Viscosity Solutions to Elliptic Partial Differential Equations

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1 Viscosity Solutions

1.1 Alexandroff Maximum Principle

We provide the problem setting

(i) $\Omega \subset \mathbb{R}^n$ bounded and connected.

(ii) Uniformly Elliptic. Given $\lambda, \Lambda > 0$, let $a_{ij}(x) \in C(\Omega)$ $s.t.$ $\lambda |\xi|^2 \leq a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \ \forall\, x \in \Omega,\, \xi \in \mathbb{R}^n$.

- Let operator L in Ω $s.t.$ $Lu \equiv a_{ij}(x) D_{ij} u \ for\ u \in C^2(\Omega)$. We call $u \in C^2(\Omega)$ a supersolution of $Lu = 0$ in Ω if $Lu \leq 0$. Notice $\forall\ \varphi \in C^2(\Omega)$ $s.t.$ $L\varphi > 0$, we have $L(u - \varphi) < 0\ in\ \Omega \implies u - \varphi$ has no local interior minimum. Hence if $\exists\ x_0 \in \Omega$ $s.t.$ $u - \varphi$ attains local mimimum, we know $L\varphi(x_0) \leq 0$.
- Geometrically, $u - \varphi$ has a local minimum at $x_0 \in \Omega \implies \varphi$ touches u from below at x_0 up to constant.

Definition 1.1 (Viscosity Solution)**.** *$f \in C(\Omega)$. We call $u \in C(\Omega)$ a viscosity **super**solution of $Lu = f$ in Ω if*

$$\forall\ x_0 \in \Omega \ and\ \forall\ \varphi \in C^2(\Omega)\ \ s.t.\ u - \varphi\ has\ a\ local\ minimum\ at\ x_0 \implies L\varphi(x_0) \leq f(x_0)$$

*$u \in C(\Omega)$ a viscosity **sub**solution of $Lu = f$ in Ω if*

$$\forall\ x_0 \in \Omega \ and\ \forall\ \varphi \in C^2(\Omega)\ \ s.t.\ u - \varphi\ has\ a\ local\ maximum\ at\ x_0 \implies L\varphi(x_0) \geq f(x_0)$$

$u \in C(\Omega)$ a viscosity solution if both viscosity supersolution and subsolution.

Now we define weakly the class of solutions to elliptic pdes.

- $\forall\ \varphi \in C^2$ at x_0, define $e_1, \cdots, e_n$ eigenvalues of Hessian $D^2\varphi(x_0)$. We see

$$L\varphi(x_0) \leq 0 \iff \sum_{i,j=1}^{n} a_{ij}(x_0) D_{ij}\varphi(x_0) \leq 0 \implies \sum_{k=1}^{n} \alpha_k e_k \leq 0 \quad for\ \lambda \leq \alpha_k \leq \Lambda$$

$$\iff \sum_{e_i > 0} \alpha_i e_i + \sum_{e_i < 0} \alpha_i e_i \leq 0 \iff \sum_{e_i > 0} \alpha_i e_i \leq \sum_{e_i < 0} \alpha_i (-e_i) \implies \lambda \sum_{e_i > 0} e_i \leq \Lambda \sum_{e_i < 0} (-e_i)$$

This is to say, at x_0, positive eigenvalues of $D^2\varphi(x_0)$ are controlled by its negative eigenvalues.

Definition 1.2 (Solution Class $\mathcal{S}(\lambda, \Lambda, f)$)**.** *$f \in C(\Omega)$. We say $u \in C(\Omega)$ belongs to $\mathcal{S}^+(\lambda, \Lambda, f)$ if*

$$\forall\ x_0 \in \Omega \ and\ \forall\ \varphi \in C^2(\Omega)\ \ s.t.\ u - \varphi\ has\ a\ local\ minimum\ at\ x_0 \implies \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i \leq f(x_0)$$

where $e_1, \cdots, e_n$ are eigenvalues of $D^2\varphi(x_0)$. Similarly, $u \in C(\Omega)$ belongs to $\mathcal{S}^-(\lambda, \Lambda, f)$ if

$$\forall\ x_0 \in \Omega \ and\ \forall\ \varphi \in C^2(\Omega)\ \ s.t.\ u - \varphi\ has\ a\ local\ maximum\ at\ x_0 \implies \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i \geq f(x_0)$$

$\mathcal{S}(\lambda,\Lambda,f) = \mathcal{S}^+(\lambda,\Lambda,f) \cap \mathcal{S}^-(\lambda,\Lambda,f)$.

