nabla_{[\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}]}\frac{\partial}{\partial x_k}&=0
\end{aligned}$$

Hence we have local coordinate representations

$$R^m_{ijk} := \frac{\partial}{\partial x_j}\Gamma^m_{ik} - \frac{\partial}{\partial x_i}\Gamma^m_{jk} + \sum_\ell \Gamma^\ell_{ik}\Gamma^m_{j\ell} - \sum_\ell \Gamma^\ell_{jk}\Gamma^m_{i\ell} \tag{57}$$

$\square$

Local Coordinates of Riemannian Curvature Tensor. For (U,ϕ) with $\phi = (x_1, \cdots, x_n)$ and

$$g = \sum_{ij} g_{ij}\, dx_i dx_j$$

with Γ^k_{ij} Christoffel symbols (46). On U, since $R \in C^\infty(M, T^0_4 M)$ is $(0,4)$-tensor

$$R = \sum_{i,j,k,\ell=1}^{n} R_{i,j,k,\ell} dx_i \otimes dx_j \otimes dx_k \otimes dx_\ell$$

and using Definition (54)

$$\begin{aligned} R_{i,j,k,\ell} &:= R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_\ell}) = g(R_\nabla(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_\ell}) \\ &= g(\sum_m R^m_{ijk}\frac{\partial}{\partial x_m}, \frac{\partial}{\partial x_\ell}) = \sum_m R^m_{ijk} g_{m\ell} \in C^\infty(U) \end{aligned}$$

Moreover, using Proposition 20.3

(a) $R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0$.

(b) $R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij}$.

$\square$

Example 20.1. *For* $\dim M = 1$ *then*

$$R = R_{1111}(dx_1 \otimes dx_1 \otimes dx_1 \otimes dx_1)$$

But this immediately implies $R_{1111} \equiv 0$ *via Bianchi identity. Hence for* $\dim M = 1$*,* $R = R_\nabla = 0$*.*

20.3 Sectional Curvature

In general, an inner product on a vector space $V \cong \mathbb{R}^n$ induces an inner product on $\Lambda^2 V$ as follows. If $\{e_1, \cdots, e_n\} \subset V$ is an ONB, then

$$\{e_i \wedge e_j \mid 1 \le i < j \le n\}$$

is an ONB of $\Lambda^2 V$.

Definition 20.4 (Sectional Curvature)**.** *Let* (M,g) *be Riemannian manifold with* R *Riemannian curvature* $(0,4)$ *tensor. Let* $p \in M$*, let* σ *be the 2* $\dim$ *subspace of* T_pM*, i.e.,* $\sigma \in Gr(2, T_pM)$*. We define the sectional curvature of* σ *to be*

$$K(p,\sigma) := \frac{R(p)(x,y,x,y)}{|x \wedge y|^2} \tag{58}$$

where x*,* y *is any basis of* σ *and*

$$|x \wedge y|^2 = \langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle^2$$

Alternatively, one may define

$$K(p,\sigma) := R(p)(e_1, e_2, e_1, e_2)$$

where e_1, e_2 *is an orthonormal basis of* σ*. Then* $K(p,\sigma) \in \mathbb{R}$ *is well-defined independent of choice of* x*,* y*,* e_1*,* e_2*.*

Remark 20.2. *Given* $\sigma \subset T_pM$ 2-dim *subspace, let* e_1*,* e_2 *be orthonormal basis and* x*,* y *any basis. If*

$$\begin{aligned} x &= ae_1 + be_2 \\ y &= ce_1 + de_2 \qquad ad - bc \neq 0 \\ \implies R(p)(x,y,x,y) &= (ad-bc)^2 R(p)(e_1,e_2,e_1,e_2) \\ |x \wedge y|^2 &= (ad-bc)^2 \end{aligned}$$

Theorem 20.1. *The Riemannian curvature tensor R on a Riemannian manifold (M, g) is determined by its sectional curvature $K(p, \sigma)$ for any $p \in M$ and for any $\sigma \in Gr(2, T_pM)$, i.e.*

$$\{R(X, Y, Z, T) \mid X,\ Y,\ Z,\ T \in \mathfrak{X}(M)\}$$

is determined by

$$\{R(X, Y, X, Y) \mid X,\ Y \in \mathfrak{X}(M)\}$$

Proof. Follows from the following lemma in linear algebra 20.2. □

Lemma 20.2 (Linear Algebra)**.** *Let V be an inner product space over $\mathbb{R}$ where $\dim_{\mathbb{R}} V = n$, e.g. $V = T_pM$. Suppose that we have two maps $r, r' \in (V^*)^{\otimes 4}$*

$$r,\ r' : V \times V \times V \times V \to \mathbb{R} \qquad (x, y, z, t) \mapsto r(x, y, z, t),\ r'(x, y, z, t)$$

$\mathbb{R}$-linear in x, y, z, t and both satisfy

(a) Bianchi identity $r(x, y, z, t) + r(y, z, x, t) + r(z, x, y, t) = 0$

(b) $r \in Sym^2(\Lambda^2 V^*)$, i.e.

