[Giorgi2025] Modern Geometry II

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Contents

1 Jacobi Fields

What is the motivation for Jacobi Fields? How geodesics vary in a manifold. This depends on the metric, and in fact, completely determined by the curvature. Let (M, g) be a Riemannian manifold. A Jacobi field $J(t)$ is a C^∞ vector field on M defined along a geodesic

$$\gamma : [0, a] \to M$$

that arises in the following way. Consider a smooth map

$$f : (-\varepsilon, \varepsilon) \times [0, a] \to M \qquad s.t. \qquad (s, t) \mapsto f(s, t) = f_s(t)$$

and

$$f(0, t) = \gamma(t)$$

We want this map to be a family of geodesics as parametrized by $s \in (-\varepsilon, \varepsilon)$, i.e., for each s

$$f_s : [0, a] \to M \qquad t \mapsto f_s(t)$$

is a geodesic. How to connect with γ? We set

$$f_0 = \gamma$$

Then we define

$$J(t) := \frac{\partial f}{\partial s}(0, t) \qquad \forall\ t \in [0, a]$$

Hence J is a vector field on γ.

1.1 Jacobi Equation

We want to derive an equation out of the motivation for J. Later we'll see this is equivalent to the definition.

Lemma 1.1. *Denote $A := (-\varepsilon, \varepsilon) \times [0, a] \subset \mathbb{R}^2$. $f : A \to M$ is smooth map, and $\frac{\partial}{\partial s}$, $\frac{\partial}{\partial t}$ are C^∞ vector fields on M.*

$$\frac{\partial f}{\partial s} := f_*(\frac{\partial}{\partial s}) \qquad \frac{\partial f}{\partial t} := f_*(\frac{\partial}{\partial t}) \quad \in C^\infty(A, f^*TM)$$

Let ∇ be Levi-Civita connection on (M, g). Denote $D := f^*\nabla$ as the pullback connection on A. Then note curvature comes up (curvature has to do with commutators of vector fields). Recall the pullback of a connection

$$D(f_*Y) = (f^*\nabla)(f_*Y) := f_*(\nabla Y) \qquad \forall\ Y \in \mathfrak{X}(A), \quad f_*Y \in C^\infty(A, f^*TM)$$

1. *Hence one computes (using that Levi-Civita connection implies ∇ symmetric)*

$$\begin{aligned} \frac{D}{ds}\frac{\partial f}{\partial t} - \frac{D}{dt}\frac{\partial f}{\partial s} &= D_{\frac{\partial}{\partial s}} f_*(\frac{\partial}{\partial t}) - D_{\frac{\partial}{\partial t}} f_*(\frac{\partial}{\partial s}) \\ &= f^*\nabla_{\frac{\partial}{\partial s}} f_*(\frac{\partial}{\partial t}) - f^*\nabla_{\frac{\partial}{\partial t}} f_*(\frac{\partial}{\partial s}) \\ &= f_*\left(\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t}\right) - f_*\left(\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s}\right) = f_*\left([\frac{\partial}{\partial s}, \frac{\partial}{\partial t}]\right) = 0 \end{aligned} \tag{1}$$

2. *We differentiate once more to see curvature comes up.*

$$\frac{D}{dt}\frac{D}{ds}f_*(\frac{\partial}{\partial t}) - \frac{D}{ds}\frac{D}{dt}f_*(\frac{\partial}{\partial t}) + D_{[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}]}f_*(\frac{\partial}{\partial t}) = (f^*R)(\frac{\partial}{\partial s}, \frac{\partial}{\partial t})(f_*\frac{\partial}{\partial t})$$

by definition of $R_{f^*\nabla}$. Now by previous computations the first term

$$\frac{D}{ds}f_*(\frac{\partial}{\partial t}) = \frac{D}{dt}f_*(\frac{\partial}{\partial s})$$

Hence

$$\frac{D^2}{dt^2}\frac{\partial f}{\partial s} - \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} = R(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t})\frac{\partial f}{\partial t} \tag{2}$$

Notice (1) and (2) are true for any C^∞ map $f : A \to M$.

3. *Now we impose our setup, in addition that f_s is a geodesic for any $s \in (-\varepsilon, \varepsilon)$, i.e.*

$$\frac{D}{dt}\frac{\partial f_s}{\partial t} = \frac{D}{dt}\frac{\partial f}{\partial t}(s,t) = 0 \qquad \forall\ s \in (-\varepsilon, \varepsilon)$$

In particular the second term in (2) vanishes. Hence we're left with two terms

$$\frac{D^2}{dt^2}\frac{\partial f}{\partial s} + R(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s})\frac{\partial f}{\partial t} = 0 \tag{3}$$

If set $s = 0$, then since we've defined

$$J(t) := \frac{\partial f}{\partial s}(0,t)$$

We let

$$\frac{\partial f}{\partial t}(0,t) = \frac{d}{dt}\gamma(t) = \gamma'(t)$$

Then (3) writes

$$\frac{D^2}{dt^2}J(t) + R(\gamma', J(t))\gamma' = 0 \qquad \forall\ t \in [0,a] \tag{4}$$

This is the Jacobi Equation.

Definition 1.1 (Jacobi Field)**.** *A C^∞ vector field $J(t)$ along a geodesic*

$$\gamma : [0,a] \to M$$

is called a Jacobi Field if it satisfies the Jacobi Equation (4).

Proposition 1.1 (Existence and Uniqueness of Jacobi Field)**.** *Let*

$$\gamma : [0,a] \to M$$

be a geodesic s.t.

$$\gamma(0) = p \qquad \gamma'(0) = v \in T_pM$$

Hence

$$\gamma(t) = \exp_p(tv)$$

is determined by the exponential map.

1. *For any u, $w \in T_pM$, there exists a unique Jacobi Field J along γ s.t.*

$$J(0) = u, \qquad \frac{D}{dt}J(0) = w$$

2. *If $J(t)$ is a Jacobi Field along γ, then there exists (6)*

$$f : (-\varepsilon, \varepsilon) \times [0,a] \to M \qquad s.t. \qquad (s,t) \mapsto f(s,t) = f_s(t)$$

and

   (a) *for any $s \in (-\varepsilon, \varepsilon)$, $f_s : [0,a] \to M$ is a geodesic.*
   (b) *$f_0 = \gamma$.*
   (c) *$\frac{\partial f}{\partial s}(0,t) = J(t)$.*

Example 1.1. *In $(\mathbb{R}^n, g_0)$, the geodesics are*

$$\gamma(t) = p + tv \qquad p, v \in \mathbb{R}^n$$

Now our Jacobi Field writes

$$J(t) = u + tw \qquad \forall\ u, w \in \mathbb{R}^n$$

and f writes

$$f(s,t) = p + su + t(v + sw)$$

for fixed p, v, u, w. f is in fact

$$f(s,t) = \exp_{p+su}(t(v + sw))$$

Proof of Proposition 1.1. 1. To get an ODE out of (4) we need to take $\{e_1, \cdots, e_n\}$ ONB of T_pM. Then we think of parallel transport. Let $e_1(t), \cdots, e_n(t)$ be the parallel transport of $e_1, \cdots, e_n$ along $\gamma(t)$, i.e.

$$\begin{cases} D_{\frac{\partial}{\partial t}} e_i(t) = 0 \\ e_i(0) = e_i \end{cases}$$

Hence $\{e_i(t)\}_{1 \le i \le n}$ forms an ONB of $T_{\gamma(t)}M$ for every $t \in [0, a]$. For any $J(t)$ as C^∞ vector fields along $\gamma(t)$, we can write

$$J(t) = \sum_{i=1}^{n} f_i(t) e_i(t)$$

for $f_i : [0, a] \to \mathbb{R}$. Then $J(t)$ is a Jacobi Field iff (4) is satisfied iff

$$\sum_{i=1}^{n} f_i''(t) e_i(t) + f_i(t) R(\gamma'(t), e_i)\gamma'(t) = 0$$

We take inner product with $e_j(t)$ for each of these and selects

$$f_j''(t) + \sum_{i=1}^{n} f_i(t) R(\gamma'(t), e_j(t), \gamma'(t), e_j(t)) = 0 \qquad \forall\ j = 1, \cdots, n \tag{5}$$

Denote

$$A_{ij}(t) := R(\gamma'(t), e_j(t), \gamma'(t), e_j(t))$$

Then we write

$$f_j''(t) + \sum_{i=1}^{n} A_{ij}(t) f_i(t) = 0$$

Hence we have

$$\frac{d^2}{dt^2}\mathbf{f} + A\mathbf{f} = 0$$

where we apply Existence and Uniqueness of ODE.

2. Set $u := J(0)$ and $w := \frac{D}{dt} J(0)$. Let

$$\lambda : (-\varepsilon, \varepsilon) \to M \qquad \lambda(s) := \exp_p(su)$$

Let $v(s)$, $w(s)$ be parallel transport along $\lambda(s)$. Define

$$f : (-\varepsilon, \varepsilon) \times [0, a] \to M \qquad f(s, t) := \exp_{\lambda(s)}(t(v(s) + sw(s))) \tag{6}$$

We need to check

(a) For each s, f_s is the unique geodesic that starts at $f_s(0) = \lambda(s)$ and with

$$f_s'(0) = v(s) + sw(s)$$

(b) $f_0(t) = \exp_{\lambda(0)}(t(v(0) + 0)) = \exp_p(tv) = \gamma(t)$.

(c) $\overline{J}(t) = \frac{\partial f}{\partial s}(0, t)$ is a Jacobi Field by our previous derivation. Check

$$\overline{J}(0) = \frac{\partial f}{\partial s}(0, 0) = \lambda'(0) = u$$
$$\frac{D}{dt}\overline{J}(0) = \frac{D}{dt}\frac{\partial f}{\partial s}(0, 0) = \frac{D}{ds}\frac{\partial f}{\partial t}(0, 0) = w(0) = w$$

where the second line follows from

$$\frac{\partial f}{\partial t}(s, 0) = v(s) + sw(s)$$
$$\frac{D}{ds}\frac{\partial f}{\partial t}(s, 0) = w(s)$$

Since they have same initial conditions, we conclude by uniqueness.

□

Remark 1.1 ($u = 0$)**.** *In the special case $u = 0$, $J(t)$ Jacobi field along $\gamma(t)$ with*

$$\gamma(0) = p \qquad \gamma'(0) = v \qquad J(0) = 0 \qquad J'(0) = w$$

Then

$$\lambda(s) = p \qquad \text{and} \qquad f(s,t) = \exp_p(t(v(s) + sw(s)))$$

and so

$$J(t) = \frac{\partial f}{\partial s}(0,t) = (d\exp_p)_{tv}(tw) \tag{7}$$

For λ fixed its easier to take derivatives.

Lemma 1.2 ($\langle J, \gamma'\rangle(t)$)**.** *Let*

$$\gamma : [0,a] \to M$$

be geodesic in M, J Jacobi field along γ. Then $\langle J, \gamma'\rangle(t)$ is linear function in t

$$\langle J, \gamma'\rangle(t) = \langle J(0), \gamma'(0)\rangle + t\langle J'(0), \gamma'(0)\rangle \qquad J'(0) := \frac{D}{dt}J(0) \tag{8}$$

Proof. Let

$$\begin{aligned} f(t) &= \langle J, \gamma'\rangle(t) \\ f'(t) &= \langle J', \gamma'\rangle(t) \qquad \text{using } D_{\frac{\partial}{\partial t}}\gamma' = 0 \\ f''(t) &= \langle J'', \gamma'\rangle(t) = \langle -R(\gamma', J)\gamma', \gamma'\rangle = 0 \qquad \text{by anti-symmetry of } R \end{aligned}$$

Hence f is a linear function in t. □

Remark 1.2 (Decomposition of J into γ' and $t\gamma'(t)$)**.** *In fact γ' and $t\gamma'(t)$ are examples of Jacobi Fields along $\gamma(t)$. Indeed*

$$\frac{D^2}{dt^2}\gamma'(t) + R(\gamma', \gamma')\gamma' = 0 \qquad \text{using } \frac{D}{dt}\gamma' = 0$$

and

$$\begin{aligned} \frac{D^2}{dt^2}(t\gamma'(t)) &= \frac{D}{dt}(\gamma'(t)) = 0 \\ R(\gamma', t\gamma')\gamma' &= 0 \end{aligned}$$

In particular given initial conditions we can explicitly write J using γ' and $t\gamma'$. For any J Jacobi field along γ we have decomposition

$$J(t) = (\langle J(0), \gamma'(0)\rangle + t\langle J'(0), \gamma'(0)\rangle)\frac{\gamma'(t)}{|\gamma'(0)|^2} + J^{\perp}(t) \qquad \text{where } \langle J^{\perp}(t), \gamma'(t)\rangle = 0$$

Hence it suffices to consider Jacobi Fields normal w.r.t. $\gamma'(t)$.

Proposition 1.2 (Killing vector field induced Jacobi Field)**.** *Let*

$$\gamma : [0,a] \to M$$

be a geodesic and let X be a Killing Vector field on M. Then

(a) The restriction $X(\gamma(s))$ of X to $\gamma(s)$ is a Jacobi Field along γ.

(b) As a consequence of above, if M is connected and there exists $p \in M$ s.t.

$$X(p) = 0 \qquad \text{and} \qquad \nabla_Y X(p) = 0 \qquad \forall\ Y \in T_pM$$

Then $X \equiv 0$ on M.

Proof. 1. We first show (a). Since X is a Killing Vector Field, its flow

$$\varphi_t : U \subset M \to M \qquad q \mapsto \varphi(t,q) = \varphi_t(q) \qquad \forall\ t \in (-\varepsilon, \varepsilon)$$

is a 1-parameter subgroup of isometries on (M, g) with $\varphi_0 = \mathrm{Id}$. The flow φ_t relates to X via

$$\varphi_t(q) \text{ is the trajectory of } X \text{ passing through } q \text{ at } t = 0 \text{ for any } q \in U$$

or in other words using X as integral curve

$$X(\varphi(t,\gamma(s))) = \frac{\partial}{\partial t}\varphi(t,\gamma(s)) = \frac{d}{dt}\varphi_t(\gamma(s))$$
$$\gamma(s) = \varphi(0,\gamma(s))$$

Since image of the geodesic γ by a family of isometries remains a geodesic,

$$\phi_t(s) = \varphi_t(\gamma(s)) \qquad \forall\ t \in (-\varepsilon, \varepsilon)$$

are a 1-parameter family of geodesics on (M, g). Thus restriction of X to γ is a variational field

$$X(\gamma(s)) = \left.\frac{d}{dt}\right|_{t=0} \phi_t(s)$$

of γ by geodesics. Hence $X(\gamma(s))$ is Jacobi Field along γ.

2. We prove (b). From (a) we know

$$J(s) := X(\gamma(s)) \qquad \forall\ s \in [0, a]$$

defines a Jacobi Field along γ. We first conduct a simple computation using Definition of pullback section

$$\frac{D}{ds}J(s) = \gamma^*\nabla_{\frac{d}{ds}}(X(\gamma(s))) = \gamma^*\nabla_{\frac{d}{ds}}(\gamma^*X(s))$$
$$= \nabla_{\gamma_*\frac{d}{ds}}X = \nabla_{\gamma'(s)}X$$

Notice assumptions imply

$$X(p) = 0 \implies X(\gamma(0)) = J(0) = 0$$

and

$$\nabla_Y X(p) = 0 \qquad \forall\ Y \in T_{\gamma(0)}M \implies \text{choosing } Y = \gamma'(0) \qquad \nabla_{\gamma'(0)}X = \frac{D}{ds}J(0) = 0$$

Hence by Existence and Uniqueness Theorem,

$$J(s) = X(\gamma(s)) \equiv 0$$

is the unique Jacobi Field along γ. But now since M is connected, for any other point $q \in M$, there exists smooth curve connecting p to q. Covering the curve by geodesic segments and applying previous argument, one obtain

$$X(q) = 0 \qquad \forall\ q \in M$$

□

1.2 Jacobi Fields on Constant Sectional Curvature Manifolds

Let (M, g) be Riemannian manifold with constant sectional curvature K. Let

$$\gamma : [0, a] \to M$$

be a normalized geodesic, i.e., $|\gamma'(t)| = 1$. Take a Jacobi field of the special case type where $u = 0$ 1.1 along γ with

$$J(0) = 0 \qquad \frac{D}{dt}J(0) = w \qquad \text{s.t.} \qquad \langle w, \gamma'(0)\rangle = 0$$

which is to say J along γ is normal w.r.t. γ'. Indeed, by Lemma (8)

$$\langle J, \gamma'\rangle(t) = \langle J(0), \gamma'(0)\rangle + t\langle w, \gamma'(0)\rangle = 0 \qquad \forall\ t \in [0, a]$$

Let V be C^∞ vector field along γ. Then using an equivalent condition for constant sectional curvature and Riemannian Curvature

$$\langle R(\gamma', J)\gamma', V\rangle = R(\gamma', J, \gamma', V) = K\left(\langle\gamma',\gamma'\rangle\langle J, V\rangle - \langle\gamma', J\rangle\langle\gamma', V\rangle\right)$$
$$= \langle KJ, V\rangle \qquad \text{using } \langle\gamma',\gamma'\rangle = 1 \text{ and } \langle\gamma', J\rangle = 0$$

Hence

$$R(\gamma', J)\gamma' = KJ$$

and our Jacobi Equation (4) writes

$$\frac{D^2}{dt^2}J + KJ = 0 \tag{9}$$

Solving Jacobi Field for constant sectional curvature with Initial $\frac{D}{dt}J(0)$. Let $w(t)$ be parallel transport of w along $\gamma(t)$ with $w(0) = w$ so

$$\langle w(t), \gamma'(t)\rangle = 0 \qquad |w(t)| = 1$$

We look for solutions of the form

$$J(t) = f(t)w(t) \qquad f : [0, a] \to \mathbb{R}$$

Then equation (9) writes, given nontrivial initial condition $\frac{D}{dt}J(0) = w$

$$\begin{aligned} \frac{D^2}{dt^2}J + KJ &= 0 \\ J(0) &= 0 \\ \frac{D}{dt}J(0) &= w \end{aligned}$$

This is equivalent to system of equations on f

$$\begin{aligned} \frac{d^2}{dt^2}f(t) + Kf(t) &= 0 \\ f(0) &= 0 \\ f'(0) &= 1 \end{aligned}$$

Now this has unique solution. So the Jacobi field

$$J = fw$$

that we find this way is the unique solution of

$$\frac{D^2}{dt^2}J + KJ = 0$$

Solutions to system of equations in f and J are given by

$$f(t) = \begin{cases} \frac{\sin(\sqrt{K}t)}{\sqrt{K}} & K > 0 \\ t & K = 0 \\ \frac{\sinh(\sqrt{-K}t)}{\sqrt{-K}} & K < 0 \end{cases}$$

$$J(t) = \begin{cases} \frac{\sin(\sqrt{K}t)}{\sqrt{K}}w(t) & K > 0 \\ tw(t) & K = 0 \\ \frac{\sinh(\sqrt{-K}t)}{\sqrt{-K}}w(t) & K < 0 \end{cases} \tag{10}$$

□

Solving Jacobi Field for constant sectional curvature with Initial $J(0)$. Similarly, if write

$$J(t) = f(t)u(t)$$

for $u(t)$ the parallel transport of u along γ, it takes initial conditions

$$\begin{aligned} \frac{D^2}{dt^2}J + KJ &= 0 \\ J(0) &= u \\ \frac{D}{dt}J(0) &= 0 \end{aligned}$$

Then this corresponds to

$$\begin{aligned} \frac{d^2}{dt^2}f(t) + Kf(t) &= 0 \\ f(0) &= 1 \\ f'(0) &= 0 \end{aligned}$$

Solutions write

$$f(t) = \begin{cases} \frac{\cos(\sqrt{K}t)}{\sqrt{K}} & K > 0 \\ 1 & K = 0 \\ \frac{\cosh(\sqrt{-K}t)}{\sqrt{-K}} & K < 0 \end{cases}$$

$$J(t) = \begin{cases} \frac{\cos(\sqrt{K}t)}{\sqrt{K}}u(t) & K > 0 \\ u(t) & K = 0 \\ \frac{\cosh(\sqrt{-K}t)}{\sqrt{-K}}u(t) & K < 0 \end{cases}$$

□

In general, it's a combination between these two solutions. What we did here is the orthogonal part in Remark 1.2.

Example 1.2 (Sphere)**.** *Take $\mathbb{S}^2$ round sphere of radius 1. Take $p = (0, 0, 1)$ to be north pole. Consider $v \in T_p\mathbb{S}^2$. The exponential map sends circles of radius ρ centered at origin to circles*

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = \sin^2(\rho),\ z = \cos(\rho)\}$$

Then let (ρ, θ) be polar coordinates on $T_p\mathbb{S}^2 = \mathbb{R}^2$. By Gauss Lemma

$$\exp_p^*(dx^2 + dy^2 + dz^2) = d\rho^2 + \sin^2 \rho d\theta^2$$

More generally, given $K > 0$, consider sphere of radius $\frac{1}{\sqrt{K}}$

$$\mathbb{S}^2(\frac{1}{\sqrt{K}}) := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = \frac{1}{K}\}$$

with constant sectional curvature K. Let $p = (0, 0, \frac{1}{\sqrt{K}})$ and the exponential map

$$\exp_p : T_p\mathbb{S}^2(\frac{1}{\sqrt{K}}) \to \mathbb{S}^2(\frac{1}{\sqrt{K}}) \qquad \{\textit{circles of radius } \rho\} \mapsto \{(x, y, z) \in \mathbb{R}^3 \mid x^2+y^2 = \frac{\sin^2(\sqrt{K}\rho)}{\sqrt{K}},\ z = \frac{\cos(\sqrt{K}\rho)}{\sqrt{K}}\}$$

Let (ρ, θ) be the polar coordinates on $\mathbb{R}^2 = T_p\mathbb{S}^2(\frac{1}{\sqrt{K}})$, then

$$\exp_p^*(dx^2 + dy^2 + dz^2) = d\rho^2 + \left(\frac{\sin(\sqrt{K}\rho)}{\sqrt{K}}\right)^2 d\theta^2 \tag{11}$$

In general, we want to define some polar coordinates on M. Let

$$F : (0, \delta) \times \mathbb{S}^{n-1} \to B_\delta(p) = \{\text{normal ball centered at } p \text{ with radius } \delta > 0\} \subset M \qquad (\rho, v) \mapsto \exp_p(\rho v)$$

We compute the differential. Then we can describe what the differential map does.

$$\begin{aligned} dF_{(\rho,v)} : T_{(\rho,v)}((0, \delta) \times \mathbb{S}^{n-1}) = \mathbb{R}\frac{\partial}{\partial \rho} \oplus T_v\mathbb{S}^{n-1} &\to T_{F(\rho,v)}M \\ (dF_{(\rho,v)})(\frac{\partial}{\partial \rho}) &= (d\exp_p)_{(\rho,v)}(v) \\ (dF_{(\rho,v)})(w) &= (d\exp_p)_{(\rho,v)}(\rho w) \qquad \text{where } w \in T_v\mathbb{S}^{n-1} = \{w \in \mathbb{R}^n \mid \langle w, v\rangle = 0\} \end{aligned}$$

Recall special case $u = 0$ yields (7). Hence in fact $(dF_{(\rho,v)})(w)$ is the Jacobi Field,

$$(dF_{(\rho,v)})(w) = (d\exp_p)_{(\rho,v)}(\rho w) = f_K(\rho)w(\rho v)$$

In particular, we've used Gauss Lemma which says exponential map is isometry

$$\begin{aligned} \langle (d\exp_p)(v), (d\exp_p)(v)\rangle &= \langle v, v\rangle = 1 \\ \langle (d\exp_p)(v), (d\exp_p)(\rho w)\rangle &= \langle v, \rho w\rangle = 0 \end{aligned}$$

Let (M, g) be our manifold with metric g.

$$F^*g = (\exp_p)^*g = d\rho^2 + f_K^2(\rho)g_{\text{can}}^{\mathbb{S}^{n-1}} = \begin{cases} d\rho^2 + \frac{\sin^2(\sqrt{K}\rho)}{K}g_{\text{can}}^{\mathbb{S}^{n-1}} & K > 0 \\ d\rho^2 + \rho^2 g_{\text{can}}^{\mathbb{S}^{n-1}} & K = 0 \\ d\rho^2 + \frac{\sinh^2(\sqrt{-K}\rho)}{-K}g_{\text{can}}^{\mathbb{S}^{n-1}} & K < 0 \end{cases}$$

1.3 Taylor Expansion of g_{ij} in Local Coordinates

In normal coordinates (as embedded in the definition of tensors)

$$g_{ij}(p) = \delta_{ij}, \qquad \frac{\partial g_{ij}}{\partial x_k}(p) = 0$$

We want to look at its Taylor Expansion.

Proposition 1.3 (Taylor Expansion of $|J(t)|^2$ in Riemannian curvature)**.** *Let (M, g) be Riemannian Manifold. $p \in M$*

$$\gamma : [0, a] \to M$$

be geodesic with

$$\gamma(0) = p \qquad \gamma'(0) = v$$

Let $J(t)$ be Jacobi Field along $\gamma(t)$ with

$$J(0) = 0 \qquad \frac{D}{dt}J(0) = w$$

Hence implies

$$\gamma(t) = \exp_p(tv) \qquad J(t) = (d\exp_p)_{tv}(tw)$$

Then

$$\begin{aligned} |J(t)|^2 = \langle w, w\rangle t^2 - \frac{1}{3}R(v, w, v, w)t^4 - \frac{1}{6}(\nabla_v R)(v, w, v, w)t^5 \\ + \left(\frac{2}{45}\langle R(v, w)v, R(v, w)v\rangle - \frac{1}{20}(\nabla_v\nabla_v R)(v, w, v, w)\right)t^6 + o(t^6) \end{aligned}$$

Proof. Let $f(t) = \langle J(t), J(t)\rangle$. Need to compute $f^{(k)}(0)$ for $0 \le k \le 6$.

$$\begin{aligned} f'(t) &= 2\langle J'(t), J(t)\rangle \\ f''(t) &= 2\langle J^{(2)}(t), J(t)\rangle + 2\langle J'(t), J'(t)\rangle \\ f^{(3)}(t) &= 2\langle J^{(3)}(t), J(t)\rangle + 6\langle J^{(2)}(t), J(t)\rangle \\ f^{(4)}(t) &= 2\langle J^{(4)}(t), J(t)\rangle + 8\langle J^{(3)}(t), J'(t)\rangle + 6\langle J^{(2)}(t), J^{(2)}(t)\rangle \\ f^{(5)}(t) &= 2\langle J^{(5)}(t), J(t)\rangle + 10\langle J^{(4)}(t), J'(t)\rangle + 20\langle J^{(3)}(t), J^{(2)}(t)\rangle \\ f^{(6)}(t) &= 2\langle J^{(6)}(t), J(t)\rangle + 12\langle J^{(5)}(t), J'(t)\rangle + 30\langle J^{(4)}(t), J^{(2)}(t)\rangle + 20\langle J^{(3)}(t), J^{(3)}(t)\rangle \end{aligned}$$

We have $J(0) = 0$, $J'(0) = w$. Notice we're about to compute

$$\begin{aligned} \frac{D}{dt}(R(\gamma', J)\gamma') &\equiv \nabla_{\gamma'}(R(\gamma', J)\gamma') \\ &= (\nabla_{\gamma'}R)(\gamma', J)\gamma' + R(\frac{D}{dt}\gamma', J)\gamma' + R(\gamma', \frac{D}{dt}J)\gamma' + R(\gamma', J)\frac{D}{dt}\gamma' \\ &= (\nabla_{\gamma'}R)(\gamma', J)\gamma' + R(\gamma', \frac{D}{dt}J)\gamma' \qquad \text{using that } \gamma \text{ is geodesic} \end{aligned}$$

where the second line follows from

$$\begin{aligned} \nabla_W(R(X, Y, Z, T)) &= (\nabla_W R)(X, Y, Z, T) + R(\nabla_W X, Y, Z, T) + \cdots + R(X, Y, Z, \nabla_W T) \qquad \forall\ T \\ \nabla_W(\langle R(X, Y)Z, T\rangle) &= (\nabla_W R)(X, Y, Z, T) + R(\nabla_W X, Y, Z, T) + \cdots + R(X, Y, Z, \nabla_W T) \\ \langle(\nabla_W(R(X, Y)Z), T\rangle &= \nabla_W(\langle R(X, Y)Z, T\rangle) - R(X, Y, Z, \nabla_W T) \qquad \text{by definition} \\ &= \langle(\nabla_W R)(X, Y)Z, T\rangle + \langle R(\nabla_W X, Y)Z, T\rangle + \langle R(X, \nabla_W Y)Z, T\rangle + \langle R(X, Y)\nabla_W Z, T\rangle \end{aligned}$$

Thus

$$\begin{aligned} J'' &= -R(\gamma', J)\gamma' \implies J''(0) = -R(v, 0)v = 0 \\ J^{(3)} &= -R'(\gamma', J)\gamma' - R(\gamma', J')\gamma' \implies J^{(3)}(0) = -R(v, w)v \\ J^{(4)} &= -R''(\gamma', J)\gamma' - 2R'(\gamma', J')\gamma' - R(\gamma', J'')\gamma' \implies J^{(4)}(0) = -2(\nabla_v R)(v, w)v \\ J^{(5)} &= -R'''(\gamma', J)\gamma' - 3R''(\gamma', J')\gamma' - 3R'(\gamma', J'')\gamma' - R(\gamma', J''')\gamma' \implies J^{(5)}(0) = -3(\nabla_v\nabla_v R)(v, w)v + R(v, R(v, w)w)w \end{aligned}$$

So

$$
\begin{aligned}
f(0) &= 0 \\
f'(0) &= 0 \\
f''(0) &= 2\langle w, w\rangle \\
f^{(3)}(0) &= 0 \\
f^{(4)}(0) &= 8\langle -R(v,w)v, w\rangle \\
f^{(5)}(0) &= 10\langle -2(\nabla_v R)(v,w)v, w\rangle \\
f^{(6)}(0) &= 12\langle -3\nabla_v\nabla_v R(v,w)v + R(v, R(v,w)v)v, w\rangle + 20\langle R(v,w)v, R(v,w)v\rangle \\
&= -36\langle \nabla_v\nabla_v R(v,w)v, w\rangle + 32\langle R(v,w)v, R(v,w)v\rangle
\end{aligned}
$$

Then

$$
\begin{aligned}
f(t) = &\frac{1}{2!}2\langle w,w\rangle t^2 - \frac{1}{4!}8\langle R(v,w)v, w\rangle t^4 - \frac{1}{5!}20\langle (\nabla_v R)(v,w)v, w\rangle t^5 \\
&+ \frac{1}{6!}\left(-36\langle \nabla_v\nabla_v R(v,w)v, w\rangle + 32\langle R(v,w)v, R(v,w)v\rangle\right) t^6 + o(t^6)
\end{aligned}
$$

□

Corollary 1.1 (Taylor Expansion of $|J(t)|^2$ in Sectional Curvature)**.** *Take v, w orthonormal, i.e.*

$$|v| = |w| = 1 \qquad \langle v, w\rangle = 0$$

Let

$$\sigma = \mathbf{Span}(v, w)$$

Then for $t > 0$

$$
\begin{aligned}
|J(t)|^2 &= t^2 - \frac{1}{3}K(\sigma)t^4 + o(t^4) \\
|J(t)| &= t\left(1 - \frac{1}{3}K(\sigma)t^2 + o(t^2)\right)^{\frac{1}{2}} = t - \frac{1}{6}K(\sigma)t^3 + o(t^3) \qquad \textit{using } (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + o(x^2)
\end{aligned}
\tag{12}
$$

Taylor Expansion for g_{ij}. Now write

$$J(t) = (d\exp_p)_{tv}(tw)$$

One has

$$f(t) = \langle J(t), J(t)\rangle = \langle (d\exp_p)_{tv}(tw), (d\exp_p)_{tv}(tw)\rangle = t^2\langle (d\exp_p)_{tv}(w), (d\exp_p)_{tv}(w)\rangle$$

By polarizing for u, $w \in T_pM$

$$
\begin{aligned}
\langle (d\exp_p)_{tv}(u), (d\exp_p)_{tv}(w)\rangle = &\langle u, w\rangle - \frac{1}{3}R(v,w,v,u)t^2 - \frac{1}{6}(\nabla_v R)(v,w,v,u)t^3 \\
&+ \left(\frac{2}{45}\langle R(v,w)v, R(v,u)v\rangle - \frac{1}{20}(\nabla_v\nabla_v R)(v,w,v,u)\right)t^4 + o(t^4)
\end{aligned}
$$

Now for $|v|$ small. One can deduce via Taylor Expansion around 0

$$
\begin{aligned}
\langle (d\exp_p)_{v}(u), (d\exp_p)_{v}(w)\rangle = &\langle u, w\rangle - \frac{1}{3}R(v,w,v,u) - \frac{1}{6}(\nabla_v R)(v,w,v,u) \\
&+ \frac{2}{45}\langle R(v,w)v, R(v,u)v\rangle - \frac{1}{20}(\nabla_v\nabla_v R)(v,w,v,u) + o(|v|^4)
\end{aligned}
$$

Let $\{e_1, \cdots, e_n\}$ as ONB basis for T_pM. Consider normal ball $B_\delta(p) \subset M$ and point $q \in B_\delta(p)$. Then q is viewed as endpoint of geodesic starting from p with velocity as linear combination of e_i. In particular

$$q = \exp_p(\sum_k x_k e_k) \in B_\delta(p) \qquad \sum_k x_k e_k \in T_pM, \text{ and } x_k \text{ are the normal coordinates associated to } \{e_1, \cdots e_n\}$$

Then

$$\left.\frac{\partial}{\partial x_i}\right|_q = (d\exp_p)_{\sum_k x_k e_k}(e_i)$$

So

$$g_{ij}(x_1, \cdots, x_n) = \langle \frac{\partial}{\partial x_i}\bigg|_q, \frac{\partial}{\partial x_j}\bigg|_q \rangle = \langle (d\exp_p)_{\sum_k x_k e_k}(e_i), (d\exp_p)_{\sum_k x_k e_k}(e_j) \rangle$$

Now apply with $v = \sum_k x_k e_k \in T_pM$ for $|x_k|$ small, and with $u = e_i$, $w = e_j$. Using the formula

$$\begin{aligned} g_{ij}(x) = \delta_{ij} &- \frac{1}{3}\sum_{k,\ell} R(e_k, e_i, e_\ell, e_j)x_k x_\ell - \frac{1}{6}\sum_{\ell,m,k} R_{\ell imj,k}x_\ell x_m x_k \\ &+ \frac{2}{45}\sum_{\ell,k,r,s,m} R_{i\ell km}R_{jrsm}x_\ell x_k x_r x_s - \frac{1}{20}\sum_{\ell,r,m,k} R_{\ell jri,mk}x_\ell x_r x_m x_k + o(|x|^4) \end{aligned}$$

We finally obtain

$$\begin{aligned} g_{ij}(x) = \delta_{ij} &- \frac{1}{3}R_{ikj\ell}(p)x_k x_\ell - \frac{1}{6}R_{ikj\ell,m}(p)x_k x_\ell x_m \\ &+ \frac{2}{45}R_{ik\ell m}(p)R_{jrsm}(p)x_k x_\ell x_r x_s - \frac{1}{20}R_{ikj\ell,rs}(p)x_k x_\ell x_r x_s + o(|x|^5) \end{aligned}$$

□

Taylor Expansion for $\det(g_{ij})$. Note

$$\det g_{ij} = \exp(\mathrm{Tr}(\log(g_{ij})))$$

One has

$$\begin{aligned} g(x) &= I + g^{(2)}(x) + g^{(3)}(x) + g^{(4)}(x) + o(|x|^4) \\ \log g(x) &= g^{(2)}(x) + g^{(3)}(x) + g^{(4)}(x) - \frac{1}{2}(g^{(2)})^2 + o(|x|^4) \qquad \text{using } \log(1+t) = t - \frac{1}{2}t^2 + o(t^2) \end{aligned}$$

Here

$$\begin{aligned} -\frac{1}{2}(g^{(2)})^2 = -\frac{1}{2}g^{(2)}_{i\ell}g^{(2)}_{\ell j} &= -\frac{1}{18}\sum_{k,\ell,r,s,m} R_{ki\ell m}R_{rmsj}x_k x_\ell x_r x_s \\ &= -\frac{1}{18}\sum_{k,\ell,r,s,m} R_{ik\ell m}R_{jrsm}x_k x_\ell x_r x_s \end{aligned}$$

Now take the trace, i.e., contracting (gives Ricci curvature)

$$\begin{aligned} \mathrm{Tr}\log g(x) = &-\frac{1}{3}\sum_{k,\ell} R_{k\ell}x_k x_\ell - \frac{1}{6}\sum_{\ell,m,k} R_{\ell m,k}x_\ell x_m x_k - \frac{1}{90}\sum_{k,\ell,r,s,m,c} R_{ck\ell m}R_{crsm}x_k x_\ell x_r x_s \\ &- \frac{1}{20}\sum_{\ell,r,m,k} R_{\ell r,mk}x_\ell x_r x_m x_k + o(|x|^5) \end{aligned}$$

Now lifting to exponential

$$e^y = 1 + y + \frac{y^2}{2} + o(|y|^2)$$

One has

$$\det g(x) = 1 - \frac{1}{3}R_{k\ell}x_k x_\ell - \frac{1}{6}R_{\ell m,k}x_\ell x_m x_k - \frac{1}{90}R_{ck\ell m}R_{crsm}x_k x_\ell x_r x_s - \frac{1}{20}R_{\ell r,mk}x_\ell x_r x_m x_\ell - \frac{1}{18}R_{k\ell}R_{mr}x_k x_\ell x_m x_r + o(|x|^4)$$

□

Proposition 1.4 (Guassian Curvature in Polar Coordinates)**.** *Let M be Riemannian manifold of dimension 2 (identified as surface). Let $B_\delta(p)$ be normal ball around $p \in M$ and consider the parametrized surface*

$$f(\rho, \theta) = \exp_p(\rho v(\theta)) \qquad \forall\ 0 < \rho < \delta \qquad -\pi < \theta < \pi$$

for $v(\theta)$ circle of radius 1 in T_pM as parametrized by the central angle θ.

1. *(ρ, θ) are coordinates in an open subset $U \subset M$ formed by the open ball minus the ray*

$$U := B_\delta(p) \setminus \{\exp_p(-\rho v(0)) \mid 0 < \rho < \delta\}$$

   These coordinates are polar coordinates at p.

Proof. It suffices to prove that

$$f : (0, \delta) \times (-\pi, \pi) \subset B_\delta(0) \subset \mathbb{R}^2 \to U \subset B_\delta(p) \subset M \qquad \text{defines a smooth diffeomorphism}$$

i.e., a bijection smooth map with smooth inverse.

(a) f as composition of smooth maps is indeed smooth in (ρ, θ) on $B_\delta(0)$.

(b) f is injective since $\exp_p$ is injective on $B_\delta(0) \subset \mathbb{R}^2$, which follows that

$$\exp_p(\rho_1 v(\theta_1)) = \exp_p(\rho_2 v(\theta_2)) \implies \rho_1 v(\theta_1) = \rho_2 v(\theta_2) \implies \rho_1 = \rho_2, \quad \theta_1 = \theta_2 \mod 2\pi$$

Since both $\theta_1,\, \theta_2 \in (-\pi,\, \pi)$ one has $\theta_1 = \theta_2$.

(c) f is surjective follows from the definition of the geodesic ball $B_\delta(p)$. By definition $\exp_p : B_\delta(0) \to B_\delta(p)$ is a diffeomorphism, hence for any

$$q \in U = B_\delta(p) \setminus \{\exp_p(-\rho v(0)) \mid 0 < \rho < \delta\}$$

There exists $w \in B_\delta(0)$ s.t.

$$\exp_p(w) = q$$

By injectivity of $\exp_p$ and excluding all possible points where $\rho v(0)$ ranging from $0 < \rho < \delta$ can map to, there must exists $\theta \neq 0 \mod 2\pi$ and $0 < \rho < \delta$ s.t.

$$\exp_p(\rho v(\theta)) = q$$

But θ has representative at $(-\pi, \pi)$.

(d) To show f is immersion, we need $\ker df_{(\rho,\theta)} = \{0\}$ for any $(\rho, \theta) \in (0, \delta) \times (-\pi, \pi)$ where

$$df_{(\rho,\theta)} : T_{(\rho,\theta)}\left((0, \delta) \times (-\pi, \pi)\right) = \mathbb{R}^2 \to T_{f(\rho,\theta)}U \cong \mathbb{R}^2$$

But using Chain rule

$$\begin{aligned} \left.\frac{\partial f}{\partial \rho}\right|_{(\rho,\theta)} &= d(\exp_p \circ (\rho v(\theta)))_{(\rho,\theta)} = (d\exp_p)_{\rho v(\theta)}(v(\theta)) \\ \left.\frac{\partial f}{\partial \theta}\right|_{(\rho,\theta)} &= d(\exp_p \circ (\rho v(\theta)))_{(\rho,\theta)} = (d\exp_p)_{\rho v(\theta)}(\rho v'(\theta)) \end{aligned}$$

Yet $v(\theta)$ and $\rho v'(\theta)$ are orthogonal, hence they span $\mathbb{R}^2$. Under $(d\exp_p)_{\rho v(\theta)}$ as isomorphism between vector spaces

$$\left\{ \left.\frac{\partial f}{\partial \rho}\right|_{(\rho,\theta)},\, \left.\frac{\partial f}{\partial \theta}\right|_{(\rho,\theta)} \right\} \qquad \text{indeed form a basis for } T_{f(\rho,\theta)}U$$

Thus the differential $df_{(\rho,\theta)}$ is injective, and hence f is immersion.

(e) By Inverse Function Theorem, and using f is bijection, f^{-1} inverse is defined everywhere on U and is smooth.

□

2. *The coefficients g_{ij} of the Riemannian metric in polar coordinates are given by*

$$g_{12} = 0, \qquad g_{11} = |\frac{\partial f}{\partial \rho}|^2 = |v(\theta)|^2 = 1, \qquad g_{22} = |\frac{\partial f}{\partial \theta}|^2$$

Proof. (a) Notice by setting $\rho = 0$, the initial radial velocity of our geodesic is one

$$\left.\frac{\partial f}{\partial \rho}\right|_{(0,\theta)} = (d\exp_p)_0(v(\theta)) = v(\theta) \implies |\left.\frac{\partial f}{\partial \rho}\right|_{(0,\theta)}| = |v| = 1$$

But a geodesic has constant speed. Hence in radial direction

$$g_{11} = \langle \left.\frac{\partial f}{\partial \rho}\right|_{(\rho,\theta)},\, \left.\frac{\partial f}{\partial \rho}\right|_{(\rho,\theta)} \rangle = |v(\theta)|^2 = 1 \qquad \forall\, \theta \in (-\pi, \pi)$$

(b) Using Gauss Lemma

$$g_{12} = \langle \frac{\partial f}{\partial \rho}\Big|_{(\rho,\theta)}, \frac{\partial f}{\partial \theta}\Big|_{(\rho,\theta)} \rangle = \langle (d\exp_p)_{\rho v(\theta)}(v(\theta)), (d\exp_p)_{\rho v(\theta)}(\rho v'(\theta))\rangle$$
$$= \langle v(\theta), \rho v'(\theta)\rangle = 0$$

Using that radial and angular velocity are orthogonal.

(c) By definition

$$g_{22} = \langle \frac{\partial f}{\partial \theta}\Big|_{(\rho,\theta)}, \frac{\partial f}{\partial \theta}\Big|_{(\rho,\theta)} \rangle = |\frac{\partial f}{\partial \theta}|^2$$

□

3. *Along the geodesic $f(\rho, 0)$, we have*

$$(\sqrt{g_{22}})_{\rho\rho} = -K(p)\rho + \tilde{R}(\rho) \qquad \textit{for some } \tilde{R} \textit{ where} \qquad \lim_{\rho\to 0} \frac{\tilde{R}(\rho)}{\rho} = 0 \tag{13}$$

and $K(p)$ the sectional curvature of M at p.

Proof. For $\theta = 0$, we make the observation that

$$\sqrt{g_{22}} = |\frac{\partial f}{\partial \theta}\Big|_{(\rho,0)}| = |(d\exp_p)_{\rho v(0)}(\rho v'(0))|$$
$$= |J(\rho)| \qquad \text{for Jacobi Field with } J(0) = 0 \text{ and } J'(0) = v'(0)$$

Then directly apply (12), for the plane spanned by $v(0)$ and $v'(0)$

$$\sigma = \mathbf{Span}\{v(0),\, v'(0)\}$$

one obtain

$$J(\rho) = \rho - \frac{1}{6}K(\sigma)\rho^3 + o(\rho^3)$$

But $\dim M = 2$, the only 2-dim subspace of T_pM is itself, so $K(\sigma) = K(p)$ is indeed the sectional curvature of M at p. Thus

$$(\sqrt{g_{22}})_{\rho\rho} = -K(p)\rho + o(\rho)$$

□

4. *In dimension 2, the sectional curvature coincides with the Gaussian Curvature.*

$$\lim_{\rho\to 0} \frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}} = -K(p)$$

Proof.

$$\frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}} = \frac{-K(p)\rho + o(\rho)}{\rho - \frac{1}{6}K(\sigma)\rho^3 + o(\rho^3)}$$
$$\lim_{\rho\to 0}\frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}} = -K(p) + \lim_{\rho\to 0}\frac{o(1)}{\rho - \frac{1}{6}K(\sigma)\rho^3 + o(\rho^3)} = -K(p)$$

where for both limits, we compute using L'Hôpital's rule. □

Corollary 1.2 (Sectional Curvature for dim = 2)**.** *Let M be Riemannian manifold of dimension 2. Let $p \in M$ and let $V \subset T_pM$ be a neighborhood of the origin where $\exp_p$ is a diffeomorphism. Let $S_r(0) \subset V$ be circle centered at the origin. Let L_r denote the length of the curve $\exp_p(S_r)$ in M. Then the sectional curvature at $p \in M$ is given by*

$$K(p) = \lim_{r\to 0} \frac{3}{\pi}\frac{2\pi r - L_r}{r^3}$$

Proof. Let $B_\delta(p)$ be normal ball around $p \in M$ s.t. $r < \delta$ and in the tangent space, $B_\delta(0) \subset V$. One parametrize the surface $\exp_p(B_\delta(0)) = B_\delta(p)$ using

$$f(\rho, \theta) = \exp_p(\rho v(\theta)) \qquad 0 < \rho < \delta, \qquad -\pi < \theta < \pi$$

Notice $f(r, \theta)$ therefore parametrizes the curve $\exp_p(S_r)$. In particular

$$|\frac{\partial}{\partial \theta} f(r, \theta)| = \sqrt{g_{22}(r, \theta)}$$

Hence the length L_r is computed via

$$L_r = \int_{-\pi}^{\pi} \sqrt{g_{22}(r, \theta)}\, d\theta$$

and since we're working with polar coordinates so the metric is radially symmetric, one obtain

$$L_r = 2\pi \sqrt{g_{22}(r)}$$

Now directly using (12)

$$\begin{aligned}
\frac{3}{\pi} \frac{2\pi r - L_r}{r^3} &= \frac{3}{\pi} \frac{2\pi r - 2\pi\sqrt{g_{22}(r)}}{r^3} = 6 \frac{r - (r - \frac{1}{6} K(p) r^3 + o(r^3))}{r^3} \\
&= K(p) + 6 \frac{o(r^3)}{r^3} \\
\lim_{r \to 0} \frac{3}{\pi} \frac{2\pi r - L_r}{r^3} &= K(p)
\end{aligned}$$

□

1.4 Conjugate Points

We study relationship between singularities of the exponential map and Jacobi Fields. Conjugate points give degeneracy of the geodesics.

Definition 1.2 (Conjugate Point)**.** *Given geodesic.*

$$\gamma : [0, a] \to M$$

Let $t_0 \in (0, a]$. The point $\gamma(t_0)$ is conjugate to $\gamma(0)$ along γ if there exists Jacobi Field J along γ s.t.

1. *$J \neq 0$ nontrivial.*
2. *$J(0) = 0 = J(t_0)$.*

We call the multiplicity of the conjugate point $\gamma(t_0)$ as the maximum number of such linearly independent Jacobi fields, i.e.

$$\text{Multiplicity}(\gamma(t_0)) := \dim\{J(t) \mid \textit{Jacobi field along } \gamma(t) \textit{ s.t. } J(0) = 0 = J(t_0)\} \geq 1$$

Remark 1.3. *Notice the multiplicity never exceeds $n - 1$. Recall that if*

$$\gamma(t) = \exp_p(tv)$$

Then $J(0) = 0$ implies

$$J(t) = (d\exp_p)_{tv}(tw)$$

and (8)

$$\langle J, \gamma' \rangle(t) = \langle J(0), \gamma'(0) \rangle + t \langle J'(0), \gamma'(0) \rangle$$

Applying to $t = t_0$ yields

$$\begin{aligned}
0 &= 0 + t_0 \langle J'(0), \gamma'(0) \rangle \\
\implies \langle J'(0), \gamma'(0) \rangle &= 0 \qquad \textit{so } \frac{D}{dt} J(0) \textit{ is perpendicular to } \gamma'(0)
\end{aligned}$$

So

$$J(t) \in \{J(t) \mid \textit{Jacobi Fields along } \gamma(t),\ J(0) = 0 \textit{ and } \langle \frac{D}{dt} J(0), \gamma'(0) \rangle = 0\} \cong \mathbb{R}^{n-1}$$

since originally one has $2n$ initial conditions to determine $J(t)$, and $J(0) = 0$ kills n while $\langle w, v\rangle = 0$ kills 1, we're left with $n-1$. In fact, if $e_1, \cdots, e_n$ are ONB of T_pM

$$e_1 = \frac{v}{|v|} = \frac{\gamma'(0)}{|\gamma'(0)|}$$

Let $J_i(t)$ be Jacobi Fields with

$$J_i(0) = 0 \qquad \frac{D}{dt}J_i(0) = e_i \qquad i \in \{1, \cdots, n\}$$

and let $J_{n+i}(t)$ be Jacobi Field s.t.

$$J_{n+i}(0) = e_i, \qquad \frac{D}{dt}J_{n+i}(0) = 0 \qquad i \in \{1, \cdots, n\}$$

So look at the space

$$\{J(t) \mid \textit{Jacobi Fields } J(0) = 0 = \langle \frac{D}{dt}J(0), \gamma'(0)\rangle\} = \text{Span}\{J_2(t), \cdots, J_n(t)\} \implies \text{Multiplicity}(\gamma(t_0)) \leq n-1$$

Remark 1.4. *If (M, g) has constant sectional curvature K. Then*

$$\{J(t) \mid \textit{Jacobi Fields } J(0) = 0\} = \text{Span}\{t\gamma'(t), f_K e_2, f_K e_3, \cdots, f_K e_n\} \cong \mathbb{R}^n$$

where

$$e_1 = \frac{\gamma'(0)}{|\gamma'(0)|},\ e_2, \cdots, e_n \qquad \textit{are ONB of } T_pM \textit{ and } e_i \textit{ are parallel transported along } \gamma(t)$$

Definition 1.3 (Conjugate Locus)**.** *Given $p \in M$, the set of first conjugate points to the point p, for all the geodesics that start at p, is the conjugate locus of p which we denote $C(p)$.*

Example 1.3. *If (M, g) has constant negative sectional curvature $K \leq 0$. Then*

$$f_K(\rho) \neq 0 \qquad \forall\ \rho \neq 0$$

This means $C(p) = \varnothing$.