Remark 1.1. *(i) Viscosity supersolution to $Lu = f$ in Ω under uniform ellipticity $\implies$ $u \in \mathcal{S}^+(\lambda,\Lambda,f)$. Viscosity subsolution to $Lu = f$ in Ω under uniform ellipticity $\implies$ $u \in \mathcal{S}^-(\lambda,\Lambda,f)$.*

(ii) $\mathcal{S}^+(\lambda,\Lambda,f)$, $\mathcal{S}^-(\lambda,\Lambda,f)$ also include solutions to fully non-linear pdes.

Example 1.1 (Pucci Equations)**.** $0 < \lambda \le \Lambda$.

- *Let $\mathcal{A}_{\lambda,\Lambda} := \{\, A$ is $n \times n$ symmetric matrix $\mid \lambda|\xi|^2 \le A_{ij}\,\xi_i\xi_j \le \Lambda|\xi|^2 \ \ \forall\ \xi \in \mathbb{R}^n\}$*
- *For M $n \times n$ symmetric, we define Pucci extremal operator*

$$\mathcal{M}^-(M) \equiv \mathcal{M}^-(\lambda,\Lambda,M) := \inf_{A\in\mathcal{A}_{\lambda,\Lambda}} A_{ij}M_{ij}, \quad \mathcal{M}^+(M) \equiv \mathcal{M}^+(\lambda,\Lambda,M) := \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} A_{ij}M_{ij}$$

If denote $e_1,\cdots,e_n$ as eigenvalues of M, we see

$$\mathcal{M}^-(\lambda,\Lambda,M) = \lambda\sum_{e_i>0} e_i + \Lambda\sum_{e_i<0} e_i, \qquad \mathcal{M}^+(\lambda,\Lambda,M) = \Lambda\sum_{e_i>0} e_i + \lambda\sum_{e_i<0} e_i$$

- *Pucci's Equations are given for $f, g \in C(\Omega)$*

$$\mathcal{M}^-(\lambda,\Lambda,M) = f, \quad \mathcal{M}^+(\lambda,\Lambda,M) = g$$

Hence $u \in \mathcal{S}^+(\lambda,\Lambda,f) \iff \mathcal{M}^-\left(\lambda,\Lambda,D^2u\right) \le f$ in viscosity sense, i.e.

$$\forall\ x_0 \in \Omega \text{ and } \forall\ \varphi \in C^2(\Omega) \text{ s.t. } u-\varphi \text{ has a local minimum at } x_0 \implies \mathcal{M}^-\left(\lambda,\Lambda,D^2\varphi(x_0)\right) \le f$$

$u \in \mathcal{S}^-(\lambda,\Lambda,f) \iff \mathcal{M}^+\left(\lambda,\Lambda,D^2u\right) \ge g$ in viscosity sense, i.e.

$$\forall\ x_0 \in \Omega \text{ and } \forall\ \varphi \in C^2(\Omega) \text{ s.t. } u-\varphi \text{ has a local maximum at } x_0 \implies \mathcal{M}^-\left(\lambda,\Lambda,D^2\varphi(x_0)\right) \ge g$$

- *For any two $n \times n$ symmetric matrices M, N*

$$\mathcal{M}^-(M) + \mathcal{M}^-(N) \le \mathcal{M}^-(M+N) \le \mathcal{M}^+(M) + \mathcal{M}^-(N) \le \mathcal{M}^+(M+N) \le \mathcal{M}^+(M) + \mathcal{M}^+(N)$$

Now we derive Alexandroff Maximum Principle for viscosity solutions. Let $v \in C(\Omega)$ for open convex set Ω.