  (b1) $r(x, y, z, t) = -r(y, x, z, t)$.

  (b2) $r(x, y, z, t) = -r(x, y, t, z)$.

  (b3) $r(z, t, x, y) = r(x, y, z, t)$.

Define $K,\ K' : Gr(2, V) \to \mathbb{R}$ s.t.

$$K(\sigma) = \frac{r(x, y, x, y)}{|x \wedge y|^2}$$
$$K'(\sigma) = \frac{r'(x, y, x, y)}{|x \wedge y|^2}$$

If $K = K'$, then $r = r'$.

Proof. Let $\Delta = r - r' \in (V^*)^{\otimes 4}$ then Δ satisfies (a) and (b1) - (b3) and

$$\Delta(x, y, x, y) = 0 \qquad \forall\ x,\ y \in V$$

We claim that

$$\Delta(x, y, z, t) = 0 \qquad \forall\ x, y, z, t \in V$$

Indeed for any $x,\ y,\ z \in V$ we have

$$\begin{aligned} 2\Delta(x, y, z, y) &= \Delta(x, y, z, y) + \Delta(z, y, x, y) \\ &= \Delta(x + z, y, x + z, y) - \Delta(x, y, x, y) - \Delta(z, y, z, y) = 0 \end{aligned}$$

Hence

$$\Delta(x, y, z, y) = 0 \qquad \forall\ x,\ y,\ z \in V$$

Now for any $x, y, z, t \in V$

$$\begin{aligned} 0 &= \Delta(x, y + t, z, y + t) - \Delta(x, y, z, y) - \Delta(x, t, z, t) \\ &= \Delta(x, y, z, t) + \Delta(x, t, z, y) \\ &= \Delta(x, y, z, t) + \Delta(z, y, x, t) \\ &= \Delta(x, y, z, t) - \Delta(y, z, x, t) \end{aligned}$$

using Bianchi we have

$$0 = \Delta(x, y, z, t) + \Delta(y, z, x, t) + \Delta(z, x, y, t) = 3\Delta(x, y, z, t)$$

□

Definition 20.5. *We say (M, g) have constant sectional curvature K_0 if for any $p \in M$ for any $\sigma \in Gr(2, T_pM)$*

$$K(p, \sigma) = K_0$$

Theorem 20.2. *(M, g) has constant sectional curvature iff*

$$R(X, Y, Z, T) = K_0(g(X, Z)g(Y, T) - g(X, T)g(Y, Z))$$

Proof. Define the RHS to be $K_0 R_0(X,Y,Z,T)$ then for any e_1, e_2 orthonormal vectors

$$R_0(e_1,e_2,e_1,e_2) = g(e_1,e_2)g(e_1,e_2) - g(e_1,e_2)^2 = 1\cdot 1 - 0^2 = 1$$

Hence

$$R_0(X,Y,Z,T) = g(X,Z)g(Y,T) - g(X,T)g(Y,Z)$$

satisfies (a) and (b1) - (b3). $\square$

Definition 20.6 (Flat). *We say a Riemannian manifold (M,g) is flat if it has constant sectional curvature* 0. *This is equivalent to saying Riemannian curvature tensor $R \equiv 0$ due to Lemma 20.2.*

Example 20.2. $(\mathbb{R}^n, g_0 = dx_1^2 + \cdots dx_n^2)$ *is flat since* $\Gamma_{ij}^k = 0 \implies R_{ijk}^\ell = 0$.

Example 20.3 (Riemannian Curvature Tensor and Sectional Curvature at $n=2$). *For Riemannian manifold (M,g) with* $\dim M = 2$. *Let (U,ϕ) be C^∞ chart on M and let (x_1,x_2) be coordinates on U. On U*

$$g = \sum_{i,j=1}^{2} g_{ij}dx_i dx_j = g_{11}dx_1^2 + 2g_{12}dx_1dx_2 + g_{22}dx_2^2$$

We have Riemannian Curvature Tensor

$$\begin{aligned}
R &= \sum_{i,j,k,\ell=1}^{2} R_{ijk\ell}dx_i \otimes dx_j \otimes dx_k \otimes dx_\ell \\
&= R_{1212}dx_1\otimes dx_2\otimes dx_1\otimes dx_2 + R_{2112}dx_2\otimes dx_1\otimes dx_1\otimes dx_2 + R_{1221}dx_1\otimes dx_2\otimes dx_2\otimes dx_1 + R_{2121}dx_2\otimes dx_1\otimes dx_2\otimes dx_1 \\
&= R_{1212}(dx_1\otimes dx_2 - dx_2\otimes dx_1)\otimes(dx_1\otimes dx_2 - dx_2\otimes dx_1) \\
&= R_{1212}(dx_1\wedge dx_2)\otimes(dx_1\wedge dx_2)
\end{aligned}$$

The only 2-dim *subspace of T_pM is itself. So sectional curvature*

$$K: M\to\mathbb{R} \qquad s.t. \qquad K(p) = K(p, T_pM) \qquad \forall\, p\in M$$

has

$$K = \frac{R(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2})}{|\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}|^2} = \frac{R_{1212}}{g_{11}g_{22}-g_{12}^2}$$