Proposition 1.5. *Let M be a Riemannian manifold with non-positive sectional curvature. Then for any $p \in M$, the conjugate locus $C(p) = \varnothing$ is empty.*

Proof. Fix any $p \in M$. Given a geodesic

$$\gamma : [0, a] \to M$$

s.t. $\gamma(0) = p$. Assume there exists nontrivial Jacobi Field J s.t.

$$J(0) = J(a) = 0$$

1. We first show that

$$\frac{d}{dt}\langle \frac{D}{dt}J, J\rangle \geq 0$$

   One calculate using that the covariant derivative $\frac{D}{dt}$ corresponds to Levi-Civita Connection (hence compatible with the metric g), and the Jacobi Equation (4).

$$\begin{aligned}\frac{d}{dt}\langle \frac{D}{dt}J, J\rangle &= \langle \frac{D^2}{dt^2}J, J\rangle + \langle \frac{D}{dt}J, \frac{D}{dt}J\rangle \\ &= -\langle R(\gamma', J(t))\gamma', J\rangle + \langle \frac{D}{dt}J, \frac{D}{dt}J\rangle\end{aligned}$$

   Notice the first term is essentially sectional curvature in the plane spanned by γ' and J with flipped sign so

$$-\langle R(\gamma', J(t))\gamma', J\rangle \geq 0$$

   due to our assumption on non-positive sectional curvature. The second term is always non-negative due to inner product structure.

2. But then

$$\langle \frac{D}{dt}J, J\rangle(a) = 0 = \langle \frac{D}{dt}J, J\rangle(0)$$

   yields

$$\langle \frac{D}{dt}J, J\rangle \equiv 0$$

3. Using compatible with the metric g again

$$\frac{d}{dt}\langle J, J\rangle = 2\langle \frac{D}{dt}J, J\rangle \equiv 0$$

Thus

$$|J|^2 = \langle J, J\rangle(t) = 0 \qquad \forall\, t \in [0, a]$$

We've reached a contradiction that J is assumed to be non-trivial. □

On the other hand, for positive sectional curvature, there could be conjugate points.

Example 1.4. *If (M, g) has constant positive sectional curvature $K > 0$. For example*

$$\mathbb{S}^n(\frac{1}{\sqrt{K}}) = \textit{sphere of radius } \frac{1}{\sqrt{K}}$$

Recall

$$J(t) = \frac{\sin(\sqrt{K}t)}{\sqrt{K}}w(t) \qquad J(0) = 0 = J(\frac{\pi}{\sqrt{K}})$$

and in the sphere

$$-p = \exp_p(\frac{\pi}{\sqrt{K}})$$

so

$$C(p) = \{-p\}$$

One can in fact relate conjugate points with singularities of the exponential map.

Proposition 1.6 (Conjugate Points and singularities of the exponential map)**.** *Let*

$$\gamma : [0, a] \to M$$

be geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$, hence

$$\gamma(t) = \exp_p(tv)$$

Then the point $q = \gamma(t_0)$ for $t_0 \in (0, a]$ is conjugate point to $p = \gamma(0)$ along γ iff

$$t_0\gamma'(0) = t_0 v$$

is a critical point of $\exp_p$, i.e., $(d\exp_p)_{t_0 v}$ is not surjective (has non-trivial kernel). Moreover

$$\textit{Multiplicity of } q \textit{ as a conjugate point of } p = \dim\ker((d\exp_p)_{t_0 v})$$

Proof. Any Jacobi Field $J(t)$ along $\gamma(t)$ s.t. $J(0) = 0$ is of the form

$$J(t) = (d\exp_p)_{tv}(tw) \qquad w := \frac{D}{dt}J(0)$$

Suppose $t_0 \neq 0$, then q is conjugate to p iff $J(t_0) = 0$ iff

$$(d\exp_p)_{t_0 v}(t_0 w) = 0$$

But $t_0 > 0$, so this vanishes iff

$$(d\exp_p)_{t_0 v}(w) = 0 \iff w \in \ker((d\exp_p)_{t_0 v})$$

Hence due to non-trivial kernel, $t_0 v$ is a critical point for $\exp_p$ via definition. □

Proposition 1.7. *Let*

$$\gamma : [0, a] \to M$$

be geodesic. Let $V_1 \in T_{\gamma(0)}M$ and $V_2 \in T_{\gamma(a)}M$. If $\gamma(a)$ is not conjugate to $\gamma(0)$ along γ, there exists a unique Jacobi Field J along γ s.t.

$$J(0) = V_1, \qquad J(a) = V_2$$

Proposition 1.8. *Let M be Riemannian manifold with constant negative sectional curvature $b < 0$. Let*

$$\gamma : [0, \ell] \to M$$

s.t. $\gamma(0) = p$ be normalized geodesics, and let $v \in T_{\gamma(\ell)}M$ s.t.

$$\langle v, \gamma'(\ell)\rangle = 0, \qquad |v| = 1$$

Then the Jacobi Field J along γ as determined by

$$J(0) = 0 \qquad J(\ell) = v$$

is given by

$$J(t) = \frac{\sinh(t\sqrt{-b})}{\sinh(\ell\sqrt{-b})} w(t) \tag{14}$$

where $w(t)$ is the parallel transport along γ of the vector

$$w(0) = \frac{u_0}{|u_0|} \qquad u_0 = (d\exp_p)^{-1}_{\ell\gamma'(0)}(v) \in T_{\gamma(0)}M \cong T_{\ell\gamma'(0)}(T_{\gamma(0)}M)$$

Proof.

1. Since M has constant negative curvature, using Proposition 1.5 we know $\gamma(\ell)$ is not conjugate point to $\gamma(0)$ along γ.
2. Using b is constant sectional curvature, our Jacobi Equation writes as in (9). Given initial data

   $$J_1(0) = 0 \qquad J_1'(0) = w(0) = \frac{u_0}{|u_0|}$$

   According to solution (10) with $b < 0$, one has

   $$J_1(t) = \frac{\sinh(\sqrt{-b}t)}{\sqrt{-b}} w(t)$$

   as the unique solution. But notice J_1 is not the solution J we seek for.
3. Since $J_1(0) = 0$, using (7) one may write J_1 as

   $$J_1(t) = (d\exp_{\gamma(0)})_{t\gamma'(0)}(tw(0))$$

   In particular one may evaluate at $t = \ell$ and obtain

   $$J_1(\ell) = (d\exp_p)_{\ell\gamma'(0)}(\ell w(0)) = (d\exp_p)_{\ell\gamma'(0)}(\ell\frac{u_0}{|u_0|})$$

   But making use of

   $$u_0 = (d\exp_p)^{-1}_{\ell\gamma'(0)}(v) \implies (d\exp_p)_{\ell\gamma'(0)}(u_0) = v$$

   so via linearity

   $$J_1(\ell) = \ell\frac{v}{|u_0|}$$

4. Finally, notice both $J_1(\ell)$ and $J(\ell)$ are expected to be in the direction of $v \in T_{\gamma(\ell)}M$. We define scaling

   $$\tilde{J}(t) := \frac{|u_0|}{\ell} J_1(t) = \frac{\sinh(\sqrt{-b}t)}{\sqrt{-b}} \frac{|u_0|}{\ell} w(t)$$

   so that

   $$\tilde{J}(0) = 0 \qquad \tilde{J}(\ell) = v$$

   Indeed due to two boundary conditions, via Existence and Uniqueness of ODE solution

   $$\tilde{J} \equiv J$$

5. It suffices to argue one has the correct scaling that matches (14). Using

   $$1 = |v| = |J(\ell)| = |\frac{\sinh(\sqrt{-b}\ell)}{\sqrt{-b}} \frac{|u_0|}{\ell} w(\ell)| = |\frac{\sinh(\sqrt{-b}\ell)}{\sqrt{-b}} \frac{|u_0|}{\ell}|$$

   Hence

   $$J(t) = \frac{\sinh(\sqrt{-b}t)}{\sqrt{-b}} \frac{|u_0|}{\ell} w(t) = \frac{\sinh(t\sqrt{-b})}{\sinh(\ell\sqrt{-b})} w(t)$$

□

Proposition 1.9 (Jacobi Fields and Conjugate Points on Locally Symmetric Spaces). *Let M be locally symmetric space. Let*

$$\gamma : [0,\infty) \to M \qquad \text{be a geodesic in } M \text{ and let} \qquad \gamma(0) = p, \qquad \gamma'(0) = v$$

Define

$$K_v : T_pM \to T_pM \qquad K_v(x) := R(v,x)v \qquad \forall\ x \in T_pM$$

Then

1. *K_v is self-adjoint.*

   Proof. For any $x,\ y \in T_pM$, using Symmetry of Riemannian Curvature Tensor

   $$\begin{aligned}\langle K_v(x), y\rangle =& \langle R(v,x)v, y\rangle = R(v,x,v,y) = R(v,y,v,x) \\ &= \langle R(v,y)v, x\rangle = \langle K_v(y), x\rangle = \langle x, K_v(y)\rangle\end{aligned}$$

   where the last equality follows by symmetric of metric. □

2. *Choose an ONB $\{e_1, \cdots, e_n\}$ of T_pM that diagonalizes K_v, i.e.*

   $$K_v(e_i) = \lambda_i e_i \qquad \forall\ i = 1, \cdots, n$$

   We extend e_i to Vector fields along γ via Parallel Transport. Then (note λ_i does not depend on t)

   $$K_{\gamma'(t)}(e_i(t)) = \lambda_i e_i(t) \qquad \forall\ t$$

   Proof. Notice $\gamma'(t)$ is the parallel transport of $\gamma'(0) = v$ along γ. Since M is locally symmetric space, given $e_i(t)$ parallel transport of e_i along γ

   $$K_{\gamma'(t)}(e_i(t)) = R(\gamma'(t), e_i(t))\gamma'(t) \qquad \text{is also parallel transport along } \gamma$$

   Thus for any $e_j(t)$ where $j \neq i$, we take covariant derivative using ∇ is Levi-Civita Connection

   $$\begin{aligned}\frac{d}{dt}\langle K_{\gamma'(t)}(e_i(t)), e_j(t)\rangle &= \langle \frac{D}{dt}K_{\gamma'(t)}(e_i(t)), e_j(t)\rangle + \langle K_{\gamma'(t)}(e_i(t)), \frac{D}{dt}e_j(t)\rangle \\ &= 0 \qquad \text{since both are parallel vector fields} \\ \langle K_{\gamma'(t)}(e_i(t)), e_j(t)\rangle &= \langle K_{\gamma'(0)}(e_i(0)), e_j(0)\rangle = \langle K_v(e_i), e_j\rangle = \lambda_i\langle e_i, e_j\rangle = 0 \qquad \forall\ t\end{aligned}$$

   Then due to choice of ONB basis

   $$K_{\gamma'(t)}(e_i(t)) = Ce_i(t) \qquad \forall\ t$$

   To determine constant

   $$\begin{aligned}\frac{d}{dt}\langle K_{\gamma'(t)}(e_i(t)), e_i(t)\rangle &= \langle \frac{D}{dt}K_{\gamma'(t)}(e_i(t)), e_i(t)\rangle + \langle K_{\gamma'(t)}(e_i(t)), \frac{D}{dt}e_i(t)\rangle = 0 \\ \langle K_{\gamma'(t)}(e_i(t)), e_i(t)\rangle &= \langle K_v(e_i), e_i\rangle = \lambda_i \\ C &= \lambda_i\end{aligned}$$

   Thus

   $$K_{\gamma'(t)}(e_i(t)) = \lambda_i e_i(t) \qquad \forall\ t$$

   □

3. *Let*

   $$J(t) := \sum_i x_i(t)e_i(t) \qquad \forall\ t \qquad \text{be Jacobi Field along } \gamma$$

   Then the Jacobi Equation is equivalent to the system of ODEs

   $$\frac{d^2}{dt^2}x_i(t) + \lambda_i x_i = 0 \qquad i = 1, \cdots, n \tag{15}$$

Proof. Recall (4) writes

$$\frac{D^2}{dt^2}J(t) + R(\gamma'(t), J(t))\gamma'(t) = 0 \qquad \forall\ t \in [0, \infty)$$

So plugging in, using product rule and using Linearity of Riemannian Curvature Tensor in $C^\infty(M)$

$$\sum_i \frac{D^2}{dt^2}(x_i(t)e_i(t)) + \sum_i R(\gamma'(t), x_i(t)e_i(t))\gamma'(t) = 0$$
$$\sum_i e_i(t)\frac{d^2}{dt^2}x_i(t) + \sum_i x_i(t)R(\gamma'(t), e_i(t))\gamma'(t) = 0$$
$$\sum_i e_i(t)\frac{d^2}{dt^2}x_i(t) + \sum_i x_i(t)K_{\gamma'(t)}(e_i(t)) = 0$$
$$\sum_i e_i(t)\frac{d^2}{dt^2}x_i(t) + \sum_i x_i(t)\lambda_i e_i(t) = 0$$
$$\frac{d^2}{dt^2}x_i(t) + \lambda_i x_i(t) = 0 \qquad \forall\ t$$

Using the fact that $\{e_1(t), \cdots, e_n(t)\}$ are ONB frames parallel w.r.t. γ. □

4. *The conjugate points of p along γ are given by*

$$\gamma(\frac{\pi k}{\sqrt{\lambda_i}}) \qquad \forall\ k \in \mathbb{Z},\ k \geq 1, \qquad \forall\ i \in \{1, \cdots, n\} \cap \{\lambda_i \textit{ is a positive eigenvalue of } K_v\ \}$$

Proof. Solving system of ODEs for (15) with

$$\vec{x}(0) = (0, \cdots, 0)$$

We wish to look for t_k s.t.

$$\vec{x}(t_k) = (0, \cdots, 0)$$

(a) In the case $\lambda_i > 0$, the general solution

$$x_i(t) = A_i \sin(\sqrt{\lambda_i}t) \qquad x_i'(0) = A_i\sqrt{\lambda_i}$$

To set $x_i(t_k) = 0$, and to keep $A_i \neq 0$ we obtain

$$\sin(\sqrt{\lambda_i}t_k) = 0 \qquad \forall\ k \implies t_k = \frac{k\pi}{\sqrt{\lambda_i}} \qquad \forall\ k \in \mathbb{Z},\ k > 0$$

Notice we omit $k = 0$ for the simple reason that it coincides with the origin of γ.

(b) In the case $\lambda_i = 0$, the general solution

$$x_i(t) = C_i t \qquad x_i'(0) = C_i$$

Setting $x_i(t_k) = 0$ but keeping $C_i \neq 0$ yields $t_k = 0$, which we omit.

(c) In the case $\lambda_i < 0$, the general solution

$$x_i(t) = D_i \sinh(\sqrt{-\lambda_i}t) \qquad x_i'(0) = D_i\sqrt{-\lambda_i}$$

Setting $x_i(t_k) = 0$ but keeping $D_i \neq 0$ yields

$$\sinh(\sqrt{-\lambda_i}t_k) = 0 \qquad \forall\ k \implies t_k = 0$$

which we omit

Hence

$$\gamma(t_k) = \gamma(\frac{\pi k}{\sqrt{\lambda_i}})$$

are precisely the conjugate points of $\gamma(0) = p$ along γ. □

2 Isometric Immersions

We want to measure the way a manifold is immersed in another. Let (M, g) and $(\overline{M}, \overline{g})$ be two Riemannian manifolds and ∇, $\overline{\nabla}$ their respective Levi-Civita connections.

Definition 2.1 (Isometric Immersion)**.**

$$f : (M, g) \to (\overline{M}, \overline{g})$$

is an isometric immersion if

1. *f is an immersion, i.e., for any $p \in M$,*

$$df_p : T_pM \to T_{f(p)}\overline{M} \qquad \textit{is injective}$$

2. *f is an isometry, i.e.*

$$f^*\overline{g} = g$$

If such f exists, then

$$n := \dim M \leq \overline{n} := \dim \overline{M}$$

Example 2.1. *The map*

$$\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^4 \qquad \mathbf{x}(\theta, \varphi) := \frac{1}{\sqrt{2}}(\cos(\theta), \sin(\theta), \cos(\varphi), \sin(\varphi)) \qquad \forall\ (\theta,\ \varphi) \in \mathbb{R}^2$$

is an immersion of $\mathbb{R}^2$ into the unit sphere $\mathbb{S}^3(1) \subset \mathbb{R}^4$, whose image $\mathbf{x}(\mathbb{R}^2)$ is a torus $\mathbb{T}^2$ with sectional curvature 0 in the induced metric.

Proof. 1. We first show $\mathbf{x}$ defines an immersion.

$$\mathbf{x}_\theta = \frac{\partial \mathbf{x}}{\partial \theta} = \frac{1}{\sqrt{2}}(-\sin(\theta), \cos(\theta), 0, 0)$$
$$\mathbf{x}_\varphi = \frac{\partial \mathbf{x}}{\partial \varphi} = \frac{1}{\sqrt{2}}(0, 0, -\sin(\varphi), \cos(\varphi))$$

Due to linear independence of $\mathbf{x}_\theta$ and $\mathbf{x}_\varphi$, clearly $d\mathbf{x}_{\theta,\varphi} = (\mathbf{x}_\theta, \mathbf{x}_\varphi)$ is injective for any $(\theta, \varphi) \in \mathbb{R}^2$. Hence $\mathbf{x}$ defines an immersion.

2. We compute

$$|\mathbf{x}(\theta, \varphi)|^2 = \frac{1}{2}\left(\cos^2(\theta) + \sin^2(\theta) + \cos^2(\varphi) + \sin^2(\varphi)\right) = 1$$

hence $\mathbf{x}$ defines an immersion into the unit sphere $\mathbb{S}^3(1) \subset \mathbb{R}^4$. Notice indeed $\mathbf{x}(\mathbb{R}^2) = \mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{2}}) = \mathbb{T}^2$ is the two dimensional torus as a set.

3. We compute the induced metric on $(\mathbf{x}(\mathbb{R}^2), \mathbf{x}^* g_0)$ where $(\mathbb{R}^4, g_0)$ is the Euclidean space.

$$\begin{aligned}\mathbf{x}^* g_0 &= \sum_{i=1}^{4} d\mathbf{x}_i^2 = \sum_{i=1}^{4} (\frac{\partial \mathbf{x}_i}{\partial \theta} d\theta + \frac{\partial \mathbf{x}_i}{\partial \varphi} d\varphi)^2 \\ &= (-\frac{1}{\sqrt{2}}\sin(\theta)d\theta)^2 + (\frac{1}{\sqrt{2}}\cos(\theta)d\theta)^2 + (-\frac{1}{\sqrt{2}}\sin(\varphi)d\varphi)^2 + (\frac{1}{\sqrt{2}}\cos(\varphi)d\varphi)^2 \\ &= d\theta^2 + d\varphi^2 = \text{Euclidean metric on } \mathbb{R}^2\end{aligned}$$

Hence $\mathbf{x}$ is in fact an isometric immersion.

4. Since we're in $(\mathbf{x}(\mathbb{R}^2), \mathbf{x}^* g_0)$ a Riemannian surface of dimension 2, the sectional curvature

$$K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2}$$

But due to the fact that induced metric is flat, $R_{1212} = 0$ hence sectional curvature is 0

$$K = 0$$

□

We want to understand how $f(M)$ can be understood as part of $\overline{M}$. Both its structure coming from g and from $\overline{g}$, i.e., tangent part, coincides due to isometry. What plays an important role is the normal part, i.e. the second fundamental form.

Definition 2.2 ($T_pM^\perp$)**.** *For any $p \in M$, one want to decompose $T_{f(p)}\overline{M}$.*

$$T_{f(p)}\overline{M} = T_pM \oplus (T_pM)^\perp$$

where we identify

$$T_pM \cong df_p(T_pM)$$

Here $\perp$ is orthogonal complement defined through $\overline{g}_{f(p)}$. More precisely

$$(T_pM)^\perp := \{v \in T_{f(p)}\overline{M} \mid \overline{g}(f(p))(v, w) = 0 \quad \forall\, w \in df_p(T_pM)\}$$

For any $v \in T_{f(p)}\overline{M}$

$$v = v^T + v^\perp \in T_pM \oplus (T_pM)^\perp$$

Definition 2.3 (Normal Bundle)**.** *The normal bundle of an isometric immersion of f is*

$$N(f) := \bigsqcup_{p\in M} (T_pM)^\perp$$

The vector bundle

$$N(f) \to M$$

is of rank $\overline{n} - n$. One pullback and decompose

$$f^*T\overline{M} = TM \oplus N(f)$$

Similarly

$$\begin{aligned} C^\infty(M, f^*T\overline{M}) &= C^\infty(M, TM) \oplus C^\infty(M, N(f)) \\ &= \mathfrak{X}(M) \oplus \mathfrak{X}(M)^\perp \\ v &= v^T + v^\perp \end{aligned}$$

where

$$\mathfrak{X}(M)^\perp := C^\infty(M, N(f))$$

Recall

Definition 2.4 (Pullback and Pushforward)**.** *For*

$$f : M \to \overline{M} \qquad C^\infty \text{ map}$$

define the pushforward

$$f_* : \mathfrak{X}(M) \to C^\infty(M, f^*T\overline{M}) \qquad X \mapsto f_*(X)(p) := df_p(X(p)) \qquad \forall\, p \in M$$

and pullback

$$f^* : \mathfrak{X}(\overline{M}) \to C^\infty(M, f^*T\overline{M}) \qquad Y \mapsto f^*Y(p) := Y(f(p)) \qquad \forall\, p \in M$$

Definition 2.5 (f-related)**.** *$X \in \mathfrak{X}(M)$ is f-related to $\overline{X} \in \mathfrak{X}(\overline{M})$ if*

$$f_*X = f^*\overline{X}$$

In other words, for any $p \in M$

$$df_p(X(p)) = \overline{X}(f(p)) \qquad \forall\, p \in M$$

In fact this can be viewed as definition for $\overline{X}$.

We use such to define the second fundamental form.

Lemma 2.1. *Suppose*

$$f : (M, g) \to (\overline{M}, \overline{g})$$

is an isometric immersion. Then

$$df_p(\nabla_X Y(p)) = (\overline{\nabla}_{\overline{X}}\overline{Y}(f(p)))^T \qquad X, Y \in \mathfrak{X}(M), \overline{X}, \overline{Y} \in \mathfrak{X}(V) \qquad \forall\, V \subset \overline{M} \text{ open neighborhood of } f(p)$$

where X is f-related to $\overline{X}$ and Y f-related to $\overline{Y}$

But what is the normal part? It is not contained in this information.

Definition 2.6 (Bilinear Form)**.** *We want to define the bilinear*

$$B : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)^{\perp} \qquad (X,\, Y) \mapsto B(X,Y)(p) := (\overline{\nabla}_{\overline{X}}\overline{Y}(f(p)))^{\perp} \tag{16}$$

In particular

$$\begin{aligned} B(X,Y)(p) &= (\overline{\nabla}_{\overline{X}}\overline{Y}(f(p)))^{\perp} \\ &= \overline{\nabla}_{\overline{X}}\overline{Y}(f(p)) - (\overline{\nabla}_{\overline{X}}\overline{Y}(f(p)))^{T} \\ &= \overline{\nabla}_{\overline{X}}\overline{Y}(f(p)) - \nabla_X Y(p) \end{aligned}$$

Remark 2.1 (Bilinear Form)**.** *Equivalently, if define*

$$D = f^*\overline{\nabla}$$

as the pullback connection on $f^*T\overline{M}$. Then one has equivalent definition for $B(X,Y)$

$$\begin{aligned} B(X,Y)(p) &= (D_X f_* Y)^{\perp}(p) \\ &= D_X f_* Y(p) - (D_X f_* Y)^{T}(p) \qquad \forall\ X,\, Y \in \mathfrak{X}(M) \end{aligned}$$

This justifies why B is well-defined. In particular, this also shows B is $C^\infty(M)$-linear in X.

Proposition 2.1 (Symmetric Bilinear Form)**.**

$$B(X,Y) = B(Y,X) \qquad \forall\ X,\, Y \in \mathfrak{X}(M)$$

Proof.

$$\begin{aligned} B(X,Y) - B(Y,X) &= (D_X f_* Y)^{\perp} - (D_Y f_* X)^{\perp} \\ &= (D_X f_* Y - D_Y f_* X)^{\perp} \\ &= (f_*([X,Y]))^{\perp} = 0 \end{aligned}$$

Since $[X,Y] \in \mathfrak{X}(M)$, then its orthogonal part is 0. ☐

Corollary 2.1. *B is $C^\infty(M)$-linear in both X and Y. In fact $B \in C^\infty(M, Sym^2 T^*M \otimes N(f))$.*

2.1 Second Fundamental Form

For any $p \in M$. Consider isometric immersion

$$f : (M, g) \to (\overline{M}, \overline{g})$$

Definition 2.7 (First Fundamental Form)**.** *The 1st fundamental form is the form we have on the tangent space of the manifold.*

$$(T_pM,\ \langle\cdot,\,\cdot\rangle_g)$$

where $g(\cdot,\cdot) = \langle\cdot,\,\cdot\rangle_g$ defined by

$$g = f^*\overline{g}$$

and is an inner product space.

What's important is the second fundamental form.

Remark 2.2. *One has 3 one-one correspondence of second fundamental forms*

1. *symmetric bilinear forms*

$$H(x,y) : T_pM \times T_pM \to \mathbb{R}$$

2. *quadratic form*

$$\mathbb{II} : T_pM \to \mathbb{R}$$

3. *Self adjoint operators*

$$S(x) : T_pM \to T_pM$$

One define using one from another

$$\mathbb{II}(x) := H(x,x)$$
$$H(x,y) := \frac{1}{2}\left(\mathbb{II}(x+y) - \mathbb{II}(x) - \mathbb{II}(y)\right)$$
$$\langle S(x), y\rangle_g := H(x,y) = \langle x, S(y)\rangle_g \qquad \forall\ x,\ y \in T_pM$$

Definition 2.8 (Second Fundamental Form)**.** *Let*

$$f : (M,g) \to (\overline{M}, \overline{g})$$

be an isometric immersion. Fix a vector on the orthogonal by Letting $\eta \in (T_pM)^\perp = (N(f))_p$*. Then for the symmetric bilinear form* B *as in* (16)*, one define*

$$H_\eta : T_pM \times T_pM \to \mathbb{R} \qquad s.t. \qquad (x,y) \mapsto H_\eta(x,y) := \langle B(x,y), \eta\rangle_{\overline{g}(f(p))}$$

Alternatively the quadratic form

$$\mathbb{II}_\eta : T_pM \to \mathbb{R} \qquad s.t. \qquad \mathbb{II}_\eta(x) := H_\eta(x,x)$$

and the self-adjoint operator

$$S_\eta : T_pM \to T_pM \qquad s.t. \qquad \langle S_\eta(x), y\rangle_{\overline{g}} = H_\eta(x,y) \tag{17}$$

These are called the second fundamental form of f *at* p *along* η*. One may write in general* $\eta \in \mathfrak{X}(M)^\perp$*.*

Proposition 2.2. *Let*

$$f : (M^n, g) \to (\overline{M}^{n+k}, \overline{g})$$

be isometric immersion and given $\eta \in \mathfrak{X}(M)^\perp$ *normal fields w.r.t.* $N(f)$*, let* S_η *be the operator associated to the second fundamental form of* f *along* η *as in* (17)

$$S_\eta : \mathfrak{X}(M) \to \mathfrak{X}(M) \qquad s.t. \qquad \langle S_\eta(X), Y\rangle_{\overline{g}} = H_\eta(X,Y)$$

Moreover we view S_η *as a tensor of order 2 given by*

$$H_\eta(X,Y) := \langle S_\eta(X), Y\rangle \qquad \forall\ X,\ Y \in \mathfrak{X}(M)$$

Notice S_η *is self-adjoint is equivalent to the tensor* H_η *being symmetric*

$$H_\eta(X,Y) = H_\eta(Y,X)$$

In fact,

$$\nabla_V S_\eta \qquad \textit{is self-adjoint} \qquad \forall\ V \in \mathfrak{X}(M)$$

Proof. We differentiate the equation

$$\langle S_\eta(X), Y\rangle = \langle X, S_\eta(Y)\rangle$$

w.r.t. $V \in \mathfrak{X}(M)$ using that ∇ is the Levi-Civita connection

$$\langle \nabla_V(S_\eta(X)), Y\rangle + \langle S_\eta(X), \nabla_V Y\rangle = \langle \nabla_V X, S_\eta Y\rangle + \langle X, \nabla_V(S_\eta(Y))\rangle$$

Notice, again by Leibniz rule

$$\nabla_V(S_\eta(X)) = (\nabla_V S_\eta)(X) + S_\eta(\nabla_V X)$$

Hence

$$\langle(\nabla_V S_\eta)(X), Y\rangle + \langle S_\eta(\nabla_V X), Y\rangle + \langle S_\eta(X), \nabla_V Y\rangle = \langle \nabla_V X, S_\eta Y\rangle + \langle X, (\nabla_V S_\eta)(Y))\rangle + \langle X, S_\eta(\nabla_V Y)\rangle$$
$$\langle(\nabla_V S_\eta)(X), Y\rangle = \langle X, (\nabla_V S_\eta)(Y))\rangle$$

Using the fact that S_η is self-adjoint, applied to vector fields $\nabla_V X$, Y, and $\nabla_V Y$, X. ☐

Lemma 2.2 (Explicit Expression)**.**

$$S_\eta(X) = -(D_X\eta)^T \qquad \forall\ X \in \mathfrak{X}(M),\ \eta \in (T_pM)^\perp \tag{18}$$

This is the tangent part of how η *changes along* X*.*

Proof.

$$\begin{aligned}\langle S_\eta(x), y\rangle &= H_\eta(x,y) = \langle B(x,y), \eta\rangle \\ &= \langle (D_X Y)^\perp, \eta\rangle = \langle D_X Y, \eta\rangle \qquad \text{since } \eta \text{ is orthogonal already} \\ &= X(\langle Y, \eta\rangle) - \langle Y, D_X \eta\rangle \qquad \text{now we use that } D \text{ is compatible with the metric} \\ &= -\langle Y, (D_X\eta)^T\rangle = -\langle (D_X\eta)^T, y\rangle \qquad \text{using } Y \in \mathfrak{X}(M) \text{ and } \eta \in (T_pM)^\perp \text{ so only tangential part is preserved}\end{aligned}$$

□

Corollary 2.2 (Shape Operator)**.** *If* $\dim \overline{M} = \dim M + 1$ *(hence there exists unique* η *s.t.* $\|\eta\| = 1$*), then one has the shape operator*

$$S_\eta(X) = -D_X \eta \tag{19}$$

Proof.

$$\begin{aligned}(D_X\eta)^\perp &= \langle D_X\eta, \eta\rangle \eta \\ &= \frac{1}{2} X(\langle \eta, \eta\rangle)\eta \qquad D \text{ is compatible with the metric} \\ &= 0\end{aligned}$$

□

Example 2.2 ($\mathbb{S}^n$)**.**

$$f : (\mathbb{S}^n, g_{can}) \hookrightarrow (\mathbb{R}^{n+1}, g)$$

For any $p \in \mathbb{S}^n$, $p = (x_1, \cdots, x_n)$ *s.t.* $\sum_{i=1}^n x_i^2 = 1$

$$\eta(p) = -p \qquad s.t. \qquad \eta \in \mathfrak{X}(\mathbb{S}^n)^\perp \qquad \textit{inward unit normal} \tag{20}$$

In particular

$$\eta(p) = -\sum_{i=1}^{n+1} x^i \left.\frac{\partial}{\partial x_i}\right|_p$$

In fact, for any $p \in \mathbb{S}^n$

$$S_\eta(p) : T_p\mathbb{S}^n \to T_p\mathbb{S}^n$$

is the identity.

Proof. We do computation in local coordinates. For any $v \in T_p\mathbb{S}^n$ s.t.

$$v = \sum_{i=1}^{n+1} a^i \left.\frac{\partial}{\partial x_i}\right|_p$$

using coordinates from the ambient manifold $\mathbb{R}^{n+1}$. What is then the shape operator? For $\overline{\nabla}$ the Levi-Civita connection on $\mathbb{R}^{n+1}$ and $D = f^*\overline{\nabla}$, we define $\overline{\eta} \in \mathfrak{X}(\mathbb{R}^{n+1})$ s.t.

$$\overline{\eta} := -\sum_{i=1}^{n+1} x^i \frac{\partial}{\partial x_i} \qquad \text{so} \qquad \overline{\eta}(p) := \eta(p) \qquad \forall\, p \in \mathbb{S}^n$$

Thus

$$\begin{aligned}S_\eta(p)(v) &= -D_v\eta = -\overline{\nabla}_v\overline{\eta} \qquad \text{where } \overline{\eta} \text{ is } f\text{-related to } \eta \text{ defined in the full ambient space } \mathbb{R}^{n+1} \text{ that restricts to } \eta \text{ on } \mathbb{S}^n \\ &= -\left.\overline{\nabla}_{\sum_i a^i \frac{\partial}{\partial x_i}} \left(-\sum_j x^j \frac{\partial}{\partial x_j}\right)\right|_p \\ &= \left.\sum_i a^i \overline{\nabla}_{\frac{\partial}{\partial x_i}} \left(\sum_j x^j \frac{\partial}{\partial x_j}\right)\right|_p \\ &= \sum_{ij} a^i \frac{\partial}{\partial x_i}(x^j) \left.\frac{\partial}{\partial x_j}\right|_p + \sum_{ij} a^i x^j\, \overline{\nabla}_{\frac{\partial}{\partial x^i}} \left.\frac{\partial}{\partial x^j}\right|_p \\ &= \sum_{ij} a^i \frac{\partial}{\partial x_i}(x^j) \left.\frac{\partial}{\partial x_j}\right|_p \qquad \text{where } \overline{\nabla}_{\frac{\partial}{\partial x^i}} \left.\frac{\partial}{\partial x^j}\right|_p = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k} = 0 \text{ due to } \Gamma = 0 \text{ on } (\mathbb{R}^{n+1}, g_0) \\ &= \sum_{ij} a^i \delta_{ij} \left.\frac{\partial}{\partial x_j}\right|_p = \sum_j a^j \left.\frac{\partial}{\partial x_j}\right|_p = v\end{aligned}$$

Therefore

$$H_\eta(x,y) = \langle B(x,y), \eta\rangle = \langle S_\eta(x), y\rangle = \langle x, y\rangle = g_{can}(x,y) \qquad \forall\ x,\ y \in \mathfrak{X}(\mathbb{S}^n)$$
$$B(x,y) = \langle x,y\rangle \eta$$

□

2.2 Gauss-Codazzi-Ricci Equations

This is about how the curvature splits into tangent and normal parts. Let

$$f : (M, g) \to (\overline{M}, \overline{g})$$

be an isometric immersion. In particular,

$$f_* : \mathfrak{X}(M) \to C^\infty(M, f^*T\overline{M}) \qquad X \mapsto f_*(X)(p) := df_p(X(p)) \qquad \text{is an injective map}$$

From now on we identify X with f_*X. For any $X,\ Y \in \mathfrak{X}(M)$ and $\eta \in \mathfrak{X}(M)^\perp$, $D := f^*\overline{\nabla}$, we look at

$$\begin{aligned} D_X Y &= (D_X Y)^T + (D_X Y)^\perp \\ &= \nabla_X Y + B(X,Y) \\ D_X \eta &= (D_X \eta)^T + (D_X \eta)^\perp \\ &= -S_\eta(X) + (D_X\eta)^\perp \end{aligned}$$

Definition 2.9. *We denote $\nabla^\perp_X \eta := (D_X\eta)^\perp$. We interpret $\nabla^\perp$ as connection on the normal bundle $\mathfrak{X}(M)^\perp$, i.e., $N(f) \to M$.*

This allows us to extend the definition of covariant derivative and connection on the normal bundle.

Definition 2.10 (Covariant Derivative)**.** *In particular, given connections ∇ on TM, T^*M and $\nabla^\perp$ on $N(f)$, $N(f)^*$, we obtain covariant derivative $\overline{\nabla}$ acting on*

$$(TM)^{\otimes r} \otimes (T^*M)^{\otimes s} \otimes (N(f))^{\otimes \ell} \otimes (N(f)^*)^{\otimes m}$$

For example

$$B(X, Y, \eta) := \langle B(X, Y), \eta\rangle$$

and in this way

$$B \in C^\infty(M, Sym^2 T^*M \otimes N(f)^*)$$

For any $X \in \mathfrak{X}(M)$, define using the compatibility condition

$$(\overline{\nabla}_X B)(Y,Z,\eta) := X(B(Y,Z,\eta)) - B(\nabla_X Y, Z, \eta) - B(Y, \nabla_X Z, \eta) - B(Y, Z, \nabla^\perp_X \eta) \qquad \forall\ Y,\ Z \in \mathfrak{X}(M),\ \eta \in \mathfrak{X}(M)^\perp \tag{21}$$

Definition 2.11 (Curvature)**.** *Similarly, for the curvature, we have full curvature*

$$\overline{R} \in \Omega^2(\overline{M}, \mathrm{End}(T\overline{M})) \qquad \textit{curvature of } \overline{\nabla}$$

and pullback curvature

$$f^*\overline{R} \in \Omega^2(M, \mathrm{End}(f^*T\overline{M})) \qquad \textit{curvature of } f^*\overline{\nabla} = D$$

and the exact curvature of the submanifold

$$R \in \Omega^2(M, \mathrm{End}(TM)) \qquad \textit{curvature of } \nabla$$

and the curvature of the orthogonal

$$R^\perp \in \Omega^2(M, \mathrm{End}(N(f))) \qquad \textit{curvature of } \nabla^\perp$$

Remark 2.3. *For $R(X, Y, \cdot, \cdot)$ where $X,\ Y \in \mathfrak{X}(M)$ but with the last two variables free*

1. *In $TM \times TM$ ($\mathfrak{X}(M) \times \mathfrak{X}(M)$) this gives Gauss Equation.*
2. *In $TM \times N(f)$ or $N(f) \times TM$ ($\mathfrak{X}(M) \times \mathfrak{X}^\perp(M)$) this gives Codazzi Equation.*
3. *In $N(f) \times N(f)$ ($\mathfrak{X}^\perp(M) \times \mathfrak{X}^\perp(M)$) this gives Ricci Equation.*

Now we introduce the equations.

Proposition 2.3 (Gauss Equation)**.** *Let X, Y, Z, $T \in \mathfrak{X}(M)$. We have Riemannian Curvature of the Ambient Manifold.*

$$\overline{R}(X,Y,Z,T) = R(X,Y,Z,T) - \langle B(X,Z), B(Y,T)\rangle + \langle B(X,T), B(Z,Y)\rangle \tag{22}$$

Proposition 2.4 (Codazzi Equation)**.** *For $\eta \in \mathfrak{X}(M)^\perp$*

$$\overline{R}(X,Y,Z,\eta) = (\overline{\nabla}_Y B)(X,Z,\eta) - (\overline{\nabla}_X B)(Y,Z,\eta) \tag{23}$$

where

$$B(X,Y,\eta) := \langle B(X,Y),\eta\rangle$$

Proposition 2.5 (Ricci Equation)**.** *For η, $\xi \in \mathfrak{X}(M)^\perp$*

$$\overline{R}(X,Y,\eta,\xi) = \langle R^\perp(X,Y)\eta,\xi\rangle + \langle [S_\eta, S_\xi]X, Y\rangle \tag{24}$$

where $R^\perp \in \Omega^2(M, \mathrm{End}(N(f)))$ is the curvature of $\nabla^\perp$.

Proof of three equations (22), (23), (24). By Definition

$$\overline{R}(X,Y,Z,T) = \langle \overline{R}(X,Y)Z, T\rangle$$
$$\overline{R}(X,Y,Z,\eta) = \langle \overline{R}(X,Y)Z, \eta\rangle$$

And

$$\overline{R}(X,Y)Z = D_Y D_X Z - D_X D_Y Z + D_{[X,Y]}Z$$

We want to write

$$\begin{aligned}
D_Y D_X Z &= D_Y(\nabla_X Z + B(X,Z)) \\
&= D_Y(\nabla_X Z) + D_Y(B(X,Z)) \\
&= \nabla_Y\nabla_X Z + B(Y,\nabla_X Z) + D_Y(B(X,Z)) \\
D_X D_Y Z &= \nabla_X\nabla_Y Z + B(X,\nabla_Y Z) + D_X(B(Y,Z)) \\
D_{[X,Y]}Z &= \nabla_{[X,Y]}Z + B([X,Y],Z)
\end{aligned}$$

Now

$$\begin{aligned}
\overline{R}(X,Y)Z &= \nabla_Y\nabla_X Z + B(Y,\nabla_X Z) + D_Y(B(X,Z)) \\
&\quad - \nabla_X\nabla_Y Z - B(X,\nabla_Y Z) - D_X(B(Y,Z)) \\
&\quad + \nabla_{[X,Y]}Z + B([X,Y],Z) \\
&= R(X,Y)Z + (B(Y,\nabla_X Z) - B(X,\nabla_Y Z) + B([X,Y],Z)) + D_Y(B(X,Z)) - D_X(B(Y,Z))
\end{aligned}$$

Let's prove Gauss Equation (22) first. We contract it with T, something tangent

$$\begin{aligned}
\overline{R}(X,Y,Z,T) &= R(X,Y,Z,T) + \langle B(Y,\nabla_X Z) - B(X,\nabla_Y Z) + B([X,Y],Z), T\rangle \\
&\quad + \langle D_Y(B(X,Z)), T\rangle - \langle D_X(B(Y,Z)), T\rangle \\
&= R(X,Y,Z,T) + 0 \qquad \text{because } B(\cdot,\cdot) \in \mathfrak{X}(M)^\perp \\
&\quad + Y(\langle B(X,Z), T\rangle) - \langle B(X,Z), D_Y T\rangle - X(\langle B(Y,Z),T\rangle) + \langle B(Y,Z), D_X T\rangle \\
&= R(X,Y,Z,T) - \langle B(X,Z), D_Y T\rangle + \langle B(Y,Z), D_X T\rangle \\
&= R(X,Y,Z,T) - \langle B(X,Z), \nabla_Y T\rangle - \langle B(X,Z), B(Y,T)\rangle + \langle B(Y,Z), \nabla_X T\rangle + \langle B(Y,Z), B(X,T)\rangle \\
&= R(X,Y,Z,T) - \langle B(X,Z), B(Y,T)\rangle + \langle B(Y,Z), B(X,T)\rangle
\end{aligned}$$

Now let's prove Codazzi (23). We contract it with η

$$\begin{aligned}
\overline{R}(X,Y,Z,\eta) &= \langle R(X,Y)Z,\eta\rangle + \langle B(Y,\nabla_X Z),\eta\rangle - \langle B(X,\nabla_Y Z),\eta\rangle + \langle B(\nabla_X Y - \nabla_Y X, Z),\eta\rangle \\
&\quad + \langle D_Y(B(X,Z)),\eta\rangle - \langle D_X(B(Y,Z)),\eta\rangle \\
&= 0 + \langle B(Y,\nabla_X Z),\eta\rangle - \langle B(X,\nabla_Y Z),\eta\rangle + \langle B(\nabla_X Y - \nabla_Y X, Z),\eta\rangle \\
&\quad + \langle D_Y(B(X,Z)),\eta\rangle - \langle D_X(B(Y,Z)),\eta\rangle \qquad \text{because } R(X,Y)Z \in \mathfrak{X}(M) \\
&= \langle B(Y,\nabla_X Z),\eta\rangle + \langle B(\nabla_X Y, Z),\eta\rangle - \langle D_X(B(Y,Z)),\eta\rangle \qquad \text{we put together all the } X \text{ derivatives} \\
&\quad - \langle B(X,\nabla_Y Z),\eta\rangle - \langle B(\nabla_Y X, Z),\eta\rangle + \langle D_Y(B(X,Z)),\eta\rangle \qquad \text{and the } Y \text{ derivatives} \\
&= \langle B(Y,\nabla_X Z),\eta\rangle + \langle B(\nabla_X Y, Z),\eta\rangle - (X(\langle B(Y,Z),\eta\rangle) - \langle B(Y,Z), D_X\eta\rangle) \\
&\quad - \langle B(X,\nabla_Y Z),\eta\rangle - \langle B(\nabla_Y X, Z),\eta\rangle + (Y(\langle B(X,Z),\eta\rangle) - \langle B(X,Z), D_Y\eta\rangle) \\
&= \langle B(Y,\nabla_X Z),\eta\rangle + \langle B(\nabla_X Y, Z),\eta\rangle - \big(X(\langle B(Y,Z),\eta\rangle) - \langle B(Y,Z), \nabla^\perp_X\eta\rangle\big) \\
&\quad - \langle B(X,\nabla_Y Z),\eta\rangle - \langle B(\nabla_Y X, Z),\eta\rangle + \big(Y(\langle B(X,Z),\eta\rangle) - \langle B(X,Z), \nabla^\perp_Y\eta\rangle\big) \\
&= -(\overline{\nabla}_X B)(Y,Z,\eta) + (\overline{\nabla}_Y B)(X,Z,\eta) \qquad \text{using definition of (21)}
\end{aligned}$$

Finally for Ricci Equation (24), by definition

$$
\begin{aligned}
\overline{R}(X,Y)\eta &= D_Y D_X \eta - D_X D_Y \eta + D_{[X,Y]}\eta \\
\text{Recall that} \quad D_X\eta &= (D_X\eta)^T + (D_X\eta)^\perp = -S_\eta(X) + \nabla^\perp_X \eta \\
(D_Y D_X \eta)^\perp &= (D_Y(-S_\eta(X) + \nabla^\perp_X\eta))^\perp = -B(Y, S_\eta(X)) + \nabla^\perp_Y \nabla^\perp_X \eta \\
(D_X D_Y \eta)^\perp &= -B(X, S_\eta(Y)) + \nabla^\perp_X \nabla^\perp_Y \eta \\
(D_{[X,Y]}\eta)^\perp &= \nabla^\perp_{[X,Y]}\eta \\
\text{Now we contract with } \xi \quad \overline{R}(X,Y,\eta,\xi) &= -\langle B(Y,S_\eta(X)),\xi\rangle + \langle \nabla^\perp_Y\nabla^\perp_X\eta,\xi\rangle + \langle B(X,S_\eta(Y)),\xi\rangle - \langle \nabla^\perp_X\nabla^\perp_Y\eta,\xi\rangle + \langle \nabla^\perp_{[X,Y]}\eta,\xi\rangle \\
&= \langle R^\perp(X,Y)\eta,\xi\rangle + \langle B(X,S_\eta(Y)),\xi\rangle - \langle B(Y,S_\eta(X)),\xi\rangle \\
\text{Recall that} \quad \langle S_\xi(X), Y\rangle &= H_\xi(X,Y) = \langle B(X,Y),\xi\rangle \\
\text{so here} \quad \langle B(X,S_\eta(Y)),\xi\rangle &= \langle S_\xi(X), S_\eta(Y)\rangle \\
\langle B(Y,S_\eta(X)),\xi\rangle &= \langle S_\xi(Y), S_\eta(X)\rangle \\
\implies \overline{R}(X,Y,\eta,\xi) &= R^\perp(X,Y,\eta,\xi) + \langle S_\xi(X), S_\eta(Y)\rangle - \langle S_\xi(Y), S_\eta(X)\rangle \\
&= R^\perp(X,Y,\eta,\xi) + \langle S_\eta\circ S_\xi(X), Y\rangle - \langle Y, S_\xi\circ S_\eta(X)\rangle \qquad \text{using } S \text{ is self-adjoint} \\
&= R^\perp(X,Y,\eta,\xi) + \langle [S_\eta, S_\xi](X), Y\rangle
\end{aligned}
$$

□

Remark 2.4 (Gauss). *If X, Y are orthonormal, then*

$$K(X,Y) = R(X,Y,X,Y)$$

the sectional curvature of Span$\{X,Y\}$ *satisfies*

$$\overline{K}(X,Y) - K(X,Y) = -\langle B(X,X), B(Y,Y)\rangle + |B(X,Y)|^2 \tag{25}$$

Example 2.3. *Consider isometric immersion of the sphere*

$$f : (\mathbb{S}^n, g_{can}) \hookrightarrow (\mathbb{R}^{n+1}, g_0)$$

Recall (20) *the unit inward normal*

$$\eta(p) = -p \in \mathfrak{X}(\mathbb{S}^n)^\perp \qquad |\eta| = 1$$

Then

$$B(X,Y) = \langle X,Y\rangle\eta$$

On our tangent space we pick X, Y orthonormal. Using (25)

$$
\begin{aligned}
\overline{K}(X,Y) - K(X,Y) &= -\langle B(X,X), B(Y,Y)\rangle + |B(X,Y)|^2 \\
0 - K(X,Y) &= -\langle \langle X,X\rangle\eta, \langle Y,Y\rangle\eta\rangle + |B(X,Y)|^2 \\
&= -\langle \eta,\eta\rangle + |\langle X,Y\rangle\eta|^2 = -1 \\
K(X,Y) &= 1
\end{aligned}
$$

Hence the sectional curvature of $\mathbb{S}^n$ is 1.

2.3 Totally Geodesic and Minimality

2.3.1 Totally Geodesic

Definition 2.12 (Totally Geodesic). *Let M be dimension n, $\overline{M}$ be dimension $n+1$*

$$f : (M,g) \to (\overline{M}, \overline{g})$$

be an isometric immersion. Let $p \in M$. We say that f is geodesic at p if the second fundamental form is 0

$$S_\eta = 0 \qquad \forall\ \eta \in (T_pM)^\perp$$

or equivalently

$$H_\eta = 0 \qquad \forall\ \eta \in (T_pM)^\perp$$

or equivalently

$$B(p) : T_pM \times T_pM \to (T_pM)^\perp \qquad \textit{is zero map}$$

The immersion f is totally geodesic if it is geodesic at any $p \in M$.

Proposition 2.6 (Transitivity of Totally geodesic)**.** *Let M be a Riemannian manifold and let $N \subset K \subset M$ be submanifolds of M. Suppose N is totally geodesic in K and that K is totally geodesic in M. Then N is totally geodesic in M.*

Proof. Since N and K are Riemannian submanifolds, consider f and g as isometric immersions

$$N \overset{f}{\hookrightarrow} K \overset{g}{\hookrightarrow} M$$

We're given that f is totally geodesic in K and g is totally geodesic in M, and want to show $g \circ f$ is totally geodesic in M.