- Convex Envelope of v in Ω is $\Gamma(v)(x) := \sup_L\{L(x) \mid L \le v \ in\ \Omega,\ L\ an\ affine\ function\}\ \forall\ x \in \Omega$. It is indeed convex function, as $\Gamma(v)(t\,x_1 + (1-t)\,x_2) \le t\,\Gamma(v)(x_1) + (1-t)\,\Gamma(v)(x_2)$ for $t \in [0,1]$, $x_1,\ x_2 \in \Omega$.
- $\{x \in \Omega \mid v(x) = \Gamma(v)(x)\}$ is Lower Contact Set of v. Points in the contact set are contact points.
- We need classical Alexandroff Maximum Principle

  Lemma 1.1. *$u \in C^{1,1}(B_1)$, with $u \ge 0$ on ∂B_1. Then with Γ_u as convex envelope of $-u^- = \min\{u, 0\}$,*

$$\sup_{B_1} u^- \le c(n)\left(\int_{B_1\cap\{u=\Gamma_u\}} \det D^2u\right)^{\frac{1}{n}}$$

Theorem 1.1 (Alexandroff Maximum Principle - Viscosity Version)**.** *$u \in \mathcal{S}^+(\lambda,\Lambda,f)$ in B_1 with $u \ge 0$ on ∂B_1 for $f \in C(\Omega)$. Then with Γ_u as convex envelope of $-u^- = \min\{u, 0\}$,*

$$\sup_{B_1} u^- \le c(n,\ \lambda,\ \Lambda)\left(\int_{B_1\cap\{u=\Gamma_u\}} \left(f^+\right)^n\right)^{\frac{1}{n}}$$

Proof. (i) We observe $\Gamma_u(x) := \sup_L\{L(x) \mid L \le \min\{u,0\} \textit{ in } B_1,\ L \textit{ an affine function}\}$. Let x_0 be contact point, *i.e.*, $u(x_0) = \Gamma_u(x_0)$. WLOG take $x_0 = 0$, and rechoose a frame where $u \ge 0$ in B_1 with $u(0) = 0$. The latter makes sense by substracting a supporting plane at $x_0 = 0$. We first show that at the contact set $x_0 = 0,\ f(0) \ge 0$. Take $h(x) = -\epsilon\frac{|x|^2}{2}$ in B_1. Then $u - h = u + \epsilon\frac{|x|^2}{2}$ has minimum at $x_0 = 0$ since $u \ge 0$ in B_1 and $u(0) = 0$. We use that $u \in \mathcal{S}^+(\lambda, \Lambda, f) \implies \lambda\sum_{e_i>0} e_i + \Lambda\sum_{e_i<0} e_i \le f(x_0)$. Here we need to compute eigenvalues for $D^2h(0)$, which are $e_i = -\epsilon\ \forall\ i$, all negative. Hence $-n\Lambda\epsilon \le f(0)$. Take $\epsilon \to 0$ gives $0 \le f(0)$.

(ii) We show that at $x_0 \in \Omega$ contact point, for some affine function L, some constant $C = C(n, \lambda, \Lambda) > 0$ and any x close to x_0 *s.t.* $\epsilon(x) \to 0$ as $x \to x_0$,

$$L(x) \le \Gamma_u(x) \le L(x) + C\{f(x_0) + \epsilon(x)\}|x - x_0|^2 \quad \forall\ x \in B_1$$