Example 20.4. *Consider* $(\mathbb{S}^2, g_{can} = d\phi^2 + \sin^2\phi\, d\theta^2)$ *for* $(\phi,\theta) = (x_1,x_2)$. *Recall Example 18.2*

$$g_{11} = 1, \qquad g_{22} = \sin^2\phi \qquad g_{12} = g_{21} = 0$$

Where

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\phi} &= 0 \\
\nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\theta} &= \nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\phi} = \cot(\phi)\frac{\partial}{\partial\theta} \\
\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\theta} &= -\sin(\phi)\cos(\phi)\frac{\partial}{\partial\phi}
\end{aligned}$$

We want to compute

$$K = \frac{R_{1212}}{g_{11}g_{22}-g_{12}^2} = \frac{R_{1212}}{\sin^2(\phi)}$$

In particular

$$\begin{aligned}
R_{1212} &= \langle R(\frac{\partial}{\partial\phi},\frac{\partial}{\partial\theta})\frac{\partial}{\partial\phi},\frac{\partial}{\partial\theta}\rangle \\
&= \langle\nabla_{\frac{\partial}{\partial\theta}}\nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\phi} - \nabla_{\frac{\partial}{\partial\phi}}\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\phi},\frac{\partial}{\partial\theta}\rangle \\
&= -\langle\nabla_{\frac{\partial}{\partial\phi}}(\cot(\phi)\frac{\partial}{\partial\theta}),\frac{\partial}{\partial\theta}\rangle \\
&= -\langle -\csc^2\phi\frac{\partial}{\partial\theta} + \cot\phi\nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}\rangle \\
&= -\langle -\csc^2(\phi)\frac{\partial}{\partial\theta} + \cot^2(\phi)\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}\rangle \\
&= \langle\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}\rangle = g_{22} = \sin^2(\phi)
\end{aligned}$$

Hence $K = 1$.

20.4 Ricci Curvature and Scalar Curvature

Definition 20.7 (Ricci Curvature)**.** *First define a symmetric $(0,2)$-tensor Q on M. For any $p \in M$, x, $y \in T_pM$ and $e_1, \cdots, e_n$ ONB of T_pM*

$$\begin{aligned} Q(p)(x,y) &:= \mathrm{Tr}\,(v \in T_pM \mapsto R(p)(x,v)y \in T_pM) \\ &= \sum_{i=1}^n \langle R(p)(x,e_i)y, e_i\rangle = \sum_{i=1}^n R(p)(x,e_i,y,e_i) = \sum_{i,j=1}^n R(p)(x, \frac{\partial}{\partial x_i}(p), y, \frac{\partial}{\partial x_j}(p))g^{ij}(p) \end{aligned} \tag{59}$$

Proof for Last Equality of (59). The last equality follows by using computations

$$\frac{\partial}{\partial x_i} = \sum_k a_{ik}e_k \qquad \frac{\partial}{\partial x_j} = \sum_\ell a_{j\ell}e_\ell$$

and g_{ij} as

$$\begin{aligned} g_{ij} &= \langle \sum_k a_{ik}e_k, \sum_\ell a_{j\ell}e_\ell\rangle = \sum_{k\ell} a_{ik}a_{j\ell}\langle e_k, e_\ell\rangle = \sum_{k=1}^n a_{ik}a_{jk} \\ g &= aa^T \\ g^{-1} &= (a^T)^{-1}a^{-1} \end{aligned}$$

Hence

$$\begin{aligned} \sum_{i,j=1}^n R(p)(x, \frac{\partial}{\partial x_i}, y, \frac{\partial}{\partial x_j})g^{ij} &= \sum_{i,j=1}^n R(p)(x, \sum_k a_{ik}e_k, y, \sum_\ell a_{j\ell}e_\ell)g^{ij} = \sum_{k,\ell} R(p)(x,e_k,y,e_\ell) \sum_{i,j=1}^n a_{ik}g^{ij}a_{j\ell} \\ &= \sum_{k,\ell} R(p)(x,e_k,y,e_\ell)(a^Tg^{-1}a)_{k\ell} = \sum_{k,\ell} R(p)(x,e_k,y,e_\ell)(a^Ta^{-T}a^{-1}a)_{k\ell} \\ &= \sum_{k,\ell} R(p)(x,e_k,y,e_\ell)\delta_{k\ell} = \sum_{k=1}^n R(p)(x,e_k,y,e_k) \end{aligned}$$

□

We also make the claim that $Q \in C^\infty(M, Sym^2T^*M)$ is symmetric tensor.