1. For any $p \in N$, by identifying $p \cong f(p) \cong g \circ f(p)$, we observe that the normal splits w.r.t. both K and M

$$\begin{aligned} T_pK &= T_pN \oplus (T_pN)^{\perp}_K \\ T_pM &= T_pK \oplus (T_pK)^{\perp}_M \\ &= T_pN \oplus (T_pN)^{\perp}_M \\ \implies (T_pN)^{\perp}_M &= (T_pN)^{\perp}_K \oplus (T_pK)^{\perp}_M \end{aligned}$$

Furthermore, for connection ∇ on N, $\overline{\nabla}$ on K and $\overline{\overline{\nabla}}$ on M, one can write bilinear forms

$$\begin{aligned} &B^K_N : \mathfrak{X}(N) \times \mathfrak{X}(N) \to \mathfrak{X}(N)^{\perp}_K && B^K_N(X,Y)(p) := (\overline{\nabla}_{\overline{X}}\overline{Y})(f(p)) - df_p(\nabla_X Y(p)) \\ &B^M_K : \mathfrak{X}(K) \times \mathfrak{X}(K) \to \mathfrak{X}(K)^{\perp}_M && B^M_K(\overline{X},\overline{Y})(f(p)) := (\overline{\overline{\nabla}}_{\overline{\overline{X}}}\overline{\overline{Y}})(g\circ f(p)) - dg_{f(p)}(\overline{\nabla}_{\overline{X}}\overline{Y}(f(p))) \\ &B^M_N : \mathfrak{X}(N) \times \mathfrak{X}(N) \to \mathfrak{X}(N)^{\perp}_M && B^M_N(X,Y)(g\circ f(p)) := (\overline{\overline{\nabla}}_{\overline{\overline{X}}}\overline{\overline{Y}})(g\circ f(p)) - d(g\circ f)_p(\nabla_X Y(p)) \end{aligned}$$

Now using f, g-related vector fields and Chain rule

$$\begin{aligned} df_p(\nabla_X Y(p)) &= (\overline{\nabla}_{\overline{X}}\overline{Y})(f(p))^T \\ dg_{f(p)}(\overline{\nabla}_{\overline{X}}\overline{Y}(f(p))) &= (\overline{\overline{\nabla}}_{\overline{\overline{X}}}\overline{\overline{Y}})(g\circ f(p))^T \\ d(g\circ f)_p(\nabla_X Y(p)) &= dg_{f(p)}(df_p(\nabla_X Y(p))) \\ &= dg_{f(p)}((\overline{\nabla}_{\overline{X}}\overline{Y})(f(p))^T) \\ &= dg_{f(p)}(\overline{\nabla}_{\overline{X}}\overline{Y}(f(p))) - dg_{f(p)}(\overline{\nabla}_{\overline{X}}\overline{Y}(f(p))^{\perp}) \\ &= (\overline{\overline{\nabla}}_{\overline{\overline{X}}}\overline{\overline{Y}})(g\circ f(p))^T - dg_{f(p)}(\overline{\nabla}_{\overline{X}}\overline{Y}(f(p))^{\perp}) \\ &= (\overline{\overline{\nabla}}_{\overline{\overline{X}}}\overline{\overline{Y}})(g\circ f(p)) - (\overline{\overline{\nabla}}_{\overline{\overline{X}}}\overline{\overline{Y}})(g\circ f(p))^{\perp} - dg_{f(p)}(\overline{\nabla}_{\overline{X}}\overline{Y}(f(p))^{\perp}) \\ B^M_N(X,Y)(g\circ f(p)) &= B^M_K(\overline{X},\overline{Y})(f(p)) + dg_{f(p)}(B^K_N(X,Y)(p)) \end{aligned}$$

2. Then since f is totally geodesic in K, $B^K_N(p) \equiv 0$, and since g is totally geodesic in M, $B^M_K(f(p)) \equiv 0$, one conclude

$$B^M_N(g\circ f(p)) \equiv 0$$

Hence by definition N is totally geodesic in M.

□

Proposition 2.7. *Let $N_1 \subset M_1$ and $N_2 \subset M_2$ be totally geodesic submanifolds of the Riemannian manifolds M_1 and M_2 respectively. Then $N_1 \times N_2$ is a totally geodesic submanifold of the product $M_1 \times M_2$ with the product metric.*

Example 2.4 ($\mathbb{S}^2 \times \mathbb{S}^2$)**.** *The sectional curvature of the Riemannian manifold $\mathbb{S}^2 \times \mathbb{S}^2$ equipped with the product metric, where $\mathbb{S}^2 \subset \mathbb{R}^3$ is the unit sphere, is non-negative. Moreover, there exists a totally geodesic, flat torus $\mathbb{T}^2$ embedded in $\mathbb{S}^2 \times \mathbb{S}^2$.*

Proof. 1. Recall $(\mathbb{S}^2, g_{\text{can}})$ is equipped with the round metric

$$g^{\mathbb{S}^2}_{\text{can}}(\phi,\theta) = d\phi^2 + \sin^2(\phi)d\theta^2$$

Hence the product metric g_{prod} on $\mathbb{S}^2 \times \mathbb{S}^2$ writes

$$\begin{aligned} g_{\text{prod}}((\phi_1,\theta_1),(\phi_2,\theta_2)) &:= g^{\mathbb{S}^2}_{\text{can}}(\phi_1,\theta_1) \oplus g^{\mathbb{S}^2}_{\text{can}}(\phi_2,\theta_2) \\ &= d\phi_1^2 + \sin^2(\phi_1)d\theta_1^2 + d\phi_2^2 + \sin^2(\phi_2)d\theta_2^2 \end{aligned}$$

Notice that

(a) When a 2-plane Π is tangent to one common copy of $\mathbb{S}^2$, then $K(\Pi) = K(\mathbb{S}^2) = 1$ equal to the sectional curvature of the sphere, which we know to be 1.

(b) When a 2-plane Π contains fixed tangent vectors from both factors of $\mathbb{S}^2$, say $X \in T\mathbb{S}^2 \times \{p\}$ and $Y \in \{p\} \times T\mathbb{S}^2$, then X and Y are orthogonal, hence independent to each other due to the product metric. Thus

$$R(X, Y, X, Y) = 0$$

and $K(\Pi) = 0$.

2. Consider

$$\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$$

define embedding

$$\mathbb{T}^2 \hookrightarrow \mathbb{S}^2 \times \mathbb{S}^2 \qquad (\theta_1, \theta_2) \to ((\frac{\pi}{2}, \theta_1), (\frac{\pi}{2}, \theta_2))$$

In view of Proposition 2.7, it suffices to prove

$$\mathbb{S}^1 \subset \mathbb{S}^2$$

is totally geodesic. Philosophically this is true because $\mathbb{S}^1$, the great circle, is preserved by the geodesic flow on $\mathbb{S}^2$. In particular, let (ϕ, θ) denote coordinates on $(\mathbb{S}^2)$ and let embedding be

$$f : \mathbb{S}^1 \hookrightarrow \mathbb{S}^2 \qquad \theta \mapsto (\frac{\pi}{2}, \theta)$$

where

$$g^{\mathbb{S}^1}_{\text{round}} = d\theta^2 = f^* g^{\mathbb{S}^2}_{\text{round}}$$

But

$$B : \mathfrak{X}(\mathbb{S}^1) \times \mathfrak{X}(\mathbb{S}^1) \to \mathfrak{X}(\mathbb{S}^1)^{\perp}_{\mathbb{S}^2} \qquad B(X, Y)(p) := (\nabla_{\overline{X}} \overline{Y}(f(p)))^{\perp}$$

and observe

$$\begin{aligned}
\Gamma^{\phi}_{\theta\theta} &= -\sin(\phi)\cos(\phi) \\
\Gamma^{\theta}_{\theta\theta} &= \cot(\phi) \\
\nabla_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\theta} &= \Gamma^{\phi}_{\theta\theta} \frac{\partial}{\partial\phi} + \Gamma^{\theta}_{\theta\theta} \frac{\partial}{\partial\theta}
\end{aligned}$$

But evaluating at $\phi = \frac{\pi}{2}$ yields

$$\nabla_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\theta} = 0$$

Hence

$$B(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}) \equiv 0$$

But this is the only chance for B to be non-zero, hence we obtain bilinear form B as a zero map, and f is thus a totally geodesic in $\mathbb{S}^2$.

□

2.3.2 Mean Curvature

A much weaker notion than totally geodesic is minimality.

Definition 2.13 (Minimal)**.** *We say f is minimal at p if trace of the second fundamental form is* 0

$$\text{Tr}(S_\eta) = 0 \qquad \forall\, \eta \in (T_pM)^{\perp}$$

In general, one can define the mean curvature.

Definition 2.14 (Mean Curvature)**.** *The mean curvature of f at p is*

$$h_\eta := \frac{1}{n}\text{Tr}(S_\eta) \qquad \forall\, \eta \in (T_pM)^{\perp} \qquad |\eta| = 1 \tag{26}$$

We define the mean curvature vector as

$$\vec{H}(p) := \frac{1}{n}\sum_{i=1}^{n} B(e_i, e_i) \in (T_pM)^{\perp} \qquad \textit{for } e_i \textit{ as orthonormal basis of } T_pM$$

Example 2.5. *Consider sphere*

$$(\mathbb{S}^n, g_{can}) \hookrightarrow (\mathbb{R}^{n+1}, g_0)$$

Recall $\eta(p) = -p$, *and*

$$B(X,Y) = \langle X, Y\rangle \eta$$

Then

$$\vec{H}(p) = \frac{1}{n}\sum_{i=1}^{n}\langle e_i, e_i\rangle \eta(p) = \frac{1}{n}\sum_{i=1}^{n}\eta(p) = \eta(p) = -p$$

Hence the mean curvature vector of the sphere is also pointing inwards, the same as normal.

2.4 Gauss Map and Local Coordinates

Definition 2.15 (Guass Map)**.** *In general, if*

$$M^n \hookrightarrow (\mathbb{R}^{n+1}, g_0)$$

s.t. there exists a unit global normal vector $N \in \mathfrak{X}(M)^{\perp}$. *Then for any* $p \in M$

$$N(p) \in (T_pM)^{\perp} \subset T_p\mathbb{R}^{n+1} = \mathbb{R}^{n+1}$$

and

$$N(p) \in \mathbb{S}^n \qquad \textit{due to} \qquad |N| = 1$$

Then

$$N : M \to \mathbb{S}^n \qquad p \mapsto N(p)$$

is called the Gauss Map. Its differential writes

$$\begin{aligned} dN_p &: T_pM \to T_{N(p)}\mathbb{S}^n = T_pM \quad (\textit{they're identified as both orthogonal to } \mathbb{R}N(p) \textit{ in } \mathbb{R}^{n+1}) \\ dN_p(v) &:= (\overline{\nabla}_v N)(p) = -S_{N(p)}(v) \end{aligned}$$

In Coordinates. Look at Gauss Map

$$\begin{array}{ccc} V \subset \mathbb{R}^n \text{ with coordinates } (u_1, \cdots, u_n) & & \\ \Big\downarrow {\scriptstyle \mathbf{x} \text{ chart}} & \searrow^{\mathbf{y} \text{ chart}} & \\ p \in M & \xrightarrow{\quad N \quad} & \mathbb{S}^n \end{array}$$

so for $p = \mathbf{x}(u_1, \cdots, u_n)$

$$dN_p : T_pM \to T_pM \qquad \mathbf{x}_i = \frac{\partial \mathbf{x}}{\partial u_i} \mapsto \mathbf{x}_i = \frac{\partial \mathbf{x}}{\partial u_i}$$

here

$$\begin{aligned} \mathbf{x}(u_1, \cdots, u_n) &= (x_1(u_1, \cdots, u_n), \cdots, x_{n+1}(u_1, \cdots, u_n)) \\ d\mathbf{x} &: T_uV \to T_{\mathbf{x}(u)}M \subset T_{\mathbf{x}(u)}\mathbb{R}^{n+1} = \mathbb{R}^{n+1} \\ \frac{\partial}{\partial u_i} &\mapsto \frac{\partial \mathbf{x}}{\partial u_i} \qquad \text{where} \qquad \frac{\partial}{\partial u_i} \cong \frac{\partial \mathbf{x}}{\partial u_i} = (\frac{\partial x_1}{\partial u_i}, \cdots, \frac{\partial x_n}{\partial u_i}) = \sum_{k=1}^{n+1}\frac{\partial x_k}{\partial u_i}\frac{\partial}{\partial x_k} \end{aligned}$$

The good thing about Gauss Map is that then we do computation for second fundamental form.

$$\begin{aligned} H_N &= \sum h_{ij} du_i du_j \\ h_{ij} &= \langle B(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}), N\rangle = \langle (\overline{\nabla}_{\frac{\partial}{\partial u_i}}\frac{\partial}{\partial u_j})^{\perp}, N\rangle = \langle \overline{\nabla}_{\frac{\partial}{\partial u_i}}\frac{\partial}{\partial u_j}, N\rangle \\ &= \langle \overline{\nabla}_{\frac{\partial}{\partial u_i}}\sum_k \frac{\partial x_k}{\partial u_j}\frac{\partial}{\partial x_k}, N\rangle = \langle \sum_k \frac{\partial^2 x_k}{\partial u_i \partial u_j}\frac{\partial}{\partial x_k}, N\rangle = \langle \mathbf{x}_{ij}, N\rangle \end{aligned}$$

□

Example 2.6 (Surface of Revolution). *Let $S \subset \mathbb{R}^3$ be the surface of revolution of $y = \cosh(z)$. Now, to do rotation we need sine and cosine.*

$$\mathbf{x}(u, v) = (\cos(u)\cosh(v), \sin(u)\cosh(v), v)$$

Then using local coordinates

$$\begin{aligned}
\mathbf{x}_u &= \frac{\partial \mathbf{x}}{\partial u} = (-\sin(u)\cosh(v), \cos(u)\cosh(v), 0) \\
\mathbf{x}_v &= \frac{\partial \mathbf{x}}{\partial v} = (\cos(u)\sinh(v), \sin(u)\sinh(v), 1)
\end{aligned}$$

This is basis for tangent space. Let's compute the first fundamental form $g = \mathbf{x}^* g_0$ (the induced metric on S).

$$\begin{aligned}
g &= Edu^2 + 2Fdudv + Gdv^2 \\
&= \langle \mathbf{x}_u, \mathbf{x}_u \rangle du^2 + 2\langle \mathbf{x}_u, \mathbf{x}_v \rangle dudv + \langle \mathbf{x}_v, \mathbf{x}_v \rangle dv^2 \\
&= \cosh^2(v) du^2 + (\sinh^2(v) + 1) dv^2 = \cosh^2(v)(du^2 + dv^2)
\end{aligned}$$

Next we compute the second fundamental form. In $\mathbb{R}^3$, normal vector is given by the cross product.

$$\begin{aligned}
N = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} &= \frac{(\cos(u)\cosh(v), \sin(u)\cosh(v), -\sinh(v)\cosh(v))}{\sqrt{\cosh^2(v) + \sinh^2(v)\cosh^2(v)}} \\
&= \frac{(\cos(u), \sin(u), -\sinh(v))}{\cosh(v)}
\end{aligned}$$

The second fundamental form writes

$$\begin{aligned}
H_N &= edu^2 + 2fdudv + gdv^2 \\
&= \langle \mathbf{x}_{uu}, N \rangle du^2 + 2\langle \mathbf{x}_{uv}, N \rangle dudv + \langle \mathbf{x}_{vv}, N \rangle dv^2
\end{aligned}$$

Notice

$$\begin{aligned}
e = \langle \mathbf{x}_{uu}, N \rangle &= \langle -\cos(u)\cosh(v)\vec{i} - \sin(u)\cosh(v)\vec{j}, \frac{\cos(u)}{\cosh(v)}\vec{i} + \frac{\sin(u)}{\cosh(v)}\vec{j} - \frac{\sinh(v)}{\cosh(v)}\vec{k} \rangle \\
&= -\cos^2(u) - \sin^2(u) = -1 \\
f = \langle \mathbf{x}_{uv}, N \rangle &= \langle -\sin(u)\sinh(v)\vec{i} + \cos(u)\sinh(v)\vec{j}, \frac{\cos(u)}{\cosh(v)}\vec{i} + \frac{\sin(u)}{\cosh(v)}\vec{j} - \frac{\sinh(v)}{\cosh(v)}\vec{k} \rangle = 0 \\
g = \langle \mathbf{x}_{vv}, N \rangle &= \langle \cos(u)\cosh(v)\vec{i} + \sin(u)\cosh(v)\vec{j}, \frac{\cos(u)}{\cosh(v)}\vec{i} + \frac{\sin(u)}{\cosh(v)}\vec{j} - \frac{\sinh(v)}{\cosh(v)}\vec{k} \rangle = 1
\end{aligned}$$

Thus the second fundamental form is

$$H_N = -du^2 + dv^2$$

By writing

$$\begin{aligned}
S_N(\frac{\partial}{\partial u}) &= a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v} \\
H_N(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}) &= -1 = \langle S_N(\frac{\partial}{\partial u}), \frac{\partial}{\partial u} \rangle = \langle a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}, \frac{\partial}{\partial u} \rangle = a\cosh^2(v) \qquad \textit{here one needs the first fundamental form} \\
a &= -\frac{1}{\cosh^2(v)} \\
H_N(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}) &= 0 = \langle S_N(\frac{\partial}{\partial u}), \frac{\partial}{\partial v} \rangle = \langle a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}, \frac{\partial}{\partial v} \rangle = b\cosh^2(v) \\
b &= 0 \\
\implies S_N(\frac{\partial}{\partial u}) &= -\frac{1}{\cosh^2(v)}\frac{\partial}{\partial u} \\
dN_p(\mathbf{x}_u) &= \frac{1}{\cosh^2(v)}\mathbf{x}_u \\
dN_p(\mathbf{x}_v) &= -\frac{1}{\cosh^2(v)}\mathbf{x}_v \\
\implies dN_p &= \frac{1}{\cosh^2(v)}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \textit{w.r.t. basis} \qquad \mathbf{x}_u, \mathbf{x}_v
\end{aligned}$$

2.5 Principal Curvature

Definition 2.16 (Principle Curvature)**.** *For*

$$f : (M^n, g) \hookrightarrow (\overline{M}^{n+1}, \overline{g})$$

isometric immersion with $\eta \in (T_pM)^\perp$ *unique up to* $\pm$. *Let*

$$S_\eta : T_pM \to T_pM$$

be self adjoint w.r.t. $\{e_i\}$ *an orthonormal basis of* T_pM *and*

$$S_\eta e_i = \lambda_i e_i$$

Then we call $\{\lambda_i\}$ *the principal curvatures. We call*

$$\det(S_\eta) = \prod_{i=1}^n \lambda_i \qquad \textit{Gauss-Kronecker Curvature}$$
$$\operatorname{Tr}(S_\eta) = \sum_{i=1}^n \lambda_i \qquad \textit{rescaled mean curvature}$$

Example 2.7. *For*

$$f : (M^2, g) \hookrightarrow (\mathbb{R}^3, g_0)$$

surface in $\mathbb{R}^3$. *Choose* $n \in T_pM^\perp$. *The sectional curvature (intrinsic)*

$$\begin{aligned} K(p) = K(e_1, e_2) &\overset{\textit{Gauss Equation}}{=} \langle B(e_1, e_1), B(e_2, e_2)\rangle - |B(e_1, e_2)|^2 \\ &= \langle \lambda_1 n, \lambda_2 n\rangle - 0 = \lambda_1\lambda_2 \end{aligned}$$

Since

$$\begin{aligned} H(e_1, e_1) = \langle B(e_1, e_1), n\rangle &= \langle S_n(e_1), e_1\rangle = \langle \lambda_1 e_1, e_1\rangle = \lambda_1 \\ \langle B(e_1, e_2), n\rangle &= \langle S_n(e_1), e_2\rangle = \lambda_1\langle e_1, e_2\rangle = 0 \end{aligned}$$

and again since we're in codimension 1

$$\langle B(e_1, e_1), B(e_2, e_2)\rangle = \langle\langle B(e_1, e_1), n\rangle n, \langle B(e_2, e_2), n\rangle n\rangle = \langle \lambda_1 n, \lambda_2 n\rangle$$

2.6 Examples

2.6.1 Hessian

In the following we discuss the Hessian.

Definition 2.17 (Hessian)**.** *Let*

$$f : \overline{M}^{n+1} \to \mathbb{R}$$

be a differentiable function. Define the Hessian Hess(f) *of* f *at* $p \in \overline{M}$ *as the linear operator*

$$\operatorname{Hess}(f) : T_p\overline{M} \to T_p\overline{M} \qquad (\operatorname{Hess}(f))(Y) := \overline{\nabla}_Y \operatorname{grad}(f) \qquad \forall\ Y \in T_p\overline{M} \tag{27}$$

where $\overline{\nabla}$ *is the Riemannian connection of* $\overline{M}$.

Lemma 2.3 (Laplacian)**.** *The Laplacian* $\overline{\Delta} f$ *is given by*

$$\overline{\Delta} f = \operatorname{Tr}(\operatorname{Hess}(f)) \tag{28}$$

Proof. By definition

$$\begin{aligned} \overline{\Delta} f &:= \operatorname{div}(\operatorname{grad}(f)) \\ &:= \operatorname{Tr}(\text{linear mapping } Y(p) \to \overline{\nabla}_Y \operatorname{grad}(f)(p) \text{ for any } p \in M) \\ &= \operatorname{Tr}(\operatorname{Hess}(f)) \end{aligned}$$

□

Lemma 2.4 (Hessian as symmetric bilinear form)**.** *For any* $X, Y \in \mathfrak{X}(\overline{M})$

$$\langle \mathrm{Hess}(f)Y, X\rangle = \langle Y, (\mathrm{Hess}(f)X\rangle \tag{29}$$

Hence Hess(f) *is self-adjoint, and determines a symmetric bilinear form on* $T_p\overline{M}$ *for any* $p \in \overline{M}$ *via*

$$\mathrm{Hess}(f)(X, Y) := \langle (\mathrm{Hess}(f))X, Y\rangle \qquad \forall\ X, Y \in T_p\overline{M} \tag{30}$$

Proof.

$$\begin{aligned}
\langle (\mathrm{Hess}(f))Y, X\rangle &= \langle \overline{\nabla}_Y \mathrm{grad}(f), X\rangle = Y(\langle \mathrm{grad}(f), X\rangle) - \langle \mathrm{grad}(f), \overline{\nabla}_Y X\rangle \\
&= Y(X(f)) - (\overline{\nabla}_Y X)(f) \qquad \text{using definition of grad}(f) \text{ and Levi-Civita is compatible with metric} \\
&= [Y, X](f) + X(Y(f)) - (\overline{\nabla}_Y X)(f) \qquad \text{using definition of Lie Bracket} \\
&= (\overline{\nabla}_X Y)(f) + X(Y(f)) \qquad \text{using Levi-Civita is symmetric} \\
&= \langle \mathrm{grad}(f), \overline{\nabla}_X Y\rangle + X(\langle \mathrm{grad}(f), Y\rangle) \\
&= \langle \overline{\nabla}_X \mathrm{grad}(f), Y\rangle = \langle Y, \overline{\nabla}_X \mathrm{grad}(f)\rangle = \langle Y, (\mathrm{Hess}(f))X\rangle
\end{aligned}$$

□

Proposition 2.8. *Let 'a' be a regular value of f, i.e., for any* $p \in f^{-1}(a)$*, f is a submersion at p. Let* $M^n \subset \overline{M}^{n+1}$ *be the hypersurface in* $\overline{M}$ *defined by*

$$M := \{p \in \overline{M} \mid f(p) = a\} = f^{-1}(a)$$

1. *The mean curvature H of* $M \subset \overline{M}$ *is given by*

$$nH = -\mathrm{div}\left(\frac{\mathrm{grad}(f)}{|\mathrm{grad}(f)|}\right) \tag{31}$$

Proof. Take an Orthonormal frame $E_1, \cdots, E_n$ and our normal vector

$$E_{n+1} := \frac{\mathrm{grad}(f)}{|\mathrm{grad}(f)|} = \eta$$

in a neighborhood p of M in $\overline{M}$. Recall H as in (26) and S_η as in (17)

$$\begin{aligned}
nH = \mathrm{Tr}(S_\eta) &= \sum_{i=1}^{n} \langle S_\eta(E_i), E_i\rangle_{\overline{g}} \\
&= -\sum_{i=1}^{n} \langle (D_{E_i}\eta)^T, E_i\rangle_{\overline{g}} \qquad (18) \\
&= -\sum_{i=1}^{n} \langle D_{E_i}\eta, E_i\rangle_{\overline{g}} = -\sum_{i=1}^{n} \langle (f^*\overline{\nabla})_{E_i}\eta, E_i\rangle_{\overline{g}} \qquad (19) \\
&= -\sum_{i=1}^{n} \langle \overline{\nabla}_{E_i}\eta, E_i\rangle_{\overline{g}} - \langle \overline{\nabla}_\eta \eta, \eta\rangle_{\overline{g}} \\
&= -\sum_{i=1}^{n+1} \langle \overline{\nabla}_{E_i}\eta, E_i\rangle \\
&= -\mathrm{div}_{\overline{M}}\eta \qquad \text{using definition of divergence} \\
&= -\mathrm{div}\left(\frac{\mathrm{grad}(f)}{|\mathrm{grad}(f)|}\right)
\end{aligned}$$

□

2. *Notice Every Embedded hypersurface* $M^n \subset \overline{M}^{n+1}$ *is locally the inverse image of a regular value. Moreover, the mean curvature H of such a hypersurface is given by*

$$H = -\frac{1}{n}\mathrm{div}(N)$$

where N is an appropriate local extension of the unit normal vector field on $M^n \subset \overline{M}^{n+1}$.

Proof. (a) Since $M \hookrightarrow \overline{M}$, there exists a smooth immersion

$$f : M \to \overline{M}$$

s.t. $f(M) \subset \overline{M}$ is homeomorphism w.r.t. subspace topology. Or using the alternative definition, for any $q \in M$, there exists a neighborhood U of q in $\overline{M}$ and a coordinate chart $\phi = (x_1, \cdots, x_{n+1})$ on U s.t.

$$\phi(M \cap U) = \phi(M) \cap \{x_{n+1} = 0\}$$

In other words

$$M \cap U = \{q \in U \mid x_{n+1}(q) = 0\}$$

It suffices to see 0 is a regular value for $f = x_{n+1}$. But for any $p \in \overline{M}$

$$df_p : T_p\overline{M} \to \mathbb{R} \qquad df_p = \frac{\partial f}{\partial x_{n+1}} dx_{n+1}$$

Then

$$\frac{\partial f}{\partial x_{n+1}} = 1$$

Hence df_p is surjective for any $p \in M \cap U$ so 0 is a regular value for $f = x_{n+1}$.

(b) For any $q \in M$, there exists neighborhood U of q in $\overline{M}$ and $a \in \mathbb{R}$ s.t.

$$U \cap M = f_U^{-1}(a)$$

for some smooth f_U and a as its regular value. Applying (31), the mean curvature H of $M \cap U \subset \overline{M}$ is

$$nH = -\text{div}(\frac{\text{grad}(f_U)}{|\text{grad}(f_U)|})$$

However one can extend the formula to neighborhood U in $\overline{M}$ because f_U is submersion on $U \cap M$, hence has non-vanishing gradient. By continuity of f_U one can extend smoothly to open neighborhood in $\overline{M}$. Now one can define a unit normal vector field N as the local extension s.t.

$$N_U := \frac{\text{grad}(f_U)}{|\text{grad}(f_U)|} \qquad \forall\, U \subset \overline{M} \qquad \text{local neighborhood s.t. } N_U \text{ is well-defined}$$

□

2.6.2 Singularity of Killing Field

Proposition 2.9. *Let X be a Killing vector field on a Riemannian manifold M. Let*

$$N = \{p \in M \mid X(p) = 0\}$$

1. *If $p \in N$ and $V \subset M$ is a normal neighborhood of p. Let $q \in N \cap V$. Then the radial segment γ joining p to q is contained in N. In particular $\gamma \cap V \subset N$.*

Proof. Let V be normal neighborhood of p, i.e., $\exp_p$ is a diffeomorphism from a subset of T_pM to V. Consider the unique radial geodesic segment $\gamma : [0, 1] \to M$ joining p and q s.t.

$$\gamma(0) = p \in N \cap V$$
$$\gamma(1) = q \in N \cap V$$

Since X is a Killing vector field, the flow of X preserves the metric of M, in particular geodesics. Let ϕ_t denote the flow of X, i.e.

$$\frac{\partial}{\partial t}\phi_t(q) = X(\phi_t(q))$$
$$\phi_0(q) = q$$

Hence ϕ_t preserves the geodesic, i.e.

$$\phi_t(\gamma(s)) = \gamma(s) \qquad \forall\, s \in [0, 1]$$

We obtain

$$X(\gamma(s)) = \frac{\partial}{\partial t}\phi_t(\gamma(s)) = \frac{\partial}{\partial t}\gamma(s) = 0 \qquad \forall\, s \in [0, 1]$$

Thus $\gamma(s) \in N$ for any s. □

2. *If $p \in N$, there exists a neighborhood $V \subset M$ of p s.t. $V \cap N$ is a submanifold of M. In particular every connected component of N is a submanifold of M.*

Proof. (a) If p is isolated, done.

(b) Otherwise let $V \subset M$ be a normal neighborhood of p s.t. there exists $q_1 \in V \cap N$. Consider the radial geodesic γ_1 joining p to q_1. If $V \cap N = \gamma_1$, by (a) we're done.

(c) Otherwise let $q_2 \in V \cap N \setminus \{\gamma_1\}$ and let γ_2 be the radial geodesic joining p to q_2. Consider

$$Q = \text{Span}\{\exp_p^{-1}(q_1), \exp_p^{-1}(q_2)\} \subset T_pM$$

and let

$$N_2 := \exp_p(Q \cap \exp_p^{-1}(V))$$

Here we denote $X_t : M \to M$ as the flow of X. Notice X_t fixes q_1 and q_2

$$X_t(q_1) = q_1 \qquad X_t(q_2) = q_2$$

In particular, since X_t preserves the geodesics, and $\exp_p^{-1}(q_1)$, $\exp_p^{-1}(q_2)$ are tangent to the geodesics joining p to q_1 and q_2 respectively

$$(dX_t)_p(\exp_p^{-1}(q_1)) = \exp_p^{-1}(q_1)$$
$$(dX_t)_p(\exp_p^{-1}(q_2)) = \exp_p^{-1}(q_2)$$

Since $Q = \text{Span}\{\exp_p^{-1}(q_1), \exp_p^{-1}(q_2)\}$ we have $(dX_t)_p$ restricted to Q is the identity. Thus $N_2 \subset V \cap N$. We proceed by picking another geodesic until the dimension of T_pM is exhausted.

□

3. *The codimension, as a submanifold of M, of a connected component N_k of N, is even.*

Proof. Recall the fact that: If a sphere has a non-vanishing differentiable vector field on it then its dimension must be odd. Now let $E_p := (T_pN_k)^\perp$ and set $V \subset M$ be a normal neighborhood of p. Let

$$N_k^\perp := \exp_p(E_p \cap \exp_p^{-1}(V))$$

For all t,

$$(dX_t)_p : E_p = (T_pN_k)^\perp \to E_p$$

so X is tangent to $N_k^\perp$. On the other hand $X \neq 0$ is tangent to the geodesic spheres of $N_k^\perp$ with center p. But by our fact above, the dimension of such geodesic sphere is odd. Hence

$$\dim N_k^\perp = \dim E_p$$

is even. □

3 Global Differential Geometry

We plan on discussing

1. Complete Manifolds
2. Hopf-Rinow Theorem
3. Hadamard Theorem

3.1 Complete Riemannian Manifolds

We always assume M is Hausdorff. Completeness for metric space means all Cauchy Sequence converges. We want to define geodesically complete, to do so one needs distance on manifolds.

Definition 3.1 (Path-Connected). *M is path-connected if for any p, $q \in M$, there exists continuous*

$$c : [0,1] \to M \qquad s.t. \qquad c(0) = p \qquad c(1) = q$$

Lemma 3.1. *If M is a connected topological manifold, then M is path connected.*

Lemma 3.2. *If M is a connected C^k manifold, there exists a C^K map*

$$c : [0,1] \to M \qquad s.t. \qquad c(0) = p \qquad c(1) = q$$

Definition 3.2 (Distance). *Let (M, g) be a connected Riemannian manifold. For every p, $q \in M$, we define the distance between p and q as infimum of the length of all curves connecting p and q*

$$d_g(p,q) := \inf\{\ell(c) \mid c : [0,1] \to M \quad \textit{piecewise smooth} \quad s.t. \quad c(0) = p \qquad c(1) = q\}$$

1. *The set is non-empty due to M is connected. Hence $d_g(p,q) \geq 0$.*
2. *We fix the metric g and denote $d(p,q)$.*

Proposition 3.1. *(M, d) defines a metric space.*

Proof. 1. Triangle Inequality. For any p, q, $m \in M$.

$$d(p,q) + d(q,m) \geq d(p,m)$$

due to composition of curves.

2. d is symmetric trivially by reversing the curve parametrization.
3. $d(p,q) \geq 0$ due to nonempty set. It suffices to check $d(p,q) = 0 \iff p = q$. We need to check $d(p,q) = 0$ implies $p = q$. We prove the contrapositive, i.e., for $p \neq q$, we want to show $d(p,q) > 0$. For this we need to use our manifold M is Hausdorff. There exists an open neighborhood U of $p \in M$ s.t. $q \notin U$. There exists r s.t. the normal ball
$$B_r(p) \subset U$$
But then $d(p,q) \geq r$ because all points at distance $\leq r$ from p are in $B_r(p)$, otherwise $q \in B_r(p)$.

□

Example 3.1. *1. On $\mathbb{R}^n$, $d(x,y) = |x - y|$.*

2. Line with two origins. Let $M = (\mathbb{R} \times \{0,1\}) / [(x,0) \sim (x,1)$ except for $x = 0]$. Then

$$\begin{aligned} d([x,0],[y,1]) &= |x-y| \qquad \forall\, x,\, y \neq 0 \\ d([x,0],[0,1]) &= |x| \\ d([0,0],[0,1]) &= 0 \end{aligned}$$

Hence we indeed need Hausdorff condition.

Remark 3.1. *1. If there exists a minimizing geodesic γ between p and q, then*

$$\ell(\gamma) = d(p,q)$$

2. The topology induced by d is the same as the original topology, i.e., the one with basis

$$\{B_r(p) \mid r > 0,\, p \in M\}$$

3. *Fix $p_0 \in M$, then*

$$f : (M, d) \to (\mathbb{R}, |\cdot|) \qquad q \mapsto d(p_0, q)$$

is continuous, In fact

$$|f(q) - f(p)| = |d(p_0, q) - d(p_0, p)| \leq |d(p, q)|$$

Then f is Lipschitz continuous.

Definition 3.3 (Geodesically Complete)**.** *A Riemannian manifold (M, g) is geodesically complete if for any $p \in M$,*

$$\exp_p(v) \qquad \text{is defined for all } v \in T_pM$$

i.e., all geodesics $\gamma(t)$ are defined for all $t \in \mathbb{R}$.

3.2 Hopf-Rinow Theorem

Hopf-Rinow says (M, g) is geodesically complete iff (M, d) is complete metric space.

Theorem 3.1 (Hopf-Rinow)**.** *The following are $(a) - (e)$ equivalent and all imply (f)*

(a) $\exp_p(v)$ *is defined for all $v \in T_pM$ at a particular point $p \in M$.*

(b) Closed and Bounded sets of (M, d) are compact.

(c) (M, d) is a complete metric space.

(d) (M, g) is geodesically complete, i.e., $\exp_q(v)$ *is defined for all $v \in T_qM$ for any $q \in M$.*

(e) There exists a sequence of compact sets $\{K_n\}$

$$K_n \subset K_{n+1} \qquad \bigcup_n K_n = M$$

s.t. if $q_n \notin K_n \ \forall\ n$ then $d(p, q_n) \to \infty$.

(f) In the above cases, for any $q \in M$ fixed there exists minimizing geodesic γ between any $p \in M$ and q, i.e.

$$\ell(\gamma) = d(p, q)$$

Example 3.2 (Counter example for (f) does not imply (a))**.** *Take $B_1(0)$ open ball in $\mathbb{R}^n$. (f) is satisfied. But* $\exp_0(v)$ *is not defined for $|v| \geq 1$. In particular $B_1(0)$ is not complete.*

Proof of Theorem 3.1 $(a) \implies (f)$. We want to find the initial velocity $v \in T_pM$ s.t. $|v| = 1$ of the geodesic γ where $\gamma(0) = p$ and $\gamma(1) = q$. In this case we want

$$\gamma(t) = \exp_p(tv) \qquad \gamma(r) = q \qquad \text{for } r = d(p, q)$$

There exists r_0 s.t. $B_{r_0}(p)$ is a normal ball at p.

1. Case one. If $r < r_0$, and $q \in B_{r_0}(p)$, then there exists a minimizing geodesic connecting p and q.

2. Case two. If $r > r_0$. The idea is to construct the curve with initial velocity step by step. Consider the map

$$f : M \to \mathbb{R} \qquad x \mapsto d(q, x)$$

which is continuous. There exists $x_0 \in S_{r_0}(p)$ the sphere of the normal ball s.t.

$$x_0 = \min_{x \in S_{r_0}(p)} f(x)$$

Note x_0 may not be unique. In particular,

$$x_0 = \exp_p(r_0 v) \qquad \text{for some unit tangent vector } v$$

Finally we can use the assumption that $\exp_p$ is defined for all $v \in T_pM$. So we define the curve

$$\gamma(t) := \exp_p(tv)$$

and I want to show that $\gamma(r) = q$. To do so we use the continuity method. We define a set

$$A = \{s \in [0, r] \mid d(\gamma(s), q) = r - s\}$$

If one can prove A is non-empty, closed and open, then since A is connected, we have $A = [0, r]$. In particular we conclude $r \in A$, and finally $d(\gamma(r), q) = r - r = 0 \implies \gamma(r) = q$.

- A is non-empty since $s = 0$ lies inside
$$d(\gamma(0), q) = d(p, q) = r - 0 = r$$
- A is closed due to closed condition.
- We're left to prove A open. We show that if $s \in A$, then there exists $\delta > 0$ s.t. $s + \delta \in A$. Since $s \in A$, one has
$$d(\gamma(s), q) = r - s$$
Consider the normal ball centered at $\gamma(s)$, of some radius δ, which is between $(0, r - s)$. Now we consider x' that minimizes the distance between the ball and q, i.e.
$$x' = \min_{x \in S_\delta(\gamma(s))} f(x)$$
Then
$$\begin{aligned} r - s = d(\gamma(s), q) &= \delta + \min\{d(q, x) \mid x \in S_\delta(\gamma(s))\} \\ &= \delta + d(q, x') \\ d(x', q) &= r - (s + \delta) \end{aligned}$$
Now by triangle inequality
$$\begin{aligned} & s + \delta \geq d(p, x') \geq d(p, q) - d(q, x') = r - r + (s + \delta) = s + \delta \\ & d(p, x') = s + \delta \\ & \implies x' = \gamma(s + \delta) \end{aligned}$$
Thus
$$d(\gamma(s + \delta), q) = r - (s + \delta) \implies s + \delta \in A$$

□

Proof of Theorem 3.1 (a) $\implies$ (b). Let $A \subset M$ closed and bounded. Then there exists $r > 0$ s.t.
$$A \subset \{x \in M \mid d(x, p) \leq r\} = \overline{B_r(p)} \subset \exp_p(\overline{B_r(0)})$$
where the latter is indeed a compact set. Hence using A closed subset of a compact set and Hausdorff topology, one knows that A is compact. □

Proof of Theorem 3.1 (b) $\implies$ (c). Start with a Cauchy Sequence $\{x_n\}$. Let $A = \overline{\{x_n\}}$ be closed and bounded. Then A is compact, and there exists a subsequence $x_{n_k} \to p_0 \in M$. Hence $x_n \to p_0$ since it's Cauchy. □

Proof of Theorem 3.1 (c) $\implies$ (d). Let $q \in M$, we want to show that $\exp_q$ is defined on T_qM. Suppose
$$\gamma : (a, s_0) \to M$$
is a normalized geodesic. Then we prove that γ can be extended to
$$\gamma : (a, s_0 + \delta) \to M$$
How do we prove? Remember we assume Cauchy sequence converges. We take a sequence that converges to s_0. Let s_n be an increasing sequence s.t. $s_n \nearrow s_0$. Then since we have normalized geodesic
$$d(\gamma(s_n), \gamma(s_m)) \leq |s_n - s_m|$$
Since $\{s_n\}$ is Cauchy, $\gamma(s_n)$ are Cauchy, and using our assumption, $\gamma(s_n) \to p_0$ in M. Then there exists $\delta > 0$ and a totally normal neighborhood V of p_0 s.t.

1. for any p_1, $p_2 \in V$, there is a minimizing geodesic between p_1 and p_2
2. for every $q \in V$,
$$\exp_q : B_{2\delta}(0) \subset T_qM \to V$$
is defined.

What is remarkable is that the δ is uniform in $q \in V$. If $\gamma(s_n)$ and $\gamma(s_m) \in V$, then γ coincides with the minimizing geodesic between $\gamma(s_n)$ and $\gamma(s_m)$. Choose s_n s.t. $\gamma(s_n) \in V$ and

$$s_0 - \delta < s_n < s_0$$

Then for the exponential map at $\gamma(s_n)$, we again center a ball at $\gamma(s_n)$ with radius 2δ, i.e.

$$\exp_{\gamma(s_n)} : B_{2\delta}(0) \to V$$

is defined. Hence $\gamma(t)$ is defined for $t \in (s_n - 2\delta, s_n + \delta)$. But $s_n + 2\delta > s_0$ by our choice. Hence γ is extended. □

Proof of Theorem 3.1 (d) $\implies$ (a). Trivial. □

Proof of Theorem 3.1 (b) $\implies$ (e). Let $K_n = \overline{B_n(p)}$. They satisfy (e). If $q_n \notin K_n$ for any n, then $d(p, q_n) \geq n$. □

Proof of Theorem 3.1 (e) $\implies$ (b). Let A be a closed and bounded set. Then there exists n s.t. $A \subset K_n$, hence A is compact. □

Corollary 3.1. *Any Riemannian metric on a compact manifold gives a complete manifold.*

Proof. Property (e) is always verified. □

Example 3.3. 1. *$\mathbb{S}^n$, $\mathbb{T}^n$ are complete.*

2. *$(\mathbb{R}^n, g_0)$ is complete.*

3. *$(B_1^n(0), g_0)$ is not complete.*

4. *Let*

$$\phi : \mathbb{R}^n \to B_1^n \qquad x \mapsto \frac{x}{\sqrt{1+|x|^2}}$$

Then the inverse writes

$$\phi^{-1} : B_1^n \to \mathbb{R}^n \qquad y \mapsto \frac{y}{\sqrt{1-|y|^2}}$$

The diffeomorphism $(\mathbb{R}^n, \phi^* g_0)$ is not complete since the ball is not complete.

5. *Any proper open subset of a complete manifold is not complete, i.e., for open embedding*

$$i : M \hookrightarrow \overline{M} \qquad i(M) \subsetneq (\overline{M}, \overline{g}) \qquad \text{open and proper}$$

Here $(M, i^*\overline{g})$ is not complete.

Definition 3.4 (Extendible)**.** *Let M, M' be connected. A Riemannian manifold (M, g) is extendible if there exists an isometric open embedding*

$$i : (M, g) \hookrightarrow (M', g') \qquad i(M) \subsetneq M'$$

Remark 3.2. *If M is compact then M is complete. If M is complete than M is non-extendible. Both converses are not true.*

1. *$(\mathbb{R}^n, g_0)$ is complete but not compact*

2. *The map*

$$\exp : \mathbb{C} \to (\mathbb{C} \setminus \{0\}, dx^2 + dy^2) \qquad z \mapsto e^z$$

gives $(\mathbb{C} \setminus \{0\}, dx^2 + dy^2)$ extendible. Hence this is incomplete. But then

$$(\mathbb{C}, \exp^*(dx^2 + dy^2))$$

is incomplete and inextendible.

Corollary 3.2. *Let (M, g) be complete Riemannian manifold. Let N be a closed submanifold. Denote*

$$i : N \to M \qquad \text{as inclusion}$$

Then

$$(N, i^* g) \qquad \text{is complete}$$

Proof. By Theorem 3.1 property (b), we need to show closed and bounded sets of N are compact. Here closed and bounded sets are w.r.t. the distance d_N given by i^*g. But

$$d_N(p,q) \geq d_M(p,q)$$

so any closed and bounded sets of N are also closed and bounded in M. So they're compact. □

We have one criterion for complete manifolds.

Proposition 3.2. *Let M and $\overline{M}$ be Riemannian manifolds and let*

$$f : M \to \overline{M} \qquad \text{be a diffeomorphism}$$

Let $\overline{M}$ be complete, and assume there exists a constant $c \geq 0$ s.t.

$$|v| \geq c|df_p(v)| \qquad \forall\, p \in M \qquad v \in T_pM$$

Then M is complete.

Proof. Let $\{p_n\}$ be a Cauchy sequence in M. By Hopf-Rinow 3.1 (c), it suffices to prove p_n converges. Notice $\{f(p_n)\}_n$ is a sequence in $\overline{M}$, and since $\overline{M}$ is complete, if we're able to show $\{f(p_n)\}_n$ is Cauchy, we have convergence of $f(p_n)$ to some point $q \in \overline{M}$. Indeed, for any $p_n \in M$, there exists totally normal neighborhood V of p_n s.t. for any $p_m \in V$, there exists γ a minimizing geodesic joining p_n and p_m, i.e.

$$\gamma : [0,1] \to M \qquad \gamma(0) = p_n \qquad \gamma(1) = p_m$$

One has

$$\begin{aligned} d_{\overline{M}}(f(p_n), f(p_m)) &\leq \int_0^1 |df_{\gamma(t)}(\gamma'(t))|\, dt \\ &\leq \int_0^1 \frac{1}{c}|\gamma'(t)|\, dt = \frac{1}{c}\ell(\gamma) \\ &= \frac{1}{c} d_M(p_n, p_m) \end{aligned}$$

But $\{p_n\}$ is Cauchy sequence in M, hence $d_M(p_n, p_m) \to 0$ so $\{f(p_n)\}$ is a Cauchy sequence. Thus there exists $q \in \overline{M}$ s.t.

$$d_{\overline{M}}(f(p_n), q) \to 0$$

Now since f is a diffeomorphism, it has a smooth inverse, hence define $p := f^{-1}(q)$ and by continuity

$$d_M(p_n, p) \to 0$$

□

3.3 Examples

We discuss further examples illustrating Hopf-Rinow.

3.3.1 Rays

Definition 3.5 (Ray)**.** *A geodesic $\gamma : [0, \infty) \to M$ in a Riemannian manifold M is a ray starting from $\gamma(0)$ if it minimizes the distance between $\gamma(0)$ and $\gamma(s)$ for any $s \in (0, \infty)$.*

Proposition 3.3. *Let M be complete and non-compact. Then for any $p \in M$, there exists a ray starting from p in M.*

Proof.

1. Since M is geodesically complete, for any $p \in M$, the exponential map $\exp_p(v)$ is defined for all $v \in T_pM$. Since M is non-compact, there exists a sequence of points $q_n \in M$ s.t. $d(p, q_n) \to \infty$.

2. Using (f) in Hopf-Rinow 3.1, for any $q_n \in M$ one can pick a minimizing geodesic γ_n between p and q_n s.t.

   $$\ell(\gamma_n) = d(p, q_n)$$

   WLOG one may parametrize γ_n using arc-length, i.e.

   $$\gamma_n(0) = p \qquad \gamma_n(d(p, q_n)) = q_n$$

3. Now consider the family of tangent vectors $\{\gamma_n'(0)\} \subset S_pM \subset T_pM$ where $|\gamma_n'(0)| = 1$ and S_pM denotes the unit sphere in T_pM. Since S_pM is compact, one may extract a convergnt subsequence $\gamma_{n_k}'(0) \to v \in S_pM$. Again since M is geodesically complete, the geodesic

$$\gamma : [0, \infty) \to M \qquad \gamma(0) = p, \qquad \gamma'(0) = v$$

exists.

4. We claim that γ is a ray. To see this, one needs to show γ minimizes the distance between p and $\gamma(s)$ for any $s \in (0, \infty)$. Now fix s, there exists k large enough s.t.

$$d(p, q_{n_k}) \geq d(p, \gamma(s))$$

hence

$$\ell(\gamma_{n_k}|_{[0,s]}) = d(p, \gamma_{n_k}(s)) \qquad \text{is length minimizing}$$

Push $k \to \infty$, since both

$$\gamma_{n_k}'(0) \to v \qquad \gamma_{n_k}(s) \to \gamma(s)$$

By continuous dependence on initial conditions

$$d(\gamma_{n_k}(s), \gamma(s)) \to 0$$

Hence $\gamma|_{[0,s]}$ is length minimizing.

□

3.3.2 Hyperbolic Plane

Definition 3.6. *A Hyperbolic Plane $H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ is equipped with Riemannian metric*

$$g_{11} = g_{22} = \frac{1}{y^2} \qquad g_{12} = 0$$

Recall the 'Minimizing' characterisation for geodesics.

Proposition 3.4. *If a piecewise differentiable curve $\gamma : [a, b] \to M$ with parameter proportional to arc length has length less or equal to any other piecewise differentiable curve joining $\gamma(a)$ and $\gamma(b)$, then γ is a geodesic in M.*

Lemma 3.3 (Geodesics of H)**.** *Geodesics of H are either*

1. *Upper semi-circles*
2. *rays $x = x_0$ for $y > 0$*

Proof. 1. We claim the segment

$$\gamma : [a, b] \to H \qquad \gamma(t) := (0, t) \qquad a > 0$$

is the image of a geodesic. Indeed, for any arc $c : [a, b] \to H$ s.t.

$$c(t) = (x(t), y(t)) \qquad c(a) = (0, a) \qquad c(b) = (0, b)$$

One has

$$\begin{aligned}\ell(c) &= \int_a^b |\frac{dc}{dt}|\, dt = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, \frac{1}{y(t)} dt \\ &\geq \int_a^b |\frac{dy}{dt}| \frac{1}{y} dt \geq \int_a^b \frac{dy}{y} = \ell(\gamma)\end{aligned}$$

Hence γ minimizes arc length for piecewise differentiable curves, and using Proposition 3.4, the image of γ is a geodesic.