As before, we choose $x_0 = 0$, and since the growth rate for L is controlled by quadratic term $|x|^2$, it suffices to prove

$$0 \le \Gamma_u(x) \le C\{f(0) + \epsilon(x)\}|x|^2 \quad \forall\ x \in B_1$$

We need to estimate for small $0 < r \ll 1$, $C_r := \frac{1}{r^2}\max_{\overline{B_r}}\Gamma_u(x)$. We first fix $r > 0$. Since Γ_u is convex function in $B_r \subset B_1$, we know Γ_u attains maximum in $\overline{B_r}$ at some point $(0, \cdots, 0, r)$ on the boundary. Notice the set $\{x \in B_1 \mid \Gamma_u(x) \le \Gamma_u(0, \cdots, 0, r)\}$ contains B_r and is convex. This is because

$$\forall\ x \in B_r,\ \Gamma_u(x) \le \max_{\overline{B_r}}\Gamma_u(x) \le \Gamma_u(0, \cdots, 0, r)$$

$$\forall\ x_1,\ x_2 \in \{\Gamma_u(x) \le \Gamma_u(0, \cdots, 0, r)\}, \textit{ and } t \in [0,1],\ \textit{we have}$$

$$\Gamma_u(tx_1 + (1-t)x_2) \le t\Gamma_u(x_1) + (1-t)\Gamma_u(x_2) \le \Gamma_u(0, \cdots, 0, r) \implies tx_1 + (1-t)x_2 \in \{\Gamma_u(x) \le \Gamma_u(0, \cdots, 0, r)\}$$

Hence it follows that $\forall\ (x',\ r) \in B_1$, we have $C_r r^2 = \Gamma_u(0, \cdots, 0, r) \le \Gamma_u(x', r)$.

(iii) Take $N > 0$ to be determined. Let $R_r = \{(x', x_n) \in B_1 \mid |x'| \le Nr,\ |x_n| \le r\}$. We construct a quadratic polynomial that touches u from below in R_r and curves up fast. Let $b > 0$ and $h(x) = (x_n + r)^2 - b|x'|^2$

- for $x_n = -r$, $h(x) = -b|x'|^2 \le 0$
- for $|x'| = Nr$, $h(x) = (x_n + r)^2 - bN^2r^2 \le 4r^2 - bN^2r^2 = (4 - bN^2)r^2 \le 0 \quad \textit{if let } b = \frac{4}{N^2}$
- for $x_n = r$, $h(x) = 4r^2 - b|x'|^2 \le 4r^2$

Now let $\widetilde{h}(x) := \frac{C_r}{4}h(x) = \frac{C_r}{4}(x_n + r)^2 - \frac{C_r}{N^2}|x'|^2$. Recall we chose $u \ge 0$ on B_1 with $u(0) = 0$, and $\Gamma_u \le u$ due to convex envelope

$$\textit{on } \partial R_r \begin{cases} \widetilde{h}(x', r) \le \frac{C_r}{4}(2r)^2 = C_r r^2 = \Gamma_u(0, \cdots, 0, r) \le \Gamma_u(x', r) \le u(x', r) & \textit{if } x_n = r \\ \widetilde{h}(x) \le 0 \le \Gamma_u(x) \le u(x) & \textit{otherwise} \end{cases}$$

$$\widetilde{h}(0) = \frac{C_r r^2}{4} > 0 = \Gamma_u(0) = u(0) \quad \textit{at } 0$$

Hence lowering $\widetilde{h}$ properly we see $u - \widetilde{h}$ has local minimum somewhere inside R_r. We compute eigenvalues of $D^2\widetilde{h}$, $e_1 = \frac{C_r}{2}$, $e_2, \cdots, e_n = -2\frac{C_r}{N^2}$. Again we use that

$$u \in \mathcal{S}^+(\lambda, \Lambda, f) \implies \lambda\frac{C_r}{2} - 2\Lambda(n-1)\frac{C_r}{N^2} \le \max_{\overline{R_r}} f$$

Choose $N = N(n, \lambda, \Lambda)$ large so that $C_r \le \frac{4}{\lambda}\max_{\overline{R_r}} f \iff \max_{\overline{B_r}}\Gamma_u(x) \le \frac{4r^2}{\lambda}\max_{\overline{R_r}} f$. Note $\max_{\overline{R_r}} f \to f(0)$ as $r \to 0$, which coincides with $\Gamma_u(x) \le C\{f(0) + \epsilon(x)\}|x|^2$ for r^2 taking the place of $|x|^2$.