Proof. Using (b3) $R_{ijk\ell} = R_{k\ell ij}$ we indeed verify Q is symmetric

$$\begin{aligned} Q(p)(x,y) &= \sum_{i=1}^n R(p)(x,e_i,y,e_i) = \sum_{i=1}^n R(p)(y,e_i,x,e_i) \\ &= Q(p)(y,x) \end{aligned}$$

□

Hence the coefficients of Q writes

$$\begin{aligned} R_{ij} &:= Q(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \sum_{k=1}^n \langle R(p)(\frac{\partial}{\partial x_i}, e_k)\frac{\partial}{\partial x_j}, e_k\rangle \\ &= \sum_{k,\ell=1}^n R(p)(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}(p), \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_\ell}(p))g^{k\ell}(p) = \sum_{k,\ell} R_{ikj\ell}g^{k\ell} \end{aligned}$$

On U

$$\begin{aligned} Q &= \sum_{i,j=1}^n R_{ij}dx_i \otimes dx_j \\ &= \sum_{i,j} R_{ij}dx_idx_j \end{aligned}$$

Here $R_{ij} = R_{ji}$ and $dx_idx_j = \frac{1}{2}(dx_i \otimes dx_j + dx_j \otimes dx_i)$. We define Ricci Curvature Tensor as

$$\mathrm{Ric} := \frac{1}{n-1}Q = \frac{1}{n-1}\sum_{i,j} R_{ij}\,dx_idx_j \in C^\infty(M, Sym^2T^*M)$$

Indeed the coefficients of Ric *in local coordinates write*

$$\mathrm{Ric}_{ij} := \mathrm{Ric}(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \frac{1}{n-1} R_{ij} = \frac{1}{n-1}\sum_{k=1}^{n} R^k_{ikj} = \frac{1}{n-1}\sum_{k,\ell=1}^{n} R_{ikj\ell} g^{k\ell}$$

Remark 20.3. *Why do we normalize by $\frac{1}{n-1}$? If (M,g) has constant sectional curvature K_0, then*

$$\begin{aligned}
R(X,Y,Z,T) &= K_0(g(X,Z)g(Y,T) - g(X,T)g(Y,Z)) \\
R_{ijk\ell} &= K_0(g_{ik}g_{j\ell} - g_{i\ell}g_{jk}) \\
R_{ik} &= \sum_{j,\ell} R_{ijk\ell} g^{j\ell} = K_0\left(\sum_{\ell} g_{ik}\sum_{j} g^{j\ell}g_{j\ell} - \sum_{\ell} g_{i\ell}\sum_{j} g_{jk}g^{j\ell}\right) \\
&= K_0\left(g_{jk}\sum_{\ell}\delta^{\ell}_{\ell} - \sum_{\ell} g_{i\ell}\delta^{\ell}_{k}\right) \\
&= K_0\left(g_{ik}n - g_{ik}\right) = (n-1)K_0 g_{ik}
\end{aligned}$$

Hence $Q = (n-1)K_0 g$ and $\mathrm{Ric} = K_0 g$.

Definition 20.8 (Scalar Curvature)**.** *Let (M,g) be Riemannian manifold. For any $p \in M$, define a linear map*

$$K(p) : T_pM \to T_pM \qquad s.t. \qquad \langle K(p)(x), y\rangle = Q_p(x,y) \qquad \forall\ x,\, y \in T_pM$$

The $(1,1)$-tensor K is self-adjoint at each point $p \in M$, i.e.

$$\langle K(p)(x), y\rangle = \langle x, K(p)(y)\rangle \qquad \forall\ x,\, y \in T_pM$$

Taking an orthonormal basis $\{e_1, \cdots, e_n\}$ of T_pM, we compute the Trace

$$\begin{aligned}
\mathrm{Tr}(K(p)) &= \sum_i \langle K(p)(e_i), e_i\rangle = \sum_i Q(p)(e_i, e_i) \\
&= \sum_{i,j=1}^{n} R(p)(e_i, e_j, e_i, e_j) = (n-1)\sum_i \mathrm{Ric}(p)(e_i, e_i)
\end{aligned}$$

Then we define scalar curvature $S \in C^\infty(M)$

$$\begin{aligned}
S(p) &:= \frac{1}{n}\sum_i \mathrm{Ric}(p)(e_i, e_i) = \frac{1}{n}\sum_{ij}\mathrm{Ric}_{ij} g^{ij} = \frac{1}{n(n-1)}\mathrm{Tr}(K(p)) \\
&= \frac{1}{n(n-1)}\sum_{ij} R_{ij} g^{ij} \\
&= \frac{1}{n(n-1)}\sum_{i,j,k} R^k_{ikj} g^{ij} \\
&= \frac{1}{n(n-1)}\sum_{i,j,k,\ell} R_{ijk\ell} g^{ik} g^{j\ell}
\end{aligned}$$

Example 20.5. *When (M,g) has constant sectional curvature K_0*

$$\begin{aligned}
\mathrm{Ric} &= K_0 g \\
S &= \frac{1}{n}\sum_{i,j}\mathrm{Ric}_{ij} g^{ij} = \frac{1}{n}\sum_{i,j} K_0 g_{ij} g^{ij} = K_0
\end{aligned}$$

Example 20.6. *For $\dim M = 2$,*

$$\begin{aligned}
R &= R_{1212}(dx_1 \wedge dx_2) \otimes (dx_1 \wedge dx_2) \\
\mathrm{Ric} &= \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} g = Kg \\
S &= \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} = K
\end{aligned}$$