2. The isometries of H are the Möbius Transforms

$$z \mapsto \frac{az + b}{cz + d} \qquad z = x + iy \qquad ad - bc = 1 \tag{32}$$

and it transforms the $0y$ axis into upper semi-circles or rays $x = x_0$ for $y > 0$. Since isometries preserve geodesics, these are geodesics. In fact they're the only geodesics. Indeed, for any $p \in H$, and any direction in T_pH, there passes either a semi-circle with center on the $0x$ axis or the circle degenerates to a ray normal to $0x$.

□

Theorem 3.2. *The Upper Half Plane $H = \mathbb{R}^2_+$ with the Lobatchevski metric g*

$$g_{11} = g_{22} = \frac{1}{y^2} \qquad g_{12} = 0$$

is complete

Proof. We want to make use of Hopf-Rinow 3.1 (a). We have to show the geodesic starting at the point $(0,1) \in \mathbb{R}^2_+$ is well-defined for all $v \in T_{(0,1)}\mathbb{R}^2_+$ for all time t. Since we require to exist for $t \geq 0$ it suffices to take $|v| = 1$.

1. If $v = (0,1)$ the geodesic is
$$\gamma(t) = (0, e^t)$$
since from Proposition 3.4
$$\ell(c) \geq \int_a^b \frac{dy}{y} = \log(b) - \log(a)$$

2. If $v = (0,-1)$ the geodesic is accordingly
$$\gamma(t) = (0, e^{-t})$$

3. If $v = (\sin(\theta), -\cos(\theta))$ we make the identification $y = iy$ in the complex field. Then we claim
$$\gamma(t) = \frac{\sin(\frac{\theta}{2})ie^t - \cos(\frac{\theta}{2})}{\cos(\frac{\theta}{2})ie^t + \sin(\frac{\theta}{2})}$$
is the geodesic with origin $i = (0,1)$ and initial velocity $v = e^{i\theta} = (\cos(\theta), \sin(\theta))$.

   (a) As in (32) $\sin^2(\frac{\theta}{2}) + \cos^2(\frac{\theta}{2}) = 1$ so image of γ is indeed a geodesic.

   (b) Compute
$$\gamma(0) = \frac{\sin(\frac{\theta}{2})i - \cos(\frac{\theta}{2})}{\cos(\frac{\theta}{2})i + \sin(\frac{\theta}{2})} = i = (0,1)$$

   (c) Compute
$$\begin{aligned}
\gamma'(t) &= -\frac{1}{(\cos(\frac{\theta}{2})ie^t + \sin(\frac{\theta}{2}))^2}\cos(\frac{\theta}{2})ie^t(\sin(\frac{\theta}{2})ie^t - \cos(\frac{\theta}{2})) + \frac{\sin(\frac{\theta}{2})ie^t}{\cos(\frac{\theta}{2})ie^t + \sin(\frac{\theta}{2})} \\
\gamma'(0) &= -\frac{1}{(\cos(\frac{\theta}{2})i + \sin(\frac{\theta}{2}))^2}\cos(\frac{\theta}{2})i(\sin(\frac{\theta}{2})i - \cos(\frac{\theta}{2})) + \frac{\sin(\frac{\theta}{2})i}{\cos(\frac{\theta}{2})i + \sin(\frac{\theta}{2})} \\
&= \frac{1}{(\cos(\frac{\theta}{2})i + \sin(\frac{\theta}{2}))^2}\left(\cos(\frac{\theta}{2})\sin(\frac{\theta}{2}) + i\cos(\frac{\theta}{2})^2 - \cos(\frac{\theta}{2})\sin(\frac{\theta}{2}) + i\sin(\frac{\theta}{2})^2\right) \\
&= i\frac{1}{(\cos(\frac{\theta}{2})i + \sin(\frac{\theta}{2}))^2} = \frac{i}{-\cos(\frac{\theta}{2})^2 + 2i\sin(\frac{\theta}{2})\cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2})^2} \\
&= \frac{i}{i\sin(\theta) - \cos(\theta)} = \frac{1}{\sin(\theta) + i\cos(\theta)} = \sin(\theta) - i\cos(\theta)
\end{aligned}$$

Since all above $\gamma(t)$ exists for all $t \geq 0$, $\exp_{(0,1)}(v)$ is defined for all $v \in T_{(0,1)}\mathbb{R}^2_+$. Hence $H = \mathbb{R}^2_+$ is geodesically complete, thus complete. □

3.3.3 Homogeneous manifold

Definition 3.7. *A Riemannian manifold M is homogeneous if for any p, $q \in M$ there exists an isometry of M which takes p to q.*

Proposition 3.5. *Any homogeneous manifold M is complete.*

Proof. By Hopf-Rinow 3.1 it suffices to show H is geodesically complete. Suppose we have a unit speed geodesic

$$c : [a,1) \to M \qquad \text{s.t.} \qquad \text{it is not extendiable to } t = 1$$

Now for any $p \in M$, due to local existence of geodesic, there exists another geodesic c_2 starting at p and $\alpha > 0$ small s.t.

$$\ell(c_2) \geq \alpha > 0$$

Let's denote

$$\delta := \min\{\frac{\alpha}{2}, \frac{1-a}{2}\} > 0$$

Since M is homogenegous, for points p and $c(1-\delta)$, there exists an isometry of M that takes p to $c(1-\delta)$. But isometry also preserves geodesics, hence our c_2 should be isometrically mapped to some geodesic of equal length with starting point $c(1-\delta)$. But

$$\ell(c_2) \geq \alpha \geq 2\delta$$

hence

$$c : [1-\delta, 1+\delta) \to M \qquad \text{is extended}$$

But this contradicts with our assumption. Thus M is complete. $\square$

3.4 Hadamard

Now going back to isomeric immersion.

Proposition 3.6. *Let*

$$f : (M, g) \to (\overline{M}, \overline{g})$$

be an isometric immersion. Then f is geodesic at $p \in M$, i.e.

$$B(x, y) = 0 \qquad \forall\ x,\ y \in T_pM$$

iff for any

$$\gamma : (-\varepsilon, \varepsilon) \to M \qquad \textit{geodesic} \qquad \gamma(0) = p$$

one has

$$\tilde{\gamma} := f \circ \gamma : (-\varepsilon, \varepsilon) \xrightarrow{\gamma} M \xrightarrow{f} \overline{M} \qquad \textit{is a geodesic on } \overline{M}$$

Proof. Assume f is geodesic at $p \in M$. Suppose γ is a geodesic in M, we want to prove $\gamma = f \circ \gamma$ is geodesic in $\overline{M}$. What is the covariant derivative of γ'?

$$\begin{aligned}(\frac{\overline{D}}{dt}\gamma')(0) &= (\frac{D}{dt}\gamma')(0) + B(\gamma'(0), \gamma'(0)) \\ &= (\frac{D}{dt}\gamma')(0) = 0 \qquad \text{since } \gamma \text{ is geodesic in } M\end{aligned}$$

On the converse, it suffices to show that $B(x, x) = 0$ for any $x \in T_pM$. Let

$$\gamma(t) = \exp_p(tx)$$

Then

$$\begin{aligned}(\frac{D}{dt}\gamma')(t) &= 0 \\ (\frac{\overline{D}}{dt}\gamma')(t) &= 0 \\ B(x, x) &= (\frac{\overline{D}}{dt}\gamma')(0) - (\frac{D}{dt}\gamma')(0) = 0\end{aligned}$$

$\square$

Recall $(\overline{M}, \overline{g})$ is Riemannian manifold of dimension $\overline{n}$ with $p \in \overline{M}$, then there exists $\varepsilon > 0$ s.t.

$$\overline{\exp}_p : B_\varepsilon(0) \subset T_pM \to \overline{M}$$

is a C^∞ embedding. Now let

$$M := \overline{\exp}_p(W \cap B_\varepsilon(0))$$

where W is a subspace of T_pM of dimension n. Then M is a n-dim submanifold of $\overline{M}$ which satisfies the two condition.

Corollary 3.3. *If $f : (M, g) \to (\overline{M}, \overline{g})$ is totally geodesic, i.e, f is geodesic for any $p \in M$. Then*

$$\exp_p = \overline{\exp}_p\big|_{V \cap T_pM}$$

where V is a neighborhood of origin of $T_p\overline{M}$ on which $\overline{\exp}_p$ is defined.

Now we give a rigidity theorem.

Theorem 3.3 (Cartan-Hadamard Theorem)**.** *Let M be a complete Riemannian manifold with $K(p,\sigma) \leq 0$ for all $p \in M$ and $\sigma \subset T_pM$ 2-planes. Then for any $p \in M$, the exponential map*

$$\exp_p : T_pM \to M \qquad \text{is a covering map}$$

In particular, if M is simply connected, then $\exp_p$ is a diffeomorphism, and hence M is diffeomorphic to $\mathbb{R}^n$.

We'll prove Theorem 3.3 via two lemmas.

Definition 3.8 (Pole)**.** *Let (M,g) be a complete Riemannian manifold. We say p is a pole if the conjugate locus $C(p) = \varnothing$ is empty, i.e.,*

$$\exp_p : T_pM \to M$$

has no critical points, hence $\exp_p$ is a local diffeomorphism.

Lemma 3.4. *Let (M,g) be a complete Riemannian Manifold with $K(p,\sigma) \leq 0$ for any $p \in M$ and $\sigma \subset T_pM$ 2-plane. Then for any $p \in M$, p is a pole.*

Remark 3.3. *Notice Lemma 3.4 does not mean if there exists $p \in M$ s.t. $K(p,\sigma) \leq 0$ for any $\sigma \subset T_pM$ then it implies p is a pole.*

Proof of Lemma 3.4. See Proposition 1.5. Compute $\langle J, J\rangle''$. □

Before we deliver the second lemma we make a remark.

Remark 3.4. *Notice that poles can exist in non-compact manifolds which have positive sectional curvature. The point $p = (0,0,0)$ of the paraboloid*

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}$$

is a pole of S. On the other hand, notice the curvature is positive.

Proof. It suffices to prove that there is no non-trivial Jacobi Field connecting any point with $p = (0,0,0)$ that vanishes on both end points. Let's parametrize S via

$$(r\cos(\theta), r\sin(\theta), r^2)$$

Then one compute the first fundamental form

$$\begin{aligned}
g &= dx^2 + dy^2 + dz^2 = d(r\cos(\theta))^2 + d(r\sin(\theta))^2 + d(r^2)^2 \\
&= (\cos(\theta)dr - r\sin(\theta)d\theta)^2 + (\sin(\theta)dr + r\cos(\theta)d\theta)^2 + (2rdr)^2 \\
&= \cos(\theta)^2dr^2 - 2r\cos(\theta)\sin(\theta)drd\theta + r^2\sin(\theta)^2d\theta^2 + \sin^2(\theta)dr^2 + 2r\sin(\theta)\cos(\theta)drd\theta + r^2\cos^2(\theta)d\theta^2 + 4r^2dr^2 \\
&= (1+4r^2)dr^2 + r^2d\theta^2
\end{aligned}$$

$$g_{11} = g_{rr} = 1 + 4r^2$$

$$g_{22} = g_{\theta\theta} = r^2$$

$$g^{11} = \frac{1}{1+4r^2}$$

$$g^{22} = \frac{1}{r^2}$$

Now we compute the Christoffel symbols.

$$\begin{aligned}
\Gamma^{\ell}_{ij} &= \frac{1}{2}\sum_{k=1}^{2} g^{\ell k}(g_{ik,j} + g_{kj,i} - g_{ij,k}) \\
\Gamma^1_{11} &= \frac{1}{2}\frac{1}{1+4r^2}8r = \frac{4r}{1+4r^2} \\
\Gamma^2_{12} = \Gamma^2_{21} &= \frac{1}{2}\frac{1}{r^2}2r = \frac{1}{r}
\end{aligned}$$

With the Christoffel symbols we solve for the geodesics. Let $(r(t), \theta(t))$ solve

$$\begin{aligned}
\frac{d^2r}{dt^2} + \frac{4r}{1+4r^2}(\frac{dr}{dt})^2 &= 0 \\
\frac{d^2\theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt} &= 0
\end{aligned}$$

Due to the symmetric structure one can assume $\theta(t) = \theta$ to be constant. Let's pause here since the ODE is difficult to solve. Instead we directly look at the Gauss curvature. Since our manifold is two dimension, one may use

$$K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2}$$

To do so, compute

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial\theta}}\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r} &= \nabla_{\frac{\partial}{\partial\theta}}(\frac{4r}{1+4r^2}\frac{\partial}{\partial r}) = \frac{4r}{1+4r^2}\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial r} \\
&= \frac{4r}{1+4r^2}(\frac{1}{r}\frac{\partial}{\partial\theta}) = \frac{4}{1+4r^2}\frac{\partial}{\partial\theta} \\
\nabla_{\frac{\partial}{\partial r}}\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial r} &= \nabla_{\frac{\partial}{\partial r}}(\frac{1}{r}\frac{\partial}{\partial\theta}) = -\frac{1}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r}(\frac{1}{r}\frac{\partial}{\partial\theta}) = 0
\end{aligned}$$

Thus

$$\begin{aligned}
R_{1212} &= \langle R(\frac{\partial}{\partial r}, \frac{\partial}{\partial\theta})\frac{\partial}{\partial r}, \frac{\partial}{\partial\theta}\rangle \\
&= \langle \nabla_{\frac{\partial}{\partial\theta}}\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r} - \nabla_{\frac{\partial}{\partial r}}\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial r}, \frac{\partial}{\partial\theta}\rangle \\
&= \langle \frac{4}{1+4r^2}\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\rangle \\
&= \frac{4r^2}{1+4r^2}
\end{aligned}$$

We proceed to compute

$$K = \frac{\frac{4r^2}{1+4r^2}}{(1+4r^2)r^2} = \frac{4}{(1+4r^2)^2} > 0$$

This concludes that the curvature is positive. Now for a radial geodesic $\gamma(t) = (r(t), \theta)$, and essentially since our manifold is 2 dimensional, the Jacobi equation writes (upon taking arc length parametrization)

$$\begin{aligned}
J''(t) + R(\gamma', J)\gamma' J(t) &= 0 \\
J'' + K(\gamma(t))J(t) &= 0 \\
J''(t) + \frac{4}{(1+4r(t)^2)^2}J(t) &= 0 \\
J''(t) + \frac{4}{(1+4t^2)^2}J(t) &= 0
\end{aligned}$$

Since $\frac{4}{(1+4t^2)^2} > 0$ and is decreasing hence the solution do not oscillate, there is no non-trivial solutions $J(t)$ that vanishes at two points. $\square$

Lemma 3.5. *Let (M, g) be a complete Riemannian manifold, and let (N, h) be another Riemannian manifold s.t. there exists*

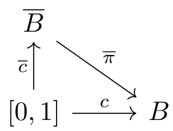
$$f : M \to N \qquad \textit{surjective and local diffeomorphism}$$

and assume for any $p \in M$, for any $v \in T_pM$, we have

$$\|df_p(v)\|_{f(p)} \geq \|v\|_p \tag{33}$$

Then we have f is a covering map.

Remark 3.5 (Path-lifting Property)**.** *If we have a path $c : [0, 1] \to B$*



Let $\overline{\pi} : \overline{B} \to B$ be a continuous surjective map, local homeomorphism with path lifting property s.t.

1. *$\overline{B}$ is locally path connected*
2. *B is locally simply connected*

Then $\overline{\pi}$ is a covering map.

Proof of Lemma 3.5. By the above fact Remark 3.5, we only need to check that f satisfies the path lifting property. Given

$$c : [0,1] \to N$$

We want to prove

(a) $\overline{c}$ can be defined. If $\overline{c}$ is defined in some small interval then one can extend it. In particular if

$$\overline{c} : [0, t_0] \to M \qquad 0 \le t_0 < 1$$

s.t.

$$f \circ \overline{c} = c$$

Then there exists a $\delta > 0$ s.t. $\overline{c}$ is defined on $[0, t_0 + \delta]$ and also satisfies

$$f \circ \overline{c} = c$$

(b) If $\overline{c}$ is now defined in

$$\overline{c} : [0, t_0) \to M \qquad 0 < t_0 \le 1$$

s.t.

$$f \circ \overline{c} = c$$

Then $\overline{c}$ can be extended to t_0

$$c : [0, t_0] \to M$$

with

$$f \circ \overline{c} = c$$

In particular $f(\overline{c}(t_0)) = c(t_0)$.

Proof of (a). Since f is local diffeomorphism, there exists U open neighborhood of $\overline{c}(t_0)$ s.t.

$$f|_U : U \to f(U) \qquad \text{is a diffeomorphism}$$

Then $f(U)$ is an open neighborhood of $c(t_0) = f(\overline{c}(t_0))$. Then there exists $\delta > 0$ s.t. the image of $(t_0 - \delta, t_0 + \delta)$ through c is contained in U. For $t \in (t_0 - \delta, t_0 + \delta)$ define

$$\overline{c}(t) := (f|_U)^{-1}(c(t))$$

since f is surjective. □

Proof of (b). In this we need (33). Let $\{t_n\}$ be a sequence $t_{n+1} > t_n$ s.t. $t_n \to t_0$. Then for any $m < n$. Compute the distance between $\overline{c}(t_n)$ and $\overline{c}(t_m)$ because we want to show $\{\overline{c}(t_n)\}$ is Cauchy.

$$\begin{aligned} d_M(\overline{c}(t_n), \overline{c}(t_m)) &\le \ell(\overline{c}|_{[t_m,t_n]}) = \int_{t_m}^{t_n} \left\| \frac{d\overline{c}}{dt}(t) \right\|_{\overline{c}(t)} dt \\ &\overset{(33)}{\le} \int_{t_m}^{t_n} \left\| df_{\overline{c}(t)}(\frac{d\overline{c}}{dt}(t)) \right\|_{c(t)} dt = \int_{t_m}^{t_n} \left\| \frac{d}{dt}(f \circ \overline{c}) \right\|_{c(t)} dt \\ &= \int_{t_m}^{t_n} \left\| \frac{d}{dt} c(t) \right\|_{c(t)} dt \le C|t_n - t_m| \qquad \text{where } C := \max_{[0,1]} |\frac{dc}{dt}(t)| \end{aligned}$$

Now $\{\overline{c}(t_n)\}$ is Cauchy. Since M is complete, by Hopf-Rinow 3.1, $\overline{c}(t_n)$ converges, so there exists $r \in M$ s.t.

$$\overline{c}(t_n) \to r$$

We define

$$r := \overline{c}(t_0)$$

It suffices to check $f(\overline{c}(t_0)) = c(t_0)$. But using continuity of f

$$f(\overline{c}(t_0)) = f(r) = f(\lim_{n\to\infty} \overline{c}(t_n)) = \lim_{n\to\infty} (f \circ \overline{c})(t_n) = \lim_{n\to\infty} c(t_n) = c(t_0)$$

□

□

Corollary 3.4 (Corollary of Lemma 3.5). *Let (M, g) be a complete Riemannian manifold. Suppose $p \in M$ is a pole. Then*

$$\exp_p : T_pM \to M$$

is a covering map. In particular, if M is simply connected, then $\exp_p$ is a diffeomorphism, and hence M is diffeomorphic to $\mathbb{R}^n$.

Proof. Since p is a pole, $\exp_p : T_pM \to M$ is a local diffeomorphism. Since M is complete Riemannian Manifold, by Hopf-Rinow 3.1

$$\exp_p : T_pM \to M \qquad \text{is surjective}$$

We define

$$\tilde{g} := \exp_p^* g$$

to be a Riemannian metric on T_pM. Then the exponential map

$$\exp_p : (T_pM, \tilde{g}) \to (M, g) \qquad \text{is a local isometry}$$

In particular

$$\left\| d\exp_p(v) \right\|_g = \|v\|_{\tilde{g}}$$

so (33) is satisfied. Now we only need to check $(T_pM, \tilde{g})$ is complete to apply Lemma 3.5. By Hopf-Rinow 3.1, we show that exp map of $(T_pM, \tilde{g})$ is defined everywhere.

$$\forall\ v \in T_0(T_pM) \cong T_pM \qquad \gamma(t) := \exp_p(tv) \qquad \forall\ t \in \mathbb{R} \qquad \text{is a geodesic in } M$$

and

$$\tilde{\gamma}(t) := tv \qquad \forall\ t \in \mathbb{R} \qquad \text{is a geodesic in } T_pM$$

One define

$$\tilde{\exp}_0 : T_0(T_pM) \to T_pM \qquad \tilde{\exp}_0(tv) := \tilde{\gamma}(t) = tv \implies \tilde{\exp}_0 : T_0(T_pM) = T_pM \to T_pM \qquad \text{is the identity}$$

In particular $\tilde{\exp}_0$ is defined everywhere at the point 0. By Hopf-Rinow 3.1 we know $(T_pM, \tilde{g})$ is complete. Hence by Lemma 3.5, $\exp_p$ is a covering map. □

Proof of Theorem 3.3. Let (M, g) be complete Riemannian manifold with $K(p, \sigma) \leq 0$. By Lemma 3.4 for any $p \in M$, p is a pole. By Corollary 3.4 we know

$$\exp_p : T_pM \to M$$

is a covering map. □

4 Space of Constant Curvature

4.1 Theorem of Cartan on Determination of the Metric by Curvature

If two Riemannian manifolds have the same Riemannian curvature, then they have the same metric. How do we compare two Riemannian manifolds of the same dimension? Let $p \in M$ and $\tilde{p} \in \tilde{M}$ with the same dimension. In particular they both have tangent space $T_pM \cong T_{\tilde{p}}\tilde{M} = \mathbb{R}^n$. Then one can cook up a map i between the tangent spaces

$$i : T_pM \to T_{\tilde{p}}\tilde{M} \qquad \text{linear isometry, i.e., sends an ONB to an ONB}$$

There exists $r > 0$ s.t.

$$\begin{aligned} \exp_p &: B_r(0) \subset T_pM \to B_r(p) \subset M \\ \tilde{\exp}_{\tilde{p}} &: B_r(0) \subset T_{\tilde{p}}\tilde{M} \to B_r(\tilde{p}) \subset \tilde{M} \end{aligned}$$

are diffeomorphisms. Now we define

$$f : B_r(p) \subset M \to B_r(\tilde{p}) \subset \tilde{M} \qquad f(q) := \tilde{\exp}_{\tilde{p}} \circ i \circ (\exp_p)^{-1}(q) \qquad f \text{ is a diffeomorphism} \tag{34}$$

Now for any $q \in B_r(p)$, there exists a unique initial velocity $v \in T_pM$ s.t. q can be reached via

$$q = \exp_p(\ell v) \qquad \ell := d(p, q)$$

Now let

$$P_{p,q} : T_pM \to T_qM \qquad \text{be the parallel transport along the geodesic } \gamma(t) := \exp_p(tv)$$

Similarly let

$$\tilde{P}_{\tilde{p},f(q)} : T_{\tilde{p}}\tilde{M} \to T_{f(q)}\tilde{M} \qquad \text{be the parallel transport along the geodesic } \tilde{\gamma}(t) := \tilde{\exp}_{\tilde{p}}(t\tilde{v})$$

Now for any $q \in B_r(p)$, define

$$\phi_q : T_qM \to T_{f(q)}\tilde{M} \qquad \phi_q := \tilde{P}_{\tilde{p},f(q)} \circ i \circ (P_{p,q})^{-1} \qquad \text{is a linear isometry} \tag{35}$$

Theorem 4.1 (Cartan)**.** *With the above notations, if for all $q \in B_r(p)$, and for all*

$$x,\ y,\ v,\ u \in T_qM$$

one has, for (35), that Riemannian curvature agrees

$$R(x, y, v, u) = \tilde{R}(\phi_q(x), \phi_q(y), \phi_q(v), \phi_q(u))$$

Then f as in (34)

$$f : B_r(p) \to B_r(\tilde{p})$$

is an isometry and

$$df_p = i$$

Remark 4.1. *This is why Riemannian curvature is so important.*

Proof of Cartan 4.1. We already know that f is a diffeomorphism. We really need to show that for all $q \in B_r(p)$ and for every $w \in T_qM$, the norm is preserved

$$\|df_p(w)\|_{f(q)} = \|w\|_q$$

Observe that

$$\begin{aligned} df_p &= d(\tilde{\exp}_{\tilde{p}} \circ i \circ (\exp_p)^{-1}) = d\tilde{\exp}_{\tilde{p}} \circ i \circ d(\exp_p)^{-1} \\ &= id_{T_{\tilde{p}}\tilde{M}} \circ i \circ (id_{T_pM})^{-1} = i \end{aligned}$$

Remark that even the identity is known, it doesn't mean f is an isometry. We need to show norms are the same. We do it through Jacobi fields. Those will allow us to use the hypothesis. We may assume $p \neq q$ and $w \neq 0$. There exists unit vector $v \in T_pM$ s.t.

$$q = \exp_p(\ell v) \qquad \ell = d(p, q) > 0$$

There exists a unique $w_0 \in T_{\ell v}(T_pM) \cong T_pM$ such that

$$(d\exp_p)_{\ell v}(w_0) = \frac{w}{\ell}$$

This exists since if $\ell v < r$ one can find the preimage due to $d\exp_p$ is a linear isomorphism. Now let

$$\gamma : [0,\ell] \to M \qquad \text{be the geodesic s.t.} \qquad \gamma(0) = p \qquad \gamma(\ell) = q$$

Look at the Jacobi Field

$$J(t) := (d\exp_p)_{tv}(tw_0)$$

Then

$$\begin{aligned} J(0) &= 0 \\ J(\ell) &= (d\exp_p)_{\ell v}(\ell w_0) = \ell (d\exp_p)_{\ell v}(w_0) = \ell \frac{w}{\ell} = w \end{aligned}$$

Now we write in coordinates. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of T_pM. We let

$$e_n := v = \gamma'(0)$$

Let $\{e_1(t), \cdots, e_n(t)\}$ be the parallel transport along the geodesic γ. Then Jacobi Field has local coordinates

$$J(t) = \sum_{i=1}^n y_i(t) e_i(t) \qquad y_i \in C^\infty([0,\ell]; M)$$

Now here the Jacobi Equation is (upon contraction)

$$J''(t) + R(\gamma', J)\gamma' = 0 \implies \frac{d^2 y_i}{dt^2} + \sum_{j=1}^n R(e_n, e_j, e_n, e_i) y_j = 0$$

Now let

$$\tilde{\gamma} : [0,\ell] \to \tilde{M} \qquad \tilde{\gamma} := f \circ \gamma \qquad \text{be geodesic so that} \qquad \tilde{\gamma}(0) = f(p) = \tilde{p} \qquad \tilde{\gamma}(\ell) = f(q)$$

Let $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$ be ONB of $T_{\tilde{p}}\tilde{M}$, and $\{\tilde{e}_1(t), \cdots, \tilde{e}_n(t)\}$ be their parallel transport along $\tilde{\gamma}$. Hence

$$\tilde{e}_i(t) = \phi_q(e_i(t))$$

Now we define

$$\tilde{J}(t) = \phi_{\gamma(t)}(J(t)) = \sum_{i=1}^n y_i(t) \tilde{e}_i(t)$$

But this is not in principle a Jacobi Field. Here we use our assumption that two Riemannian curvatures are the same. Hence

$$\begin{aligned} &\frac{d^2 y_i}{dt^2} + \sum_{j=1}^n \tilde{R}(\tilde{e}_n, \tilde{e}_j, \tilde{e}_n, \tilde{e}_i) y_j \\ &= \frac{d^2 y_i}{dt^2} + \sum_{j=1}^n R(e_n, e_j, e_n, e_i) y_j \\ &= 0 \end{aligned}$$

Hence $\tilde{J}(t)$ is Jacobi field with $\tilde{J}(0) = 0$. What about its length?

$$\left\|\tilde{J}(t)\right\|_{\gamma(t)} = \sqrt{\sum_{i=1}^n y_i^2} = \|J(t)\|_{\gamma(t)} \qquad \forall\ t$$

Now $\tilde{J}(t)$ is a Jacobi Field along $\tilde{\gamma}(t)$ with

$$\begin{aligned} \tilde{J}'(0) &= \sum_{i=1}^n y_i'(0) \tilde{e}_i(0) \\ &= \sum_{i=1}^n i(y_i'(0) e_i) = i(w_0) \\ \tilde{J}(t) &= (d\tilde{\exp}_{\tilde{p}})_{ti(v)}(ti(w_0)) \\ \tilde{J}(\ell) &= (d\tilde{\exp}_{\tilde{p}})_{\ell i(v)} \circ i \circ (\ell w_0) \\ &= (d\tilde{\exp}_{\tilde{p}})_{\ell i(v)} \circ i \circ ((d\exp_p)_{\ell v})^{-1}(w) \qquad \text{using definition of } w_0 \\ &= d(\tilde{\exp}_{\tilde{p}} \circ i \circ \exp_p^{-1})_{\exp_p(\ell v)}(w) \\ &= d(f)_q(w) \qquad \text{using } q = \exp_p(\ell v) \end{aligned}$$

Thus

$$\|df_q(w)\| = \left\|\tilde{J}(\ell)\right\| = \|J(\ell)\| = \|w\|$$

□

Corollary 4.1. *Let (M, g) and $(\tilde{M}, \tilde{g})$ be two Riemannian manifolds of dimension n, with the same constant sectional curvature, in particular*

$$R(x, y, u, v) = K_0(g(x, u)g(y, v) - g(x, v)g(y, u))$$

Let $p \in M$ and $\tilde{p} \in \tilde{M}$. Let $\{e_1, \cdots, e_n\}$ be ONB of T_pM and $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$ ONB of $T_{\tilde{p}}\tilde{M}$. Then there exists U open neighborhood of p in M and $\tilde{U}$ open neighborhood of $\tilde{p}$ in $\tilde{M}$ and an isometry

$$f : U \to \tilde{U} \qquad f(p) = \tilde{p} \qquad df_p(e_i) = \tilde{e}_i$$

Proof. Choose

$$i : T_pM \to T_{f(p)}\tilde{M} \qquad i(e_j) = \tilde{e}_j$$

and

$$f = \exp_p \circ i \circ (\tilde{\exp}_{\tilde{p}})^{-1}$$

□

4.2 Conformal Deformation of the Curvature

Let (M, g) be a Riemannian manifold. Look at

Definition 4.1 (Conformal Deformation)**.**

$$\tilde{g} = e^{2f} g$$

for some $f \in C^\infty(M)$ smooth function on the manifold M. This is known as a conformal change(deformation).

We denote ∇ as Levi-Civita connection of g and $\tilde{\nabla}$ as Levi-Civita connection of $\tilde{g}$. Using the expression for $\tilde{g}(\tilde{\nabla}_X Y, Z)$ we get

$$\tilde{\nabla}_X Y = \nabla_X Y + X(f)Y + Y(f)X - g(X, Y)\text{grad}(f) \qquad \text{where } g(\text{grad}(f), Y) := df(Y)$$

Proposition 4.1. *Let g be a Riemannian metric on a manifold M and let $\tilde{g} := e^{2f}g$ where $f \in C^\infty(M)$. Let ∇ and $\tilde{\nabla}$ denote respectively the Levi-Civita connections on (M, g) and $(M, \tilde{g})$. Then for any X, $Y \in \mathfrak{X}(M)$ one has*

$$\tilde{\nabla}_X Y = \nabla_X Y + X(f)Y + Y(f)X - g(X, Y)\text{grad}(f) \qquad \textit{where } g(\text{grad}(f), Y) := df(Y)$$

Proof. Let's prove using local coordinates. Denote

$$\tilde{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} := \sum_{\ell=1}^{n} \tilde{\Gamma}_{ij}^{\ell} \frac{\partial}{\partial x_\ell}$$

where

$$\begin{aligned}
\tilde{\Gamma}_{ij}^{\ell} &= \frac{1}{2}\sum_{k=1}^{n} \tilde{g}^{\ell k}(\tilde{g}_{ik,j} + \tilde{g}_{kj,i} - \tilde{g}_{ij,k}) \\
&= \frac{1}{2}\sum_{k=1}^{n} e^{-2f} g^{\ell k}(\frac{\partial}{\partial x_j}(e^{2f}g_{ik}) + \frac{\partial}{\partial x_i}(e^{2f}g_{kj}) - \frac{\partial}{\partial x_k}(e^{2f}g_{ij})) \\
&= \frac{1}{2}\sum_{k=1}^{n} e^{-2f} g^{\ell k}\left(2e^{2f}\frac{\partial f}{\partial x_j}g_{ik} + e^{2f}g_{ik,j} + 2e^{2f}\frac{\partial f}{\partial x_i}g_{kj} + e^{2f}g_{kj,i} - 2e^{2f}\frac{\partial f}{\partial x_k}g_{ij} - e^{2f}g_{ij,k}\right) \\
&= \sum_{k=1}^{n} g^{\ell k}\left(\frac{\partial f}{\partial x_j}g_{ik} + \frac{\partial f}{\partial x_i}g_{kj} - \frac{\partial f}{\partial x_k}g_{ij}\right) + \Gamma_{ij}^{\ell} \\
&= \Gamma_{ij}^{\ell} + \delta_i^{\ell}\frac{\partial f}{\partial x_j} + \delta_j^{\ell}\frac{\partial f}{\partial x_i} - \sum_{k=1}^{n} g^{\ell k} g_{ij}\frac{\partial f}{\partial x_k}
\end{aligned}$$

Now

$$
\begin{aligned}
\tilde{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} &:= \sum_{\ell=1}^{n} \tilde{\Gamma}_{ij}^{\ell} \frac{\partial}{\partial x_\ell} \\
&= \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + \sum_{\ell=1}^{n} \left(\delta_i^\ell \frac{\partial f}{\partial x_j} + \delta_j^\ell \frac{\partial f}{\partial x_i} - \sum_{k=1}^{n} g^{\ell k} g_{ij} \frac{\partial f}{\partial x_k} \right) \frac{\partial}{\partial x_\ell} \\
&= \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i} + \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} - g_{ij} \sum_{\ell=1}^{n} \sum_{k=1}^{n} g^{\ell k} \frac{\partial f}{\partial x_k} \frac{\partial}{\partial x_\ell} \\
&= \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i} + \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} - g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \text{grad}(f)
\end{aligned}
$$

Hence this is true for coordinate basis. In general let

$$
\begin{aligned}
X &= \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \qquad a_i \in C^\infty(U) \\
Y &= \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j} \qquad b_j \in C^\infty(U)
\end{aligned}
$$

in local charts. We compute

$$
\begin{aligned}
\tilde{\nabla}_X Y &= \tilde{\nabla}_{\sum_i a_i \frac{\partial}{\partial x_i}} (\sum_j b_j \frac{\partial}{\partial x_j}) \\
&= \sum_i a_i \left(\sum_j \left(\frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} + b_j \tilde{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \right) \right) \\
&= \sum_i a_i \left(\sum_j \left(\frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} + b_j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + b_j \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i} + b_j \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} - b_j g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \text{grad}(f) \right) \right) \\
&= \sum_i a_i \left(\sum_j \left(\nabla_{\frac{\partial}{\partial x_i}} (b_j \frac{\partial}{\partial x_j}) + b_j \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i} + b_j \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} - b_j g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \text{grad}(f) \right) \right) \\
&= \nabla_{\sum_i a_i \frac{\partial}{\partial x_i}} (\sum_j b_j \frac{\partial}{\partial x_j}) + \sum_j b_j \frac{\partial f}{\partial x_j} \sum_i a_i \frac{\partial}{\partial x_i} + \sum_i a_i \frac{\partial f}{\partial x_i} \sum_j b_j \frac{\partial}{\partial x_j} - g(\sum_i a_i \frac{\partial}{\partial x_i}, \sum_j b_j \frac{\partial}{\partial x_j}) \text{grad}(f) \\
&= \nabla_X Y + Y(f) X + X(f) Y - g(X,Y) \text{grad}(f)
\end{aligned}
$$

□

Remark 4.2. *If f is a constant, then $\tilde{g} = kg$ is constant times g. In this case*

$$
\begin{aligned}
\tilde{\nabla}_X Y &= \nabla_X Y \\
\tilde{R}(X,Y)Z &= R(X,Y)Z \\
\tilde{R}(X,Y,Z,W) &= k^2 R(X,Y,Z,W) \\
\tilde{\text{Ric}} &= \text{Ric} \\
\tilde{S} &= k^{-2} S
\end{aligned}
$$

But what happens in general?

Definition 4.2 (Kulkarni-Nomizu Product)**.** *For S, T symmetric 2-tensors on M, $(S \circ T)$ gives a 4-tensor on M. We define their Kulkarni-Nomizu Product as*

$$
(S \circ T)(X,Y,Z,W) = S(X,Z)T(Y,W) + S(Y,W)T(X,Z) - S(X,W)T(Y,Z) - S(Y,Z)T(X,W)
$$

Proposition 4.2.

$$
\begin{aligned}
(S \circ T)(X,Y,Z,W) &= -(S \circ T)(Y,X,Z,W) \qquad \textit{anti-symmetric in first two components} \\
&= -(S \circ T)(X,Y,W,Z) \qquad \textit{anti-symmetric in second two components} \\
&= (S \circ T)(Z,W,X,Y) \qquad \textit{symmetric w.r.t. the two sets of components}
\end{aligned}
$$

4.2.1 Riemannian Curvature Deformation

Let R denote curvature tensor of g and $\tilde{R}$ denote curvature tensor of $\tilde{g}$.

Theorem 4.2. *If (M, g) has constant sectional curvature κ, the Riemannian Curvature writes*

$$R = \frac{1}{2}\kappa g \circ g \tag{36}$$

Under conformal deformation, we have

$$\tilde{R} = e^{2f}(R - (\mathrm{Hess}(f)) \circ g + (df \otimes df) \circ g - \frac{1}{2}|df|^2 g \circ g) \tag{37}$$

where $\mathrm{Hess}(f) = \sum_{i,j=1}^{n} f_{;ij} dx_i dx_j$*,* $f_{,ij}$ *is covariant derivative w.r.t.* g*. And*

$$|df|^2 = \sum_{i,j} g^{ij} f_{;i} f_{;j}$$

Example 4.1 (Hyperbolic Space: Upper Half Space Model)**.**

Definition 4.3 (Upper Half Hyperbolic Space)**.** *We take*

$$\mathcal{H}^n := \{(y_1, \cdots, y_n) \in \mathbb{R}^n \mid y_n > 0\}$$

and metric

$$\tilde{g} := \frac{dy_1^2 + \cdots dy_n^2}{y_n^2} = e^{2f} g_0$$

where $g_0 = dy_1^2 + \cdots dy_n^2$ *is the Euclidean metric. Here*

$$e^{2f} = \frac{1}{y_n^2} \implies f = -\log(y_n)$$

We compute

$$\begin{aligned} df &= d(-\log(y_n)) = -\frac{dy_n}{y_n} \\ df \circ df &= \frac{dy_n \circ dy_n}{y_n} = \frac{dy_n^2}{y_n} \\ |df|^2 &= \frac{1}{y_n^2} \\ \mathrm{Hess}(f) &= \frac{1}{y_n^2} dy_n^2 \end{aligned}$$

Then we apply the formula (37)

$$\begin{aligned} \tilde{R} &= \frac{1}{y_n^2}\left(R - \frac{1}{y_n^2} dy_n^2 \circ g_0 + \frac{dy_n^2}{y_n^2} \circ g_0 - \frac{1}{2}\frac{1}{y_n^2} g_0 \circ g_0\right) \\ &= -\frac{1}{2}(\frac{1}{y_n^2} g_0) \circ (\frac{1}{y_n^2} g_0) \\ &= -\frac{1}{2}\tilde{g} \circ \tilde{g} \end{aligned}$$

Hence $(\mathcal{H}^n, \tilde{g})$ *has constant sectional curvature* -1 *using* (36).

Example 4.2 (Hyperbolic Space: Unit Disk Model)**.**

Definition 4.4 (Unit Disk Hyperbolic Space)**.** *We take*

$$D^n := \{(u_1, \cdots, u_n) \in \mathbb{R}^n \mid |u| < 1\}$$

and metric

$$\tilde{g} := \frac{4}{(1 - |\vec{u}|^2)^2}(du_1^2 + \cdots + du_n^2) = e^{2f} g_0$$

where

$$\begin{aligned}
g_0 &= du_1^2 + \cdots + du_n^2 \\
e^{2f} &= \frac{4}{(1-|\vec{u}|^2)^2} \\
e^f &= \frac{2}{(1-|\vec{u}|^2)} \\
f &= \log 2 - \log(1-|\vec{u}|^2)
\end{aligned}$$

We compute

$$\begin{aligned}
f_i &= -\frac{-2u_i}{1-|\vec{u}|^2} = \frac{2u_i}{1-|\vec{u}|^2} \\
df &= \frac{\sum_{i=1}^n 2u_i du_i}{1-|\vec{u}|^2} \\
df \otimes df &= \sum_{i,j} \frac{4u_i u_j du_i du_j}{(1-|\vec{u}|^2)^2} \\
|df|^2 &= \frac{4|\vec{u}|^2}{(1-|\vec{u}|^2)^2} \\
f_{ij} &= \frac{2\delta_{ij}(1-|\vec{u}|^2) + 4u_i u_j}{(1-|\vec{u}|^2)^2} = \frac{2\delta_{ij}}{1-|\vec{u}|^2} + \frac{4u_i u_j}{(1-|\vec{u}|^2)^2} \\
\mathrm{Hess}(f) &= 2\frac{\sum du_i^2}{1-|\vec{u}|^2} + \frac{4\sum_{i,j} u_i u_j du_i du_j}{(1-|\vec{u}|^2)^2}
\end{aligned}$$

So we apply (37)

$$\begin{aligned}
\tilde{R} &= e^{2f}\left(R - (2\frac{\sum du_i^2}{1-|\vec{u}|^2} + \frac{4\sum_{i,j} u_i u_j du_i du_j}{(1-|\vec{u}|^2)^2}) \circ g_0 + (\sum_{i,j} \frac{4u_i u_j du_i du_j}{(1-|\vec{u}|^2)^2}) \circ g_0 - \frac{1}{2}\frac{4|\vec{u}|^2}{(1-|\vec{u}|^2)^2} g_0 \circ g_0\right) \\
&= e^{2f}\left(-2\frac{\sum_i du_i^2}{1-|\vec{u}|^2} \circ g_0 - \frac{2|\vec{u}|^2}{(1-|\vec{u}|^2)^2} g_0 \circ g_0\right) \\
&= -2\frac{1}{(1-|\vec{u}|^2)^2} e^{2f}\left(1-|\vec{u}|^2 + |\vec{u}|^2\right) g_0 \circ g_0 \\
&= -\frac{2}{(1-|\vec{u}|^2)^2} e^{2f} g_0 \circ g_0 \\
&= -\frac{1}{2}(e^{2f} g_0) \circ (e^{2f} g_0) \qquad \textit{using } \frac{2}{(1-|\vec{u}|^2)^2} = \frac{1}{2}e^{2f} \\
&= -\frac{1}{2}\tilde{g} \circ \tilde{g}
\end{aligned}$$

Thus $(D^n, \tilde{g})$ *has constant sectional curvature* -1*.*

One look at a non-hyperbolic example.

Example 4.3. *Given any positive constant* $K > 0$*, define a Riemannian metric* g_K *on* $\mathbb{R}^n$ *by*

$$g_K = \frac{4\sum_{i=1}^n dx_i^2}{(1+K|x|^2)^2}$$

Then

1. $(\mathbb{R}^n, g_K)$ *has constant sectional curvature* K*.*

   Proof. Let $f \in C^\infty(M)$ s.t.

$$g_K = \frac{4\sum_{i=1}^n dx_i^2}{(1+K|x|^2)^2} = e^{2f} g_0$$

where g_0 denotes the flat metric. Hence

$$\begin{aligned} g_0 &= \sum_{i=1}^n dx_i^2 \\ e^{2f} &= \frac{4}{(1+K|x|^2)^2} \\ e^f &= \frac{2}{1+K|x|^2} \\ f &= \log 2 - \log(1+K|x|^2) \end{aligned}$$

We compute

$$\begin{aligned} f_i &= -\frac{2Kx_i}{1+K|x|^2} \\ df &= -\frac{\sum_{i=1}^n 2Kx_i dx_i}{1+K|x|^2} \\ df\otimes df &= \frac{\sum_{i,j=1}^n 4K^2 x_i x_j dx_i dx_j}{(1+K|x|^2)^2} \\ |df|^2 &= \frac{4K^2|x|^2}{(1+K|x|^2)^2} \\ f_{ij} &= \frac{-2K\delta_{ij}(1+K|x|^2)+4K^2x_ix_j}{(1+K|x|^2)^2} = -\frac{2K\delta_{ij}}{1+K|x|^2}+\frac{4K^2x_ix_j}{(1+K|x|^2)^2} \\ \text{Hess}(f) &= -\frac{2K\sum_{i=1}^n dx_i^2}{1+K|x|^2}+\frac{4K^2\sum_{i,j=1}^n x_ix_jdx_idx_j}{(1+K|x|^2)^2} \end{aligned}$$

Now we apply (37) so that

$$\begin{aligned} R_K &= e^{2f}\left(R-\left(-\frac{2K\sum_{i=1}^n dx_i^2}{1+K|x|^2}+\frac{4K^2\sum_{i,j=1}^n x_ix_jdx_idx_j}{(1+K|x|^2)^2}\right)\circ g_0+\left(\frac{\sum_{i,j=1}^n 4K^2x_ix_jdx_idx_j}{(1+K|x|^2)^2}\right)\circ g_0-\frac{1}{2}\frac{4K^2|x|^2}{(1+K|x|^2)^2}g_0\circ g_0\right) \\ &= e^{2f}\left(\frac{2K\sum_{i=1}^n dx_i^2}{1+K|x|^2}\circ g_0-\frac{2K^2|x|^2}{(1+K|x|^2)^2}g_0\circ g_0\right) \\ &= \frac{2}{(1+K|x|^2)^2}e^{2f}(K+K^2|x|^2-K^2|x|^2)g_0\circ g_0 \\ &= \frac{1}{2}K\frac{4}{(1+K|x|^2)^2}e^{2f}g_0\circ g_0 \\ &= \frac{1}{2}K(e^{2f}g_0)\circ(e^{2f}g_0) \\ &= \frac{1}{2}Kg_K\circ g_K \end{aligned}$$

Thus constant sectional curvature equals $K>0$ from (36). □

2. *$(\mathbb{R}^n, g_K)$ is not complete.*

Proof. Consider the radial path

$$\gamma(t) := tv \qquad \text{where } v\in\mathbb{R}^n \text{ and } |v|=1 \qquad \forall\, t\geq 0$$

Fix any $R>0$, we compute the length

$$\begin{aligned} \ell(\gamma|_{[0,R]}) &= \int_0^R \sqrt{g_{K\gamma(t)}(\gamma'(t),\gamma'(t))}\,dt \\ &= \int_0^R \sqrt{g_{K tv}(v,v)}\,dt \\ &= \int_0^R \sqrt{\frac{4\sum_{i=1}^n v_i^2}{(1+Kt^2)^2}}dt \\ &= \int_0^R \frac{2}{1+Kt^2}\,dt \\ &= \frac{2}{\sqrt{K}}\arctan(\sqrt{K}R) \end{aligned}$$

Now

$$\lim_{R\to\infty} \ell(\gamma|_{[0,R]}) = \frac{2}{\sqrt{K}}\frac{\pi}{2} = \frac{\pi}{\sqrt{K}}$$

so we conclude the radial path has finite length. Thus consider the sequence

$$x_n := nv$$

we observe

$$\begin{aligned} d_K(x_n, x_m) &= \int_n^m \frac{2}{1+Kt^2}\,dt \\ &= \frac{2}{\sqrt{K}}\arctan(\sqrt{K}m) - \frac{2}{\sqrt{K}}\arctan(\sqrt{K}n) \to 0 \end{aligned}$$

as n, $m \to \infty$ hence x_n is a Cauchy sequence. However this sequence diverges in $(\mathbb{R}^n, g_0)$, and in particular, since length of the radial path is finite, the sequence x_n does not converge to a point in $(\mathbb{R}^n, g_K)$. □

4.2.2 Ricci Curvature Deformation

Let Ric be Ricci Curvature of g and $\tilde{\text{Ric}}$ be Ricci of $\tilde{g}$. Then

$$(n-1)\tilde{\text{Ric}}(X,Y) = \tilde{g}^{k\ell}\tilde{R}(X, \frac{\partial}{\partial x_k}, Y, \frac{\partial}{\partial x_\ell}) = e^{-2f} g^{k\ell}\tilde{R}(X, \frac{\partial}{\partial x_k}, Y, \frac{\partial}{\partial x_\ell}) \tag{38}$$

In general if S is a symmetric 2-tensor

$$\begin{aligned} g^{k\ell}(S\circ g)(X, \frac{\partial}{\partial x_k}, Y, \frac{\partial}{\partial x_\ell}) &= S(X,Y)g^{k\ell}g(\partial_k,\partial_\ell) + g^{k\ell}S(\partial_k,\partial_\ell)g(X,Y) - g^{k\ell}S(X,\partial_k)g(Y,\partial_\ell) - g^{k\ell}S(Y,\partial_\ell)g(X,\partial_k) \\ &= nS(X,Y) + \text{Tr}(S)g(X,Y) - S(X,Y) - S(X,Y) \\ &= (n-2)S(X,Y) + \text{Tr}(S)g(X,Y) \qquad \text{since } g^{k\ell}g(Y,\partial_\ell) = Y^k \end{aligned}$$

Therefore writing

$$\tilde{R} = e^{2f}(R + U\circ g) \qquad \text{where } U = df\otimes df - \text{Hess}(f) - \frac{1}{2}|df|^2 g$$

Then

$$\begin{aligned} (n-1)\tilde{\text{Ric}} &= e^{-2f}g^{k\ell}e^{2f}\left(R(X,\partial_k,Y,\partial_\ell) + (U\circ g)(X,\partial_k,Y,\partial_\ell)\right) \\ &= (n-1)\text{Ric}(X,Y) + (n-2)U(X,Y) + \text{Tr}(U)g(X,Y) \\ &= (n-1)\text{Ric}(X,Y) + (n-2)(df\otimes df - \text{Hess}(f) - \frac{1}{2}|df|^2 g)(X,Y) + (|df|^2 - \Delta f - \frac{n}{2}|df|^2)g(X,Y) \end{aligned}$$

notice

$$-\frac{1}{2}(n-2) + 1 - \frac{n}{2} = -n + 2 = 2 - n$$

Thus we havee formula

$$\tilde{\text{Ric}} = \text{Ric} + \frac{n-2}{n-1}\left(df\otimes df - \text{Hess}(f) - |df|^2 g\right) - \frac{\Delta f}{n-1}g \tag{39}$$

4.2.3 Scalar Curvature Deformation

We write

$$\begin{aligned} n\tilde{S} = \tilde{g}^{k\ell}\tilde{\text{Ric}}_{k\ell} &= e^{-2f}g^{k\ell}\left(\text{Ric}_{k\ell} + \frac{n-2}{n-1}(df\otimes df - \text{Hess}(f) - |df|^2 g)_{k\ell} - \frac{\Delta f}{n-1}g_{k\ell}\right) \\ &= e^{-2f}\left(nS + \frac{n-2}{n-1}(|df|^2 - \Delta f - n|df|^2) - \Delta f\frac{n}{n-1}\right) \\ &= e^{-2f}nS - (n-2)|df|^2 - 2\Delta f \qquad \text{since } -n+2-n = -2n+2 \end{aligned}$$

Now we have formula

$$\tilde{S} = e^{-2f}\left(S - \frac{n-2}{n}|df|^2 - \frac{2}{n}\Delta f\right) \tag{40}$$

Notice this is Elliptic Problem if we want to give conditions on S. We further define for $n \geq 3$

$$\begin{aligned} u &= e^{\frac{n-2}{2}f} \\ e^{2f} &= (e^{\frac{n-2}{2}f})^{\frac{4}{n-2}} = u^{\frac{4}{n-2}} \\ \tilde{g} &= u^{\frac{4}{n-2}} g \\ \log u &= \frac{n-2}{2} f \\ f &= \frac{2}{n-2} \log(u) \\ df &= \frac{2}{n-2} \frac{du}{u} \\ |df|^2 &= \frac{4}{(n-2)^2} \frac{|du|^2}{u} \\ f_{;i} &= \frac{2}{n-2} \frac{u_{;i}}{u} \\ f_{;ij} &= \frac{2}{n-2} \left(\frac{u_{;ij}}{u} - \frac{u_{;i}u_{;j}}{u^2} \right) \\ \Delta_g f &= \frac{2}{n-2} \left(\frac{\Delta_g u}{u} - \frac{|du|^2}{u^2} \right) \end{aligned}$$

Hence

$$\begin{aligned} \tilde{S} &= u^{-\frac{4}{n-2}} \left(S - \frac{n-2}{n} \frac{4}{(n-2)^2} \frac{|du|^2}{u} - \frac{2}{n} \frac{2}{n-2} \left(\frac{\Delta_g u}{u} - \frac{|du|^2}{u^2} \right) \right) \\ &= u^{-\frac{4}{n-2}} \left(S - \frac{4}{n(n-2)} \frac{\Delta_g u}{u} \right) \\ &= u^{-\frac{4}{n-2}} u^{-1} \left(Su - \frac{4}{n(n-2)} \Delta_g u \right) \\ &= u^{-\frac{n+2}{n-2}} \left(Su - \frac{4}{n(n-2)} \Delta_g u \right) \end{aligned}$$

we derived the Yamabe Equation

$$u^{\frac{n+2}{n-2}} \tilde{S} - Su + \frac{4}{n(n+2)} \Delta_g u = 0 \tag{41}$$

We denote $\text{scal} = n(n-1)S = g^{ik} g^{j\ell} R_{ijk\ell}$. Then

$$\begin{aligned} u^{\frac{n+2}{n-2}} \frac{1}{n(n-1)} \tilde{\text{scal}} - \frac{1}{n(n-1)} \text{scal} u + \frac{4}{n(n-2)} \Delta_g u = 0 \\ \frac{4(n-1)}{n-2} \Delta_g u - \text{scal} u + \tilde{\text{scal}} u^{\frac{n+2}{n-2}} = 0 \end{aligned} \tag{42}$$

Proposition 4.3 (Yamabe Conjecture)**.** *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Then there exists a metric $\tilde{g}$ that is conformal to g, i.e.,*

$$\tilde{g} = e^{2f} g \qquad \textit{for some } f \in C^\infty(M)$$

and it has constant scalar curvature. In particular

$$\frac{4(n-1)}{n-2} \Delta_g u - \text{scal} u + C u^{\frac{n+2}{n-2}} = 0 \qquad \textit{for } C \in \mathbb{R}$$

Remark 4.3. *In $n = 2$, the uniformization theorem says that any compact manifold (M, g) is conformal to one that is constant sectional curvature.*

Proof of Yamabe Conjecture 4.3. Consider the Einstein-Hilbert Action.

$$\int_M R_g \text{dvol}_g$$

and the normalized Einstein-Hilbert Action

$$\mathcal{E}(g) := \frac{\int_M R_g \text{dvol}_g}{\text{vol}(M, g)^{\frac{n-2}{2}}}$$

so that

$$\mathcal{E}(\lambda^2 g) = \mathcal{E}(g) \qquad \forall\ \lambda \in \mathbb{R}$$

The critical points of Einstein-Hilbert Action are Einstein manifolds, i.e.

$$\mathrm{Ric}(g) = \Lambda g$$

We define the Yamabe Invariant

$$Y(M, g) := \inf\{\mathcal{E}(\tilde{g}) \mid \tilde{g} \text{ conformal to } g\} = \inf\{\mathcal{E}(u^{\frac{4}{n-2}} g) \mid u \in C^\infty(M),\ u > 0\}$$

Theorem 4.3 (Aubin)**.** *For $n = \dim M$*

$$Y(M, g) \leq Y(\mathbb{S}^n, g_{can})$$

One needs three theorems.