(iv) By above we have $\Gamma_u(x) \in C^{1,1}$ in B_1 and

$$\det D^2\Gamma_u(x) \le C(n,\lambda,\Lambda)(f(x))^n \quad a.e.\ x \in \{u = \Gamma_u\}$$

Apply Lemma 1.1 to Γ_u.

□

1.2 Harnack Inequality

We build up ingredients starting from Calderon-Zygmund. Recall we're in $\mathbb{R}^n$.

- Let Q_1 be unit cube. Cut into 2^n equally sized cubes, take as first generation.
- Do the same cutting for the smaller cubes. Repeat. Cubes from all generations are called dyadic cubes.
- Any $(k+1)$-generation cube Q comes from k-generation $\widetilde{Q}$, as predecessor of Q.

Lemma 1.2 (Calderon-Zygmund Decomposition). *$f \in L^1(Q_1)$, $f \ge 0$, and $\alpha > \frac{1}{|Q_1|}\int_{Q_1} f$ is fixed constant. Then $\exists$ sequence of nonoverlapping dyadic cubes $\{Q_j\} \subset Q_1$ s.t.*

$$f(x) \le \alpha \quad a.e.\ in\ Q_1 \setminus \bigcup_j Q_j, \qquad \alpha \le \frac{1}{|Q_j|}\int_{Q_j} f dx \le 2^n \alpha \quad \forall\ j$$

Proof. (i) Cut Q_1 into 2^n dyadic cubes. We design algorithm to keep cube Q if $\alpha \le \frac{1}{|Q|}\int_Q f$. For others keep cutting, and continue the process. Let $\{Q_j\}$ be the sequence of cubes we've kept. Note such process is infinite, *i.e.*, for any generation, there must exist some cube that needs to be cut. This is because if $\exists$ some generation *s.t.* all cubes are kept, then it's predecessor must be kept, by induction from the base case $\alpha > \frac{1}{|Q_1|}\int_{Q_1} f$, which contradicts it being cut.

(ii) Also, any predecessor Q of Q_j that we've kept has to satisfy $\frac{1}{|Q|}\int_Q f dx < \alpha$. But $|Q| = 2^n |Q_j|$, so for Q_j we've kept, $\alpha \le \frac{1}{|Q_j|}\int_{Q_j} f \le 2^n \frac{1}{|Q|}\int_Q f dx \le 2^n\alpha$.

(iii) Let $F = Q_1 \setminus \bigcup_j Q_j$, and $\forall\ x \in F$, by our choice of $\{Q_j\}$, there exists a subsequence of cubes $Q^i \ni x$ *s.t.*

$$\frac{1}{|Q^i|}\int_{Q^i} f < \alpha \quad and \quad diam\left(Q^i\right) \to 0 \quad as\ i\ \to \infty$$

By Lebesgue density theorm, $f \le \alpha \quad a.e.\ in\ F$.

□

Corollary 1.1. *Suppose measurable sets $A \subset B \subset Q_1$ satisfy:*

(i) $|A| < \delta$ for some $\delta \in (0,1)$

(ii) $\forall\ Q$ dyadic cube, $|A \cap Q| \ge \delta |Q| \implies \widetilde{Q} \subset B$ for $\widetilde{Q}$ predecessor of Q.

Then we have $|A| \le \delta |B|$.

Proof. Apply Lemma 1.2 to $f = \chi_A$ choosing $\alpha = \delta > \frac{1}{|Q|}\int_Q \chi_A = |A|$, we have a sequence of cubes $\{Q_j\}$ *s.t.*

$$\chi_A(x) \le \delta \quad a.e.\ in\ Q_1 \setminus \bigcup_j Q_j, \qquad \delta \le \frac{1}{|Q_j|}|A \cap Q_j| \le 2^n\delta \quad \forall\ j$$

But notice $\delta \in (0,1)$, so $\chi_A(x) \le \delta$ *a.e. in* $Q_1 \setminus \bigcup_j Q_j \iff \chi_A \equiv 0$ *a.e. in* $Q_1 \setminus \bigcup_j Q_j \iff A \subset \bigcup_j Q_j$ up to set of measure zero. Also notice reason why next generation of cubes occur is $\frac{1}{|\widetilde{Q_j}|}\int_{\widetilde{Q_j}} \chi_A = \frac{1}{|\widetilde{Q_j}|}\left|A \cap \widetilde{Q_j}\right| < \delta$.