We carry out the calculation

$$\begin{aligned}
S &= \frac{1}{2}\left(R_{1212}g^{11}g^{22} + R_{2112}g^{21}g^{12} + R_{1221}g^{12}g^{21} + R_{2121}g^{22}g^{11}\right) \\
&= \frac{1}{2}\left(R_{1212}g^{11}g^{22} - R_{1212}g^{21}g^{12} - R_{1212}g^{12}g^{21} + R_{1212}g^{22}g^{11}\right) \\
&= R_{1212}g^{11}g^{22} - R_{1212}(g^{12})^2 = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} = K
\end{aligned}$$

21 Covariant Derivative of Tensors

Proposition 21.1 (Covariant Derivative on Tensor)**.** *Consider an affine connection ∇ on C^∞ manifold M. Given $X \in \mathfrak{X}(M)$*

$$\nabla_X : \mathfrak{X}(M) \to \mathfrak{X}(M) \qquad Y \mapsto \nabla_X Y$$

defined on $(1,0)$-tensors. Then ∇_X has a unique extension $\nabla_X : C^\infty(M, T^r_s M) \to C^\infty(M, T^r_s M)$ to any (r,s)-tensors s.t.

(i) ∇_X *is* $\mathbb{R}$*-linear.*

(ii) $\nabla_X(c(S)) = c(\nabla_X S)$ *for any c contraction.*

(iii)

$$\nabla_X(S \otimes T) = \nabla_X S \otimes T + S \otimes \nabla_X T$$

Proof. For $(0,0)$-tensor, for any $f \in C^\infty(M)$ and $Y \in \mathfrak{X}(M)$, we need

$$\begin{aligned}
\nabla_X(fY) &= X(f)Y + f\nabla_X Y \\
\nabla_X(fY) &= \nabla_X(f \otimes Y) = (\nabla_X f) \otimes Y + f \otimes \nabla_X Y \\
&= (\nabla_X f)Y + f\nabla_X Y \\
\implies \nabla_X f &= X(f)
\end{aligned}$$

For $(0,1)$-tensors, for any $\alpha \in \Omega^1(M)$, $Y \in \mathfrak{X}(M)$

$$\begin{aligned}
X(\alpha(Y)) &= \nabla_X(\alpha(Y)) = \nabla_X(c(\alpha \otimes Y)) = c(\nabla_X(\alpha \otimes Y)) \\
&= c((\nabla_X \alpha) \otimes Y + \alpha \otimes \nabla_X Y) \\
&= (\nabla_X \alpha)(Y) + \alpha(\nabla_X Y) \\
\implies (\nabla_X \alpha)(Y) &= X(\alpha(Y)) - \alpha(\nabla_X Y)
\end{aligned} \tag{60}$$

It is good to compare with Lie Derivative

$$(L_X \alpha)(Y) = X(\alpha(Y)) - \alpha(L_X Y)$$

Now for any (r,s)-tensor, for T $(0,s)$-tensor, $Y_1, \cdots, Y_r \in \mathfrak{X}(M)$

$$(\nabla_X T)(Y_1, \cdots, Y_s) = X(T(Y_1, \cdots, Y_s)) - \sum_{i=1}^{s} T(Y_1, Y_2, \cdots, Y_{i-1}, \nabla_X Y_i, Y_{i+1}, \cdots, Y_s) \tag{61}$$

again, compare with Lie Derivative as in Lemma 11.6

$$(L_X T)(Y_1, \cdots, Y_s) = X(T(Y_1, \cdots, Y_s)) - \sum_{i=1}^{s} T(Y_1, Y_2, \cdots, Y_{i-1}, [X, Y_i], Y_{i+1}, \cdots, Y_s)$$

□

Definition 21.1 (Covariant Derivative of (r,s)-tensor)**.**

$$\nabla : C^\infty(M, T^r_s M) \to C^\infty(M, T^r_{s+1} M) \qquad T \mapsto \nabla T$$

s.t. for any $X_1, \cdots, X_{s+1} \in \mathfrak{X}(M)$ we have

$$(\nabla T)(X_1, \cdots, X_s, X_{s+1}) = (\nabla_{X_{s+1}} T)(X_1, \cdots, X_s) \tag{62}$$

and $\nabla_{X_{s+1}}$ satisfies (i) - (iii) as in Proposition 21.1. Note we have $(r, s+1)$-tensor on LHS and (r,s)-tensor on RHS.

Theorem 21.1 (2nd Bianchi Identity)**.** *Let (M,g) be Riemannian manifold. Let R be Riemannian curvature tensor $(0,4)$-tensor. Apply ∇ Levi-Civita connection so that ∇R is $(0,5)$-tensor with*

$$\nabla R(X,Y,Z,T,W) + \nabla R(X,Y,T,W,Z) + \nabla R(X,Y,W,Z,T) = 0$$

Definition 21.2 (Locally Symmetric Space)**.** *Let (M,g) be Riemannian manifold. Let ∇ be the Levi-Civita connection on M. M is locally symmetric space if*

$$\nabla R = 0 \qquad \textit{for } R \textit{ Riemannian curvature tensor (54) of } M$$

Proposition 21.2 (Locally Symmetric Space)**.** *Let (M, g) be be Riemannian manifold.*