Theorem 4.4 (1976 Yamabe-Trudinger-Aubin)**.** *If $Y(M, g) < Y(\mathbb{S}^n, g_{can})$ for $n = \dim M$, then the Yamabe Conjecture 4.3 holds.*

Theorem 4.5 (Aubin)**.** *If (M, g) is of dimension ≥ 6 and not locally conformally flat. Then $Y(M, g) < Y(\mathbb{S}^n, g_{can})$ for $n = \dim M$.*

Theorem 4.6 (1984 Schoen)**.** *If (M, g) has dimension $3, 4, 5$ or (M, g) is locally conformally flat, then $Y(M, g) < Y(\mathbb{S}^n, g_{can})$ unless (M, g) is conformal to $(\mathbb{S}^n, g_{can})$.*

Combining above, the Yamabe Conjecture is proved. □

4.3 Geodesics of Hyperbolic Space

Recall

Definition 4.5 (Unit Disc Model)**.** (D^n, h)

$$D^n := \{u \in \mathbb{R}^n \mid |u| < 1\} \qquad h := \frac{4 \sum_{i=1}^n du_i^2}{(1 - |u|^2)^2}$$

Definition 4.6 (Upper Half Space Model)**.** $(\mathcal{H}^n, g)$

$$\mathcal{H}^n := \{y \in \mathbb{R}^n \mid y_n > 0\} \qquad g := \frac{\sum_{i=1}^n dy_i^2}{y_n^2}$$

One needs the following lemma to show (D^n, h) is complete.

Lemma 4.1. *Let (M, g) be a Riemannian manifold and let $\sigma : M \to M$ be an isometry. Denote*

$$M^\sigma := \{x \in M \mid \sigma(x) = x\} \qquad \text{as the set of fixed points of } \sigma$$

Suppose M^σ is non-empty and is a submanifold of M. Then M^σ is a totally geodesic submanifold of M.

Proof. Since $M^\sigma \neq \varnothing$, there exists $p \in M^\sigma$. Due to local existence, for given $v \in T_p M^\sigma$, there exists $\varepsilon > 0$ s.t.

$$\gamma : [0, \varepsilon) \to M^\sigma \qquad \gamma(0) = p \qquad \gamma'(0) = v$$

is geodesic in M. Since $\sigma(x) = x$, and using that σ is an isometry

$$\sigma \circ \gamma(0) = \sigma(p) = p \qquad (\sigma \circ \gamma)'(0) = d\sigma_p \circ \gamma'(0) = d\sigma_p(v) = v$$

and thus by uniqueness of geodeiscs, σ fixes geodesics. Hence

$$\sigma \circ \gamma = \gamma \implies \gamma \subset M^\sigma$$

Thus γ is a geodesic in M^σ. Hence any geodesic in M starting in $p \in M^\sigma$, $v \in T_p M^\sigma$ remains a geodesic in M^σ. This is equivalent to say M^σ is a totally geodesic submanifold of M. □

Proposition 4.4. *(D^n, h) is complete.*

Proof. By Hopf-Rinow 3.1, it suffices to show that $\exp_0$ is defined on the whole tangent space of T_0D^n. Note

$$h(0) = 4\sum_{i=1}^{n} du_i^2$$

Now we simplify it further by rotating. Given initial velocity at 0. For any $v \in T_0D^n$ unit length, there exists $A \in O(n)$ s.t.

$$Av = (\frac{1}{2}, 0, \cdots, 0) \qquad \text{and} \qquad A : D^n \to D^n \qquad \text{for } A \in O(n) \text{ is an isometry}$$

It suffices to show that the geodesic γ with $\gamma(0) = 0$ and $\gamma'(0) = (\frac{1}{2}, 0, \cdots, 0)$ is defined for all times. This is to say

$$\gamma'(0) = \frac{1}{2} \frac{\partial}{\partial u_1}\Big|_0$$

Consider

$$\sigma : D^n \to D^n \qquad (u_1, \cdots, u_n) \mapsto (u_1, -u_2, -u_3, \cdots, -u_n) \qquad \sigma = \text{diag}(-1, 1, 1, \cdots, 1) \in O(n) \qquad \text{an isometry of } D^n$$

Now the fixed points of σ are

$$(D^n)^\sigma = \{(u_1, 0, \cdots, 0) \in D^n \mid u_1 \in [-1, 1]\}$$

Since $(D^n)^\sigma$ is a totally geodesic submanifold. We need to prove that the geodesic lives forever. Denote

$$\beta : (-1, 1) \to D^n \qquad t \mapsto (t, 0, \cdots, 0)$$

This is a curve β with the same image as a geodesic.

$$\beta'(t) = (1, 0, \cdots, 0) \qquad |\beta'(t)|^2 = \frac{4}{(1-t^2)^2} \qquad |\beta'(t)| = \frac{2}{1-t^2}$$

To reparametrize it in arc length

$$\begin{aligned}
s(t_0) &= \int_0^{t_0} |\beta'(t)|\, dt = \int_0^{t_0} \frac{2}{1-t^2}\, dt = \int_0^{t_0} \left(\frac{1}{1+t} + \frac{1}{1-t}\right) dt = \log(1+t) - \log(1-t)\Big|_0^{t_0} \\
&= \log(\frac{1+t_0}{1-t_0}) \\
e^{s(t_0)} &= \frac{1+t_0}{1-t_0} \\
t_0 &= \frac{e^{s(t_0)} - 1}{e^{s(t_0)} + 1} \\
t &= \frac{e^{s(t)} + 1}{e^{s(t)} - 1} = \tanh(\frac{s(t)}{2}) = \tanh(\frac{s}{2})
\end{aligned}$$

Now

$$\gamma : \mathbb{R} \to D^n \qquad s \mapsto (t, 0, \cdots, 0) = (\tanh(\frac{s}{2}), 0, \cdots, 0) \qquad \gamma(0) = 0 \qquad \gamma'(0) = (\frac{1}{2}, 0, \cdots, 0)$$

Hence the Disc Model is complete. □

Remark 4.4. *In general,*

$$\exp_0 : T_0D^n \to D^n \qquad \sum_i a_i \frac{\partial}{\partial u_i} \mapsto \begin{cases} 0 & \vec{a} = 0 \\ \tanh(|\vec{a}|\frac{s}{2})\frac{\vec{a}}{|\vec{a}|} & \vec{a} \neq 0 \end{cases}$$

Now we want to find geodesics on $\mathcal{H}^n$ where $\gamma(0) = p$ and $\gamma'(0) = v$. We need reduction to 2-dim. For $\vec{y} = (\xi, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ we want to define

$$\phi_{A,b}(\xi, t) := (A\xi + \vec{b}, t) \qquad A \in O(n-1), \qquad \vec{b} \in \mathbb{R}^{n-1} \qquad \text{is an isometry}$$

We may assume $p = (0, \cdots, 0, y)$ and $\vec{v} = (0, \cdots, 0, a, b)$. Now

$$\sigma = \begin{pmatrix} -I_{n-2} & 0 \\ 0 & 1 \end{pmatrix}$$

The fixed points are

$$(\mathcal{H}^n)^\sigma = \{(0, \cdots, 0, a, b) \in \mathcal{H}^n\} \cong (\mathcal{H}^2, \frac{dx^2 + dy^2}{y^2})$$

But for the $\mathcal{H}^2$ the isometries are given by

$$\mathrm{PSL}(2, \mathbb{R}) \cup \sigma \mathrm{PSL}(2, \mathbb{R})$$

where

$$\mathrm{PSL}(2, \mathbb{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \mid ad - bc = 1\}/\{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$$

and

$$\sigma : \mathcal{H}^2 \to \mathcal{H}^2 \qquad (x, y) \mapsto (-x, y)$$

4.4 Space Forms

Definition 4.7 (Space Form)**.** *A Space form is a connected complete Riemannian Manifold with constant sectional curvature.*

Theorem 4.7. *Let (M^n, g) be a Space Form of Dimension n. Let $(\tilde{M}, \tilde{g})$ be its universal cover, i.e., $(\tilde{M}, \tilde{g})$ is simply connected, complete with constant sectional curvature. Then $(\tilde{M}, \tilde{g})$ is isometric either to (up to rescaling $K_{\lambda 2g} = \frac{1}{\lambda^2} K_g$*

1. *$(\mathcal{H}^n, g)$ with $K = -1$*
2. *$(\mathbb{R}^n, g_0)$ with $K = 0$*
3. *$(\mathbb{S}^n, g_{can})$ with $K = 1$.*

Proof. 1. Case $K = -1, 0$. Let

$$\Delta_n := \begin{cases} \mathcal{H}^n & K = -1 \\ \mathbb{R}^n & K = 0 \end{cases}$$

Given two exponential maps

$$\begin{aligned} \exp_{\tilde{p}} : & T_{\tilde{p}}\tilde{M} \to \tilde{M} \\ \exp_p : & T_0\Delta \to \Delta_n \end{aligned}$$

Let any $\tilde{p} \in \tilde{M}$, and any point $p \in \Delta_n$, define any linear isometry

$$i : T_{\tilde{p}}\tilde{M} \to T_0\Delta$$

The same setup as Cartan's Theorem. Since $K \leq 0$, by Hadamard Theorem 3.3, $\exp_{\tilde{p}}$ and $\exp_p$ are diffeomorphisms. Since K is constant sectional curvature, by Cartan's Theorem 4.1

$$f := \exp_p \circ i \circ \exp_{\tilde{p}}^{-1}$$

is an isometry.

2. Case $K = 1$. Again take any $\tilde{p} \in \tilde{M}$, $p \in \mathbb{S}^n$ linear isometry. One has similar diagram

$$\begin{aligned} \exp_p : & T_p\mathbb{S}^n \to \mathbb{S}^n \\ i : & T_p\mathbb{S}^n \to T_{\tilde{p}}\tilde{M} \\ \exp_{\tilde{p}} : & T_{\tilde{p}}\tilde{M} \to \tilde{M} \end{aligned}$$

Then we define

$$f_p = (\exp_{\tilde{p}}) \circ i \circ (\exp_p)^{-1} : \mathbb{S}^n \setminus \{-p\} \to \tilde{M}$$

Since K is constant, f is a local isometry. Let's take another $p' \in \mathbb{S}^n \setminus \{\pm p\}$. Then define

$$i' \equiv df_{p'} : T_{p'}\mathbb{S}^n \to T_{f(p')}\tilde{M}$$

which is another linear isometry. Denote $\tilde{p'} := f(p')$. Let's define another f' s.t.

$$f' := (\exp_{\tilde{p'}}) \circ i \circ (\exp_{p'})^{-1} : \mathbb{S}^n \setminus \{-p'\} \to \tilde{M}$$

This is another local isometry. Notice

$$\begin{aligned} f(p') &= f'(p') = \tilde{p'} \\ df_{p'} &= df'_{p'} = i' \end{aligned}$$

One has lemma.

Lemma 4.2. *If given two Riemannian Manifolds (M,g), (N,h) where M connected, and f_1, f_2*

$$f_1,\ f_2 : (M,g) \to (N,h)$$

smooth maps, and local isometires. Also assume there exists $p \in M$ s.t.

$$\begin{aligned} f_1(p) &= f_2(p) \\ (df_1)_p &= (df_2)_p : T_pM \to T_qN \end{aligned}$$

Then $f_1 = f_2$.

Proof. Take the set

$$A := \{q \in M \mid f_1(q) = f_2(q),\ (df_1)_q = (df_2)_q\} \subset M$$

Notice

(a) $A \neq \varnothing$ because $p \in A$.
(b) A is closed by definition.

Now for any $q \in A$, there exists $r > 0$ s.t.

(a) the exponential map

$$\exp_q : B_r(0) \subset T_qM \to B_r(q) \subset M \qquad \text{is a diffeomorphism}$$

(b) f_1, f_2 maps isometrically $B_r(q)$ to $B_r(\tilde{q})$ where

$$\tilde{q} = f_1(q) = f_2(q)$$

Now

$$\begin{aligned} \exp_{\tilde{q}} \circ (df_1)_q &= f_1 \circ \exp_q \\ \exp_{\tilde{q}} \circ (df_2)_q &= f_2 \circ \exp_q \\ (df_1)_q = (df_2)_q &= i \end{aligned}$$

So we have diagram that commutes

$$f_1 = \exp_{\tilde{q}} \circ i \circ (\exp_q)^{-1} = f_2 \qquad \text{on } B_r(q)$$

Thus $B_r(q) \subset A$ and so A is open. □

Via Lemma 4.2

$$f = f' : \mathbb{S}^n \setminus \{-p, -p'\} \to \tilde{M}$$

How do we put the points back? Define

$$h(x) := \begin{cases} f(x) & x \in \mathbb{S}^n \setminus \{-p\} \\ f'(x) & x \in \mathbb{S}^n \setminus \{-p'\} \end{cases}$$

Then clearly h is a local isometry. Now h is surjective, $h(\mathbb{S}^n)$ is closed, nonempty, and also open by completeness of $\tilde{M}$. Thus $h(\mathbb{S}^n) = \tilde{M}$. Hence h is an isometry using Lemma 3.5.

□

Now by the Theorem, (M^n, g) complete with constant sectional curvature either $0, \pm 1$. Then (M,g) is isometric to $(\tilde{M}/\Gamma, \hat{g})$ where $\tilde{M}$ is either $\mathcal{H}^n$, $\mathbb{R}^n$ or $\mathbb{S}^n$, and Γ is a subgroup of discrete isometries acting in a fully discontinuous way. $\hat{g}$ is the only metric s.t.

$$(\tilde{M}, \tilde{g}) \to (\tilde{M}/\Gamma, \hat{g}) \qquad \text{is a local isometry}$$

Recall that

$$\text{Isom}(\mathcal{H}^n, g) \cong O(n,1)$$

since one can realize $(\mathcal{H}^n, g)$ as a submanifold of $(\mathbb{R}^{n,1}, -dx_0^2 + dx_1^2 + \cdots dx_n^2)$. And

$$\begin{aligned} \text{Isom}(\mathbb{R}^n, g_0) &\cong O(n) \rtimes \mathbb{R}^n \qquad x \mapsto Ax + \vec{b} \\ \text{Isom}(\mathbb{S}^n, g_{can}) &\cong O(n+1) \end{aligned}$$

Proposition 4.5. *M^n complete Riemannian manifold with $K = \pm 1$, $n = 2m$. Then M^n is isometric to $\mathbb{S}^n$ or $\mathbb{R}P^n = \mathbb{S}^n/\{\pm 1\}$.*

Proof. $M \cong \tilde{M}/\Gamma = \mathbb{S}^n/\Gamma$ for $\Gamma \subset O(n+1) = O(2m+1)$ discrete subgroup. Let $\gamma \in \Gamma$ be its eigenvalue

$$\{e^{-i\theta_1}, e^{i\theta_1}, \cdots, e^{-i\theta_k}, e^{i\theta_k}, 1, 1, \cdots, 1, -1, \cdots, -1\} \qquad 2k + r + s = 2m$$

and that $\det \gamma = (-1)^s$.

1. If $r > 0$ then there exists $x \in \mathbb{S}^n$ s.t. $\gamma(x) = x$ so upon free group action, $\gamma = id$. Then $M \cong \mathbb{S}^n$.
2. If $r = 0$, γ^2 has eigenvalues

   $$\{e^{-2i\theta_1}, e^{2i\theta_1}, \cdots, e^{-2i\theta_k}, e^{2i\theta_k}, 1, 1, \cdots, 1\}$$

   Thus by first case, $\gamma^2 = id$ so eigenvalues $\{-1, \cdots, -1\}$ and $\gamma = -id$. Thus $M \cong \mathbb{S}^n \setminus \{\pm 1\}$.

□

Remark 4.5. *If n is odd, there are some more possibilities, For example $\mathbb{S}^3 \setminus \mathbb{Z}_q$ lens space has $K = 1$.*

Example 4.4 (Lens Space)**.** *We identify $\mathbb{R}^4$ with $\mathbb{C}^2$ via*

$$(x_1, x_2, x_3, x_4) \mapsto (x_1 + ix_2, x_3 + ix_4)$$

Let

$$S^3 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$$

and consider

$$h : S^3 \to S^3 \qquad h(z_1, z_2) := (e^{\frac{2\pi i}{q}} z_1, e^{\frac{2\pi i r}{q}} z_2) \qquad \forall\ (z_1, z_2) \in S^3$$

where q and r are relatively prime integers for $q > 2$. Then

$$G = \{\mathrm{Id}, h, h^2, \cdots, h^{q-1}\}$$

is a group of isometries of the sphere S^3 with the usual metric, which operates in a totally discontinuous manner. The manifold (S^3/G) is called a lens space.

Proof.

1. Our first claim is that G forms a cyclic-q group. Indeed, since $h^q(z_1, z_2) = \left(e^{2\pi i} z_1, e^{2\pi i r} z_2\right) = (z_1, z_2)$, the order of h divides q. Because $\gcd(q, r) = 1$, the smallest k for which $e^{\frac{2\pi i r k}{q}} = 1$ is $k = q$. Thus, h has order q, and $G \cong \mathbb{Z}/q\mathbb{Z}$.
2. Next we see each element of G is an isometry on S^3. The standard metric on $S^3 \subset \mathbb{C}^2$ is induced by the Hermitian inner product on $\mathbb{C}^2$. For $h^k \in G$

   $$h^k(z_1, z_2) = \left(e^{\frac{2\pi i k}{q}} z_1,\ e^{\frac{2\pi i r k}{q}} z_2\right).$$

   The map h^k is a unitary transformation because $|e^{\frac{2\pi i k}{q}}| = 1$. Unitary transformations preserve the Hermitian inner product and hence the metric on S^3.
3. Finally we verify that the group G acts properly discontinuously on S^3. To show the action is properly discontinuous:

   (a) To see freeness, suppose h^k fixes a point (z_1, z_2). Then:

   $$e^{\frac{2\pi i k}{q}} z_1 = z_1 \quad \text{and} \quad e^{\frac{2\pi i r k}{q}} z_2 = z_2.$$

   If $z_1 \neq 0$, then $e^{\frac{2\pi i k}{q}} = 1 \implies k \equiv 0 \mod q$. Similarly, if $z_2 \neq 0$, $k \equiv 0 \mod q$. Since $\gcd(q, r) = 1$, the only solution is $k = 0$, so the action is free.

   (b) For a finite group, proper discontinuity follows automatically since every point $p \in S^3$ has a neighborhood U such that $h^k(U) \cap U = \emptyset$ for all $h^k \neq \mathrm{Id}$.

□

4.5 Conformal Maps

Definition 4.8. *Let V, W be finite dimensional vector spaces equipped with an inner product. We say that a linear map $L : V \to W$ is a linear conformal map if*

1. *L is a linear isomorphism*
2. *and the angles are preserved, i.e.*

$$\frac{\langle L(v_1), L(v_2)\rangle_W}{|L(v_1)|_W |L(v_2)|_W} = \frac{\langle v_1, v_2\rangle_V}{|v_1|_V |v_2|_V} \qquad \forall\ v_1,\ v_2 \in V \setminus \{0\}$$

i.e., $\cos($*angle between* $L(v_1)$ *and* $L(v_2)) = \cos($*angle between* v_1 *and* $v_2)$.

Lemma 4.3. *Let $L : V \to W$ be a linear isomorphism. Then the followings are equivalent*

1. *L is a conformal map.*
2. *There exists $\lambda \in \mathbb{R}_+$ s.t. $|L(v)|_W = \lambda |v|_V$ for any $v \in V$.*
3. *There exists λ_+ s.t. $\langle L(v), L(w)\rangle_W = \lambda^2 \langle v, w\rangle_V$ for any $v,\ w \in V$.*

Definition 4.9 (Conformal Map)**.** *Let (M, g), (N, h) be two Riemannian manifolds. A C^∞ function $f : M \to N$ map is conformal w.r.t. g and h if for any $p \in M$*

$$df_p : T_pM \to T_{f(p)}N$$

is a linear conformal map.

Remark 4.6. *By Lemma 4.3, f is a conformal map iff*

$$\begin{cases} f \text{ is a local diffeomorphism} \\ f^*h = \lambda^2 g \end{cases}$$

Here

$$\lambda : M \to (0, \infty)$$

C^∞ function is called the conformal factor.

Remark 4.7. *A local Isometry is a conformal map with $\lambda = 1$. In particular,*

$$\begin{cases} \text{local isometry} & \implies \text{conformal map} \implies \text{local diffeomorphism} \\ f^*h = g & \not\Longleftarrow\ f^*h = \lambda^2 g \not\Longleftarrow \text{ unless } n = 1 \end{cases}$$

Example 4.5. 1. *Dilations.*

$$f : \mathbb{R}^n \to \mathbb{R}^n \qquad x \mapsto \lambda x \qquad \lambda > 0$$

Then

$$f^*g_0 = f^*(dx_1^2 + \cdots dx_n^2) = \lambda^2(dx_1^2 + \cdots + dx_n^2) = \lambda^2 g_0$$

and for any $x \in \mathbb{R}^n$

$$df_x : \mathbb{R}^n \to \mathbb{R}^n$$

is represented by λI_d. f is an orientation-preserving conformal map of $\mathbb{R}^n$.

2. *Inversion.*

$$f : \mathbb{R}^n \setminus \{x_0\} \to \mathbb{R}^n \setminus \{x_0\} \qquad |f(x) - x_0| \cdot |x - x_0| = 1$$

where in particular

$$f(x) - x_0 = \frac{1}{|x - x_0|} \frac{x - x_0}{|x - x_0|}$$

Here $\frac{x-x_0}{|x-x_0|}$ gives the direction of $\vec{x} - \vec{x_0}$ and $\frac{1}{|x-x_0|}$ gives the length. Rewriting gives

$$f(x) = x_0 + \frac{x - x_0}{|x - x_0|^2}$$

Now for any $v \in T_x(\mathbb{R}^n \setminus \{0\}) = T_x\mathbb{R}^n \cong \mathbb{R}^n$

$$\begin{aligned}(df_x)(v) &= \frac{v|x-x_0|^2 - (x-x_0)2\langle v, x-x_0\rangle}{|x-x_0|^4} \\ &= \frac{1}{|x-x_0|^2}\left(v - \frac{2\langle v, x-x_0\rangle}{|x-x_0|^2}(x-x_0)\right) \end{aligned} \tag{43}$$

Taking the square

$$\begin{aligned}|df_x(v)|^2 &= \frac{1}{|x-x_0|^4}\left(|v|^2 - \frac{4\langle v, x-x_0\rangle}{|x-x_0|^2}\langle v, x-x_0\rangle + \frac{4\langle v, x-x_0\rangle^2}{|x-x_0|^4}|x-x_0|^2\right) \\ &= \frac{1}{|x-x_0|^4}|v|^2\end{aligned}$$

Hence f *is a conformal map with*

$$f^*g_0 = \frac{1}{|x-x_0|^4}g_0$$

From the formula of inversion (43),

(a) *if* $\langle v, x-x_0\rangle = 0$ *then*

$$df_x(v) = \frac{1}{|x-x_0|^2}v$$

(b) *If* $v \in \mathbb{R}(x-x_0)$ *in the span, say* $v = \xi(x-x_0)$ *then*

$$df_x(v) = \frac{1}{|x-x_0|^2}\left(\xi(x-x_0) - \frac{2\langle \xi(x-x_0), x-x_0\rangle(x-x_0)}{|x-x_0|^2}\right) = -\xi\frac{x-x_0}{|x-x_0|^2} = -\frac{1}{|x-x_0|^2}v$$

So $df_x : \mathbb{R}^n \to \mathbb{R}^n$ *is represented by*

$$\frac{1}{|x-x_0|^2}\begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Hence f *is an orientation reversing conformal map.*

Theorem 4.8 (Liouville)**.** *Let* $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ *be a conformal map with respect to* g_0*,* $n \geq 3$*. Let* U *be connected. Then* f *is the reflection to* U *of* F *where* F *is a composition of isometries, dilation and inversion, at most one of each.*

4.5.1 Examples in Lower Dimensions

$n = 1$ Let

$$f : (a,b) \to (\mathbb{R}, dx^2) \qquad x \mapsto f(x)$$

It is a diffeomorphism, hence $f'(x) \neq 0$. Then

$$f^*g_0 = f^*(dx^2) = (f'(x))^2 dx^2$$

is a conformal map with conformal factor $f'(x)$. In particular, a local diffeomorphism is always a conformal map in dimension 1.

$n = 2$ Let

$$f : U \subset \mathbb{R}^2 \to \mathbb{R}^2 \qquad (x,y) \mapsto f(x,y) = (u(x,y), v(x,y))$$

Now the differential

$$df_{(x,y)} : \mathbb{R}^2 \to \mathbb{R}^2$$

is represented by metric

$$df_{(x,y)} = \begin{pmatrix} \frac{\partial u}{\partial x}(x,y) & \frac{\partial u}{\partial y}(x,y) \\ \frac{\partial v}{\partial x}(x,y) & \frac{\partial v}{\partial y}(x,y) \end{pmatrix}$$

If f is conformal, then necessarily $\det(df_{(x,y)}) \neq 0$. We have two cases

(a) If $\det(df_{(x,y)}) > 0$ we have model $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Then f is orientation preserving.

$$df_x = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

If we satisfy the Cauchy-Riemann Equations

$$u_x = v_y$$
$$v_x = -u_y$$

Then $\det(df_{(x,y)}) > 0$. This means if we construct $w(z) = w(x+iy) = u(x,y) + iv(x,y)$, then

$$\frac{\partial}{\partial \overline{z}} w = 0 \qquad \text{where} \qquad \frac{\partial}{\partial \overline{z}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

This corresponds to f being holomorphic. It doesn't have to be composition of isometries, dilations or inversions.

(b) If $\det(df_{(x,y)}) < 0$ we have model $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$. Then f is orientation reversing. We want

$$u_x = -v_y$$
$$u_y = v_x$$

This corresponds to

$$\frac{\partial f}{\partial z} = 0 \qquad \text{where} \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

hence f is anti-holomorphic.

However the group generated by isometries, dilations and inversions in $\mathbb{R}^2$ is given by

$$\mathrm{PSL}(2,\mathbb{C}) \cup \sigma\mathrm{PSL}(2,\mathbb{C})$$

where

$$\mathrm{PSL}(2,\mathbb{C}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}) \mid ad - bc = 1\}/\{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\} \qquad z \mapsto \frac{az+b}{cz+d}$$

and

$$\sigma(z) := -\overline{z}$$

In complex coordinates,

(a) Isometries of $\mathbb{R}^2$ are

$$\mathbb{R} \ltimes O(2) = \mathbb{R} \ltimes SO(2) \bigsqcup \mathbb{R} \ltimes O(2) \qquad \{z \mapsto e^{i\theta}z + z_0\} \cup \{z \mapsto e^{i\theta}\overline{z} + \overline{z_0}\}$$

(b) Dilations are of the form $z \mapsto \lambda z$ for $\lambda > 0$

(c) Inversion w.r.t. $z_0 \in \mathbb{C}$ are of the form

$$z \mapsto z_0 + \frac{z - z_0}{|z - z_0|^2} = z_0 + \frac{1}{\overline{z} - \overline{z_0}}$$

Theorem 4.9. *The isometries of $\mathcal{H}^n$ are restrictions to $\mathcal{H}^n \subset \mathbb{R}^n$ of the conformal transformations of $\mathbb{R}^n$ that take $\mathcal{H}^n$ into itself for $n \geq 2$.*

4.6 Riemannian Submersion and Horizontal Lift

Definition 4.10. *We recall some definitions.*

1. *A differentiable mapping*

$$f : \overline{M}^{n+k} \to M^n$$

is a submersion if f is surjective, and for all $\overline{p} \in \overline{M}$ the differential

$$df_{\overline{p}} : T_{\overline{p}}\overline{M} \to T_{f(p)}M$$

has rank n.

2. *In this case, for all $p \in M$, the fiber*
$$f^{-1}(p) = F_p$$
is a submanifold of $\overline{M}$ and a tangent vector of $\overline{M}$ tangent to some F_p, $p \in M$ is called a vertical vector of the submersion.

3. *If in addition, $\overline{M}$ and M have Riemannian metrics, the submersion f is Riemannian if for all $p \in \overline{M}$, the differential*
$$df_p : T_p\overline{M} \to T_{f(p)}M$$
preserves lengths of vectors orthogonal to F_p.

Definition 4.11 (Horizontal Lift)**.** *Let*
$$f : \overline{M} \to M$$
be a Riemannian submersion.

1. *A vector $\overline{x} \in T_{\overline{p}}\overline{M}$ is horizontal if it is orthogonal to the fiber. The tangent space hence admits a decomposition*
$$T_{\overline{p}}\overline{M} = (T_{\overline{p}}\overline{M})^h \oplus (T_{\overline{p}}\overline{M})^v$$
where $(T_{\overline{p}}\overline{M})^h$ denotes the subspace of horizontal vectors and $(T_{\overline{p}}\overline{M})^v$ denotes the subspace of vertical vectors.

2. *If $X \in \mathfrak{X}(M)$, the horizontal lift $\overline{X}$ of X is the horizontal field defined by*
$$df_{\overline{p}}(\overline{X}(p)) := X(f(p)) \qquad \forall\ p \in \overline{M}$$

Proposition 4.6. 1. *The horizontal lift $\overline{X}$ is differentiable.*

Proof. (a) Since f is a submersion, by the Rank Theorem, for any $\overline{p} \in \overline{M}$, there exist neighborhoods $U \subset \overline{M}$ of $\overline{p}$ and $V \subset M$ of $f(\overline{p})$ such that $f|_U : U \to V$ is locally given by projection $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$, i.e.,
$$\pi(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}) = (x_1, \ldots, x_n).$$
In these coordinates, the vertical subspace $(T_{\overline{p}}\overline{M})^v$ corresponds to $\ker d\pi$, and the horizontal subspace $(T_{\overline{p}}\overline{M})^h$ is the orthogonal complement.

(b) Let $X \in \mathfrak{X}(M)$ be a smooth vector field. Locally on V, X can be written as
$$X = \sum_{i=1}^{n} a_i(x)\frac{\partial}{\partial x_i}$$
where $a_i \in C^\infty(V)$. The horizontal lift $\overline{X}$ in $U \subset \overline{M}$ is then
$$\overline{X} = \sum_{i=1}^{n}(a_i \circ f)\frac{\partial}{\partial x_i}$$
since the horizontal vectors in U are spanned by $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$. Because $a_i \circ f$ are smooth on U and the basis vectors $\frac{\partial}{\partial x_i}$ are smooth, $\overline{X}$ is smooth on U.

(c) The horizontal distribution is globally defined and smooth because:
- The vertical distribution $\ker df$ is a smooth subbundle of $T\overline{M}$.
- The horizontal distribution $(T\overline{M})^h$, being its orthogonal complement with respect to the Riemannian metric, is also a smooth subbundle.

Thus, the horizontal lift $\overline{X}$ is globally smooth, as its local coordinate representations agree on overlaps due to the smoothness of the metric and df. Since $\overline{X}$ is smooth in local coordinates and the horizontal distribution is globally smooth, $\overline{X}$ is differentiable everywhere on $\overline{M}$.

□

2. *Let ∇ and $\overline{\nabla}$ be the Riemannian connections of M and $\overline{M}$ respectively. Then*
$$\overline{\nabla}_{\overline{X}}\overline{Y} = \overline{\nabla_X Y} + \frac{1}{2}[\overline{X}, \overline{Y}]^v \qquad \forall\ X, Y \in \mathfrak{X}(M)$$
where Z^v is the vertical component of Z.

Proof. (a) Let X, Y, $Z \in \mathfrak{X}(M)$ and let $T \in \mathfrak{X}(\overline{M})$ be a vertical field. Then for any $p \in \overline{M}$

$$\langle \overline{X}(\overline{p}), T(\overline{p})\rangle \overset{\text{isometry}}{=} \langle df_{\overline{p}}(\overline{X}(\overline{p})), df_{\overline{p}}(T(\overline{p}))\rangle \overset{\text{horizontal lift}}{=} \langle X(f(\overline{p})), 0\rangle = 0$$

since T is vertical ($df_{\overline{p}}(T) = 0$). Similarly, $\langle \overline{Y}, T\rangle = \langle \overline{Z}, T\rangle = 0$.

(b) Also since f is a Riemannian submersion and using horizontal lift

$$\begin{aligned}\overline{X}\langle \overline{Y}, \overline{Z}\rangle &= \overline{X}\left(\langle df(Y), df(Z)\rangle\right) = \overline{X}\left(\langle Y, Z\rangle \circ f\right)\\ &= df(\overline{X})\left(\langle Y, Z\rangle\right) = X\langle Y, Z\rangle\end{aligned}$$

(c) For any $T \in \mathfrak{X}(\overline{M})$ vertical field, using horizontal lifts are f-related

$$df[\overline{X}, T] = [df(\overline{X}), df(T)] = [X, 0] = 0$$

Also using definition of f-related

$$[X, Y] = [df\overline{X}, df\overline{Y}] = df[\overline{X}, \overline{Y}]$$

(d) For any $T \in \mathfrak{X}(\overline{M})$ vertical field, since $\overline{X}$ and $\overline{Y}$ are both horizontal

$$T\langle \overline{X}, \overline{Y}\rangle = \langle \nabla_T \overline{X}, \overline{Y}\rangle + \langle \overline{X}, \nabla_T \overline{Y}\rangle = 0$$

(e) Thus concluding from above

$$\begin{aligned}&\langle [\overline{X}, \overline{Y}], \overline{Z}\rangle = \langle df[\overline{X}, \overline{Y}], df\overline{Z}\rangle = \langle [X, Y], Z\rangle\\ &\langle [\overline{X}, T], \overline{Y}\rangle = 0\end{aligned}$$

(f) Finally using the formula for Riemannian connection as a function of the metric

$$\begin{aligned}2\langle \overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z}\rangle &= \overline{X}\langle \overline{Y}, \overline{Z}\rangle + \overline{Y}\langle \overline{X}, \overline{Z}\rangle - \overline{Z}\langle \overline{X}, \overline{Y}\rangle + \langle [\overline{X}, \overline{Y}], \overline{Z}\rangle + \langle [\overline{Z}, \overline{X}], \overline{Y}\rangle + \langle [\overline{Z}, \overline{Y}], \overline{X}\rangle\\ &= X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle + \langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle + \langle [Z, Y], X\rangle\\ &= 2\langle \nabla_X Y, Z\rangle\\ 2\langle \overline{\nabla}_{\overline{X}}\overline{Y}, T\rangle &= \overline{X}\langle \overline{Y}, T\rangle + \overline{Y}\langle \overline{X}, T\rangle - T\langle \overline{X}, \overline{Y}\rangle + \langle [\overline{X}, \overline{Y}], T\rangle + \langle [T, \overline{X}], \overline{Y}\rangle + \langle [T, \overline{Y}], \overline{X}\rangle\\ &= \langle [\overline{X}, \overline{Y}], T\rangle + \langle [T, \overline{X}], \overline{Y}\rangle + \langle [T, \overline{Y}], \overline{X}\rangle\\ &= \langle [\overline{X}, \overline{Y}], T\rangle\end{aligned}$$

(g) Thus

$$\begin{aligned}\langle \overline{\nabla}_{\overline{X}}\overline{Y}, Z\rangle &= \langle \overline{\nabla}_{\overline{X}}\overline{Y}, Z^h\rangle + \langle \overline{\nabla}_{\overline{X}}\overline{Y}, Z^v\rangle \qquad \forall\, Z \in \mathfrak{X}(\overline{M})\\ &= \langle \overline{\nabla_X Y}, Z\rangle + \frac{1}{2}\langle [\overline{X}, \overline{Y}]^v, Z\rangle\end{aligned}$$

□

3. *Observe that $[\overline{X}, \overline{Y}]^v(p)$ depends only on $\overline{X}(p)$ and $\overline{Y}(p)$.*

Proof. Since $[\overline{X}, \overline{Y}]^v$ is the vertical component, it suffices to consider how it inner products with $T \in \mathfrak{X}(\overline{M})$. From (b) we deduce

$$\begin{aligned}\frac{1}{2}\langle [\overline{X}, \overline{Y}]^v, T\rangle &= \langle \overline{\nabla}_{\overline{X}}\overline{Y}, T\rangle - \langle \overline{\nabla_X Y}, T\rangle\\ &= \langle [\overline{X}, \overline{Y}], T\rangle + \langle \overline{\nabla}_{\overline{Y}}\overline{X}, T\rangle - \langle \overline{\nabla_X Y}, T\rangle\\ &= \langle [\overline{X}, \overline{Y}], T\rangle + \frac{1}{2}\langle [\overline{Y}, \overline{X}], T\rangle - \langle \overline{\nabla_X Y}, T\rangle\\ &= \langle [\overline{X}, \overline{Y}], T\rangle + \frac{1}{2}\langle [\overline{Y}, \overline{X}], T\rangle\end{aligned}$$

Using that $\overline{\nabla_X Y}$ is the horizontal lift, hence lies in the horizontal field. Yet th RHS is now only dependent on $\overline{X}(p)$ and $\overline{Y}(p)$. □

Proposition 4.7 (Curvature of Riemannian Submersion)**.** *Let $f : \overline{M} \to M$ be a Riemannian Submersion. Let X, Y, Z, $W \in \mathfrak{X}(M)$ and $\overline{X}, \overline{Y}, \overline{Z}, \overline{W}$ be there horizontal lifts. Let R and $\overline{R}$ be the curvature tensors on M and $\overline{M}$ respectively.*

1. *Then*

$$\langle \overline{R}(\overline{X},\overline{Y})\overline{Z},\overline{W}\rangle = \langle R(X,Y)Z,W\rangle - \frac{1}{4}\langle [\overline{X},\overline{Z}]^v,[\overline{Y},\overline{W}]^v\rangle + \frac{1}{4}\langle [\overline{Y},\overline{Z}]^v,[\overline{X},\overline{W}]^v\rangle - \frac{1}{2}\langle [\overline{Z},\overline{W}]^v,[\overline{X},\overline{Y}]^v\rangle$$

Proof. First observe

$$\overline{X}\langle \overline{\nabla}_{\overline{Y}}\overline{Z},\overline{W}\rangle = \overline{X}\langle \overline{\nabla_Y Z} + \frac{1}{2}[\overline{Y},\overline{Z}]^v,\overline{W}\rangle = \overline{X}\langle \overline{\nabla_Y Z},\overline{W}\rangle = X\langle \nabla_Y Z, W\rangle$$

Thus leveraging Levi-Civita connection

$$\begin{aligned}\langle \overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{Y}}\overline{Z},\overline{W}\rangle &= \overline{X}\langle \overline{\nabla}_{\overline{Y}}\overline{Z},\overline{W}\rangle - \langle \overline{\nabla}_{\overline{Y}}\overline{Z},\overline{\nabla}_{\overline{X}}\overline{W}\rangle \\ &= X\langle \nabla_Y Z,W\rangle - \langle \overline{\nabla_Y Z} + \frac{1}{2}[\overline{Y},\overline{Z}]^v, \overline{\nabla_X W} + \frac{1}{2}[\overline{X},\overline{W}]^v\rangle \\ &= X\langle \nabla_Y Z,W\rangle - \langle \nabla_Y Z,\nabla_X W\rangle - \frac{1}{4}\langle [\overline{Y},\overline{Z}]^v,[\overline{X},\overline{W}]^v\rangle \\ &= \langle \nabla_X\nabla_Y Z,W\rangle - \frac{1}{4}\langle [\overline{Y},\overline{Z}]^v,[\overline{X},\overline{W}]^v\rangle\end{aligned}$$

On the other hand for any $T \in \mathfrak{X}(\overline{M})$ vertical

$$\begin{aligned}\langle \overline{\nabla}_T\overline{X},\overline{Y}\rangle &= \langle \overline{\nabla}_{\overline{X}}T,\overline{Y}\rangle + \langle [T,\overline{X}],\overline{Y}\rangle = \overline{X}\langle T,\overline{Y}\rangle - \langle T,\overline{\nabla}_{\overline{X}}\overline{Y}\rangle + \langle [T,\overline{X}],\overline{Y}\rangle \\ &= -\langle T,\overline{\nabla}_{\overline{X}}\overline{Y}\rangle\end{aligned}$$

Thus directly applying above

$$\begin{aligned}\langle \overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{Z},\overline{W}\rangle &= \langle \overline{\nabla}_{[\overline{X},\overline{Y}]^h}\overline{Z},\overline{W}\rangle + \langle \overline{\nabla}_{[\overline{X},\overline{Y}]^v}\overline{Z},\overline{W}\rangle \\ &= \langle \nabla_{[X,Y]}Z,W\rangle - \langle [\overline{X},\overline{Y}]^v,\overline{\nabla}_{\overline{Z}}\overline{W}\rangle \\ &= \langle \nabla_{[X,Y]}Z,W\rangle - \langle [\overline{X},\overline{Y}]^v,[\overline{Z},\overline{W}]\rangle - \langle [\overline{X},\overline{Y}]^v,\overline{\nabla}_{\overline{W}}\overline{Z}\rangle \\ &= \langle \nabla_{[X,Y]}Z,W\rangle - \langle [\overline{X},\overline{Y}]^v,[\overline{Z},\overline{W}]\rangle - \frac{1}{2}\langle [\overline{W},\overline{Z}],[\overline{X},\overline{Y}]^v\rangle \\ &= \langle \nabla_{[X,Y]}Z,W\rangle - \frac{1}{2}\langle [\overline{X},\overline{Y}]^v,[\overline{Z},\overline{W}]\rangle \\ &= \langle \nabla_{[X,Y]}Z,W\rangle - \frac{1}{2}\langle [\overline{X},\overline{Y}]^v,[\overline{Z},\overline{W}]^v\rangle\end{aligned}$$

Thus combining all above

$$\begin{aligned}\langle \overline{R}(\overline{X},\overline{Y})\overline{Z},\overline{W}\rangle &= \langle \overline{\nabla}_{\overline{Y}}\overline{\nabla}_{\overline{X}}\overline{Z},\overline{W}\rangle - \langle \overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{Y}}\overline{Z},\overline{W}\rangle + \langle \overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{Z},\overline{W}\rangle \\ &= \langle \nabla_Y\nabla_X Z,W\rangle - \frac{1}{4}\langle [\overline{X},\overline{Z}]^v,[\overline{Y},\overline{W}]^v\rangle - \langle \nabla_X\nabla_Y Z,W\rangle \\ &+ \frac{1}{4}\langle [\overline{Y},\overline{Z}]^v,[\overline{X},\overline{W}]^v\rangle + \langle \nabla_{[X,Y]}Z,W\rangle - \frac{1}{2}\langle [\overline{X},\overline{Y}]^v,[\overline{Z},\overline{W}]^v\rangle \\ &= \langle R(X,Y)Z,W\rangle - \frac{1}{4}\langle [\overline{X},\overline{Z}]^v,[\overline{Y},\overline{W}]^v\rangle + \frac{1}{4}\langle [\overline{Y},\overline{Z}]^v,[\overline{X},\overline{W}]^v\rangle - \frac{1}{2}\langle [\overline{Z},\overline{W}]^v,[\overline{X},\overline{Y}]^v\rangle\end{aligned}$$

□

2. *For σ the plane generated by the orthonormal vectors X, $Y \in \mathfrak{X}(M)$ and $\overline{\sigma}$ the plane generated by $\overline{X}$, $\overline{Y}$, we have*

$$K(\sigma) = \overline{K}(\overline{\sigma}) + \frac{3}{4}|[\overline{X},\overline{Y}]^v|^2 \geq \overline{K}(\overline{\sigma}) \tag{44}$$

Proof. Since X and Y are orthonormal and using f is isometry, $\overline{X}$ and $\overline{Y}$ are orthonormal

$$\begin{aligned}\overline{K}(\overline{\sigma}) &= \overline{R}(\overline{X},\overline{Y},\overline{X},\overline{Y}) \\ &= \langle R(X,Y)X,Y\rangle - \frac{1}{4}\langle [\overline{X},\overline{X}]^v,[\overline{Y},\overline{Y}]^v\rangle + \frac{1}{4}\langle [\overline{Y},\overline{X}]^v,[\overline{X},\overline{Y}]^v\rangle - \frac{1}{2}\langle [\overline{X},\overline{Y}]^v,[\overline{X},\overline{Y}]^v\rangle \\ &= \langle R(X,Y)X,Y\rangle - \frac{3}{4}\langle [\overline{X},\overline{Y}]^v,[\overline{X},\overline{Y}]^v\rangle \\ &= K(\sigma) - \frac{3}{4}|[\overline{X},\overline{Y}]^v|^2\end{aligned}$$

□

Example 4.6 (Curvature of the Complex Projective Space)**.** *Define a Riemannian metric on $\mathbb{C}^{n+1} \setminus \{0\}$ in the following way: If $Z \in \mathbb{C}^{n+1} \setminus \{0\}$ and V, $W \in T_Z(\mathbb{C}^{n+1} \setminus \{0\})$*

$$\langle V, W\rangle_Z = \frac{Real(V,W)}{(Z,Z)}$$

The metric $\langle \cdot, \cdot\rangle$ restricted to $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1} \setminus \{0\}$ coincides with the metric induced from $\mathbb{R}^{2n+2}$. Notice for all $0 \leq \theta \leq 2\pi$

$$e^{i\theta} : \mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}$$

is an isometry, hence it is possible to define a Riemannian metric on $P^n(\mathbb{C})$ s.t. the submersion f is Riemannian. Show that in this metric, the sectional curvature of $P^n(\mathbb{C})$ is given by

$$K(\sigma) = 1 + 3\cos^2(\varphi)$$

where σ is generated by the orthonormal pair X, Y

$$\cos(\varphi) = \langle \overline{X}, i\overline{Y}\rangle$$

and $\overline{X}$, $\overline{Y}$ are the horizontal lifts of X and Y. In particular,

$$1 \leq K(\sigma) \leq 4$$

Proof. Let Z be the position vector describing $\mathbb{S}^{2n+1}$. Since

$$\left.\frac{d}{d\theta}\right|_{\theta=0} e^{i\theta} Z = iZ$$

We know $iZ \in T_Z(\mathbb{S}^{2n+1})$ and is vertical. Now let $\overline{\nabla}$ be the Riemannian connection of $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$ and let X, $Y \in \mathfrak{X}(P^n(\mathbb{C}))$. Choose

$$\alpha : (-\varepsilon, \varepsilon) \to \mathbb{S}^{2n+1}$$

s.t.

$$\alpha(0) = Z \qquad \alpha'(0) = \overline{X}$$

Then

$$\begin{aligned}(\overline{\nabla}_{\overline{X}} iZ)_Z &= \left.\frac{d}{dt}\right|_{t=0} iZ \circ \alpha(t) \\ &= \left.\frac{d}{dt}\right|_{t=0} i\alpha(t) = i\alpha'(0) = i\overline{X}\end{aligned}$$

Thus

$$\begin{aligned}\langle [\overline{X}, \overline{Y}], iZ\rangle &= \langle \overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X}, iZ\rangle \\ &= \overline{X}\langle \overline{Y}, iZ\rangle - \langle \overline{Y}, \overline{\nabla}_{\overline{X}} iZ\rangle - \overline{Y}\langle \overline{X}, iZ\rangle + \langle \overline{X}, \overline{\nabla}_{\overline{Y}} iZ\rangle \\ &= -\langle i\overline{X}, \overline{Y}\rangle + \langle i\overline{Y}, \overline{X}\rangle \\ &= 2\cos(\varphi)\end{aligned}$$

Notice, since iZ spans the vertical subspace at Z, the vertical component $[\overline{X}, \overline{Y}]^v$ must be proportional to iZ. Assume

$$[\overline{X}, \overline{Y}]^v = ciZ$$

Then

$$\begin{aligned}\langle [\overline{X}, \overline{Y}], iZ\rangle = c|iZ|^2 = 2\cos(\varphi) \\ c = 2\cos(\varphi)\end{aligned}$$

Thus

$$|[\overline{X}, \overline{Y}]^v|^2 = \langle 2\cos(\varphi) iZ, 2\cos(\varphi) iZ\rangle = 4\cos^2(\varphi)$$

Also notice the sphere $\mathbb{S}^{2n+1}$ has constant sectional curvature 1. Now using (44)

$$\begin{aligned}K(\sigma) &= \overline{K}(\overline{\sigma}) + \frac{3}{4}|[\overline{X}, \overline{Y}]^v|^2 \\ &= 1 + 3\cos^2(\varphi)\end{aligned}$$

□

5 Variations of Energy

5.1 Minimizing Arc Length

Let (M, g) be a Riemannian manifold, p, $q \in M$. Denote

$$\Omega_{p,q} := \{c : [0,1] \to M \mid \text{piecewise } C^\infty,\ c(0) = p \text{ and } c(1) = q\}$$

Definition 5.1 (Arc-length Functional)**.** *We define the Arc-length Functional*

$$L : \Omega_{p,q} \to \mathbb{R} \qquad c \mapsto L(c) := \int_0^1 |c'(t)| dt$$

1. Observe that we have lower bound
$$d(p,q) = \inf_{c \in \Omega_{p,q}} L(c)$$
2. We want to find $\delta \in \Omega_{p,q}$ with
$$L(\delta) \leq L(c)$$
for any $c \in \Omega_{p,q}$.
3. However δ is not unique, because up to reparametrization
$$\delta \circ \phi$$
for
$$\phi : [0,1] \to [0,1]$$
has the same length.