Now by assumption (ii), since $\delta |Q_j| \le |A \cap Q_j|$, we have $\widetilde{Q_j} \subset B \ \forall \ j$, so

$$A \subset \bigcup_j \widetilde{Q_j} \subset B$$

upon relabelling $\widetilde{Q_j}$ so they're nonoverlapping, we get

$$|A| \le \sum_i \left| A \cap \widetilde{Q^i} \right| \le \delta \sum_i \left| \widetilde{Q^i} \right| \le \delta |B|$$

□

Now we prove lemmas that lead to Harnack Inequality. Let Q_r denote cube with side length $r \ge 0$. The following is key ingredient: If solution is small somewhere in Q_3, then it's under control in a good portion of Q_1.

Lemma 1.3. *$u \in \mathcal{S}^+ (\lambda, \Lambda, f)$ in $B_{2\sqrt{n}}$ for $f \in C\left(B_{2\sqrt{n}}\right)$. Then $\exists\ \epsilon_0 > 0$, $\mu \in (0,1)$ and $M > 1$ depending only on n, λ, Λ s.t. if*

$$u \ge 0 \ in \ B_{2\sqrt{n}}, \quad \inf_{Q_3} u \le 1, \quad \|f\|_{L^n\left(B_{2\sqrt{n}}\right)} \le \epsilon_0$$

we have $|\{u \le M\} \cap Q_1| > \mu$.

Proof. (i) Note $B_{1/4} \subset Q_1 \subset Q_3 \subset B_{2\sqrt{n}}$. Define g in $B_{2\sqrt{n}}$ by

$$g(x) := -M \left(1 - \frac{|x|^2}{4n}\right)^\beta \quad for\ large\ \beta > 0\ to\ be\ determined\ and\ some\ M > 0$$

We note that $g = 0$ on $\partial B_{2\sqrt{n}}$. We also choose M according to β so that $g \le -2$ in Q_3.
Let $w = u + g$ in $B_{2\sqrt{n}}$. We wish to show that by choosing β large, we have

$$w \in \mathcal{S}^+ (\lambda, \Lambda, f) \quad in \ B_{2\sqrt{n}} \setminus Q_1$$

The idea is to construct function g that is concave outside Q_1 so the contact set of $w = u+g$, *i.e.*, correction of u by g, occurs in Q_1. In fact, we localize where contact sets occur by choosing suitable functions.

(ii) Suppose φ is quadratic polynomial with property $w - \varphi$ has a local minimum at $x_0 \in B_{2\sqrt{n}}$. Rewrite to see $w - \varphi = u - (\varphi - g)$ has local minumum at x_0. By assumption $u \in \mathcal{S}^+(\lambda, \Lambda, f) \iff \mathcal{M}^-\left(\lambda, \Lambda, D^2 u\right) \le f$ in viscosity sense for $\mathcal{M}^-$ Pucci extremal operator, we have

$$\mathcal{M}^-\left(\lambda, \Lambda, D^2\varphi(x_0)\right) + \mathcal{M}^-\left(\lambda, \Lambda, -D^2 g(x_0)\right) \le \mathcal{M}^-\left(\lambda, \Lambda, D^2\varphi(x_0) - D^2 g(x_0)\right) \le f(x_0)$$

In order to show that $\mathcal{M}^-\left(\lambda, \Lambda, D^2\varphi(x_0)\right) \le f(x_0) \ \forall \ x_0 \in B_{2\sqrt{n}} \setminus Q_1$, we omit a portion in $B_{2\sqrt{n}}$ by choosing β large so that

$$\mathcal{M}^-\left(\lambda, \Lambda, -D^2 g(x_0)\right) \ge 0 \quad \forall \ x_0 \in B_{2\sqrt{n}} \setminus B_{1/4}$$