1. *Let M be locally symmetric space and*
$$\gamma : [0, \ell) \to M \qquad \text{be geodesic of } M$$
For any X, Y, Z parallel vector fields along γ
$$R(X, Y)Z \qquad \text{is also a parallel vector field along } \gamma$$
2. *If M is locally symmetric, connected, and $\dim M = 2$, then M has constant sectional curvature.*
3. *If M has constant sectional curvature, then M is locally symmetric space.*

Local Coordinates. Consider an affine ∇ connection on a C^∞ manifold M with (U, ϕ) $\phi = (x_1, \cdots, x_n)$ C^∞ chart.
$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x_k}$$
for $\Gamma^k_{ij} \in C^\infty(U)$. For cotangent bundle
$$\nabla_{\frac{\partial}{\partial x_i}} dx_j = \sum_k \left(\nabla_{\frac{\partial}{\partial x_i}} dx_j \right) (\frac{\partial}{\partial x_k}) dx_k$$
Where for $\alpha \in \Omega^1(M)$, $\alpha = a_i dx_i$ and $a_i = \alpha(\frac{\partial}{\partial x_i})$. We have
$$\left(\nabla_{\frac{\partial}{\partial x_i}} dx_j \right) (\frac{\partial}{\partial x_k}) = \frac{\partial}{\partial x_i} \left(dx_j(\frac{\partial}{\partial x_k}) \right) - dx_j \left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_k} \right) = -\Gamma^j_{ik}$$
where
$$dx_j(\frac{\partial}{\partial x_k}) = \delta_{jk} \qquad \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_k} = \sum_\ell \Gamma^\ell_{ik} \frac{\partial}{\partial x_\ell}$$
Hence for T (r, s)-tensor with $e_i = \frac{\partial}{\partial x_i}$, $e^j = dx_j$ we have
$$\nabla_{e_i} e_j = \Gamma^k_{ij} e_k \qquad \nabla_{e_i} e^j = -\Gamma^j_{ik} e^k \tag{63}$$
For general (r, s)-tensors we write in local coordinates
$$T = T^{i_1, \cdots, i_r}_{j_1, \cdots, j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}$$
where $T^{i_1, \cdots, i_r}_{j_1, \cdots, j_s} \in C^\infty(U)$. So $\nabla T \in C^\infty(M, T^r_{s+1} M)$ is $(r, s+1)$-tensor with
$$\nabla T = (\nabla T)^{i_1, \cdots, i_r}_{j_1, \cdots, j_{s+1}} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s} \otimes e^{j_{s+1}}$$
Define
$$T^{i_1, \cdots, i_r}_{j_1, \cdots, j_s; k} := (\nabla T)^{i_1, \cdots, i_r}_{j_1, \cdots, j_s, k} = (\nabla_{e_k} T)^{i_1, \cdots, i_r}_{j_1, \cdots, j_s}$$
We want to express
$$T^{i_1, \cdots, i_r}_{j_1, \cdots, j_s; k}$$
in terms of $T^{i_1, \cdots, i_r}_{j_1, \cdots, j_s}$ and Γ^k_{ij}. Using Leibniz rule for Covariant Derivative (61)
$$\begin{aligned}
\nabla_{e_k} T &= \nabla_{e_k} \left(T^{i_1, \cdots, i_r}_{j_1, \cdots, j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s} \right) \\
&= e_k \left(T^{i_1, \cdots, i_r}_{j_1, \cdots, j_s} \right) e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s} \\
&+ \sum_{\alpha=1}^r T^{i_1, \cdots, i_r}_{j_1, \cdots, j_s} e_{i_1} \otimes \cdots \otimes e_{i_{\alpha-1}} \otimes \nabla_k e_{i_\alpha} \otimes e_{i_{\alpha+1}} \otimes \cdots \otimes \left(e^{j_1} \otimes \cdots \otimes e^{j_s} \right) \\
&+ \sum_{\beta=1}^s T^{i_1, \cdots, i_r}_{j_1, \cdots, j_s} (e_{i_1} \otimes \cdots \otimes e_{i_r}) \otimes e^{j_1} \otimes \cdots \otimes e^{j_{\beta-1}} \otimes \nabla_k e^{j_\beta} \otimes e^{j_{\beta+1}} \otimes \cdots \otimes e^{j_s}
\end{aligned}$$
Then we switch $\nabla_k e_{i_\alpha} = \Gamma^\ell_{k i_\alpha} e_\ell$ and $\nabla_k e^{j_\beta} = -\Gamma^{j_\beta}_{k\ell} e^\ell$ as in (63) so
$$\begin{aligned}
&\nabla_{e_k} T \\
&= \left(e_k (T^{i_1, \cdots, i_r}_{j_1, \cdots, j_s}) + \Gamma^{i_\alpha}_{k\ell} T^{i_1, \cdots, i_{\alpha-1}, \ell, i_{\alpha+1}, \cdots i_r}_{j_1, \cdots, j_s} - \Gamma^\ell_{k, j_\beta} T^{i_1, \cdots, i_r}_{j_1, \cdots, j_{\beta-1}, \ell, j_{\beta+1}, \cdots, j_s} \right) e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}
\end{aligned}$$