Observe by Cauchy-Schwarz

$$L(c)^2 = \left(\int_0^1 |c'(t)|\, dt \right)^2 \leq \int_0^1 1 dt \int_0^1 |c'(t)|^2\, dt = \int_0^1 |c'(t)|^2\, dt$$

Notice $=$ iff $|c'(t)| =$ constant.

Definition 5.2 (Energy Functional)**.**

$$E : \Omega_{p,q} \to \mathbb{R} \qquad c \mapsto E(c) := \int_0^1 |c'(t)|^2 dt$$

1. Now suppose we have a minimizer δ s.t.
$$L(\delta) \leq L(c) \qquad \forall\ c \in \Omega_{p,q}$$
Then there exists only one reparametrization ϕ s.t. $\tilde{\delta} = \delta \circ \phi$ has constant velocity $|\tilde{\delta}'(t)| =$ constant.
2. Then for any $c \in \Omega_{p,q}$
$$E(c) \geq L(c)^2 \geq L(\delta)^2 = L(\tilde{\delta})^2 = E(\tilde{\delta})$$
Now $\tilde{\delta}$ is unique and a minimizer of the energy functional.

Now given $c \in \Omega_{p,q}$ we want to compute the differential of the energy functional.

$$dE_c : T_c\Omega_{p,q} \to \mathbb{R}$$

We need to discuss variations.

Definition 5.3 (Variation (formal))**.** *Let*

$$f : (-\varepsilon, \varepsilon) \to \Omega_{p,q} \qquad s \mapsto f_s$$

be a curve in the space of curves $\Omega_{p,q}$ s.t. $f_0 = c$. f is called a proper variation of c.

1. *Then we define*
$$V(t) := \left.\frac{d}{ds}\right|_{s=0} f_s(t) \in T_c\Omega_{p,q}$$
This is vector field along c, called the variation field of f. Notice $V(0) = V(1) = 0$.

2. *Now we define the first variation of energy as*

$$dE_c(V) = \frac{d}{ds}\bigg|_{s=0} E(f_s)$$

3. *If c is a critical point of dE_c, i.e.*

$$dE_c(V) = 0 \qquad \forall\ V \in T_c\Omega_{p,q}$$

Then we define the second variation of energy as

$$d^2E_c(V,V) = \frac{d^2}{ds^2}\bigg|_{s=0} E(f_s)$$

In fact we give a precise definition

Definition 5.4 (Variation)**.** *Let $c : [0,a] \to M$, $a > 0$ be piecewise C^∞ curve. A variation of c is a continuous map*

$$f : (-\varepsilon, \varepsilon) \times [0,a] \to M \qquad (s,t) \mapsto f(s,t)$$

s.t.

1. *$f(0,t) = c(t)$.*
2. *There exists $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = a$ s.t.*

$$f|_{(-\varepsilon,\varepsilon)\times[t_n,t_{n+1}]}$$

is smooth.

We also define the following

1. *We say that f is proper if*

$$f(s,0) = c(0) \qquad \textit{and} \qquad f(s,a) = c(a) \qquad \forall\ s \in (-\varepsilon, \varepsilon)$$

2. *Given a curve*

$$f_s : [0,a] \to M \qquad s \mapsto f_s(t) := f(s,t)$$

We denote

$$g_t : (-\varepsilon, \varepsilon) \to M \qquad s \mapsto g_t(s) := f(s,t)$$

as the transverse curve.

3. *We define*

$$V(t) := \frac{d}{ds}\bigg|_{s=0} f_s(t)$$

as a variation field. f proper implies $V(0) = V(a) = 0$.

Conversely we have

Proposition 5.1 (Construction of Variation)**.** *Let*

$$c : [0,a] \to M$$

be piecewise C^∞ curve, and $V(t) \not\equiv 0$ be piecewise C^∞ vector field along c. Then there exists variation

$$f : (-\varepsilon, \varepsilon) \times [0,a] \to M$$

of c s.t.

$$V(t) = \frac{\partial f}{\partial s}(0,t)$$

Proof. By compactness of $[0,a]$, there exists $\delta > 0$ s.t.

$$\exp_{c(t)}(v)$$

is defined for $|v| < \delta$, $v \in T_{c(t)}M$, for any $t \in [0,a]$. Now we want to consider

$$\varepsilon := \frac{\delta}{\max\limits_{t\in[0,a]} |V(t)|}$$

Define

$$f : (-\varepsilon, \varepsilon) \times [0,a] \to M \qquad (s,t) \mapsto \exp_{c(t)}(sV(t))$$

This is well-defined because $|sv(t)| < \delta$ for $|s| < \varepsilon$. Here

1. $f(0,t) = \exp_{c(t)}(0) = c(t)$
2. $\frac{\partial f}{\partial s}(0,t) = V(t)$

□

Remark 5.1. *If $V(0) = V(a) = 0$, then we can choose f s.t. $f(s,0) = \exp_{c(0)}(0) = c(0)$ is proper.*

We have the following application of variation.

Proposition 5.2. *Let M be a complete Riemannian manifold, and let $N \subset M$ be a closed submanifold of M. Let $p_0 \in M$, $p_0 \notin N$ and let $d(p_0, N)$ be distance from p_0 to N. Then there exists a point $q_0 \in N$ s.t.*

$$d(p_0, q_0) = d(p_0, N)$$

and that a minimizing geodesic which joins p_0 to q_0 is orthogonal to N at q_0.

Proof.

1. Since N is closed and $p_0 \notin N$, the distance

$$d(p_0, N) := \inf_{q \in N} d(p_0, q) > 0$$

Let $\{q_n\} \subset N$ be a minimizing sequence such that $d(p_0, q_n) \to d(p_0, N)$. For sufficiently large n, all q_n lie within the closed geodesic ball $\overline{B}(p_0, d(p_0, N) + 1)$. By the Hopf-Rinow theorem, this ball is compact in the complete manifold M. Hence, $\{q_n\}$ has a convergent subsequence $q_{n_k} \to q_0$. Since N is closed, $q_0 \in N$. Continuity of the distance function gives

$$d(p_0, q_0) = \lim_{k \to \infty} d(p_0, q_{n_k}) = d(p_0, N).$$

Thus, q_0 realizes the minimum distance. Furthermore since M is complete, thee exists a minimizing geodesic which joins p_0 and q_0.

2. Let $\gamma : [0,1] \to M$ be a minimizing geodesic from p_0 to q_0 s.t. $\gamma(0) = p_0$, $\gamma(1) = q_0$ with unit-speed parametrization. Suppose for contradiction that $\gamma'(1)$ is not orthogonal to $T_{q_0}N$. Then there exists $v \in T_{q_0}N$ with $\langle \gamma'(1), v \rangle \neq 0$. Consider a smooth curve $\alpha : (-\epsilon, \epsilon) \to N$ with $\alpha(0) = q_0$ and $\alpha'(0) = v$. Consider the variation

$$\gamma : (-\epsilon, \epsilon) \times [0,1] \to M \qquad (s,t) \mapsto \gamma_s(t)$$

where

(a) $\gamma_0(t) = \gamma(t)$
(b) $\gamma_s(0) = p_0$ fixed point, so $\frac{d}{ds}\gamma_s(0) = 0$.
(c) $\gamma_s(1) = \alpha(s)$, so $\frac{d}{ds}\gamma_s(1) = \alpha'(s)$.

Notice the arc-length writes

$$L[\gamma_s] := \int_0^1 \left\| \frac{\partial \gamma_s}{\partial t} \right\| (t)\, dt$$

so

$$\begin{aligned}
\frac{d}{ds} L[\gamma_s] &= \int_0^1 \frac{d}{ds} \left(\langle \frac{\partial \gamma_s}{\partial t}, \frac{\partial \gamma_s}{\partial t} \rangle \right)^{\frac{1}{2}} dt \\
&= \int_0^1 \frac{1}{\left\| \frac{\partial \gamma_s}{\partial t} \right\|} \langle \frac{D}{ds} \frac{\partial \gamma_s}{\partial t}, \frac{\partial \gamma_s}{\partial t} \rangle dt \\
&= \int_0^1 \frac{1}{\left\| \frac{\partial \gamma_s}{\partial t} \right\|} \langle \frac{D}{dt} \frac{\partial \gamma_s}{\partial s}, \frac{\partial \gamma_s}{\partial t} \rangle dt \qquad \text{Gauss Lemma} \\
\left.\frac{d}{ds}\right|_{s=0} L[\gamma_s] &= \int_0^1 \frac{1}{\|\gamma'(t)\|} \langle \frac{D}{dt} V(t), \gamma'(t) \rangle dt \qquad V(t) := \left.\frac{d}{ds}\right|_{s=0} \gamma_s(t) \text{ denotes variational field} \\
&= \int_0^1 \frac{d}{dt} \langle V(t), \gamma'(t) \rangle - \langle V(t), \frac{D}{dt}\gamma'(t) \rangle dt \qquad \text{product rule, and unit speed parametrizaiton} \\
&= \int_0^1 \frac{d}{dt} \langle V(t), \gamma'(t) \rangle dt \qquad \text{using } \gamma \text{ is geodesic} \\
&= \langle V(1), \gamma'(1) \rangle - \langle V(0), \gamma'(0) \rangle \\
&= \langle \alpha'(0), \gamma'(1) \rangle = \langle v, \gamma'(1) \rangle
\end{aligned}$$

The first variation of arc length for the variation $\gamma_s(t)$ yields

$$\frac{d}{ds}\bigg|_{s=0} L[\gamma_s] = \int_0^1 |\gamma_s(t)| dt = \langle \gamma'(1), v\rangle.$$

If $\langle \gamma'(1), v\rangle \neq 0$, this derivative is non-zero, implying shorter paths from p_0 to $\alpha(s)$ for small $|s|$. This contradicts the minimality of q_0. Hence, $\gamma'(1)$ must be orthogonal to $T_{q_0}N$.

□

5.2 Formulas for First and Second Variations

Definition 5.5 (Energy function of Variation)**.** *Let f be a variation of c. Then the energy function of f is*

$$E : (-\varepsilon, \varepsilon) \to \mathbb{R} \qquad s \mapsto E(s) := \int_0^a |\frac{\partial f}{\partial t}(s,t)|^2 dt = E(f_s)$$

which coincides with the energy functional of f_s. We can see this as the map

$$E : \{\text{piecewise smooth curves } c : [0,a] \to M\} \to \mathbb{R} \qquad c \mapsto E(c) := \int_0^a |c'(t)|^2 dt$$

with differential

$$dE_c(V) = \frac{d}{ds}\bigg|_{s=0} E(\beta(s)) = E'(0)$$

where $\beta(s)$ is a curve in the space of curves with $\beta(0) = c$, $\beta'(0) = V$.

5.2.1 First Variation

Proposition 5.3 (Formula for First Variation of Energy)**.**

$$\frac{1}{2}\frac{d}{ds}\bigg|_{s=0} E(s) = -\int_0^a \langle V(t), \frac{D}{dt}\frac{dc}{dt}\rangle dt + \sum_{i=1}^k \langle V(t_i), \frac{dc}{dt}(t_i^-) - \frac{dc}{dt}(t_i^+)\rangle + \langle V(a), \frac{dc}{dt}(a)\rangle - \langle V(0), \frac{dc}{dt}(0)\rangle \qquad (45)$$

Proof. Compute

$$E(s) = \int_0^a \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle dt = \sum_{i=0}^k \int_{t_i}^{t_{i+1}} \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle dt$$

So

$$\begin{aligned}
\frac{d}{ds}\left(\int_{t_i}^{t_{i+1}} \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle dt\right) &= 2\int_{t_i}^{t_{i+1}} \langle \frac{D}{ds}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle dt = 2\int_{t_i}^{t_{i+1}} \langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle dt \\
&= 2\int_{t_i}^{t_{i+1}} \left(\frac{d}{dt}\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle - \langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\rangle\right) dt \\
&= 2\,\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle\bigg|_{t_i^+}^{t_{i+1}^-} - 2\int_{t_i}^{t_{i+1}} \langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\rangle dt \\
\frac{1}{2}E'(s) &= \sum_{i=0}^k \langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle\bigg|_{t_i^+}^{t_{i+1}^-} - \int_0^a \langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\rangle dt
\end{aligned}$$

Now we evaluate at $s = 0$. In particular

$$\begin{aligned}
\frac{\partial f}{\partial t}\bigg|_{s=0} &= c'(t) \\
\frac{\partial f}{\partial s}\bigg|_{s=0} &= V(t)
\end{aligned}$$

So

$$\frac{1}{2}E'(0) = \sum_{i=0}^k \left(\langle V(t_{i+1}^-), c'(t_{i+1}^-)\rangle - \langle V(t_i^+), c'(t_i^+)\rangle\right) - \int_0^a \langle V(t), \frac{D}{dt}\frac{dc}{dt}\rangle dt$$

Rearranging yields (45).

□

In fact critical points of

$$E:\Omega_{p,q}\to\mathbb{R}$$

are critical points.

Proposition 5.4. *Let*

$$c:[0,a]\to M$$

be piecewise C^∞. Then for any proper variation of c,

$$\frac{d}{ds}E(0)=0 \iff c \qquad \textit{is a geodesic}$$

Proof. 1. $\Longleftarrow$. For any proper variation of c,

$$V(0)=V(a)=0$$

If c is a geodesic, then c is small so that

$$\frac{dc}{dt}(t_i^+)=\frac{dc}{dt}(t_i^-)$$

and $\frac{D}{dt}\frac{dc}{dt}=0$. By first variation formula (45)

$$E'(0)=0$$

2. $\Longrightarrow$. We consider the following

(a) Let

$$V(t)=g(t)\frac{D}{dt}\frac{dc}{dt}$$

be C^∞ smooth curve with

$$g(t)=\sin(\frac{\pi(t-t_i)}{t_{i+1}-t_i}) \qquad \forall\ t\in[t_i,t_{i+1}]$$

In particular g vanishes at each t_i. Hence

$$V(0)=V(a)=V(t_i)=0$$

Then V is the variational field of a proper variation of c. By our assumption

$$E'(0)=0$$

By our first variation formula (45)

$$\begin{aligned}0&=\frac{1}{2}E'(0)=-\int_0^a\langle V(t),\frac{D}{dt}\frac{dc}{dt}\rangle dt=-\int_0^a g(t)|\frac{D}{dt}\frac{dc}{dt}|^2dt\\&=-\sum_{i=0}^k\int_{t_i}^{t_{i+1}}g(t)|\frac{D}{dt}\frac{dc}{dt}|^2dt\geq 0\\0&=\int_{t_i}^{t_{i+1}}g(t)|\frac{D}{dt}\frac{dc}{dt}|^2dt \qquad \forall\ i\\0&=g(t)|\frac{D}{dt}\frac{dc}{dt}|^2 \qquad \forall\ t\in[t_i,t_{i+1}]\\0&=\frac{D}{dt}\frac{dc}{dt} \qquad \forall\ t\in(t_i,t_{i+1})\end{aligned}$$

In particular, c must be a piecewise geodesic. So in particular, for any V s.t. $V(0)=V(a)=0$ we have

$$\frac{1}{2}E'(0)=\sum_{i=1}^k\langle V(t_i),\frac{dc}{dt}(t_i^-)-\frac{dc}{dt}(t_i^+)\rangle=0$$

(b) We choose $\overline{V}(t)$ such that

$$\overline{V}(0)=\overline{V}(a)=0$$

and $\overline{V}(t_i)=\frac{dc}{dt}(t_i^-)-\frac{dc}{dt}(t_i^+)$. It has an associated proper variation satisfying

$$0=\frac{1}{2}E'(0)=\sum_{i=1}^k|\frac{dc}{dt}(t_i^-)-\frac{dc}{dt}(t_i^+)|^2$$

$$\frac{dc}{dt}(t_i^-)=\frac{dc}{dt}(t_i^+)$$

Hence c is smooth. Because c is smooth and $\frac{D}{dt}\frac{dc}{dt}=0$ we know c is a geodesic.

□

5.2.2 Second Variation

Proposition 5.5 (Formula for Second Variation of Energy)**.**

$$\frac{1}{2}\frac{d^2}{ds^2}\bigg|_{s=0} E(s) = \langle \frac{D}{ds}\frac{\partial f}{\partial s}(0,a), \gamma'(a)\rangle - \langle \frac{D}{ds}\frac{\partial f}{\partial s}(0,0), \gamma'(0)\rangle + \langle V(a), \frac{D}{dt}V(a)\rangle - \langle V(0), \frac{D}{dt}V(0)\rangle$$
$$+ \sum_{i=1}^{k} \langle V(t_i), \frac{D}{dt}V(t_i^-) - \frac{D}{dt}V(t_i^+)\rangle - \int_0^a \langle V(t), \frac{D^2}{dt^2}V(t) + R(\gamma', V)\gamma'\rangle dt \tag{46}$$

Proof. Recall from the proof of first variation, we have

$$\frac{1}{2}E'(s) = \sum_{i=0}^{k} \langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle \bigg|_{t=t_i^+}^{t=t_{i+1}^-} - \int_0^a \langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\rangle dt$$

We take a derivative.

$$\frac{1}{2}E''(s) = \sum_{i=0}^{k} \langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle \bigg|_{t=t_i^+}^{t=t_{i+1}^-} + \sum_{i=0}^{k} \langle \frac{\partial f}{\partial s}, \frac{D}{ds}\frac{\partial f}{\partial t}\rangle \bigg|_{t=t_i^+}^{t=t_{i+1}^-}$$
$$- \int_0^a \langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\rangle dt - \int_0^a \langle \frac{\partial f}{\partial s}, \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t}\rangle dt$$

Notice

$$\begin{aligned}\frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} &= \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial t} + R(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s})\frac{\partial f}{\partial t} \\ &= \frac{D}{dt}\frac{D}{dt}\frac{\partial f}{\partial s} + R(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s})\frac{\partial f}{\partial t} \qquad \text{Gauss lemma}\end{aligned}$$

So

$$\frac{1}{2}E''(s) = \sum_{i=0}^{k} \langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle \bigg|_{t=t_i^+}^{t=t_{i+1}^-} + \sum_{i=0}^{k} \langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial s}\rangle \bigg|_{t=t_i^+}^{t=t_{i+1}^-}$$
$$- \int_0^a \langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\rangle dt - \int_0^a \langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{D}{dt}\frac{\partial f}{\partial s} + R(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s})\frac{\partial f}{\partial t}\rangle dt$$

Assume that $f(0,t) = \gamma(t)$ is a geodesic. Then

$$\begin{aligned}\frac{D}{dt}\frac{\partial f}{\partial t}(0,t) &= \frac{D}{dt}\gamma'(t) = 0 \\ \frac{\partial f}{\partial s}(0,t) &= V(t) \\ \frac{\partial f}{\partial t}(0,t) &= \gamma'(t)\end{aligned}$$

Then

$$\frac{1}{2}E''(0) = \sum_{i=0}^{k} \langle \frac{D}{ds}\frac{\partial f}{\partial s}(0,t), \gamma'(t)\rangle \bigg|_{t=t_i^+}^{t=t_{i+1}^-} + \sum_{i=0}^{k} \langle V(t), \frac{D}{dt}V(t)\rangle \bigg|_{t=t_i^+}^{t=t_{i+1}^-}$$
$$- \int_0^a \langle \frac{D}{ds}\frac{\partial f}{\partial s}(0,t), \frac{D}{dt}\gamma'(t)\rangle dt - \int_0^a \langle V(t), \frac{D^2}{dt^2}V(t) + R(\gamma', V)\gamma'\rangle dt$$
$$\overset{\gamma \text{ smooth}}{=} \langle \frac{D}{ds}\frac{\partial f}{\partial s}(0,a), \gamma'(a)\rangle - \langle \frac{D}{ds}\frac{\partial f}{\partial s}(0,0), \gamma'(0)\rangle + \langle V(a), \frac{D}{dt}V(a)\rangle - \langle V(0), \frac{d}{dt}V(0)\rangle$$
$$+ \sum_{i=1}^{k} \langle V(t_i), \frac{D}{dt}V(t_i^-) - \frac{D}{dt}V(t_i^+)\rangle - \int_0^a \langle V(t), \frac{D^2}{dt^2}V(t) + R(\gamma', V)\gamma'\rangle dt$$

□

Proposition 5.6. *If f is proper variation of a geodesic, then*

$$\frac{1}{2}E''(0) = \int_0^a \left(\langle \frac{D}{dt}V(t), \frac{D}{dt}V(t) - R(\gamma', V, \gamma', V) \right) dt \tag{47}$$

Proof. In particular, if f is proper, then for any s

$$f(s,0) = \gamma(0)$$
$$f(s,a) = \gamma(a)$$
$$V(0) = V(a) = 0$$
$$\frac{D^2}{ds^2}f(0,0) = \frac{D^2}{ds^2}f(0,a) = 0$$

Then

$$\frac{1}{2}E''(0) = -\int_0^a \langle V(t), \frac{D^2}{dt^2}V(t) + R(\gamma', V)\gamma' \rangle dt + \sum_{i=1}^{k} \langle V(t_i), \frac{D}{dt}V(t_i^-) - \frac{D}{dt}V(t_i^+) \rangle$$

Now applying Integration by Parts,

$$\langle \frac{DV}{dt}, \frac{DV}{dt} \rangle = \frac{d}{dt}\left(\langle V(t), \frac{D}{dt}V(t) \rangle \right) - \langle V(t), \frac{D^2}{dt^2}V(t) \rangle$$

$$\begin{aligned}\sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \langle \frac{DV}{dt}, \frac{DV}{dt} \rangle dt &= \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \frac{d}{dt}\left(\langle V(t), \frac{D}{dt}V(t) \rangle \right) - \int_0^a \langle V(t), \frac{D^2}{dt^2}V(t) \rangle dt \\ &= \sum_{i=0}^{k} \langle V(t_i), \frac{D}{dt}V(t_i) \rangle \Big|_{t=t_i^+}^{t=t_{i+1}^-} - \int_0^a \langle V(t), \frac{D^2}{dt^2}V(t) \rangle dt\end{aligned}$$

□

Corollary 5.1. *Now let V be a piecewise C^∞ vector field along $\gamma(t)$. $V(0) = V(a) = 0$. $v, w \in T_\gamma \Omega_{p,q}$. If $\gamma(t)$ is a geodesic, then it is a critical point of*

$$E : \Omega_{p,q} \to \mathbb{R}$$

so

$$\begin{aligned}\frac{1}{2}\frac{d^2}{ds^2}E(0) &= \frac{1}{2}\text{Hess}(E)(\gamma)(v,w) \\ &\overset{(47)}{=} \int_0^a \left(\langle \frac{Dv}{dt}, \frac{Dw}{dt} \rangle - R(\gamma', v, \gamma', w) \right) dt\end{aligned}$$

We look at one example.

Proposition 5.7. *Let M^n be an orientable Riemannian manifold with positive curvature and even dimension. Let γ be a closed geodesic in M, i.e., γ is an immersion of the circle $\mathbb{S}^1$ in M that is geodesic at all of its points. Then γ is homotopic to a closed curve whose length is strictly less than that of γ.*

Proof. Let $\gamma : [0, L] \to M$ be a closed geodesic parameterized by arc length, where $L = \text{Length}(\gamma)$.

1. The normal bundle $N(\gamma)$ consists of vectors orthogonal to $\gamma'(t)$ along γ. Since $\dim M =$ n is even, then using γ is 1-dim, we know $N(\gamma)$ has odd $n-1$ dimension. Now consider the parallel transport along γ

$$P_\gamma : T_{\gamma(0)}M^\perp \to T_{\gamma(0)}M^\perp$$

which is an element of $SO(n-1)$. In odd dimensions, every orientation-preserving orthogonal transformation P_γ has at least 1 eigenvalue 1. Indeed, this is because the characteristic polynomial of P_γ has real coefficients, so in odd dimensions, there exists at least one real eigenvalue. Since P_γ preserves orientation hence has determinant 1, the real eigenvalue must be 1. Now we pick $v \in T_{\gamma(0)}M^\perp$ as eigenvector of P_γ with eigenvalue 1. We parallel transport v along γ to define

$$V(t) := P_{\gamma|_{[0,t]}}(v)$$

By our construction, $V(t)$ is parallel so

$$\nabla_{\gamma'(t)}V(t) = 0 \qquad \forall\ t$$

and using closedness of γ, $V(L) = V(0) = v$. Also using the normal bundle, $V(t)$ is orthogonal to $\gamma'(t)$ for all t.

2. Define a variation of γ by:

$$f:(-\varepsilon,\varepsilon)\times[0,L]\to M \qquad (s,t)\mapsto f(s,t)\equiv f_s(t):=\exp_{\gamma(t)}(sV(t))$$

for $\epsilon>0$ is small. We compute

$$\begin{aligned} f(s,0)=f(s,L)&=\exp_{\gamma(0)}(sv)\\ V(0)&=\left.\frac{d}{ds}\right|_{s=0} f_s(0)=d(\exp_{\gamma(0)})_0(v)=v=V(L)\\ \frac{D}{ds}\frac{\partial f}{\partial s}(0,0)&=\frac{D}{ds}\frac{\partial f}{\partial s}(0,L)=0\end{aligned}$$

The energy functional is

$$E(f_s)=\int_0^L \|f_s(t)\|^2dt$$

By above computations, the second variation at $s=0$ thus shares the same formula as the proper variation case (47)

$$E''(0)=\int_0^L \left(\|\nabla_{\gamma'}V\|^2-\langle R(V,\gamma')\gamma',V\rangle\right)dt,$$

where R is the Riemann curvature tensor. Since V is parallel ($\nabla_{\gamma'}V=0$), this simplifies to

$$E''(0)=-\int_0^L \langle R(V,\gamma')\gamma',V\rangle dt.$$

3. By positive curvature, $\langle R(V,\gamma')\gamma',V\rangle>0$ for all t. Hence

$$E''(0)=-\int_0^L \underbrace{\langle R(V,\gamma')\gamma',V\rangle}_{>0}dt<0.$$

Since $E''(0)<0$, there exists $s>0$ such that $E(f(s,\cdot))<E(\gamma)$. For small s, the length $\text{Length}(f(s,\cdot))$ satisfies

$$\text{Length}(f(s,\cdot))\le\sqrt{2E(f(s,\cdot))}<\sqrt{2E(\gamma)}=\text{Length}(\gamma).$$

Thus, $f(s,\cdot)$ is a closed curve homotopic to γ with strictly shorter length.

□

We have another example.

Proposition 5.8. *Let N_1, N_2 be two closed disjoint submanifolds of a compact Riemannian manifold,*

1. *The distance between N_1 and N_2 is attained by a geodesic γ perpendicular to both N_1 and N_2.*

   Proof. The proof adapts the argument in Proposition 5.2 to both submanifolds N_1 and N_2.

   (a) Since M is compact and N_1,N_2 are closed, the distance

   $$d(N_1,N_2)=\inf\{d(p,q)\mid p\in N_1,q\in N_2\}$$

   is attained by some $p_0\in N_1$ and $q_0\in N_2$. Let $\gamma:[0,1]\to M$ be a minimizing geodesic from p_0 to q_0, parameterized by arc length.

   (b) Suppose $\gamma'(0)$ is not orthogonal to $T_{p_0}N_1$. Then there exists $v\in T_{p_0}N_1$ with $\langle\gamma'(0),v\rangle\neq0$. Construct a variation $\gamma_s(t)$ where $\gamma_s(0)$ moves along a curve $\alpha(s)\subset N_1$ with $\alpha'(0)=v$, while $\gamma_s(1)=q_0$. The first variation of length is

   $$\left.\frac{d}{ds}\right|_{s=0} L[\gamma_s]=\langle v,\gamma'(0)\rangle\neq0.$$

   This implies shorter paths exist for small $|s|$, contradicting the minimality of γ. Hence, $\gamma'(0)\perp T_{p_0}N_1$.

   (c) Similarly, suppose $\gamma'(1)$ is not orthogonal to $T_{q_0}N_2$. Take $w\in T_{q_0}N_2$ with $\langle\gamma'(1),w\rangle\neq0$. Define a variation $\gamma_s(t)$ where $\gamma_s(1)$ moves along a curve $\beta(s)\subset N_2$ with $\beta'(0)=w$, while $\gamma_s(0)=p_0$. The first variation

   $$\left.\frac{d}{ds}\right|_{s=0} L[\gamma_s]=\langle w,\gamma'(1)\rangle\neq0,$$

   again contradicting minimality. Thus, $\gamma'(1)\perp T_{q_0}N_2$.

Therefore, the minimizing geodesic γ is perpendicular to both N_1 and N_2 at its endpoints p_0 and q_0. $\square$

2. *For any orthogonal variation $h(t,s)$ of γ, with $h(0,s) \in N_1$ and $h(\ell,s) \in N_2$, we have the expression for formula of the second variation*

$$\frac{1}{2}E''(0) = I_\ell(V,V) + \langle V(\ell), S^{(2)}_{\gamma'(\ell)}V(\ell)\rangle - \langle V(0), S^{1}_{\gamma'(0)}(V(0))\rangle$$

where V is the variational vector and $S^{(i)}_{\gamma'}$ is the linear map associated to the second fundamental form of N_i in the direction of γ', $i = 1,2$.

Proof. Let $\gamma : [0,\ell] \to M$ be a minimizing geodesic between N_1 and N_2, parameterized by arc length. Let $h(t,s)$ be a smooth variation of γ such that

- $h(t,0) = \gamma(t)$ for all t,
- $h(0,s) \in N_1$ and $h(\ell,s) \in N_2$ for all s.

Let $V(t) = \frac{\partial h}{\partial s}\big|_{s=0}$ be the variational vector field, which is orthogonal to γ'.

(a) The energy functional is $E(s) = \int_0^\ell \langle \frac{\partial h}{\partial t}, \frac{\partial h}{\partial t}\rangle\, dt$. Using formula (46), its second derivative at $s = 0$ is

$$\frac{1}{2}E''(0) = \int_0^\ell \left(\langle \frac{D}{dt}V, \frac{D}{dt}V\rangle - \langle R(\gamma',V)\gamma',V\rangle\right) dt + \left\langle \frac{D}{dt}V, V\right\rangle\Big|_0^\ell$$

Indeed, using orthogonality of V with γ', first few terms vanish. After integrating $\int_0^\ell \langle \frac{D}{dt}V, \frac{D}{dt}V\rangle\, dt$ by parts, we obtain

$$\int_0^\ell \left\langle \frac{D}{dt}V, \frac{D}{dt}V\right\rangle dt = \left\langle \frac{D}{dt}V, V\right\rangle\Big|_0^\ell - \int_0^\ell \left\langle \frac{D^2}{dt^2}V, V\right\rangle dt.$$

For a geodesic γ, $\frac{D^2}{dt^2}V = R(\gamma',V)\gamma'$, so substituting back yields the result.

(b) At $t = 0$ and $t = \ell$, the variational vector field V is tangent to N_1 and N_2, respectively. The second fundamental forms $S^{(1)}$ and $S^{(2)}$ encode the normal curvature of N_1 and N_2 via

- At $t = 0$: $\langle \frac{D}{dt}V(0), V(0)\rangle = \left\langle S^{(1)}_{\gamma'(0)}V(0), V(0)\right\rangle$.
- At $t = \ell$: $\langle \frac{D}{dt}V(\ell), V(\ell)\rangle = \left\langle S^{(2)}_{\gamma'(\ell)}V(\ell), V(\ell)\right\rangle$.

(c) Substituting the boundary terms into $E''(0)$

$$\frac{1}{2}E''(0) = I_\ell(V,V) + \left\langle V(\ell), S^{(2)}_{\gamma'(\ell)}V(\ell)\right\rangle - \left\langle V(0), S^{(1)}_{\gamma'(0)}V(0)\right\rangle,$$

where $I_\ell(V,V) = \int_0^\ell (\langle \nabla_t V, \nabla_t V\rangle - \langle R(\gamma',V)\gamma',V\rangle)\, dt$.

$\square$

Proposition 5.9. *Let $\tilde{M}$ be a complete simply connected Riemannian manifold, with curvature $K \le 0$. Let*

$$\gamma : (-\infty,\infty) \to \tilde{M}$$

be a normalized geodesic and let $p \in \tilde{M}$ be a point which does not belong to γ. Let

$$d(s) := d(p,\gamma(s))$$

1. *Consider the minimizing geodesic*

$$\sigma_s : [0,d(s)] \to \tilde{M}$$

joining p to $\gamma(s)$, that is,

$$\sigma_s(0) = p, \qquad \sigma_s(d(s)) = \gamma(s)$$

Consider the variation

$$h(t,s) = \sigma_s(t)$$

Then

(a) $\frac{1}{2}E'(s) = \frac{3}{2}\langle \gamma'(s), \sigma_s'(d(s))\rangle$.

Proof. We assume $\langle \frac{\partial h}{\partial t}, \frac{\partial h}{\partial t}\rangle(t,s)$ is integrable w.r.t. t and integrates to $H(t,s)$, then formally

$$\begin{aligned}
E(s) &= \int_0^{d(s)} \langle \frac{\partial h}{\partial t}, \frac{\partial h}{\partial t}\rangle dt = H((d(s),s) - H(0,s) \\
E'(s) &= H_t(d(s),s)d'(s) + H_s(d(s),s) - H_s(0,s) \\
&= \langle \frac{\partial h}{\partial t}, \frac{\partial h}{\partial t}\rangle(d(s),s)d'(s) + 2\int_0^{d(s)} \langle \frac{D}{ds}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial t}\rangle dt \\
&= \langle \sigma_s'(d(s)), \sigma_s'(d(s))\rangle d'(s) + 2\int_0^{d(s)} \langle \frac{D}{ds}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial t}\rangle dt \\
&= \langle \gamma'(s), \sigma_s'(d(s))\rangle + 2\int_0^{d(s)} \langle \frac{D}{ds}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial t}\rangle dt \\
\int_0^{d(s)} \langle \frac{D}{ds}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial t}\rangle dt &= \int_0^{d(s)} \langle \frac{D}{dt}\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\rangle dt \\
&= \int_0^{d(s)} \frac{d}{dt}\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\rangle - \langle \frac{\partial h}{\partial s}, \frac{D}{dt}\frac{\partial h}{\partial t}\rangle dt \\
&= \langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\rangle(d(s)) - \langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\rangle(0) \qquad \text{using } \sigma_s \text{ is minimizing geodesic} \\
&= \langle \gamma'(s), \sigma_s'(d(s))\rangle
\end{aligned}$$

Here I suspect the original answer is wrong and should lead to

$$\frac{1}{2}E'(s) = \frac{3}{2}\langle \gamma'(s), \sigma_s'(d(s))\rangle$$

□

(b) $\frac{1}{2}E''(s) > 0.$

Proof. From the first variation, it suffices to differentiate

$$\langle \gamma'(s), \sigma_s'(d(s))\rangle.$$

Differentiating again:

$$\frac{d}{ds}\langle \gamma'(s), \sigma_s'(d(s))\rangle = \langle \gamma'(s), \frac{D}{ds}\sigma_s'(d(s))\rangle,$$

since $\gamma''(s) = 0$. The term $\frac{D}{ds}\sigma_s'(d(s))$ is computed via the Jacobi field $J(t) = \frac{\partial h}{\partial s}(t,s)$ along σ_s. By the second variation formula

$$\frac{1}{2}E''(s) = I(J,J) + \langle \gamma'(s), \frac{D}{ds}J(d(s))\rangle,$$

where $I(J,J)$ is the index form. Since $K \leq 0$ and σ_s is minimizing, $I(J,J) > 0$ unless J is parallel. But J is not parallel thus

$$E''(s) > 0.$$

□

2. *Then conclude*

(a) from Step 1 (a) that s_0 is a critical point of d iff $\langle \gamma'(s_0), \sigma_s'(d(s_0))\rangle = 0$

Proof. Indeed, $\langle \gamma'(s_0), \sigma_s'(d(s_0))\rangle = 0$ iff $E'(s_0) = 0$. But also note

$$E(s) = d(s) = d(p, \gamma(s))$$

as energy of the minimizing geodesic σ_s. Thus differentiation and substitution yields the result. □

(b) from Step 1 (b) that d has a unique critical point, which is a minimum.

Proof. $E''(s) > 0$, and since $E(s) = d(s)$, it follows that $d''(s) > 0$. This implies $d(s)$ is strictly convex. On a complete simply connected manifold with $K \leq 0$, the distance function $s \mapsto d(p, \gamma(s))$ is proper (coercive) because $\gamma(s)$ escapes to infinity as $|s| \to \infty$. This is done by Hadamard's Theorem 3.3. A strictly convex, coercive function on $\mathbb{R}$ has exactly one critical point, which is a global minimum. Hence, $d(s)$ attains a unique minimum at its critical point s_0. □

3. *From step 2, it follows that*

   Theorem 5.1. *If $\tilde{M}$ is complete, simply connected and has curvature $K \leq 0$, then a point off the geodesic γ of $\tilde{M}$ can be connected by a unique geodesic perpendicular to γ.*

   Show by examples that the condition on the curvature and the condition of the simple connectivity are essential.

   Proof. (a) If $K \leq 0$ is dropped, let $\tilde{M}$ be a sphere, let p be north pole and γ be its equator, then there are infinitely many perpendicular lines of minimal length.

   (b) If simple connectedness is dropped, let $\tilde{M}$ be an infinite cylinder and p be any point. Let γ be a straight line passing through the antipodal point of p. Then there are two lines reaching minimal length.

   □

5.3 Bonnet-Myers Theorem

Now we look at some clever applications.

Theorem 5.2 (Bonnet-Myer)**.** *Suppose we have some complete Riemannian manifold (M^n, g). Suppose that there exists $r > 0$ s.t. either of the following is satisfied*

1. *Myer. For any $p \in M$ and $v \in T_pM$ unit, the Ricci*

$$\mathrm{Ric}_p(v, v) \geq \frac{1}{r^2}$$

2. *Bonnet. Or for any $p \in M$, for any 2−plane σ in T_pM, the sectional curvature*

$$K(p, \sigma) \geq \frac{1}{r^2}$$

Then M is compact, and

$$\mathrm{diam}(M, g) := \sup_{p,q \in M} d(p, q) \leq \pi r$$

Proof. It suffices to prove for Myer.

1. By contradiction, suppose that $\mathrm{diam}(M, g) > \pi r$. Then there exists two points p, $q \in M$ s.t.

$$d(p, q) =: \ell > \pi r$$

   Since the manifold is complete, there exists also a normalized geodesic that connects these two points p, q. Then our γ

$$\gamma : [0, \ell] \to M \qquad \gamma(0) = p \qquad \gamma(\ell) = q$$

   We want to apply the second variation formula so we want to find a variation that gives us a contradiction.

2. We construct a variation by imposing a vector field along this curve. We define it by using the O.N. frame of the tangent space. Let $\{e_1, \cdots, e_n\}$ be ONB of T_pM where $e_n := \gamma'(0)$. Then parallel transport them. Let $e_i(t)$ be the parallel transport of e_i along γ. We define our variational field (the one that saturates the sphere)

$$\begin{aligned} &V_i(t) := \sin(\frac{\pi t}{\ell})e_i(t) \qquad i = 1, \cdots, n-1 \\ &V_i(0) = V_i(\ell) = 0 \end{aligned}$$

   Thus we have a proper variation associated to the Variational Field, i.e., V_i are the variational field of f_i of γ.

3. Now let $E_i(s)$ be the energy of $f_i(s, t)$, the proper variation associated to V_i. Let's compute

$$\begin{aligned} E_i(s) &:= \int_0^\ell |\frac{\partial f_i}{\partial t}(s, t)|^2 \geq \frac{1}{\ell}\ell(f_i)^2 \qquad \text{Cauchy Schwarz} \\ &\geq \frac{1}{\ell}\ell(\gamma)^2 \qquad \text{since } \gamma \text{ is geodesic} \\ &= E(\gamma) = E(0) \end{aligned}$$

Now that γ is geodesic and V proper, we know $E_i'(0) = 0$ and $E_i''(0) \geq 0$ this is indeed a minimum. By the second variation formula (47)

$$\begin{aligned}
\frac{1}{2}E_i''(0) &= -\int_0^\ell \left(\langle \frac{D^2 V_i}{dt^2}, V_i \rangle + R(\gamma', V_i, \gamma', V_i) \right) dt \qquad \text{no boundary terms because all are piecewise smooth} \\
&= \int_0^\ell \left(\frac{\pi^2}{\ell^2} \sin^2(\frac{\pi t}{\ell}) - \sin^2(\frac{\pi t}{\ell}) R(e_n, e_i, e_n, e_i) \right) dt \qquad \forall\, i = 1, \cdots, n-1 \\
\frac{1}{n-1}\sum_{i=1}^{n-1} \frac{1}{2}E_i''(0) &= \int_0^\ell \left(\frac{\pi^2}{\ell^2} \sin^2(\frac{\pi t}{\ell}) - \sin^2(\frac{\pi t}{\ell}) \mathrm{Ric}_p(e_n, e_n) \right) dt \\
&\leq \int_0^\ell \left(\frac{\pi^2}{\ell^2} - \frac{1}{r^2} \right) \sin^2(\frac{\pi t}{\ell}) dt < 0 \qquad \text{since } \pi r < \ell
\end{aligned}$$

Now this contradicts $E_i''(0) = 0$.

Since M is totally bounded and complete, M is compact. □

Remark 5.2.
- *Bonnet's assumption implies Myer's assumption. Indeed*

$$\mathrm{Ric}_p(v, v) = \frac{1}{n-1}\sum_{i=1}^{n-1} R(e_i, v, e_i, v) = \frac{1}{n-1}\sum_{i=1}^{n-1} K(p, Span(e_i, v)) \qquad \{e_1, \cdots, e_{n-1}, v\} \text{ O.N.B. of } T_pM$$

- *The inequalities are sharp.* $\mathbb{S}^n$ *satisfies*

$$\mathrm{Ric}_p(v, v) = \frac{1}{r^2}, \qquad K(p, \sigma) = \frac{1}{r^2}, \qquad \mathrm{diam} = \pi r$$

Theorem 5.3 (Cheng-Shiohama)**.** *In fact*

$$\begin{aligned} \mathrm{Ric}_p(v, v) &\geq \frac{1}{r^2} \\ \mathrm{diam} &= \pi r \end{aligned}$$

implies that

$$(M^n, g) \cong (\mathbb{S}^n, g_{can})$$

- *It is necessary that* K, Ric *are bounded away from* 0*! For example*

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}$$

is complete but

$$K = \frac{4}{(1 + x^2 + y^2)^2} > 0, \qquad \inf_{p \in S} K(p) = 0$$

and S is indeed not compact!

Corollary 5.2. *If (M^n, g) is a complete Riemannian manifold with* $\mathrm{Ric}_p(v, v) \geq \frac{1}{r^2}$*. Then the first fundamental group $\pi_1(M)$ is finite.*

Proof. Let $(\tilde{M}, \tilde{g})$ be the universal cover. Then $\tilde{\mathrm{Ric}}_p \geq \frac{1}{r^2}$. By Myer's Theorem 5.2, $\tilde{M}$ is compact, so for any $p \in M$, $\pi^{-1}(p)$ is a discrete set in a compact manifold, so that its finite. $|\pi_1(M)| = \#\pi^{-1}(p) < \infty$. □

Example 5.1. *Introduce a complete Riemannian metric on* $\mathbb{R}^2$*. Prove that*

$$\lim_{r \to \infty} \left(\inf_{x^2 + y^2 \geq r^2} K(x, y) \right) \leq 0$$

where $(x, y) \in \mathbb{R}^2$ and $K(x, y)$ is the Gaussian curvature of the given metric at (x, y).

Proof. Assume for contradiction that

$$\lim_{r \to \infty} \left(\inf_{x^2 + y^2 \geq r^2} K(x, y) \right) > 0$$

Then, there exists $\epsilon > 0$ and $R > 0$ such that

$$K(x, y) \geq \epsilon \quad \text{for all} \quad x^2 + y^2 \geq R^2$$

Consider the closed subset $M = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \geq R^2\}$ with the induced metric. Since $(\mathbb{R}^2, g)$ is complete, M is also a complete Riemannian manifold. By construction, $K \geq \epsilon > 0$ on M. Now the Bonnet-Myers Theorem 5.2 states that a complete connected Riemannian manifold with Ricci curvature bounded below by some strictly positive constant is compact and has finite diameter. In dimension 2, the Ricci curvature coincides with the Gaussian curvature. Thus, M must be compact. However, M is homeomorphic to $\mathbb{R}^2 \setminus B_R(0)$, which is non-compact as it contains unbounded sequences (e.g., $(n, 0)$ for $n > R$). We reach a contradiction. Therefore,

$$\lim_{r\to\infty} \left(\inf_{x^2+y^2 \geq r^2} K(x,y) \right) \leq 0$$

□

5.4 Synge-Weinstein Theorem

Theorem 5.4 (Weinstein)**.** *Let (M^n, g) be a compact oriented Riemannian manifold with positive sectional curvature. Suppose*

$$f : (M,g) \to (M,g)$$

is an isometry s.t. f preserves(reverses) the orientation if $n = \dim M$ is even (odd). Then f has a fixed point, i.e., there exists $p \in M$ s.t. $f(p) = p$.

Proof. Suppose that f has no fixed points. Consider

$$h : M \to \mathbb{R} \qquad q \mapsto d(q, f(q))$$

continuous function on M. Since M is compact, there exists $p \in M$ s.t.

$$h(p) = \min_{q\in M} h(q)$$

i.e.,

$$\ell := d(p, f(p)) = \min_{q\in M} d(q, f(q)) > 0$$

Since M is compact, M is complete, as usual we take the normalizing geodesic between them, i.e., there exists γ normalized geodesic s.t.

$$\gamma(0) = p \qquad \gamma(\ell) = f(p)$$

Now consider the two velocity vectors $\gamma'(0)$ and $\gamma'(\ell)$. We need two claims that gives a contradiction.

1. Claim 1. For f as in our assumption

$$df_p : T_pM \to T_{f(p)}M$$

sends $\gamma'(0) \mapsto \gamma'(\ell)$.

Proof. Indeed, let $p' := \gamma(t')$. We look at the distances between p' and $f(p')$.

$$\begin{aligned} d(p', f(p')) &\leq d(p', f(p)) + d(f(p), f(p')) \\ &= d(p', f(p)) + d(p, p') \qquad \text{using } f \text{ is an isometry} \\ &\leq \ell - t' + t' = \ell = d(p, f(p)) \end{aligned}$$

But on the other hand, $d(p, f(p))$ is the minimum. Thus we have

$$d(p', f(p')) = d(p, f(p)) = d(p', f(p)) + d(f(p), f(p')) = \ell(\gamma|_{[t',\ell]}) + \ell((f\circ\gamma)|_{[0,t']})$$

Hence γ and $f \circ \gamma$ are normalized geodesics. Thus

$$\gamma'(\ell) = (f \circ \gamma)'(0) = df_p(\gamma'(0))$$

□

2. Claim 2. There exists a parallel vector field $V(t)$ along $\gamma(t)$ s.t. $|V(t)| = 1$ and $\langle V(t), \gamma'(t)\rangle = 0$.