To do so, we first need to calculate Hessian of g

$$D_{ij} g(x) = \frac{M}{2n}\beta \left(1 - \frac{|x|^2}{4n}\right)^{\beta-1} \delta_{ij} - \frac{M}{4n^2}\beta(\beta-1)\left(1 - \frac{|x|^2}{4n}\right)^{\beta-2} x_i x_j$$

Let $x = (|x|, 0, \cdots, 0)$, then eigenvalues of $-D^2 g(x)$ are given by

$$e^+(x) = \frac{M}{2n}\beta\left(1 - \frac{|x|^2}{4n}\right)^{\beta-2}\left(\frac{2\beta-1}{4n}|x|^2 - 1\right) multiplicity\, 1 \qquad e^-(x) = -\frac{M}{2n}\beta\left(1 - \frac{|x|^2}{4n}\right)^{\beta-1} multiplicity\, n{-}1$$

We choose β large so for $|x| \geq \frac{1}{4}$, $e^+(x) > 0$ and $e^-(x) < 0$. Hence for $|x| \geq \frac{1}{4}$,

$$\mathcal{M}^-\left(\lambda, \Lambda, -D^2 g(x_0)\right) = \lambda e^+(x) + (n-1)\Lambda e^-(x)$$

$$= \frac{M}{2n}\beta\left(1 - \frac{|x|^2}{4n}\right)^{\beta-2}\left\{\lambda\left(\frac{2\beta-1}{4n}|x|^2 - 1\right) - (n-1)\Lambda\left(1 - \frac{|x|^2}{4n}\right)\right\} \geq 0$$

if choose β large depending only on λ, Λ, n. Hence $w \in \mathcal{S}^+(\lambda, \Lambda, f)$ *in* $B_{2\sqrt{n}} \setminus Q_1$, or equivalently,

$$w \in \mathcal{S}^+(\lambda, \Lambda, f + \eta) \quad in\ B_{2\sqrt{n}}$$

for some $\eta \in C_0^\infty(Q_1)$ and $0 \leq \eta \leq C(n, \lambda, \Lambda)$.

(iii) Apply Theorem1.1 to w in $B_{2\sqrt{n}}$. Note by assumption, $w = u + g \geq 0$ on $\partial B_{2\sqrt{n}}$ and since $\inf_{Q_3} u \leq 1$, $g \leq -2$ in Q_3, we have $\inf_{Q_3} w = \inf_{Q_3}(u+g) \leq -1 \iff \sup_{Q_3} w^- \geq 1$. Hence by choice of η

$$1 \leq \sup_{Q_3} w^- \leq C\left(\int_{B_{2\sqrt{n}} \cap \{w = \Gamma_w\}} (|f| + \eta)^n\right)^{\frac{1}{n}} \leq C\|f\|_{L^n\left(B_{2\sqrt{n}}\right)} + C\left|\{w = \Gamma_w\}\right] \cap Q_1|^{\frac{1}{n}}$$

Recall $\|f\|_{L^n\left(B_{2\sqrt{n}}\right)} \leq \epsilon_0$, so taking ϵ_0 small enough we have

$$\frac{1}{2} \leq C\left|\{w = \Gamma_w\}\right] \cap Q_1|^{\frac{1}{n}}$$

Finally we notice $w = \Gamma_w$ convex envelope of $-w^- = \min\{w, 0\} \implies w \leq 0 \implies u(x) \leq -g(x) \leq M$.

$$\frac{1}{2} \leq C\left|\{u \leq M\}\right] \cap Q_1|^{\frac{1}{n}} \implies choose\ \mu \in \left(0, \left(\frac{1}{2C}\right)^n\right)$$

□

Next we prove the complement lemma suggesting power decay of distribution functions under same assumptions.