Hence we have formula

$$T^{i_1,\cdots,i_r}_{j_1,\cdots,j_s;k} = e_k(T^{i_1,\cdots,i_r}_{j_1,\cdots,j_s}) + \sum_{\ell,\alpha}^{r} \Gamma^{i_\alpha}_{k\ell} T^{i_1,\cdots,i_{\alpha-1},\ell,i_{\alpha+1},\cdots i_r}_{j_1,\cdots,j_s} - \sum_{\ell,\beta}^{s} \Gamma^{\ell}_{k,j_\beta} T^{i_1,\cdots,i_r}_{j_1,\cdots,j_{\beta-1},\ell,j_{\beta+1},\cdots,j_s} \tag{64}$$

where $e_k = \frac{\partial}{\partial x_k}$. $\square$

Lemma 21.1. *Let ∇ be affine connection on a smooth manifold M. Then ∇ is symmetric iff for any $f \in C^\infty(M)$, the $(0,2)$-tensor ∇df is symmetric, i.e.*

$$(\nabla df)(X,Y) = (\nabla df)(Y,X) \qquad \forall\ X,\ Y \in \mathfrak{X}(M)$$

Proof. Using (60), since $df \in \Omega^1(M)$ for any $f \in C^\infty(M)$, for any $X,\ Y \in \mathfrak{X}(M)$, using Definition (62)

$$\begin{aligned}(\nabla df)(Y,X) &:= \nabla_X df(Y) = X(df(Y)) - df(\nabla_X Y)\\ &= X(Y(f)) - (\nabla_X Y)(f)\end{aligned}$$

Now assume ∇ is symmetric.

$$\begin{aligned}(\nabla df)(Y,X) &= X(Y(f)) - (\nabla_X Y)(f) = X(Y(f)) - ((\nabla_Y X)(f) - [Y,X](f))\\ &= X(Y(f)) - X(Y(f)) + Y(X(f)) - (\nabla_Y X)(f)\\ &= Y(X(f)) - (\nabla_Y X)(f) = (\nabla df)(X,Y)\end{aligned}$$

On the other hand assume $(\nabla df)(Y,X) = (\nabla df)(X,Y)$. Then

$$\begin{aligned}0 &= (\nabla df)(Y,X) - (\nabla df)(X,Y) = (X(Y(f)) - (\nabla_X Y)(f)) - (Y(X(f)) - (\nabla_Y X)(f))\\ &= [X,Y](f) + \nabla_Y X(f) - \nabla_X Y(f) \qquad \forall\ f \in C^\infty(M)\end{aligned}$$

$\square$

For (M,g) Riemannian manifold with ∇ Levi-Civita connection.

Lemma 21.2. *∇ is compatible with g implies*

$$\begin{aligned}(\nabla g)(X,Y,Z) &= (\nabla_Z g)(X,Y) = Z(g(X,Y)) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) = 0 \qquad \forall\ X,\ Y,\ Z \in \mathfrak{X}(M)\\ \implies \nabla g &= 0\\ g_{ij;k} &= 0 \qquad \forall\ i,\ j,\ k\end{aligned}$$

as an answer to (40).

In fact, for $f \in \mathfrak{X}(M)$, we denote

$$f_{;i} = e_i(f) = \frac{\partial f}{\partial x_i}$$

and

$$\nabla f = f_{;i} e^i = \sum_i \frac{\partial f}{\partial x_i} dx_i = df$$

Definition 21.3 (Gradient)**.** *For $f \in C^\infty(M)$, we define vector field* grad$f \in \mathfrak{X}(M)$ *s.t.*

$$g\left(\mathrm{grad} f, X\right) = df(X) = X(f)$$

with $\mathrm{grad} f = \sum_j (\mathrm{grad} f)^j e_j$*, then*

$$f_{;j} = e_j(f) = df(e_j) = \langle \mathrm{grad} f, e_j\rangle = \sum_i (\mathrm{grad} f)^i g_{ij}$$

Therefore

$$df = f_{;i} e^i = \sum_i \frac{\partial f}{\partial x_i} dx_i$$

$$\mathrm{grad} f = f^i_{;} e_i = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i} \tag{65}$$

where $f^i_{;} = g^{ij} f_{;j}$.