Proof. Let $P : T_pM \to T_{f(p)}M$ be the parallel transport along γ (P is orientation preserving).

$$T_pM \underset{df_p}{\overset{P}{\to}} T_{f(p)}M$$

$P(\gamma'(0)) = \gamma'(\ell)$ since $\gamma'(t)$ is parallel to $\gamma(t)$. Define

$$A := P^{-1} \circ df_p : T_pM \to T_pM$$

Then $A \in O(n)$ and $\det(A) = (-1)^n$. In particular -1 if n odd and 1 if n even. Note

$$\begin{aligned} A(\gamma'(0)) &= P^{-1}(df_p(\gamma'(0))) \\ &= P^{-1}(\gamma'(\ell)) \qquad \text{use Claim 1} \\ &= \gamma'(0) \qquad \text{so } \gamma'(0) \text{ is an eigenvector with eigenvalue 1} \end{aligned}$$

Let W be the orthogonal complement of $\mathbb{R}\gamma'(0)$ in T_pM, i.e.

$$T_pM = \mathbb{R}\gamma'(0) \oplus W$$

Consider

$$B := A|_W : W \to W \cong \mathbb{R}^{n-1} \qquad B \in O(n-1) \qquad \det(B) = (-1)^n$$

Recall that if $C \in O(m)$ and 1 is not an eigenvalue, then $\det(C) = (-1)^n$. So if $C \in O(m)$ s.t. $\det(C) = (-1)^{m+1}$, then 1 is an eigenvalue. Thus our B has to have 1 as an eigenvalue. Now let $v \in W$ be the associated eigenvector and take $|v| = 1$

$$Bv = v$$

Let $V(t)$ be the parallel transport of v along $\gamma(t)$. Then since the parallel transport doesn't change the norms, and since

$$\langle v, \gamma'(0)\rangle = 0 \qquad \langle v, v\rangle = 1$$

Then

$$\langle V(t), \gamma'(t)\rangle = 0 \qquad \langle V(t), V(t)\rangle = 1$$

Thus

$$\begin{aligned} (P^{-1} \circ df_p)(V(0)) &= (P^{-1} \circ df_p)(v) = Av \\ &= v = V(0) \end{aligned}$$

Hence in particular

$$df_p(V(0)) = P(V(0)) = V(\ell)$$

Finally we define

$$h : (-\varepsilon, \varepsilon) \times [0, \ell] \to M \qquad (s,t) \mapsto h(s,t) := \exp_{\gamma(t)}(sV(t))$$

In particular each

$$s \mapsto h(s,t)$$

is a geodesic. Let

$$\begin{aligned} \alpha(s) &:= h(s,0) = \exp_p(sv) \\ \beta(s) &:= h(s,\ell) = \exp_{f(p)}(s\ df_p(v)) \end{aligned}$$

These are themselves geodesics. As

$$\beta(0) = f(p) \qquad \beta'(0) = V(\ell) = df_p(V(0))$$

In fact we conclude here that

$$\beta = f \circ \alpha$$

Now we consider a curve for fixed s. For fixed $s \in (-\varepsilon, \varepsilon)$, consider

$$h_s : [0, \ell] \to M$$

a smooth curve from $\alpha(s)$ to $\beta(s)$. Now the energy writes

$$\begin{aligned} E(s) = \int_0^\ell |\frac{\partial h}{\partial t}(s,t)|^2 dt &\overset{\text{Cauchy Schwarz}}{\geq} \frac{1}{\ell}\left(|\frac{\partial h}{\partial t}(s,t)|dt\right)^2 = \frac{1}{\ell}\ell(h_s)^2 \\ &\geq \frac{1}{\ell} d(\alpha(s), f \circ \alpha(s))^2 \geq \frac{1}{\ell} d(p, f(p))^2 = \ell = E(0) \end{aligned}$$

Thus we've built this nice vector field. $\square$

Since the original p and $f(p)$ have shortest distance, all variations get longer. Here is where we'll get our contradiction. We use positive curvature. We push off the curve along a parallel field, then necessarily has decreasing energy. For negative curvature, pushing off the geodesic increases energy. Now to make this rigorous. What have we shown? Back to our setup: we have γ a geodesic, and

$$\langle V(t), \gamma'(t)\rangle = 0 \qquad \forall\, t \implies E'(0) = 0$$

and because

$$E(s) \geq E(0) \qquad \forall\, s$$

Thus $E(0)$ must be a local minimum, so $E''(0) \geq 0$. But we have the second variation formula (46)

$$\begin{aligned}
\frac{1}{2}E''(0) &= -\int_0^\ell \langle \frac{D^2V}{dt^2} + R(\gamma', V)\gamma', V\rangle dt + \langle \frac{D^2h}{ds^2}(0,\ell), \gamma'(\ell)\rangle - \langle \frac{D^2h}{ds^2}(0,0), \gamma'(0)\rangle \\
&\quad + \langle \frac{DV}{dt}(\ell), V(\ell)\rangle - \langle \frac{DV}{dt}(0), V(0)\rangle \\
&= -\int_0^\ell \langle R(\gamma', V)\gamma', V\rangle dt + \langle \frac{D^2h}{ds^2}(0,\ell), \gamma'(\ell)\rangle - \langle \frac{D^2h}{ds^2}(0,0), \gamma'(0)\rangle \qquad \frac{DV}{dt} \equiv 0 \text{ since } V \text{ is parallel} \\
&= -\int_0^\ell \langle R(\gamma', V)\gamma', V\rangle dt \qquad \text{because } s \mapsto h(s,t) \text{ is geodesic}
\end{aligned}$$

Since γ', V has length 1, $\gamma' \perp V$, we know $\{V(t), \gamma'(t)\}$ span the $\pi(s)$ 2−plane, whose sectional curvature is strictly negative

$$\frac{1}{2}E''(0) = -\int_0^\ell \langle R(\gamma', V)\gamma', V\rangle dt = -\int_0^\ell k(\pi(t))dt < 0$$

Thus we have a contradiction.

□

Remark 5.3. *This assumption in fact excludes the case of the sphere.*

$$A : \mathbb{S}^n \to \mathbb{S}^n \qquad p \mapsto -p$$

Then this is the opposite of the orientation requirement on the isometry, i.e., A is orientation preserving if n is odd and orientation reversing if n is even.

Corollary 5.3 (Synge)**.** *(M^n, g) compact with positive sectional curvature*

1. *if M orientable, n even, then $\pi_1(M) = 1$*
2. *if n odd, M is orientable.*

Proof. 1. For universal cover, $\tilde{M}$ is complete with $K \geq c > 0$, then by Myers 5.2 $\tilde{M}$ is compact. We want to show $M = \tilde{M}$. If not,

$$\tilde{M} \to M$$

choose $\varphi \neq \text{id}$, $\varphi : \tilde{M} \to \tilde{M}$ transformation (so φ has no fixed points). By M orientable, φ preserves orientation. But this contradicts to Weinstein Theorem 5.4.

2. If M is not orientable, there exists orientation double cover

$$\tilde{M} \to M$$

now $\phi : \tilde{M} \to \tilde{M}$ is orientation reversing without fixed points, so this is contradiction to Weinstein 5.4.

□

Remark 5.4. *$\mathbb{R}P^n = \mathbb{S}^n/\{antipodal\}$ is orientable iff n is odd.*

5.5 Index Forms

Give (M, g) a compact manifold, p, $q \in M$. Pick

$$\gamma : [0, a] \to M$$

normalized geodesic and $|\gamma'| = 1$. We can look at other curves connecting the two points. Recall

$$\Omega_{p,q} := \{c : [0, a] \to M \mid \text{piecewise } C^\infty \ c(0) = p,\ c(a) = q\}$$

Now we look at the energy functional on this space

$$E : \Omega_{p,q} \to \mathbb{R} \qquad c \mapsto \int_0^a |\frac{dc}{dt}|^2 dt$$

We know that γ is a critical point of E, i.e., for every

$$V \in T_\gamma\Omega_{p,q} = \{V \text{ piecewise } C^\infty \text{ vector field along } \gamma,\ V(0) = V(a) = 0\}$$

we have

$$dE_\gamma(V) = 0$$

The Hessian of E at the point γ is

$$\text{Hess}E_\gamma(V, W) = 2I_\gamma(V, W) \qquad \text{index form}, \qquad \forall\ V,\ W \in T_\gamma\Omega_{p,q}$$

Definition 5.6 (Index Form)**.** *The index form is*

$$I_\gamma(V, W) := \int_0^a \langle\frac{DV}{dt}, \frac{DW}{dt}\rangle - R(\gamma', V, \gamma', W)dt$$

In general, suppose W is finite dimensional vector space, and

$$B : W \times W \to \mathbb{R} \qquad \text{symmetric bilinear}$$

We define

Definition 5.7 (Index of B)**.** *Index of B is $\dim(W_-)$ where $W_- \subset W$ is a maximal subspace s.t. $B|_{W_-}$ is negative definite.*

1. *Null space of B is $V_0 = \{v \in W \mid B(v, w) = 0 \quad \forall\ w \in W\}$. Nullity($B$) is $\dim V_0$.*
2. *B is non degenerate if Nullity(B) $= 0$.*

Theorem 5.5. *Let $V \in T_\gamma\Omega_{p,q}$.*

1. *$V \in$ Null Space of I_γ iff V is a Jacobi Field.*
2. *Nullity(I_γ) $> 0 \iff q = \gamma(a)$ is a conjugate point of $p = \gamma(0)$ along γ and*

$$\textit{Nullity}(I_\gamma) = \dim\{\textit{Jacobi Fields } V \textit{ along } \gamma \textit{ s.t. } V(0) = V(a) = 0\} < \infty$$

3. *Furthermore,*

$$\textit{Index}(I_\gamma) = \#\{\textit{conjugate point } \gamma(t), \textit{ for } 0 < t < a\} \qquad \textit{counted with multiplicity}$$

5.5.1 Morse Theory

Some facts: (M, g), p, q generic, are not conjugate.

Theorem 5.6. *For $(\mathbb{S}^n, g)$ any metric, if p, q are not conjugate, then there exists ∞-many geodesic connecting them, i.e. the energy functional*

$$E : \Omega_{p,q} \to \mathbb{R}$$

has ∞-many critical points.

This is done in topology. Idea is to take X compact manifold with complicated topology. A Morse function

$$f : X \to \mathbb{R}$$

must have many critical points.

Definition 5.8 (Morse Function)**.** *A Morse function is s.t.* Hess$f$ *is not degenerate at critical points, so there's no nullities.*

If $f : X$ Morse function with n_i critical points of index i, then f is obtained from CW decomposition with n_k k-cells. Once we have the space with this decomposition, we have homology of X, $H_k(X)$. Once we have homology sometimes we can reverse the thing.

Example 5.2. *A Morse Function on* $\mathbb{T}^2$ *has at least* 1 *index* 0 *critical point, at least* 2 *index* 1 *critical points, and* 1 *index* 3 *critical point.*

Proof Sketch. Notice

$$E : \Omega_{p,q} \to \mathbb{R}$$

is Morse exactly when p, q are not conjugate. And

$$H_i(\Omega_{p,q}, \mathbb{S}^n) = \begin{cases} \mathbb{Z} & (n-1) \mid i \\ 0 & \text{otherwise} \end{cases}$$

Hence E has a critical point of index i for every $(n-1) \mid i$. In particular there's infinitely many. □

6 Lorentizan Geometry

Definition 6.1 (Lorentizan Manifold)**.** *A Lorentizan Manifold (M, g) is a differentiable manifold of dimension $1+n$ endowed with a Lorentizan metric. A Lorentizan metric is a differentiable assignment of a symmetric, non-degenerate, bilinear form g_p with signature $(-,+,\cdots,+)$ in T_pM for any $p \in M$.*

Definition 6.2 (SpaceTime)**.** *A spacetime is a non-orientiable Lorentizan Manifold of dimension $1+3$.*

Remark 6.1. *Not all manifolds admit Lorentizan metric. This has to do with the fact that, M admits Lorentizan metric iff M admits a non-vanishing vector field (The vector field that gives the direction of time). Hence M is either non-compact, or compact with $\chi(M) = 0$ Euler Characteristic.*

Now g_p is not positive definite, and for any $p \in M$,

$$(T_pM, g_p) \underset{\text{isometric}}{\cong} (\mathbb{R}^{1+n}, -(dx_0)^2 + (dx_1)^2 + \cdots + (dx_n)^2) \qquad \text{Minkowski Spacetime}$$

Since this is an isometry, there exists a basis fo Tangent Space T_pM denotes

$$e_0,\ e_1, \cdots, e_n$$

s.t. the metric g_p at point p equals

$$g_p(e_\alpha, e_\beta) = m_{\alpha\beta} \qquad m_{\alpha\beta} := \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

In particular, for every $v = \sum_\alpha v^\alpha e_\alpha \in T_pM$, we know that

$$g_p(v, v) = -(v^0)^2 + (v^1)^2 + \cdots (v^n)^2$$

Now this imposes a trichotomy on T_pM.

Definition 6.3 (Spacelike, Null, Timelike)**.** *We say that $v \in T_pM$ is*

1. *spacelike if $g_p(v, v) > 0$*
2. *lightlike/null if $g_p(v, v) = 0$*
3. *timelike if $g_p(v, v) < 0$*

The latter two cases $g_p(v, v) \leq 0$ are called causal.

Definition 6.4 (Lightcone)**.** *The vectors satisfying*

$$(v^0)^2 = (v^1)^2 + \cdots (v^n)^2$$

span a double cone $C_p \subset T_pM$. The interior of the cone C_p contains timelike vectors, while the exterior of the cone C_p contains spacelike vectors. In general a Minkowski metric writes

$$-c^2dt^2 + dx^2 + dy^2 + dz^2$$

and the lightcone has slope $\pm c$. Nothing with mass travels to the exterior of the lightcone. Information/light/gravity travels as the speed of light, hence on the cone. Anything with mass stays in the interior of the cone.

Definition 6.5 (Time-Orientation; Future Directed)**.** *A time-orientation of (M, g) is a continuous choice of a component of timelike vectors at $p \in M$ (future directed). A curve*

$$\alpha : I \to M$$

is future-directed if

$$\alpha'(t) \in T_{\alpha(t)}M \qquad \text{is a future-directed timelike vector} \qquad \forall\ t \in I$$

The proper time τ of an observer (timelike curve) is defined to be the parametrization of its timelike curve such that

$$g_{\alpha(t)}(\alpha'(t), \alpha'(t)) = -1 \qquad \forall\ t \in I$$

This is really the arclength parametrization.

Remark 6.2. *Null curves satisfy*

$$g_{\alpha(t)}(\alpha'(t), \alpha'(t)) = 0 \qquad \forall\, t \in I$$

One can still define an affine parametrization s.t.

$$\nabla_{\alpha'(t)}\alpha'(t) = 0$$

for null-geodesics.

Definition 6.6 (Submanifold)**.** *Let N be a submanifold of M. Then N is called*

1. *Spacelike if $g|_{T_xN}$ is positive-definite so $(N, g|_{T_xN})$ is a Riemannian manifold.*
2. *Null if $g|_{T_xN}$ is degenerate, i.e., the first entry is 0. Lightcones are Null Hypersurfaces.*
3. *Timelike if $(N, g|_{T_xN})$ is Lorentizan*

Remark 6.3. *If a submanifold N is Hypersurface, then N is called*

1. *spacelike if normal vector is timelike*
2. *null if normal vector is null*
3. *timelike if normal vector is spacelike*

Definition 6.7 (Causal Future)**.** *Let $S \subset M$, then the Causal future of S*

$$J^+(S) := \{\text{all points in } M \text{ that can be connected to } S \text{ by a future-directed causual curve}\}$$

The meaning of $J^+(S)$ is the part that S can influence. This is the only part that S can send information to. Alternatively,

$$I^+(S) := \{\text{all points in } M \text{ that can be connected to } S \text{ by a future-directed timelike curve}\}$$

Definition 6.8 (Cauchy Surface)**.** *A spacelike hypersurface Σ is a Cauchy Surface if every inextendible causal curve intersects it exactly once. A spacetime with a Cauchy surface is called globally hyperbolic.*

Remark 6.4.

1. *A globally hyperbolic spacetime is always homeomorphic to $\Sigma \times \mathbb{R}$.*
2. *In a globally hyperbolic spacetime, there exists a global time function t s.t.*

   $$\{t = constant\}$$

   are spacelike hypersurface
3. *In a globally hyperbolic spacetime, there exists a timelike geodesic connecting $x, y \in M$, $y \in I^+(x)$.*

Definition 6.9 (Cauchy Development)**.** *Let Σ be a spacelike hypersurface. Then the Cauchy Development of Σ is the biggest globally hyperbolic subset of M that admits Σ as a Cauchy Surface.*

Example 6.1. *Consider the manifold $\mathbb{R}_t \times \mathbb{R}_r$ equipped with the Lorentzian metric*

$$g = -(1+r^2)dt^2 + \frac{1}{1+r^2}dr^2$$

Then

1. *Show the manifold is timelike geodesically complete, i.e., all inextendible timelike geodesics can be defined on all $\mathbb{R}$.*

   Proof. We start by finding the geodesics for the manifold $(\mathbb{R}_t \times \mathbb{R}_r, g)$. We compute, denoting 0 as t coordinate and 1 as r coordinate

   $$\begin{aligned}
   g_{00} &= -(1+r^2) \\
   g_{11} &= \frac{1}{1+r^2} \\
   g_{00,1} &= -2r \\
   g_{11,1} &= -\frac{2r}{(1+r^2)^2} \\
   \Gamma^1_{00} &= \frac{1}{2}g^{11}(-g_{00,1}) = r(1+r^2) \\
   \Gamma^0_{01} = \Gamma^0_{10} &= \frac{1}{2}g^{00}g_{00,1} = \frac{r}{1+r^2} \\
   \Gamma^1_{11} &= \frac{1}{2}g^{11}g_{11,1} = -\frac{r}{1+r^2}
   \end{aligned}$$

Denote τ as parametrization for the geodesic. The geodesic equation with coordinates $(x^0, x^1) = (t, r)$ hence writes

$$\frac{d^2x^0}{d\tau^2} + 2\Gamma^0_{01}\frac{dx^0}{d\tau}\frac{dx^1}{d\tau} = 0$$
$$\frac{d^2t}{d\tau^2} + \frac{2r(\tau)}{1+r^2(\tau)}\frac{dt}{d\tau}\frac{dr}{dt} = 0$$
$$\frac{d^2x^1}{d\tau^2} + \Gamma^1_{00}(\frac{dx^0}{d\tau})^2 + \Gamma^1_{11}(\frac{dx^1}{d\tau})^2 = 0$$
$$\frac{d^2r}{d\tau^2} + r(1+r^2)(\frac{dt}{d\tau})^2 - \frac{r}{1+r^2}(\frac{dr}{d\tau})^2 = 0$$

From there we observe that there is a conserved quantity:

$$E = (1+r^2)\frac{dt}{d\tau} \implies \frac{dt}{d\tau} = \frac{E}{1+r^2}. \tag{48}$$

For timelike geodesics, $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$:

$$-(1+r^2)\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{1+r^2}\left(\frac{dr}{d\tau}\right)^2 = -1.$$

Substituting $\frac{dt}{d\tau}$ from (48):

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - 1 - r^2. \tag{49}$$

This is a harmonic oscillator equation. For $E^2 \geq 1$, the solution is:

$$r(\tau) = \sqrt{E^2-1}\cos(\tau+\phi), \quad \text{where } \phi \text{ is a phase constant.} \tag{50}$$

When $E^2 = 1$, $r(\tau) = 0$ (static geodesic). Substitute (50) into (48):

$$\frac{dt}{d\tau} = \frac{E}{1+(E^2-1)\cos^2(\tau+\phi)}.$$

Integrating gives:

$$t(\tau) = \frac{E}{\sqrt{E^2-1}}\arctan\left(\sqrt{E^2-1}\tan(\tau+\phi)\right) + t_0. \tag{51}$$

As $\tau \to \pm\infty$, $t(\tau)$ grows unboundedly. For $E^2 = 1$, $t(\tau) = \tau + t_0$. The solutions (50) and (51) are smooth and defined for all $\tau \in \mathbb{R}$. The affine parameter τ covers $\mathbb{R}$, and no geodesic encounters a singularity or boundary in finite τ. Thus, all timelike geodesics are complete.

□

2. *Consider the time orientation s.t. ∂_t is future-directed. Is the above metric globally hyperbolic?*

Proof. (a) The metric g is static, and ∂_t is a timelike Killing vector field. The hypersurfaces $\Sigma_c = \{t = c\}$ are spacelike everywhere since their normal vector ∂_t is timelike. The spacetime is strongly causal because the absence of closed causal curves is guaranteed by the staticity and the $\mathbb{R}_t$ factor.

(b) The surfaces $\Sigma_c = \{t = c\}$ are natural candidates. To verify if they are Cauchy, we check if all inextendible causal curves intersect Σ_c exactly once. Null geodesics satisfy $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$. From earlier results:

$$\dot{r} = \pm E, \quad \dot{t} = \frac{E}{1+r^2},$$

with solutions:

$$r(\lambda) = \pm E\lambda + r_0, \quad t(\lambda) = \frac{1}{E}\arctan(E\lambda + r_0) + C.$$

As $\lambda \to \pm\infty$, $t(\lambda) \to \pm\frac{\pi}{2E} + C$. Thus, null geodesics asymptote to finite t-values and do not cross all Σ_c hypersurfaces. For example, a null geodesic with $t(\lambda) \to \frac{\pi}{2E} + C$ as $\lambda \to \infty$ never intersects Σ_c for $c > \frac{\pi}{2E} + C$.

(c) Since inextendible null geodesics do not intersect all Σ_c hypersurfaces, no Σ_c is a Cauchy surface. Hence, the spacetime is not globally hyperbolic.

□

6.1 Null Hypersurface

Definition 6.10 (Null Hypersurface)**.** *A hypersurface $\mathcal{H}$ of M is null if for every $x \in M$, the normal vector $L \in T_xM$ to $T_x\mathcal{H}$ is null, i.e.*

$$g_x(L, L) = 0 \qquad g_x(L, X) = 0 \qquad \forall\ X \in T_x\mathcal{H}$$

Remark 6.5 (Both Normal and Tangent)**.** *Since* $\dim T_x\mathcal{H} = n$*, and*

$$\langle L \rangle^{\perp} = T_x\mathcal{H}$$

and since L is null, we only have one extra direction that is allowed. Necessarily $L \in T_x\mathcal{H}$. Thus L is both normal and tangent to $\mathcal{H}$ itself at $x \in M$.

Remark 6.6. *The integral curves of the null line bundle $\langle L_x \rangle$ for any $x \in \mathcal{H}$ are null geodesics (null generators).*

$$\begin{aligned} g(\nabla_L L, X) &= D_L(g(L, X)) - g(L, \nabla_L X) \qquad \forall\ X \in T\mathcal{H} \\ &= 0 - g(L, \nabla_X L) - g(L, [L, X]) \\ &= -g(L, \nabla_X L) \\ &= -\frac{1}{2} D_X(g(L, L)) \\ &= 0 \qquad \forall\ X \in T\mathcal{H} \end{aligned}$$

Thus $\nabla_L L$ is normal to $\mathcal{H}$, hence spanned by L

$$\nabla_L L = fL$$

we can always rescale $\tilde{L} = kL$ s.t.

$$\nabla_{\tilde{L}} \tilde{L} = 0$$

i.e.

$$\begin{aligned} 0 &= \nabla_{kL} kL = k\nabla_L(kL) = k(k\nabla_L L + L(k)L) \\ 0 &= L(k) + kf \end{aligned}$$

Solving for the ODE yields the result.

Example 6.2. *1. Lightcone*

2. $\{u = constant\}$ for any u s.t. $g(\nabla u, \nabla u) = 0$.

6.2 Einstein Equations

Definition 6.11. *A spacetime in GR is a triple*

$$(N^4, \overline{g}_{\mu\nu}, T_{\mu\nu})$$

where N^4 is a 4− dim manifold, $\overline{g}_{\mu\nu}$ is a Lorentizan metric, $T_{\mu\nu}$ is a $(0, 2)-$spacetime tensor, called the energy-momentum tensor, s.t. they satisfy the Einstein Equation (52).

Where does Einstein Equation come from? Consider the Action

$$S(\overline{g}_{\mu\nu}) = \int_N (\mathcal{L}_G + 8\pi\mathcal{L}_M)$$

where $\mathcal{L}_G$ is Einstein Lagrangian and $\mathcal{L}_M$ is the matter Lagrangian.

$$\mathcal{L}_G := \overline{R}\sqrt{-\det(\overline{g})} \qquad \overline{R} \text{ denotes the scalar curvature}$$

We compute the variation

$$\frac{\delta \mathcal{L}_G}{\delta \overline{g}_{\mu\nu}} = \sqrt{-\det(\overline{g})}\,\overline{G}^{\mu\nu} \qquad \overline{G}^{\mu\nu} := \overline{R}^{\mu\nu} - \frac{1}{2} g^{\mu\nu}\overline{R} \qquad \text{the Einstein Tensor}$$

The variation of the Matter Lagrangian is

$$\frac{\delta \mathcal{L}_M}{\delta \overline{g}_{\mu\nu}} = T^{\mu\nu}\sqrt{-\det(\overline{g})}$$

Thus to find a critical point for the Action S, we obtain the Einstein's Equation (1915)

$$\begin{aligned} \overline{G}^{\mu\nu} &= 8\pi T^{\mu\nu} \\ \overline{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\overline{R} &= 8\pi T_{\mu\nu} \end{aligned} \tag{52}$$

LHS is curvature of the spacetime while RHS is the matter. We wish to understand what kind of equation this is.

Remark 6.7 (Bianchi Identity)**.**

$$\nabla_\mu G^{\mu\nu} = 0$$

According to Einstein this implies

$$\nabla_\mu T^{\mu\nu} = 0$$

So we expect some sort of conservation law.

Lemma 6.1. *Show that the second Bianchi Identity for the Riemann Tensor implies that the Einstein Tensor*

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

is divergence-free, i.e.

$$\nabla^\mu G_{\mu\nu} = 0$$

Proof. Denote ∇ as the Levi-Civita Connection. Bianchi's Second Identity gives

$$\nabla_\alpha R^\gamma{}_{\beta\mu\nu} + \nabla_\mu R^\gamma{}_{\beta\nu\alpha} + \nabla_\nu R^\gamma{}_{\beta\alpha\mu} = 0$$

Contracting once gives

$$\begin{aligned} 0 &= \nabla_\alpha R^\alpha{}_{\beta\mu\nu} + \nabla_\mu R^\alpha{}_{\beta\nu\alpha} + \nabla_\nu R^\alpha{}_{\beta\alpha\mu} \\ &= \nabla_\alpha R^\alpha{}_{\beta\mu\nu} - \nabla_\mu R_{\beta\nu} + \nabla_\nu R_{\beta\mu} \end{aligned}$$

Contracting again gives

$$\begin{aligned} 0 &= \nabla^\alpha R_{\alpha\beta\mu}{}^\beta - \nabla_\mu R^\beta_\beta + \nabla^\beta R_{\beta\mu} \\ &= \nabla^\alpha R_{\alpha\mu} - \nabla_\mu R + \nabla^\beta R_{\beta\mu} \\ &= 2(\nabla^\alpha R_{\alpha\mu} - \frac{1}{2}\nabla_\mu R) \qquad \text{relabelling} \end{aligned}$$

Thus

$$\nabla^\mu (R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = 0$$

□

Remark 6.8.

1. *(52) is a Tensorial equation, of physical meaning and should not depend on coordinates. This has infinite degree of freedom (gauge). We also choose a gauge to start with studying the equation.*
2. $g = \begin{pmatrix} 1 & & & \\ 5 & 2 & & \\ 6 & 8 & 3 & \\ 7 & 9 & 10 & 4 \end{pmatrix}$ *we have 10 unknowns, 6 equations, and 4 gauge* $\{x^\mu\}$*.*

Example 6.3. *Take* $T \equiv 0$*. We get*

$$\begin{aligned} \overline{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\overline{R} &= 0 \\ \overline{R} - \frac{1}{4}4\overline{R} &= 0 \qquad \textit{take trace} \\ \mathrm{Ric}_{\mu\nu} &= 0 \end{aligned} \tag{53}$$

Einstein Vacuum Equation.

Example 6.4. *Let*

$$T = 2F \otimes F - \frac{1}{2} g |F|^2$$

where F is a 2-form that satisfies Maxwell Equations

$$dF = 0 \qquad \text{div} F = 0$$

Then this is Einstein Maxwell Equation.

Definition 6.12. *We say T satisfies local energy condition if*

$$T(v, v) \geq 0 \qquad \forall\ v \qquad \textit{timelike}$$

It is only understood in 1952 (Choquet-Bruhat) that this is a Cauchy Problem, i.e., GR is a well-posed theory. With appropriate initial data, there exists unique solution.

6.3 Cauchy Problem

Initial data set

$$\big(M^3 \text{ 3-dim manifold},\ g_{ij} \text{ Riemannian metric},\ h_{ij} \text{ symmetric 2-tensor},\ \mu \text{ local energy density},\ J^i \text{ local momentum density}\big) \tag{54}$$

Motivation: consider $N^4 = \mathbb{R} \times M$ with coordinates (t, x_μ). How would one construct $\overline{g}$?

$$\overline{g} := -dt^2 + g''(t) dx_i dx_j$$

We use convention

$$1 \leq i, j \leq 3$$
$$0 \leq \alpha, \beta \leq 3$$

so

$$g_{ij} = g_{ij}(0) \qquad \text{Riemannian metric}$$
$$h_{ij} = \frac{1}{2} \frac{d}{dt}\bigg|_{t=0} g_{ij}(t) \qquad \text{symmetric 2-tensor, 2nd fundamental form w.r.t. } \frac{\partial}{\partial t}$$
$$\mu = 8\pi T(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})$$
$$J^i = 8\pi T(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^j}) g^{ij}$$

More generally we want N^4 with a time function

$$\langle dt, dt \rangle < 0$$

and we normalize to take

$$\langle e_0, e_0 \rangle = -1$$

Then

$$\overline{R}_{00} + \frac{1}{2} \overline{R} = 8\pi\mu$$
$$\overline{R}_{0i} = 8\pi J_i$$

This is the Einstein Equation under the setup. But this is not everything for the initial data. Since M is submanifold of N, they need to be compatible s.t. they satisfy Gauss-Codazzi. Recall Gauss Equation (22)

$$\begin{aligned}
\overline{R}(X, Y, Z, T) &= R(X, Y, Z, T) - \langle B(Y, T), B(X, Z) \rangle + \langle B(X, T), B(Y, Z) \rangle \\
\overline{R}_{ijk\ell} &= R_{ijk\ell} + h_{j\ell} h_{ik} - h_{i\ell} h_{jk} \\
\overline{R} &= \overline{R}_{0\mu 0\nu} \overline{g}^{\mu\nu} \overline{g}^{00} + R_{\alpha 0 \beta 0} \overline{g}^{\alpha\beta} \overline{g}^{00} + \overline{R}_{ijk\ell} g^{ik} g^{j\ell} \\
&= -2\overline{R}_{00} + (R_{ijk\ell} + h_{j\ell} h_{ik} - h_{i\ell} h_{jk}) g^{ik} g^{j\ell} \\
&= -2\overline{R}_{00} + R + (\text{Tr}(h))^2 - |h|^2 \\
8\pi\mu = \overline{R}_{00} + \frac{1}{2}\overline{R} &= \frac{1}{2}(R + (\text{Tr}(h))^2 - |h|^2)
\end{aligned}$$

Recall Codazzi (23)

$$\begin{aligned}
\overline{R}(X,Y,Z,\eta) &= (\overline{\nabla}_Y B)(X,Z,\eta) - (\overline{\nabla}_X B)(Y,Z,\eta) \qquad \eta := e_0 \\
\overline{R}_{ijk0} &= -h_{ik;j} + h_{jk;i} \\
8\pi J_j = \overline{R}_{0j} &= \overline{R}_{ijk0} g^{ik} = (-h_{ik;j} + h_{jk;i}) g^{ik} \\
&= -(\mathrm{Tr}(h))_{;j} + h_{jk;i} g^{ik}
\end{aligned}$$

Thus we have constraint equations

$$8\pi\mu = \frac{1}{2}(R + (\mathrm{Tr}(h))^2 - |h|^2) \tag{55}$$

$$8\pi J_j = -(\mathrm{Tr}(h))_{;j} + h_{jk;i} g^{ik} \tag{56}$$

Theorem 6.1 (Choquet-Bruhat 1952)**.** *Given an initial data set (54), there exists a unique spacetime $(N^4, \overline{g})$ solution to the Einstein Vacuum Equation (53) s.t.*

$$i : M^3 \hookrightarrow N^4$$

is an isometric immersion with

$$i^* \overline{g} = g$$

and h its 2nd fundamental form.

Sketch of Proof. We need to fully use our Gauge freedom. We choose harmonic gauge (harmonic coordinates) s.t.

$$H^\mu = \Box_g x^\mu = \nabla_\alpha \nabla^\alpha x^\mu$$

This quantity explicitly, is

$$\begin{aligned}
0 = H^\mu &= \sum_{\alpha,\beta} \frac{1}{\sqrt{-\det(g)}} \partial_\alpha \left(\sqrt{-\det(g)} g^{\alpha\beta} \partial_\beta x^\mu \right) \\
&= \sum_{\alpha,\beta} \frac{1}{\sqrt{-\det(g)}} \partial_\alpha \left(\sqrt{-\det(g)} g^{\alpha\mu} \right) \\
&= \sum_{\alpha} \left(\partial_\alpha g^{\alpha\mu} + \frac{1}{2} g^{\alpha\mu} \sum_{\rho,\sigma} g^{\rho\sigma} \partial_\alpha g_{\rho\sigma} \right)
\end{aligned}$$

On the other hand

$$R_{\mu\nu} = -\frac{1}{2} \sum_{\alpha,\beta} g^{\alpha\beta} \left(-2\partial_\beta \partial_{(\nu} g_{\mu)\alpha} + \partial_\alpha \partial_\beta g_{\mu\nu} + \partial_\mu \partial_\nu g_{\alpha\beta} \right) + F_{\mu\nu}(g, \partial g)$$

Then we have reduced Einstein Equation

$$\begin{aligned}
R^H_{\mu\nu} &= R_{\mu\nu} + \sum_\alpha g_{\alpha(\mu} \partial_{\nu)} H^\alpha \\
&= -\frac{1}{2} \sum_{\alpha\beta} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} + F(g, \partial g)
\end{aligned}$$

This is a quasilinear wave equation.

$$G_{\mu\nu} = R^H_{\mu\nu} - \frac{1}{2} R^H g_{\mu\nu} - \sum_\alpha (g_{\alpha(\mu} \partial_{\nu)} H^\alpha - \frac{1}{2} g_{\mu\nu} \partial_\alpha H^\alpha) \overset{\text{want}}{=} 0$$

Thus setting

$$\begin{aligned}
G_{\mu\nu} &= 0 \\
R^H_{\mu\nu} &= 0
\end{aligned}$$

We have

$$0 = \sum_{\rho,\mu,\alpha} -\frac{1}{2} g_{\alpha\nu} g^{\rho\mu} \partial_\rho \partial_\mu H^\alpha + \text{l.o.t.}$$

So H satisfies a wave equation. If

$$H^\mu = \partial_t H^\mu = 0$$

initially then $H \equiv 0$ everywhere. Indeed if $H = 0$ initially, then

$$\left.\frac{d}{dt}\right|_{t=0} H = 0 \qquad \text{consequence of Codazzi} \implies G_{\mu\nu}(\partial_t)^\mu = 0$$

so $H \equiv 0$ then $R^H = 0$ satisfies $\mathrm{Ric} = 0$. $\square$

6.4 Positive Mass Theorem

Definition 6.13 (Asymptotically Flat)**.** *Any initial data set $(M^3, g_{ij}, h_{ij}, \mu, J)$ is asymptotically flat if*

1. *M is orientable and (M, g) is complete, and for some compact set K and some Ball B*

$$M \setminus K \cong \mathbb{R}^3 \setminus B$$

This fixes the topology.

2. *On $M \setminus K \cong \mathbb{R}^3 \setminus B$*

$$g = g_{ij} dx^i dx^j$$

defines (x_1, x_2, x_3) the coordinates on $\mathbb{R}^3$. Let

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

then on the metric

$$g_{ij} = \delta_{ij} + a_{ij}, \qquad a_{ij} = O(\frac{1}{r}), \qquad \frac{\partial}{\partial x_k} a_{ij} = O(\frac{1}{r^2})$$

On the fundamental form

$$h_{ij} = O(\frac{1}{r^2})$$

What is not asymptotically flat? The big bang where it is expanding. But even in the universe if it expand, if we study how two blackholes merge, then it is asymptotically flat because how the universe expands doesn't matter as it is negligible. This is isolated system.

Definition 6.14. *If we have asymptotically flat initial data set, we define*

1. *mass/energy*

$$E := \frac{1}{16\pi} \lim_{r \to \infty} \int_{S_r} \sum_{i,j} (\partial_i g_{ij} - \partial_j g_{ii}) N^j dA$$

2. *linear momentum*

$$P_i := \frac{1}{8\pi} \lim_{r \to \infty} \int_{S_r} \sum_j (h_{ij} - (\mathrm{Tr}(h)) g_{ij}) N^j dA$$

where

$$S_r := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = r^2\}$$

and

$$N^j := \frac{x^j}{r} \qquad \textit{exterior unit normal}$$

and dA is the area element of g_{ij}.

One can show this is well-defined and doesn't depend on the coordinates nor the foliation. Mass is hard to define locally, but if one go faraway this definition works.

Example 6.5. *If $M = \mathbb{R}^3$, then*

$$P_i = \frac{1}{8\pi} \lim_{r \to \infty} \int_{S_r} \langle X, \nu \rangle d\sigma \qquad X_j := h_{ij} - (\mathrm{Tr}(h)) g_{ij} \qquad \textit{fixing } i$$

We could apply the divergence theorem and write

$$\begin{aligned} P_i &= \frac{1}{8\pi} \lim_{r \to \infty} \int_{B_r} \mathrm{div}(X) dV \\ &= \frac{1}{8\pi} \lim_{r \to \infty} \int_{B_r} T_{0i} dV \qquad \textit{by constraint equations} \end{aligned}$$

Some history

1. Schoen-Yau 1979-1981
2. Witten 1982

Theorem 6.2 (Positive Mass Theorem)**.** *Let (M^3, g, h, μ, J) be asymptotically flat initial data set satisfying the constraint equations (55), (56) and the dominant energy condition. Then for any asymptotically flat ends*

$$E_\ell - |P_\ell| \geq 0$$

Moreover, if there is equality

$$E_\ell = |P_\ell|$$

for some ℓ, then M has only one end, is diffeomorphic to $\mathbb{R}^3$, and

$$(M^3, g_{ij}, h_{ij}) \overset{isometrically}{\hookrightarrow} (\mathbb{R}^{1+3}, (-dx^0)^2 + \sum_{i=1}^{n} (dx^i)^2)$$

The gravitational energy of an isolated system is nonnegative, and $= 0$ iff there does not exist gravitating object (Minkowski).

6.5 Null Structure Equations

Let S be 2-dim spacelike hypersurface of a 4-dim Lorentizan manifold.

1. A spacetime can be foliated by null hypersurfaces, either outgoing (outgoing C null cone) or ingoing/incoming (ingoing $\underline{C}$ null cone).
2. We can choose a null frame

   $$\{e_1, e_2, e_3, e_4\} \qquad \{e_1, e_2\} \text{ tangent to } S, \qquad \{e_3, e_4\} \text{ null}$$

   s.t. e_3 is in the direction of ingoing null cone, e_4 is in the direction of outgoing null cone and

   $$g(e_3, e_3) = g(e_4, e_4) = 0, \qquad g(e_3, e_4) = -2, \qquad g(e_3, e_a) = g(e_4, e_a) = 0, \qquad g(e_a, e_b) = \delta_{ab} \qquad a, b \in \{1, 2\}$$

3. In general we want to define the Christoffel symbols

   $$\nabla_{e_\mu} e_\nu = \sum_\lambda \Gamma^\lambda_{\mu\nu} e_\lambda$$

   and we're interested in those with at least one e_3 or e_4. They are completely determined by the following coefficients (all the quantities are tensors on the sphere, but we only care about e_3 and e_4). For any $a, b \in \{1, 2\}$

   $$\begin{aligned} \chi_{ab} &= g(\nabla_{e_a} e_4, e_b) \\ \underline{\chi_{ab}} &= g(\nabla_{e_a} e_3, e_b) \\ \eta_a &= g(\nabla_{e_3} e_4, e_a) \\ \underline{\eta_a} &= g(\nabla_{e_4} e_3, e_a) \\ \omega &= -g(\nabla_{e_4} e_4, e_3) \\ \underline{\omega} &= -g(\nabla_{e_3} e_3, e_4) \\ \xi_a &= g(\nabla_{e_a} e_4, e_3) \\ \underline{\xi}_a &= g(\nabla_{e_a} e_3, e_4) = -\xi_a \end{aligned}$$

   Theses are tensors on the surface S.

   Remark 6.9. *(a) χ, $\underline{\chi}$ are the projection of the 2nd fundamental forms of the embedding of S w.r.t. e_4/e_3. In fact*

   $$\begin{aligned} H(X, Y) &= \frac{1}{2}\chi(X, Y)e_3 + \frac{1}{2}\underline{\chi}(X, Y)e_4 \\ &= \nabla_X Y - \not\nabla_X Y \end{aligned}$$

   (b) Observe that this is the same computation we did to show that the second fundamental form is symmetric.

   $$\chi(X, Y) - \chi(Y, X) = g(D_X e_4, Y) - g(D_Y e_4, X) = g((e_4, [X, Y])$$

   So χ symmetric iff $[X, Y] \perp e_4$ iff $[X, Y] \in Span\langle e_3, e_4\rangle^\perp$ iff $\langle e_3, e_4\rangle^\perp$ is integrable (in Frobenius sense). In the case of Kerr χ is not symmetric.

(c) *We can decompose into shear and expansion*

$$\chi_{ab} = \hat{\chi}_{ab} + \frac{1}{2}\text{Tr}(\chi)g_{ab}$$
$$\underline{\chi}_{ab} = \hat{\underline{\chi}}_{ab} + \frac{1}{2}\text{Tr}(\underline{\chi})g_{ab}$$

Example 6.6. *For standard spheres of radius r in Minkowski spacetime*

$$\begin{aligned}\hat{\chi} = \hat{\underline{\chi}} &= 0 \\ \text{Tr}(\chi) &= \frac{2}{r} \\ \text{Tr}(\underline{\chi}) &= -\frac{2}{r}\end{aligned}$$

4. Now we decompose the curvature. For $a, b \in \{1, 2\}$

$$\begin{aligned}\alpha_{ab} &= R(e_a, e_4, e_b, e_4) \\ \underline{\alpha}_{ab} &= R(e_a, e_3, e_b, e_3) \\ \beta_a &= R(e_a, e_4, e_3, e_4) \\ \underline{\beta}_a &= R(e_a, e_3, e_3, e_4) \\ \rho &= R(e_3, e_4, e_3, e_4) \\ \sigma &= \not{\epsilon}^{ab} R_{ab34} = (*R)_{3434} \qquad \text{where } (*R)_{\alpha\beta\gamma\delta} := \epsilon_{\mu\nu\alpha\beta} R^{\mu\nu}{}_{\gamma\delta}\end{aligned}$$

where ϵ is the spacetime volume form w.r.t. g and $\not{\epsilon}$ is volume form on the spheres S w.r.t. induced metric $\not{g}$. Observe that $\text{Ric}(g) = 0$ implies

$$\text{Tr}(\alpha) = \text{Tr}(\underline{\alpha}) = 0$$

5. The null structure equation are propagation equations for Christoffel symbols (comes from definition of curvature)

$$\begin{aligned}\nabla_4 \Gamma &= R + \Gamma \cdot \Gamma + \not{\nabla}\Gamma \\ \nabla_3 \Gamma &= R + \Gamma \cdot \Gamma + \not{\nabla}\Gamma\end{aligned}$$

Example 6.7. *One has Raychaduri Equations, Codazzi Equations, and the Gauss curvature of S*

$$\begin{aligned}\nabla_4 \text{Tr}(\chi) &= -\frac{1}{2}(\text{Tr}\chi)^2 + \omega \text{Tr}(\chi) - |\hat{\chi}|^2 \qquad \textit{Raychaduri Equations} \\ \nabla_3 \text{Tr}(\underline{\chi}) &= -\frac{1}{2}(\text{Tr}(\underline{\chi}))^2 + \underline{\omega}\text{Tr}(\underline{\chi}) - |\hat{\underline{\chi}}|^2 \\ \nabla_4 \hat{\chi} &= \omega\hat{\chi} - \text{Tr}(\chi)\hat{\chi} - \alpha \qquad \textit{Codazzi Equations} \\ \nabla_3 \hat{\underline{\chi}} &= \underline{\omega}\hat{\underline{\chi}} - \text{Tr}(\underline{\chi})\hat{\underline{\chi}} - \underline{\alpha} \\ \text{div}\chi - d\text{Tr}(\chi) + \chi \cdot \xi - (\text{Tr}(\chi)) \cdot \xi &= -\beta \\ K &= \hat{\chi} \cdot \hat{\chi} - \text{Tr}(\chi)\text{Tr}(\underline{\chi}) - \rho \qquad \textit{Gauss curvature of S}\end{aligned}$$

6.6 Trapped Surfaces

This is a gateway to blackhole.

Definition 6.15. *A closed spacelike surface S is called trapped if the expansions are negative*

$$\text{Tr}(\chi) < 0 \qquad \text{Tr}(\underline{\chi}) < 0 \qquad \forall\, p \in S$$

Remark 6.10. *This condition implies that the area of the surface decreases when deformed along the two null directions. As time passes, the sphere is forced to shrink. Let τ be affine parameter along e_4. Then*

$$\begin{aligned}\frac{d}{d\tau} g(e_a, e_b) &= g(\nabla_{e_4} e_a, e_b) + g(e_a, \nabla_{e_4} e_b) = g(\nabla_{e_a} e_4, e_b) + g(e_a, \nabla_{e_b} e_4) \\ &= 2\chi_{ab} \\ \frac{d}{d\tau}\sqrt{\det(g)} &= \frac{1}{2}\sqrt{-\det(g)} g^{ab} \frac{d}{d\tau} g_{ab} = \text{Tr}(\chi)\sqrt{-\det(g)}\end{aligned}$$

Theorem 6.3 (Penrose Incompleteness Theorem)**.** *Let (M, g) be a $3+1$-dim globally hyperbolic Lorentizan manifold with a non-compact Cauchy surface whose Ricci curvature satisfies* $\text{Ric}(v, v) \geq 0$ *for any null vector v (dominant energy condition). If (M, g) has a closed trapped surface, then (M, g) is future geodesically incomplete.*

Sketch of Proof.

1. Jacobi Fields. Consider our surface S with the outgoing null cone generated by S, denote as C. A Jacobi field X on C is called a normal Jacobi field if

$$\mathcal{L}_{e_4} X = 0$$

The Jacobi field measures the displacement of the geodesics.

2. Focal Points. Given $p \in S$ on the surface and

$$q = \gamma_p(\tau_*)$$

for γ_p the geodesic along C starting at p. q is a focal point to p if there exists a non-trivial normal Jacobi Field along C s.t. $J(\tau_*) = 0$.

3. To show existence of focal points, Raychaduri Equations are used. For geodesic $L = e_4$ and $\underline{L} = e_3$ ($D_L L = D_{\underline{L}} \underline{L} = 0$ implies $\omega = \underline{\omega} = 0$)

$$\begin{aligned} L(\text{Tr}\chi) &= -\frac{1}{2}(\text{Tr}(\chi))^2 - |\hat{\chi}|^2 - \text{Ric}(L, L) \leq 0 \\ \underline{L}(\text{Tr}(\underline{\chi})) &= -\frac{1}{2}(\text{Tr}(\underline{\chi}))^2 - |\underline{\hat{\chi}}|^2 - \text{Ric}(\underline{L}, \underline{L}) \leq 0 \end{aligned}$$

Due to assumptions on $\text{Ric}(v, v) \leq 0$. Hence $\text{Tr}(\chi)$ and $\text{Tr}(\underline{\chi})$ will remain negative if they are negative initially.

4. Existence of Focal points with a trapped surface. If $\text{Tr}(\chi) < 0$ at a point $p \in S$, then p has a focal point. We know by Raychaduri

$$\begin{aligned} \text{Tr}(\chi)(0) &= -\kappa < 0 \\ L(\text{Tr}(\chi)) &\leq 0 \end{aligned}$$

Thus $\text{Tr}(\chi(\tau)) < 0$ for any τ. We solve

$$\begin{aligned} L(\text{Tr}(\chi)) &\leq -\frac{1}{2}(\text{Tr}(\chi))^2 \\ \frac{1}{(\text{Tr}(\chi))^2} L(\text{Tr}(\chi)) &\leq -\frac{1}{2} \\ L(-\frac{1}{\text{Tr}(\chi)}) &\leq -\frac{1}{2} \\ -\frac{1}{\text{Tr}(\chi)} &\leq -\frac{1}{-\kappa} - \frac{1}{2}\tau = \frac{1}{\kappa} - \frac{\tau}{2} \\ -\text{Tr}(\chi)(\tau) &\geq \frac{1}{\frac{1}{\kappa} - \frac{\tau}{2}} \\ \text{Tr}(\chi)(\tau) &\leq \frac{1}{\frac{\tau}{2} - \frac{1}{\kappa}} \end{aligned}$$

Hence there exists some $\tau_\kappa = \frac{2}{\kappa}$ s.t. $\text{Tr}(\chi)(\tau) \overset{\tau \to \tau_\kappa}{\to} -\infty$. Hence one can construct

$$J^4 = M^A_B (e_4)^B \implies \det(M(\tau_*)) = 0$$

5. Now we sketch the proof. Suppose (M, g) is geodesically complete. Take our trapped surface, take $p \in S$ and so $\text{Tr}(\chi) < 0$. Then γ_p is defined for all times. But also there exists a focal point between $\frac{2}{\kappa}$. Then the boundary of the future of S is contained in the null-cone generated by the null-cone of this γ

$$\partial J^+(S) \subseteq \bigcup \gamma_p([0, \frac{2}{\kappa}]) \cup \underline{\gamma_p}([0, \frac{2}{\kappa}]) \qquad \underline{\gamma_p} \text{ is tangent to } \underline{C}$$

Thus $p \in \partial J^+(S)$ iff p lies on a null geodesic starting orthogonally from S and not containing focal points. But RHS is compact due t focal points. Thus $\partial J^+(S)$ is compact. But we have a non-compact C Cauchy surface. For each $q \in \partial J^+(S)$, the integral curve of T timelike vector through q intersects the Cauchy surface Σ exactly once via φ homeomorphism. Now $\varphi(\partial J^+(S))$ is open and closed in Σ and so

$$\varphi(\partial J^+(S)) = \Sigma$$

but Σ is non-compact and we reach a contradiction.

□

6.7 Blackholes

Definition 6.16 (Null-Infinity)**.** *The future null-infinity $\mathcal{I}^+$ consists of all ideal limit points of null geodesics, which reach arbitrarily large spatial distances.*

Remark 6.11. *Any asymptotically flat spacetime admits a null infinity.*

Remark 6.12. *A perturbation of Minkowski has complete $\mathcal{I}^+$ (Christodoulou–Klainerman)*

Definition 6.17 (Blackhole)**.** *A Blackhole B is a region of spacetime that cannot send signals to $\mathcal{I}^+$, i.e.,*

$$B = M \setminus J^-(\mathcal{I}^+)$$

The boundary of blackhole ∂B is called the event horizon.

The presence of a trapped surface implies the existence of a blackhole.

Proposition 6.1. *If S is trapped, then S cannot lie on $J^-(\mathcal{I}^+)$.*

Proof. Suppose $S \subset J^-(\mathcal{I}^+)$, then there exists $p \in \mathcal{I}^+$ s.t. $p \in \partial J^+(S)$. But then p lies on a null generator of S, and γ must be complete. But S is trapped, so all null generators of $\partial J^+(S)$ has affine length. □

6.7.1 Schwarzschild: Spherical Symmetry

By imposing spherical symmetry, we reduce to 2 degree of freedom. $SO(3)$ acts isometrically. Orbits of this group are

$$r^2(d\theta^2 + \sin^2(\theta)d\varphi^2)$$

For $M > 0$ constant

$$\mathcal{M} = \mathbb{R}_t \times (2M, \infty)_r + \mathbb{S}^2$$

then the Schwarzschild solutions write

$$g_M = -(1 - \frac{2M}{r})dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2)$$

1. This is indeed a solution to $\mathrm{Ric} = 0$ Einstein Vacuum Equation, as 1-parameter family. This is spacetime of an isolated body of mass M! This is general relativity version of Newtonian theory. This spacetime does not have matter! The effective mass M comes out of the gravity.
2. $\frac{\partial}{\partial t}$ is a Killing vector field for g_M, i.e.

   $$(\mathcal{L}_{\frac{\partial}{\partial t}} g)_{\mu\nu} = \partial_t g_{\mu\nu} = 0$$

   When this happens, we say the metric is stationary, so the metric does not depend on time.
3. as $r \to \infty$, the metric goes to

   $$-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2) = \text{Minkowski}$$

   Hence Schwarzschild solution is Asymptotically flat.

Theorem 6.4 (Birkhoff)**.** *The Schwarzchild metric is the only spherically symmetric solution to* $\mathrm{Ric}(g) = 0$.

Notice $r = 2M$ is not a singularity.

Example 6.8. *Consider the metric*

$$g = -t^2dt^2 + dx^2$$

on $M = (0, \infty)_t \times \mathbb{R}_x$. This metric looks singular at $t = 0$. But take

$$\tilde{t} := \frac{1}{2}t^2 \qquad t > 0$$

Then

$$\tilde{g} = -d\tilde{t}^2 + dx^2$$

this is Minkowski. Hence $t = 0$ is just a coordinate singularity (since we change coordinates we can just extend past $t = 0$).

We introduce ingoing Eddington-Finkelstein coordinates

$$\begin{aligned} r_* &:= r + 2M \log(r - 2M) \\ v &= t + r_* \end{aligned}$$

Thus

$$\tilde{g} = -(1 - \frac{2M}{r})dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2)$$

Thus for $r = 2M$ this is regular

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2(\theta) \end{pmatrix}$$

Now

$$\tilde{\mathcal{M}} = \mathbb{R}_v \times (0, \infty)_r \times \mathbb{S}^2$$

One can easily verify that $\{r = 2M\}$ is a null hypersurface. In fact $\{r = c\}$ is timelike if $r > 2M$ and spacelike if $r < 2M$. $\{r = 0\}$ is a singularity (crashing singularity)

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{C_M}{r^6} \to \infty \qquad r \to 0$$

At $r = 0$, any future directed causal geodesic starting at $r < 2M$ will reach $r = 0$ in finite proper time. We cannot extend $\tilde{g}$ past $r = 0$ as a C^2 metric. In fact it cannot be extended as a C^0 metric (Sbierski 2018). But one can possibly do Kruskal extension. What is a null geodesic flow? For example take timelike hypersurface $\{r = 3M\}$ (photon sphere). There are null geodesics 'trapped' here.

6.7.2 Geodesic Flow in Schwarzschild

Let

$$r(\tau) := (t(\tau), r(\tau), \theta(\tau), \varphi(\tau))$$

be a geodesic. Since sphercial symmetry, we can assume that γ lies on $\theta = \frac{\pi}{2}$. We have the following constants of motion

1. $g(\gamma', \gamma') = \kappa = \begin{cases} 0 & \text{null} \\ -1 & \text{timelike} \end{cases}$

2. $g(\gamma', \frac{\partial}{\partial t}) = E$ Energy/Mass. This is constant since

$$\nabla_{\gamma'}(g(\gamma', \frac{\partial}{\partial t})) = 0 \qquad \nabla_{(M}(\frac{\partial}{\partial t})_{\nu)} = 0$$

3. $g(\gamma', \frac{\partial}{\partial t}) = L$ angular motion.

These are actually enough constants of motion so that the Hamilton-Jacobi equation is completely integrable. In particular, for $r = r(\tau)$

$$\begin{aligned} \frac{d^2r}{d\tau^2} + V(r) = E^2 \\ V(r) := \frac{1}{2}\kappa - \kappa\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3} \end{aligned}$$

By studying this ODE we get lots of information about the geodesic.

1. For example for timelike geodesic ($\kappa = -1$) there exists stable circular orbits for $r > 6M$ and unstable circular orbits for $3M < r < 6M$.

2. For null geodesics $\kappa = 0$

$$V(r) = \frac{L^2}{2r^3}(r - 2M)$$

V has a maximum at $r = 3M$ so there exists unstable circular orbits at $r = 3M$.

6.7.3 Kerr Black Hole

The Kerr Black Hole (1963) is a 2-parameter family $M \in \mathbb{R}$, $a \in \mathbb{R}$ for $|a| \leq M$. Define

$$\Delta := r^2 - 2Mr + a^2$$
$$\rho^2 := r^2 + a^2\cos^2(\theta)$$
$$r_t := M + \sqrt{M^2 - a^2}$$

And our manifold is

$$\mathcal{M} = \mathbb{R}_t \times (r_t, \infty)_r \times \mathbb{S}^2$$

Our metric is

$$g_{M,a} := g_{tt}dt^2 + g_{t\varphi}dtd\varphi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\varphi\varphi}d\varphi^2$$

with

$$g_{tt} = -(1 - \frac{2Mr}{\rho^2})$$
$$g_{rr} = \frac{\rho^2}{\Delta}$$
$$g_{\theta\theta} = \rho^2$$
$$g_{t\varphi} = -\frac{2Mr}{\varphi^2}a\sin^2(\theta)$$
$$g_{\varphi\varphi} = (r^2 + a^2 + \frac{2Mra^2}{\varphi^2}\sin^2(\theta))\sin^2(\theta)$$

This is still stationary. Axis-symmetric, but not spherical-symmetric. a stands for angular momentum.

1. This is solution to
$$\mathrm{Ric}(g_{\mu\nu}) = 0$$
2. $a = 0$ reduces to Schwarzschild. For $|a| \leq M$, Δ has roots so there exists black hole. For $|a| = M$, Δ has double root $\Delta = (r - M)^2$, this is extremal black hole.
3. Asymptotically flat.
4. Symmetries. ∂_t, ∂_φ the killing tensor $K_{\mu\nu}$ satisfies $\nabla_{(\mu}K_{\nu\alpha)} = 0$. For any γ geodesic
$$K_{\mu\nu}\gamma'^\mu\gamma'^\nu = \text{constant}$$
hence geodesic flow is integrable
5. superradiance. ∂_t is timelike for $r \gg 1$.
$$g(\partial_t, \partial_t) > 0$$
close to $\mathcal{H}^+$
$$M + \sqrt{M^2 - a^2} < r < M + \sqrt{M^2 - a^2\cos^2(\theta)}$$
Note Penrose process.
6. Trapped Null geodesics. In a full interval $[r_1, r_2]$ roots of
$$r(r - 3M)^2 - 4a^2M$$
but still unstable.

There is conjecture: Kerr is the only asymptotically flat stationary solution of Einstein Vacuum equation (no hair theorem).

6.8 Wave Equation

We consider Wave Equation on Minkowski Spacetime $(\mathbb{R}^{1,3}, m)$.

$$\Box_m\phi = -\partial_t^2\phi + \partial_x^2\phi + \partial_y^2\phi + \partial_z^2\phi$$

We want to formulate the Wave Equation as Cauchy Problem. The natural prescribed conditions are

$$\phi(t = 0, x) = f(x)$$
$$\partial_t\phi(t = 0, x) = g(x)$$

We have formula. There exists unique smooth solution $\phi : \mathbb{R}^{1,3} \to \mathbb{R}$ s.t.

$$\phi(t,x) = \frac{1}{4\pi t^2}\int_{|y-x|=t}(tg(y) + f(y) + \sum_i (\partial_{y_i} f)(y^i - x^i))dS(y)$$

Some remarks

1. We almost don't expect this to hold true in the curved spacetime. This is special to Minkowski. (Not robust enough)
2. Our data can only propagate in a finite speed.
3. Sharp Huygen's Principle in $3+1$ dimension.

What are robust things? The energy estimate. We first discuss energy estimate for wave in Minkowski. We start with the equation and try to use the idea of Noether : Symmetry gives conservation. The $\partial_t \phi$ is very important!

$$\begin{aligned}
0 &= -\Box\phi\partial_t\phi \\
&= (\partial_t^2\phi - \Delta\phi)\partial_t\phi \\
&= \frac{1}{2}\partial_t(|\partial_t\phi|^2) - \mathrm{div}_x(\partial_t\phi\nabla_x\phi) + \nabla_x\partial_t\phi\cdot\nabla_x\phi \\
&= \frac{1}{2}\partial_t(|\partial_t\phi|^2) - \mathrm{div}_x(\partial_t\phi\nabla_x\phi) + \partial_t\nabla_x\phi\cdot\nabla_x\phi \qquad \text{using } (\mathbb{R}^{1,3}, m) \text{ is flat (if not this commutation gives lower order terms)} \\
&= \frac{1}{2}\partial_t(|\partial_t\phi|^2 + |\nabla\phi|^2) - \mathrm{div}_x(\partial_t\phi\nabla_x\phi)
\end{aligned}$$

Then we want to integrate this in the region $[0,T]\times\mathbb{R}^3$, assuming ϕ decays fast enough as $|x|\to\infty$. We get

$$\begin{aligned}
\int_{\{t=T\}}(|\partial_t\phi|^2 + |\nabla\phi|^2)d^3x &= \int_{\{t=0\}}(|\partial_t\phi|^2 + |\nabla\phi|^2)d^3x \\
&= \|g\|^2_{L^2(\mathbb{R}^3)} + \|f\|^2_{\dot{H}^1(\mathbb{R}^3)}
\end{aligned}$$

Now we discuss Wave Equation on a general spacetime $(\mathcal{M}, g)$.

$$\Box_g\psi = g^{\mu\nu}\nabla_\mu\partial_\nu\psi = 0$$

Energy-Momentum Tensor method. What is the Energy-Momentum tensor? This is a symmetric 2-tensor defined associated to ψ

$$\begin{aligned}
T_{\mu\nu}[\psi] &= \partial_\mu\psi\partial_\nu\psi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi \qquad \text{here } \mathcal{L}[\psi] \equiv g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi \\
T &= d\psi\otimes d\psi - \frac{1}{2}g^{-1}(d\psi, d\psi)g
\end{aligned}$$

Proposition 6.2 (Local Conservation Law)**.**

$$\nabla^\mu T_{\mu\nu}[\psi] = (\Box_g\psi)\partial_\nu\psi$$

In particular if $\Box_g\psi = 0$*, then*

$$\nabla^\mu T_{\mu\nu} = 0$$

Proof.

$$\begin{aligned}
\nabla^\mu T_{\mu\nu} &= \nabla^\mu\partial_\mu\psi\partial_\nu\psi + \partial_\mu\psi\nabla^\mu\partial_\nu\psi - \frac{1}{2}g_{\mu\nu}(\nabla^\mu\partial_\alpha\psi\partial^\alpha\psi + \partial_\alpha\psi\nabla^\mu\partial^\alpha\psi) \\
&= \Box_g\psi\partial_\nu\psi + \partial_\mu\psi\nabla^\mu\partial_\nu\psi - \nabla_\nu\partial_\alpha\psi\partial^\alpha\psi \\
&= \Box_g\psi\partial_\nu\psi
\end{aligned}$$

□

Corollary 6.1. *If X is a spacetime vector field (thinking of X as multiplier) on (M, g), then $T_{\mu\nu}X^\mu$ has divergence $\nabla^\mu(T_{\mu\nu}X^\nu)$.*

Proof. Note $(\mathcal{L}_X g)^{\mu\nu} = \nabla^\mu X^\nu + \nabla^\nu X^\mu$ and define deformation tensor ${}^{(X)}\Pi^{\mu\nu}$

$$\begin{aligned}
\nabla^\mu(T_{\mu\nu}X^\nu) &= \nabla^\mu T_{\mu\nu}X^\nu + T_{\mu\nu}\nabla^\mu X^\nu \\
&= \Box\psi\partial_\nu\psi X^\nu + T_{\mu\nu}\frac{1}{2}(\nabla^\mu X^\nu + \nabla^\nu X^\mu) \\
\nabla^\mu(T_{\mu\nu}X^\nu) &= \Box_g\psi X(\psi) + T_{\mu\nu}\,{}^{(X)}\Pi^{\mu\nu}
\end{aligned}$$

□

The heuristic is $\mathcal{L}[g_{\mu\nu}, \psi, \partial\psi]$ is given.

$$S[g,\psi] := \int_{\mathcal{M}} \mathcal{L} d\mu_g$$

Say g is given, local diffeomorphism is like a local symmetry. Then this gives us local conservation.

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}\mathcal{L}$$

is divergence-free if ψ solves the Euler-Lagrange Equations. Note

$$\mathcal{L}[g,\psi] := g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi$$

Proposition 6.3. *For X, Y future casual vectors, then $T(X,Y) \geq 0$. For X, Y timelike vectors, we have concrete lower bound*

$$T(X,Y) \geq c\sum_{\mu=1}^{3+1} |\partial_\mu\psi|^2$$

for $c > 0$.

For Minkowski, apply divergence theorem to $T_{\mu\nu}[\psi](\partial_t)^\nu$ between boundaries $\{t=0\}$ and $\{t=T\}$ which we take to be spacelike ($n = \partial_t$). ${}^{\partial_t}\Pi = 0$ if $\Box_g\psi = 0$, then

$$0 < \int_{\{t=T\}} T(\partial_t, n) = \int_{\{t=0\}} T(\partial_t, n)$$

This is true because ∂_t and n are timelike, hence $\{t=T\}$ is spacelike.

Proof of 6.3. Take null vectors L, $\underline{L}$ and normalized s.t. $g(L,\underline{L}) = -2$ then

$$\begin{aligned}
T(L,L) &= |L\psi|^2 \\
T(\underline{L},\underline{L}) &= |\underline{L}\psi|^2 \\
T(L,\underline{L}) &= |e_1\psi|^2 + |e_2\psi|^2
\end{aligned}$$

□

We discuss Divergence Theorem on Lorentzian manifold. Recall Stoke's Theorem. Let M be an oriented n-manifold with boundary ∂M and let ω be a smooth $(n-1)$-form on M with compact support. Then the Stoke's Theorem says

$$\int_M d\omega = \int_{\partial M} \omega$$

This is idea of Poincaré Duality. If suffices to give a metric structure. In our case we want to give the manifold a Lorentzian structure. Then the volume form of (M,g) is

$$\epsilon := \sqrt{-\det(g)}dx^0 \wedge dx^1 \wedge \cdots \wedge dx^n$$

The interior derivative is the following: for a k-form ω and a vector field X, we define $(k-1)$-form $i_X\omega$ as contraction

$$i_X\omega(v_1, \cdots, v_{k-1}) = \omega(X, v_1, \cdots, v_l)$$

Lemma 6.2 (Cartan's)**.**

$$d(i_X\epsilon) = (\text{div}X)\epsilon$$

Now we plug into Stoke's Theorem with the setting that ∂M (n dim) being timelike or spacelike (we exclude null pieces). If $e_1, \cdots, e_n$ is an ONB of $T_p(\partial M)$, and $N, e_1, e_2, \cdots, e_n$ is an O.N.B. of T_pM ($n+1$ dim). Then for $\omega = i_X\epsilon$, we have

$$\int_M d(i_X\epsilon) = \int_{\partial M} i_X\epsilon$$

On the other hand by Cartan's

$$\int_M \operatorname{div}(X)\epsilon = \int_M d(i_X\epsilon) = \int_{\partial M} i_X\epsilon$$

The point is to make sense of what the RHS is. Let's try to compute

$$\begin{aligned} i_X\epsilon(e_1, e_2, \cdots, e_n) &= \epsilon(X, e_1, \cdots, e_n) \\ &= \epsilon\left(\frac{g(X,N)N}{g(N,N)} + \sum_i g(X, e_i)e_i, e_1, \cdots, e_n\right) \\ &= \frac{g(X,N)}{g(N,N)}\epsilon(N, e_1, \cdots, e_n) \qquad \text{using volume form is tensor} \\ &= \frac{g(X,N)}{g(N,N)}\epsilon_{\partial M} \qquad \text{this is how we define the volume form on the boundary} \end{aligned}$$

Plugging this in and we see the following

$$\int_M \operatorname{div}(X)\epsilon = \int_M d(i_X\epsilon) = \int_{\partial M} i_X\epsilon = \int_{\partial M} \frac{g(X,N)}{g(N,N)}\epsilon_{\partial M}$$

We take our $X := T(\partial_t, \cdot)^{\#}$. We really take the normal to be inward $-N = -\partial_t$. Now integrating

$$\int_M \operatorname{div}(X) = \int_{\{t=T\}} T(\partial_t, -N) + \int_{\{t=0\}} T(\partial_t, N)$$

Thus we get

$$\int_{\{t=T\}} T(\partial_t, M) + \int_M \operatorname{div}(T(\partial_t, \cdot)) = \int_{\{t=0\}} T(\partial_t, N)$$

If in practice we can show $\int_M \operatorname{div}(T(\partial_t, \cdot)) > 0$ then we can bound our energy.

In general, for future timelike X

$$\int_{\Sigma_t} T \cdot X \cdot n \sim \int_{\Sigma_t} \sum_\alpha (\partial_\alpha\psi)^2 = E(t)$$

Thus

$$E(t) + \int_R \Box\psi X(\psi) + T_{\mu\nu}\,{}^{(X)}\Pi^{\mu\nu} = E(0)$$

We want to choose X and ψ solving the equation so that the middle term has a sign.

On Schwarzschild, thinking of event horizon as $\{r = 2M\}$, $X = \partial_t$ we verify

$$T(\partial_t, n_{\Sigma_t}) \sim (\partial_t\psi)^2 + (1 - \frac{2M}{r})(\partial_r\psi)^2 + |\nabla\!\!\!/\psi|^2$$

We still have a symmetry

$${}^{\partial_t}\Pi^{\mu\nu} = 0$$

1. For $\Box_g\psi = 0$ we have Energy Estimate

   $$E(t) \leq E(0)$$

   This middle term can be resolved by Red shift Method (Dafermos-Rodnianski).

2. We also have Morawetz Estimate. Want to choose X s.t.

   $$T_{\mu\nu}\,{}^{(X)}\Pi^{\mu\nu} \geq 0$$

   On Schwarzschild near photon sphere, we have

   $$\int_R (\partial_r\psi)^2 + \psi^2 + (r-3M)^2((\partial_t\psi)^2 + |\nabla\!\!\!/\psi|^2) \lesssim E(0)$$

   This is the trapping phenomenon. This we cannot resolve because it is some physics. Alternatively one can go to higher derivatives.

In Kerr, however, ∂_t is not timelike everywhere outside Black hole. This is the Ergo Region.

A Final

Problem 1

Consider

$$\mathbb{S}^n := \{(x_1, \cdots, x_{n+1}) \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$$

equipped with the Riemannian metric g_{can} induced from the Euclidean metric on $\mathbb{R}^{n+1}$. Define

$$f : \mathbb{S}^n \to \mathbb{R} \qquad f(x_1, \cdots, x_{n+1}) := x_{n+1}$$

Then for $t \in (-1, 1)$,

$$M_t := f^{-1}(t)$$

is an $(n-1)$-dim C^∞ submanifold of $\mathbb{S}^n$. Let g_t be the Riemannian metric on M_t induced from $(\mathbb{S}^n, g_{\text{can}})$. Let H_t be the second fundamental form of M_t in $(\mathbb{S}^n, g_{\text{can}})$ w.r.t. the unit normal

$$\nu := \frac{\text{grad} f}{|\text{grad} f|}$$

Problem A.1. *Show that for a fixed $t \in (-1, 1)$,*

$$H_t = \lambda(t) g_t$$

for some constant $\lambda(t) \in \mathbb{R}$.

Answer A.1. *First we note*

$$i : (M_t, g_t) \hookrightarrow (\mathbb{S}^n, g_{can})$$

is an isometric immersion. Denote ∇ as connection on (M_t, g_t) and $\nabla^{\mathbb{S}^n}$ as connection on $(\mathbb{S}^n, g_{can})$. Now by definition, to compute the second fundamental form, we want to compute for any X, $Y \in \mathfrak{X}(M_t)$ and $p \in M_t$

$$H_t(X, Y)(p) := \langle S_\nu(X), Y\rangle_{g_t(p)}$$

where S_ν has the explicit expression

$$S_\nu(X) := -(D_X \nu)^T$$

where $D := i^* \nabla^{\mathbb{S}^n}$

1. *Let's begin by understanding the unit normal ν. To do so we need to know* $\text{grad} f \equiv \text{grad}^{M_t} f$ *w.r.t. M_t. But one can partition the full gradient* $\text{grad}^{\mathbb{R}^{n+1}}$ *into radial direction and spherical direction. Thus for any $x \in \mathbb{S}^n$, using $f(x_1, \cdots, x_{n+1}) = x_{n+1}$ so the full gradient is $e_{n+1} = (0, \cdots, 0, 1)$, we write*

$$\begin{aligned} \text{grad}^{\mathbb{S}^n} f(x) &= \text{grad}^{\mathbb{R}^{n+1}} f(x) - \langle \text{grad}^{\mathbb{R}^{n+1}} f(x), x\rangle x \\ &= e_{n+1} - \langle e_{n+1}, x\rangle x \end{aligned}$$

   In particular, when restricted to M_t, $x_{n+1} = t$ so

$$\text{grad}^{M_t} f(x) = e_{n+1} - tx$$

   so the unit normal ν restricted to M_t writes

$$\begin{aligned} \nu(x) &= \frac{e_{n+1} - tx}{\|e_{n+1} - tx\|} = \frac{e_{n+1} - tx}{\sqrt{1 - 2t\langle e_{n+1}, x\rangle + t^2 |x|^2}} \\ &= \frac{e_{n+1} - tx}{\sqrt{1 - t^2}} \qquad \textit{using } x \in \mathbb{S}^n \textit{ and } M_t := f^{-1}(t) \end{aligned}$$

2. *Notice M_t is $(n-1)$-dim submanifold of $\mathbb{S}^n$ so S_ν as the shape operator is*

$$S_\nu(X) = -D_X \nu$$

   We make observation that in the spherical direction, differentiating position vector x w.r.t. $X \in \mathfrak{X}(\mathbb{S}^n)$ yields the vector field unchanged, in particular

$$D_X(x) = X$$

   Hence

$$\begin{aligned} S_\nu(X) &= -\frac{1}{\sqrt{1 - t^2}} D_X(e_{n+1} - tx) \\ &= \frac{t}{\sqrt{1 - t^2}} D_X(x) = \frac{t}{\sqrt{1 - t^2}} X \end{aligned}$$

3. Now we obtain

$$\begin{aligned} H_t(X,Y) &= \langle S_\nu(X), Y\rangle_{g_t} \\ &= \frac{t}{\sqrt{1-t^2}} g_t(X,Y) \end{aligned}$$

Thus for each $t \in (-1,1)$, $\lambda(t) := \frac{t}{\sqrt{1-t^2}} \in \mathbb{R}$ *is the constant.*

Problem A.2. *Find* $\lambda(t)$ *for all* $t \in (-1,1)$.

Answer A.2. *As in the previous problem we found*

$$\lambda(t) := \frac{t}{\sqrt{1-t^2}} \qquad \forall\ t \in (-1,1)$$

Problem A.3. *When is* M_t *totally geodesic?*

Answer A.3. *Recall* M_t *is totally geodesic if the second fundamental form vanishes for any normal vector, at any point* $p \in M_t$. *Thus we equate*

$$\lambda(t) = \frac{t}{\sqrt{1-t^2}} = 0$$

and find this is only possible for $t = 0$. *Since the codimension is 1, the normal vector* ν *spans the whole* $(TM_t)^\perp$ *at each point on* M_t. *Then indeed* M_0 *is totally geodesic for the vanishing* $\lambda(0) = 0$.

Problem 2

Let (x, y) be coordinates of $\mathbb{R}^2$, and let $z = x + iy$.

Problem A.4. *Let*

$$H^2 := \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

be the upper half plane, equipped with the Riemannian metric

$$g = \frac{dx^2 + dy^2}{y^2} = -\frac{4dzd\overline{z}}{|z - \overline{z}|^2}$$

Prove that the map

$$f : z \mapsto \frac{az + b}{cz + d}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$ *is an isometry of* (H^2, g).

Answer A.4. *To show isometry, let's compute the pullback metric* f^*g.

$$\begin{aligned} f^*g &= -\frac{4}{|\frac{az+b}{cz+d} - \frac{a\overline{z}+b}{c\overline{z}+d}|^2} d(\frac{az + b}{cz + d}) d(\frac{a\overline{z} + b}{c\overline{z} + d}) \\ &= -\frac{4}{|\frac{ad(z-\overline{z})+bc(\overline{z}-z)}{c^2|z|^2+cd(z+\overline{z})+d^2}|^2} d(\frac{az + b}{cz + d}) d(\frac{a\overline{z} + b}{c\overline{z} + d}) \end{aligned}$$

Let's compute the two differentials first.

$$\begin{aligned} d(\frac{az + b}{cz + d}) &= (\frac{a}{cz + d} - \frac{acz + bc}{(cz + d)^2}) dz \\ &= \frac{ad - bc}{(cz + d)^2} dz \\ &= \frac{1}{(cz + d)^2} dz \qquad \textit{using } \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \\ d(\frac{a\overline{z} + b}{c\overline{z} + d}) &= \frac{ad - bc}{(c\overline{z} + d)^2} d\overline{z} \\ &= \frac{1}{(c\overline{z} + d)^2} d\overline{z} \end{aligned}$$

Thus it suffices to compute

$$\begin{aligned} f^*g &= -\frac{4}{|\frac{ad(z-\overline{z})+bc(\overline{z}-z)}{c^2|z|^2+cd(z+\overline{z})+d^2}|^2} \frac{1}{(cz + d)^2} \frac{1}{(c\overline{z} + d)^2} dzd\overline{z} \\ &= -\frac{4}{|ad(z - \overline{z}) + bc(\overline{z} - z)|^2} dzd\overline{z} \\ &= -\frac{4}{|z - \overline{z}|^2 |ad - bc|^2} dzd\overline{z} \\ &= -\frac{4dzd\overline{z}}{|z - \overline{z}|^2} = g \end{aligned}$$

Notice that positive determinant yields invertibility of the matrix, hence f *itself is a diffeomorphism. Thus using* $ad - bc = 1$ *makes* f *an isometry of* (H^2, g).

Problem A.5. *Let*

$$D^2 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

be equipped with the Riemannian metric

$$h = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2} = \frac{4dzd\overline{z}}{(1 - |z|^2)^2}$$

Prove that the map

$$f : z \mapsto \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}}$$

where $\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(1, 1)$ *is an isometry of* (D^2, h).

Answer A.5. *Recall that* $\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(1,1)$ *if* $|\alpha|^2 - |\beta|^2 = 1$*. Let's compute pullback* f^*h

$$\begin{aligned} f^*h &= \frac{4}{(1 - |\frac{\alpha z+\beta}{\overline{\beta}z+\overline{\alpha}}|^2)^2} d(\frac{\alpha z+\beta}{\overline{\beta}z+\overline{\alpha}}) d(\frac{\overline{\alpha z}+\overline{\beta}}{\beta\overline{z}+\alpha}) \\ &= \frac{4|\overline{\beta}z+\overline{\alpha}|^4}{(|\overline{\beta}z+\overline{\alpha}|^2 - |\alpha z+\beta|^2)^2} d(\frac{\alpha z+\beta}{\overline{\beta}z+\overline{\alpha}}) d(\frac{\overline{\alpha z}+\overline{\beta}}{\beta\overline{z}+\alpha}) \end{aligned}$$

We compute the differentials

$$\begin{aligned} d(\frac{\alpha z+\beta}{\overline{\beta}z+\overline{\alpha}}) &= (\frac{\alpha}{\overline{\beta}z+\overline{\alpha}} - \frac{\alpha z+\beta}{(\overline{\beta}z+\overline{\alpha})^2}\overline{\beta})dz \\ &= \frac{|\alpha|^2 - |\beta|^2}{(\overline{\beta}z+\overline{\alpha})^2}dz \\ &= \frac{1}{(\overline{\beta}z+\overline{\alpha})^2}dz \qquad \textit{using } |\alpha|^2 - |\beta|^2 = 1 \\ d(\frac{\overline{\alpha z}+\overline{\beta}}{\beta\overline{z}+\alpha}) &= \frac{1}{(\beta\overline{z}+\alpha)^2}d\overline{z} \\ d(\frac{\alpha z+\beta}{\overline{\beta}z+\overline{\alpha}})d(\frac{\overline{\alpha z}+\overline{\beta}}{\beta\overline{z}+\alpha}) &= \frac{1}{(|\overline{\beta}z+\overline{\alpha}|^2)^2}dzd\overline{z} = \frac{1}{|\overline{\beta}z+\overline{\alpha}|^4}dzd\overline{z} \end{aligned}$$

Thus

$$\begin{aligned} f^*h &= \frac{4}{(|\overline{\beta}z+\overline{\alpha}|^2 - |\alpha z+\beta|^2)^2}dzd\overline{z} \\ &= \frac{4}{(|\beta|^2|z|^2 + \overline{\alpha}\beta\overline{z} + \overline{\beta}z\alpha + |\alpha|^2 - |\alpha|^2|z|^2 - \beta\overline{\alpha z} - \overline{\beta}\alpha z - |\beta|^2)^2}dzd\overline{z} \\ &= \frac{4}{(1-|z|^2)^2}dzd\overline{z} = h \qquad \textit{using } |\alpha|^2 - |\beta|^2 = 1 \end{aligned}$$

Notice positive determinant makes f *a diffeomorphism. Thus* f *is an isometry of* (D^2, h)*.*

Problem 3

Let (M, g) be a complete Riemannian manifold of dimension $n \geq 2$. Suppose that there exists constants $a > 0$ and $c \geq 0$ s.t. for all pairs of points in M and for all minimizing geodesics $\gamma(s)$, parametrized by arc length s joining these points, we have

$$\mathrm{Ric}(\gamma'(s)) \geq a + \frac{\partial f}{\partial s}$$

along γ, where f is a function of s satisfying $|f(s)| \leq c$ along γ.

Problem A.6. *Show that M is compact.*

Answer A.6. *Take any two points $p, q \in M$, $p \neq q$. Since M is complete, by Hopf-Rinow there exists a minimizing geodesic connecting p and q with arc-length parametrization, denoted as γ*

$$\gamma : [0, L] \to M \qquad \gamma(0) = p, \qquad \gamma(L) = q$$

Our job would be to prove L has an upper bound uniform in p, q, hence L is totally bounded. Combining with M being complete, this yields M is compact.

We construct a variation by imposing a vector field along the curve γ. Let $\{e_1, \cdots, e_n\}$ be an ONB of T_pM where we choose $e_n := \gamma'(0)$. We do a parallel transport and denote $\{e_i(t)\}$ as the parallel transport of e_i along γ. We define our variation as

$$V_i(t) := \sin(\frac{\pi t}{L})e_i(t), \qquad i = 1, \cdots, n-1$$

Notice under such definition

$$V_i(0) = V_i(L) = 0 \qquad \forall\ i = 1, \cdots, n-1$$

Thus we have a family of proper variations $\{h_i\}_{i=1}^{n-1}$ associated to the variational field

$$h_i : (-\varepsilon, \varepsilon) \times [0, L] \to M \qquad (s, t) \mapsto h_i(s, t)$$

s.t.

$$\begin{aligned} h_i(0, t) &= \gamma(t) \\ \frac{\partial h_i}{\partial s}(0, t) &= V_i(t) \\ h_i(s, 0) &= \gamma(0) = p \qquad \textit{due to proper variation} \end{aligned}$$

For these variations we define the energy as

$$E_i(s) := \int_0^L |\frac{\partial h_i}{\partial t}(s, t)|^2 dt$$

Notice

$$\begin{aligned} E_i(s) &:= \int_0^L |\frac{\partial h_i}{\partial t}(s, t)|^2 \geq \frac{1}{L}\ell(h_i)^2 \qquad \textit{Cauchy Schwarz} \\ &\geq \frac{1}{L}\ell(\gamma)^2 \qquad \textit{since } \gamma \textit{ is geodesic} \\ &= E(\gamma) = E(0) \end{aligned}$$

Now that γ is geodesic and V_i are proper, we know $E_i'(0) = 0$, so this is indeed a minimum. Thus $E_i''(0) \geq 0$. Since h_i are proper variations, the second variation formula writes

$$\begin{aligned} \frac{1}{2}E_i''(0) &= -\int_0^L \left(\langle\frac{D^2 V_i}{dt^2}, V_i\rangle(t) + R(\gamma', V_i, \gamma', V_i)\right) dt \\ &= -\int_0^L \left(-(\frac{\pi^2}{L^2})\langle\sin(\frac{\pi t}{L})e_i(t), \sin(\frac{\pi t}{L})e_i(t)\rangle + \sin(\frac{\pi t}{L})^2 R(\gamma', e_i, \gamma', e_i)\right) dt \\ &= \int_0^L \left(\frac{\pi^2}{L^2}\sin(\frac{\pi t}{L})^2 - \sin(\frac{\pi t}{L})^2 R(e_n, e_i, e_n, e_i)\right) \\ \frac{1}{2}\frac{1}{n-1}\sum_{i=1}^{n-1} E_i''(0) &= \int_0^L \left(\frac{\pi^2}{L^2}\sin(\frac{\pi t}{L})^2 - \sin(\frac{\pi t}{L})^2 \mathrm{Ric}_p(e_n, e_n)\right) dt \\ &\leq \int_0^L \left(\frac{\pi^2}{L^2}\sin(\frac{\pi t}{L})^2 - \sin(\frac{\pi t}{L})^2 (a + \frac{\partial f}{\partial t}(t))\right) dt \end{aligned}$$

We compute the integral

$$
\begin{aligned}
\int_0^L \frac{\pi^2}{L^2}\sin(\frac{\pi t}{L})^2 dt &= \frac{\pi^2}{2L^2}\int_0^L (1-\cos(2\frac{\pi t}{L}))dt \\
&= \frac{\pi^2}{2L} - \frac{\pi^2}{2L^2}\,\frac{L}{2\pi}\sin(2\frac{\pi t}{L})\Big|_0^L = \frac{\pi^2}{2L}
\end{aligned}
$$

$$
-\int_0^L a\sin(\frac{\pi t}{L})^2 dt = -\frac{aL}{2}
$$

$$
\begin{aligned}
\int_0^L \frac{\partial f}{\partial t}(t)\sin(\frac{\pi t}{L})^2 dt &= \sin(\frac{\pi t}{L})^2 f(t)\Big|_0^L - \int_0^L \frac{2\pi}{L}\sin(\frac{\pi t}{L})\cos(\frac{\pi t}{L})f(t)dt \\
&= -\int_0^L \frac{\pi}{L}\sin(\frac{2\pi t}{L})f(t)dt
\end{aligned}
$$

$$
|\int_0^L \frac{\partial f}{\partial t}(t)\sin(\frac{\pi t}{L})^2 dt| \le \pi c
$$

Combining above and imposing $E''(0) \ge 0$ *yields*

$$
\begin{aligned}
0 &\le \frac{\pi^2}{2L} - \frac{aL}{2} + \pi c \\
\frac{a}{2}L^2 - \pi c L - \frac{\pi^2}{2} &\le 0
\end{aligned}
$$

Using that $a > 0$ *so LHS is quadratic polynomial with positive opening, if the root exists, L has upper bound*

$$
C = C(a,c)
$$

independent of p *and* q*. More precisely*

$$
\begin{aligned}
L &= \frac{\pi c \pm \sqrt{\pi^2 c^2 + \pi^2 a}}{a} = \pi\frac{c \pm \sqrt{c^2+a}}{a} \\
&\le \pi\frac{c+\sqrt{c^2+a}}{a} =: C(a,c) < \infty
\end{aligned}
$$

Problem A.7. *Calculate an estimate for the diameter of* M*, and observe that if* f*,* $c = 0$*, we obtain the Theorem of Bonnet-Myers.*

Answer A.7. *Recall*

$$
\operatorname{diam}(M,g) := \sup_{p,q\in M} d(p,q)
$$

and we already obtained

$$
L = d(p,q) \le \pi\frac{c+\sqrt{c^2+a}}{a} < \infty \qquad \forall\ p,\, q \in M
$$

Thus we can upper bound the diameter of M *by*

$$
\operatorname{diam}(M,g) \le \pi\frac{c+\sqrt{c^2+a}}{a}
$$

If $c = 0$ *and* $f = 0$*, then*

$$
\operatorname{diam}(M,g) \le \pi\frac{\sqrt{a}}{a}
$$

Notice $\frac{\partial f}{\partial t} = 0$ *and so*

$$
\operatorname{Ric}(\gamma'(t)) \ge a + \frac{\partial f}{\partial s} = a =: \frac{1}{r^2}
$$

then

$$
\pi\frac{\sqrt{a}}{a} = \pi r
$$

and we recover the Bonnet-Myers Theorem.

Problem 4

Let (x^1, x^2, x^3) be coordinates on $\mathbb{R}^3$. Given any $\rho > 0$, define

$$S_\rho := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sum_{i=1}^{3}(x^i)^2 = \rho^2\}$$

Let

$$(r, \phi, \theta) \in (0, \infty) \times [0, \pi) \times [0, 2\pi)$$

be spherical coordinates on $\mathbb{R}^3$, i.e.

$$\begin{aligned} x^1 &= r\sin(\phi)\cos(\theta) \\ x^2 &= r\sin(\phi)\sin(\theta) \\ x^3 &= r\cos(\phi) \end{aligned}$$

The Euclidean metric on $\mathbb{R}^3$ is given by

$$g_0 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = dr^2 + r^2(d\phi^2 + \sin^2(\phi)d\theta^2)$$

Problem A.8. *Let*

$$\begin{aligned} d\Omega^1 &:= dx^2 \wedge dx^3 \\ d\Omega^2 &:= dx^3 \wedge dx^1 \\ d\Omega^3 &:= dx^1 \wedge dx^2 \end{aligned}$$

and let

$$i_\rho : S_\rho \hookrightarrow \mathbb{R}^3$$

be the inclusion. Show that

$$i_\rho^*(d\Omega^i) = i_\rho^*(x^i r\sin(\phi)d\phi \wedge d\theta)$$

Answer A.8. *Let's compute using brute force*

$$\begin{aligned}
i_\rho^*(d\Omega^1) &= d(\rho\sin(\phi)\sin(\theta)) \wedge d(\rho\cos(\phi)) \\
&= (\rho\cos(\phi)\sin(\theta)d\phi + \rho\sin(\phi)\cos(\theta)d\theta) \wedge (-\rho\sin(\phi)d\phi) \\
&= \rho^2\sin^2(\phi)\cos(\theta)d\phi \wedge d\theta = \rho\sin(\phi)(\rho\sin(\phi)\cos(\theta))d\phi \wedge d\theta \\
&= i_\rho^*(r\sin(\phi)x^1 d\phi \wedge d\theta) \\
i_\rho^*(d\Omega^2) &= d(\rho\cos(\phi)) \wedge d(\rho\sin(\phi)\cos(\theta)) \\
&= (-\rho\sin(\phi)d\phi) \wedge (\rho\cos(\phi)\cos(\theta)d\phi - \rho\sin(\phi)\sin(\theta)d\theta) \\
&= \rho^2\sin^2(\phi)\sin(\theta)d\phi \wedge d\theta \\
&= i_\rho^*(r\sin(\phi)x^2 d\phi \wedge d\theta) \\
i_\rho^*(d\Omega^3) &= d(\rho\sin(\phi)\cos(\theta)) \wedge d(\rho\sin(\phi)\sin(\theta)) \\
&= (\rho\cos(\phi)\cos(\theta)d\phi - \rho\sin(\phi)\sin(\theta)d\theta) \wedge (\rho\cos(\phi)\sin(\theta)d\phi + \rho\sin(\phi)\cos(\theta)d\theta) \\
&= \rho^2\cos^2(\theta)\sin(\phi)\cos(\phi)d\phi \wedge d\theta + \rho^2\sin^2(\theta)\sin(\phi)\cos(\phi)d\phi \wedge d\theta \\
&= \rho^2\sin(\phi)\cos(\phi)d\phi \wedge d\theta \\
&= i_\rho^*(r\sin(\phi)x^3 d\phi \wedge d\theta)
\end{aligned}$$

Problem A.9. *Consider an asymptotically flat Riemannian metric on $\mathbb{R}^3$ of the form*

$$g = u(r)^2dr^2 + r^2(d\phi^2 + \sin^2(\phi)d\theta^2)$$

where $u(r) > 0$ is a C^∞ function in r, and there exists constants $M \geq 0$ and $R > 0$ such that

$$u(r) = 1 + \frac{M}{r} + o(\frac{1}{r}) \qquad \forall\ r > R$$

Prove that

$$\lim_{\rho\to\infty} \frac{1}{16\pi}\int_{S_\rho}\sum_{i,j=1}^{3}(\partial_j g_{ij} - \partial_i g_{jj})d\Omega^i = M$$

Answer A.9. *This first thing to notice is g_{ij} is small perturbation of δ_{ij}, i.e.*

$$
\begin{aligned}
g &= (u(r)^2 - 1)dr^2 + g_0 \\
&= (\frac{2M}{r} + o(\frac{1}{r}))dr^2 + g_0
\end{aligned}
$$

Notice

$$
\begin{aligned}
dr^2 &= d(\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2})d(\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}) \\
&= \frac{1}{r^2}(x^1dx^1 + x^2dx^2 + x^3dx^3)^2 \\
&= \frac{x^ix^j}{r^2}dx^idx^j
\end{aligned}
$$

So

$$
\begin{aligned}
g_{ij} &= \delta_{ij} + \frac{1}{r^2}(\frac{2M}{r} + o(\frac{1}{r}))x^ix^j \\
\partial_j g_{ij} &= (-\frac{6M}{r^4}\frac{x^j}{r} + o(\frac{1}{r^4})\frac{x^j}{r})x^ix^j + (\frac{2M}{r^3} + o(\frac{1}{r^3}))(x^i + x^i\delta_{ij}) \\
&= (-\frac{6M}{r^5} + o(\frac{1}{r^5}))x^i(x^j)^2 + (\frac{2M}{r^3} + o(\frac{1}{r^3}))(x^i + x^i\delta_{ij}) \\
\partial_i g_{jj} &= (-\frac{6M}{r^4}\frac{x^i}{r} + o(\frac{1}{r^4})\frac{x^i}{r})(x^j)^2 + (\frac{2M}{r^3} + o(\frac{1}{r^3}))2x^j\delta_{ij} \\
&= (-\frac{6M}{r^5} + o(\frac{1}{r^5}))x^i(x^j)^2 + (\frac{2M}{r^3} + o(\frac{1}{r^3}))2x^j\delta_{ij} \\
\partial_j g_{ij} - \partial_i g_{jj} &= (\frac{2M}{r^3} + o(\frac{1}{r^3}))(x^i + x^i\delta_{ij} - 2x^j\delta_{ij}) \\
\sum_{i,j=1}^{3}(\partial_j g_{ij} - \partial_i g_{jj}) &= \sum_{i=1}^{3}(\frac{2M}{r^3} + o(\frac{1}{r^3}))(3x^i + x^i - 2x^i) \\
&= \sum_{i=1}^{3}(\frac{2M}{r^3} + o(\frac{1}{r^3}))2x^i
\end{aligned}
$$

Now let's integrate against the volume forms

$$
\begin{aligned}
\sum_{i,j=1}^{3}(\partial_j g_{ij} - \partial_i g_{jj})d\Omega^i &= \sum_{i=1}^{3}(\frac{2M}{r^3} + o(\frac{1}{r^3}))2(x^i)^2 r\sin(\phi)d\phi \wedge d\theta \\
&= (\frac{2M}{r^3} + o(\frac{1}{r^3}))2r^3\sin(\phi)d\phi \wedge d\theta \\
\int_{S_\rho}\sum_{i,j=1}^{3}(\partial_j g_{ij} - \partial_i g_{jj})d\Omega^i &= 4M\int_0^{2\pi}\int_0^{\pi}\sin(\theta)d\phi d\theta + o(1) \\
&= 8M\pi\int_0^{\pi}\sin(\phi)d\phi + o(1) \\
&= 8M\pi\,(-\cos(\phi))|_0^{\pi} + o(1) \\
&= 16M\pi + o(1)
\end{aligned}
$$

$$
\lim_{\rho\to 0}\frac{1}{16\pi}\int_{S_\rho}\sum_{i,j=1}^{3}(\partial_j g_{ij} - \partial_i g_{jj})d\Omega^i = M
$$

Problem 5

Consider the metric

$$h = -x^2 dt^2 + dx^2$$

on $N = (-\infty, \infty)_t \times (0, \infty)_x$.

Problem A.10. *Is $x = 0$ a curvature singularity?*

Answer A.10. *In order to see whether $x = 0$ is curvature singularity, let's just compute the Christoffel symbols, and then the curvature and see if it blows up at $x = 0$.*

$$\begin{aligned} h_{00} &= -x^2 \\ h^{00} &= -\frac{1}{x^2} \\ h_{11} &= h^{11} = 1 \\ h_{00,1} &= -2x \end{aligned}$$

using $\Gamma^\ell_{ij} = \frac{1}{2}\sum_{k=0}^{1} h^{\ell k}\left(h_{ik,j} + h_{kj,i} - h_{ij,k}\right)$ we obtain

$$\begin{aligned} \Gamma^0_{01} &= \Gamma^0_{10} = \frac{1}{x} \\ \Gamma^1_{00} &= x \end{aligned}$$

Then we want to compute $R^m_{ijk} = \frac{\partial}{\partial x_j}\Gamma^m_{ik} - \frac{\partial}{\partial x_i}\Gamma^m_{jk} + \sum_\ell \Gamma^\ell_{ik}\Gamma^m_{j\ell} - \sum_\ell \Gamma^\ell_{jk}\Gamma^m_{i\ell}$ we see

$$\begin{aligned} R^1_{010} &= \frac{\partial}{\partial x}\Gamma^1_{00} - \frac{\partial}{\partial t}\Gamma^1_{10} + \Gamma^1_{00}\Gamma^1_{11} - \Gamma^0_{10}\Gamma^1_{00} \\ &= 1 - \frac{1}{x}x = 0 \end{aligned}$$

and all other Riemannian tensor components vanish trivially. Thus the manifold (N, h) is flat, and has no curvature singularity.

Problem A.11. *Can you isometrically embed (N, h) into a larger manifold?*

Answer A.11. *This part of the solution is based on [Wal84] page 149 - 151. One can make a series of change of variables to isometrically embed (N, h) into $(\mathbb{R}^2, -dT^2 + dX^2)$. In each 'change of variable' we denote $\tilde{h}$ as the new metric and h as the original metric, abuse of notation. One start by computing null geodesics. The null condition reads*

$$0 = -x^2\dot{t}^2 + \dot{x}^2$$

where the dot denotes derivative w.r.t. affine parameter. Rearranging yields

$$\left(\frac{dt}{dx}\right)^2 = \frac{1}{x^2}$$

solving for the ODE gives

$$t = \pm\log(x) + C$$

where $+$ correspondences to outgoing geodesic, and $-$ to ingoing geodesic. Hence we change into null coordinates

$$\begin{aligned} u &= t - \log(x) \\ v &= t + \log(x) \end{aligned}$$

and our metric takes the form

$$\begin{aligned} \tilde{h} &= -\exp(v-u)dudv \\ &= -x^2 d(t - \log(x))d(t + \log(x)) \\ &= -x^2(dt - \frac{1}{x}dx)(dt + \frac{1}{x}dx) \\ &= -x^2(dt^2 - \frac{1}{x^2}dx^2) = h \end{aligned}$$

But this still corresponds to the region $x > 0$ in N. To extend beyond $x = 0$, we define

$$\begin{aligned} U &= -e^{-u} \\ V &= e^{v} \end{aligned}$$

Thus our metric again writes

$$\begin{aligned}\tilde{h} &= -dUdV = -d(-e^{-u})d(e^{v}) \\ &= -e^{-u}e^{v}dudv = h\end{aligned}$$

Now there is no longer singularity at $U = 0$ or $V = 0$ so we extend via

$$\begin{aligned}T &= \frac{U+V}{2} \\ X &= \frac{V-U}{2}\end{aligned}$$

and obtain

$$\begin{aligned}\tilde{h} &= -dT^2 + dX^2 \\ &= -d(\frac{U+V}{2})^2 d(\frac{V-U}{2})^2 \\ &= -dUdV = h\end{aligned}$$

Hence composing all change of variables, $(N, h) \hookrightarrow (\mathbb{R}^2, -dT^2 + dX^2)$. If one is unhappy about this we can also write out the change of variable directly

$$\begin{aligned}x &= (X^2 - T^2)^{\frac{1}{2}} \\ t &= \tanh^{-1}(\frac{T}{X})\end{aligned}$$

or equivalently

$$\begin{aligned}T &= x\sinh(t) \\ X &= x\cosh(t)\end{aligned}$$

Now we directly verify

$$\begin{aligned}-dT^2 + dX^2 &= -(\sinh(t)dx + x\cosh(t)dt)^2 + (\cosh(t)dx + x\sinh(t)dt)^2 \\ &= -\sinh^2(t)dx^2 - 2x\sinh(t)\cosh(t)dxdt - x^2\cosh^2(t)dt^2 + \cosh(t)^2dx^2 + 2x\sinh(t)\cosh(t)dxdt + x^2\sinh^2(t)dt^2 \\ &= -x^2dt^2 + dx^2\end{aligned}$$

Hence this embedding is isometric.

References

[dC92] M.P. do Carmo. *Riemannian Geometry*. Mathematics (Boston, Mass.). Birkhäuser, 1992.

[Wal84] Robert M. Wald. *General Relativity*. Chicago Univ. Pr., Chicago, USA, 1984.