Definition 21.4 (Divergence)**.** *For $Y \in \mathfrak{X}(M)$ $(1,0)$-tensor, we define smooth function* $\mathrm{div}Y \in C^\infty(M)$ *s.t.*

$$\mathrm{div}(Y)(p) = \mathrm{Tr}\left(v \in T_pM \mapsto \nabla_v Y \in T_pM\right) = c(\nabla Y)$$

For $Y = Y^i e_i$, $\nabla Y = Y^i_{;j} e_i \otimes e^j$ where $Y^i_{;j} = e_j(Y^i) + \Gamma^i_{jk} Y^k$ as in (64). Therefore

$$\mathrm{div}(Y) = Y^i_{;i} = e_i(Y^i) + \Gamma^i_{ik} Y^k = \sum_i \frac{\partial}{\partial x_i} Y^i + \sum_{i,k=1}^n \Gamma^i_{ik} Y^k \tag{66}$$

Lemma 21.3. *Given $Y \in \mathfrak{X}(M)$ and* $\mathrm{div}Y$ *as in (66)*

$$\mathrm{div}Y = \frac{1}{\sqrt{\det(g)}} \sum_i \frac{\partial}{\partial x_i}\left(\sqrt{\det(g)} Y^i\right) \tag{67}$$

Proof. Using Jacobi's Formula

$$\frac{\partial}{\partial x_i}(\det(g)) = \det(g) \mathrm{Tr}(g^{-1} \frac{\partial g}{\partial x_i})$$

We look at

$$\begin{aligned}
\sum_{i=1}^n \Gamma^i_{ik} &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n g^{ij}(g_{ij,k} + g_{kj,i} - g_{ik,j}) = \frac{1}{2} \sum_{i,j=1}^n g^{ij} \frac{\partial}{\partial x_k} g_{ij} + \frac{1}{2}\left(\sum_{ij} g^{ij} g_{kj,i} - g^{ji} g_{jk,i}\right) \\
&= \frac{1}{2} \sum_{i,j=1}^n g^{ij} \frac{\partial}{\partial x_k} g_{ij} = \frac{1}{2} \mathrm{Tr}(g^{-1} \frac{\partial}{\partial x_k} g) = \frac{1}{2} \frac{1}{\det(g)} \frac{\partial}{\partial x_k}(\det(g)) \\
&= \frac{1}{2} \frac{\partial}{\partial x_k} \log(\det(g)) = \frac{\partial}{\partial x_k} \log(\sqrt{\det(g)}) = \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x_k}\left(\sqrt{\det(g)}\right)
\end{aligned}$$

Hence

$$\begin{aligned}
\mathrm{div}(Y) &= \sum_i \frac{\partial}{\partial x_i} Y_i + \sum_{i,k} \Gamma^i_{ik} Y^k \\
&= \sum_k \frac{\partial}{\partial x_k} Y_k + \sum_k \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x_k}\left(\sqrt{\det(g)}\right) Y^k \\
&= \frac{1}{\sqrt{\det(g)}} \sum_i \frac{\partial}{\partial x_i}\left(\sqrt{\det(g)} Y^i\right)
\end{aligned}$$

□

Definition 21.5 (Hessian)**.** *For $f \in C^\infty(M)$, define $(0,2)$-tensor* $\mathrm{Hess}f \in C^\infty(M, T^0_2 M)$

$$\mathrm{Hess}(f) = \nabla\nabla f = \nabla df$$

hence $\mathrm{Hess}f \in C^\infty(M, Sym^2 T^*M)$ *symmetric $(0,2)-$tensor s.t.*

$$\begin{aligned}
\mathrm{Hess}(f)(X,Y) &= (\nabla df)(X,Y) = (\nabla_Y df)(X) = Y(df(X)) - df(\nabla_Y X) \\
&= YX(f) - (\nabla_Y X)f \\
&= XY(f) - (\nabla_X Y)f \\
&= \mathrm{Hess}(f)(Y,X)
\end{aligned}$$

Where $\nabla_X Y - \nabla_Y X = [X,Y]$ and we've used ∇ compatibility with the metric. Define $f_{;ij}$ s.t.

$$\nabla\nabla f = \nabla df = \nabla(f_{;i} e^i) = \sum_{i,j} f_{;ij} e^i \otimes e^j$$

so one may calculate

$$f_{;ij} = e_j(f_{;i}) - \Gamma^k_{ij} f_k = \sum_{i,j} \frac{\partial f}{\partial x_i \partial x_j} - \sum_k \Gamma^k_{ij} \frac{\partial f}{\partial x_k} \tag{68}$$

Definition 21.6 (Laplacian)**.** *For $f \in C^\infty(M)$, define smooth function $\Delta f \in C^\infty(M)$ s.t.*

$$\Delta f := \operatorname{div}(\operatorname{grad} f) = \operatorname{div}(f_;^i e_i) = f_{;\ i}^{i} = f_{;ij} g^{ij}$$

For $e_i = \frac{\partial}{\partial x_i}$ we have

$$\Delta f = \sum_{i,j} g^{ij} \left(\frac{\partial \partial f}{\partial x_i \partial x_j} - \sum_k \Gamma_{ij}^k \frac{\partial f}{\partial x_k} \right)$$

For $g_{ij} = \delta_{ij}$ we recover

$$\Delta f = \sum_i \frac{\partial^2 f}{\partial {x_i}^2}$$

Lemma 21.4. *In local coordinates, for $f \in C^\infty(M)$*

$$\Delta f = \frac{1}{\sqrt{\det(g)}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{\det(g)} g^{ij} \frac{\partial f}{\partial x_j} \right) \tag{69}$$

Proof. Using $\Delta f = \operatorname{div}(\operatorname{grad} f)$ where

$$\operatorname{grad} f = \sum_{ij} g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}$$

plugging in (69) we have the result. □