Lemma 1.4. *$u \in \mathcal{S}^+(\lambda, \Lambda, f)$ in $B_{2\sqrt{n}}$ for $f \in C\left(B_{2\sqrt{n}}\right)$. Then $\exists\ \epsilon_0, \epsilon > 0$, and $C > 0$ constants depending only on n, λ, Λ s.t. if*

$$u \geq 0\ in\ B_{2\sqrt{n}}, \quad \inf_{Q_3} u \leq 1, \quad \|f\|_{L^n\left(B_{2\sqrt{n}}\right)} \leq \epsilon_0$$

we have $|\{u \geq t\} \cap Q_1| \leq Ct^{-\epsilon}\ \forall\ t > 0$.

Proof. (i) We wish to show under above assumptions, for any $k \in \mathbb{Z}_+$ and M, μ from Lemma 1.3

$$\left|\{u > M^k\} \cap Q_1\right| \leq (1-\mu)^k$$

This makes sense if take $M^k = t$, then $k = \frac{\log t}{\log M}$, so $(1-\mu)^k = \left((1-\mu)^{\frac{1}{\log M}}\right)^{\log t} \implies \epsilon = -\frac{log(1-\mu)}{\log M} > 0$.

(ii) For $k = 1$, this is direct result from Lemma 1.3. Suppose $k-1$ holds, *i.e.*,

$$\left|\{u > M^{k-1}\} \cap Q_1\right| \leq (1-\mu)^{k-1}$$

We set $A := \{u > M^k\} \cap Q_1$ and $B := \{u > M^{k-1}\} \cap Q_1$, we will show $|A| \leq (1-\mu)|B|$ by Corollary 1.1.

- Indeed $A \subset B \subset Q_1$
- Since $M > 1$, $|A| < |\{u > M\} \cap Q_1| \leq 1 - \mu$ by Lemma 1.3.

- We hope to show if $Q = Q_r(x_0)$ is a cube centered at x_0 with side length r in Q_1 *s.t.*

$$|A \cap Q| \geq (1-\mu)|Q|$$

then $\widetilde{Q} \cap Q_1 \subset B$ for $\widetilde{Q} = Q_{3r}(x_0)$. Suppose not, then $\exists\ \tilde{x} \in \widetilde{Q} = Q_{3r}(x_0)$ *s.t.* $u(\tilde{x}) \leq M^{k-1}$. Consider transformation

$$x = x_0 + ry \quad for\ y \in Q_1 \implies x \in Q = Q_r(x_0)$$

and the function $\tilde{u}$ defined on $B_{2\sqrt{n}} \supset Q_1$

$$\tilde{u}(y) := \frac{1}{M^{k-1}} u(x) = \frac{1}{M^{k-1}} u(x_0 + ry)$$

We know $u \geq 0$ in $B_{2\sqrt{n}} \implies \tilde{u} \geq 0$ in $B_{2\sqrt{n}}$, and $\inf_{y \in Q_3} \tilde{u}(y) \leq 1$ due to existence of $\tilde{x} \in \widetilde{Q} = Q_{3r}(x_0)$. Also, defining

$$\tilde{f}(y) := \frac{r^2}{M^{k-1}} f(x) \quad \forall\ y \in B_{2\sqrt{n}}$$

We see, since $0 < r < 1$ and $M > 1$

$$\left\|\tilde{f}\right\|_{L^n\left(B_{2\sqrt{n}}\right)} \leq \frac{r}{M^{k-1}} \|f\|_{L^n\left(B_{2\sqrt{n}}\right)} \leq \|f\|_{L^n\left(B_{2\sqrt{n}}\right)} \leq \epsilon_0$$

And obviously $\tilde{u} \in \mathcal{S}^+\left(\lambda, \Lambda, \tilde{f}\right)$, by Lemma 1.3, we have for $\mu \in (0,1)$

$$\mu < |\{\tilde{u}(y) \leq M\} \cap Q_1| = r^{-n}\left|\{u(x) \leq M^k\} \cap Q\right| \implies \mu|Q| < |A^c \cap Q| \implies |Q| < |Q|$$

We've reached a contradiction.

□

We obtain C^0 estimate for u on $Q_{1/4}$ using the above 2 lemmas.

Lemma 1.5. *$u \in \mathcal{S}(\lambda, \Lambda, f)$ in $Q_{4\sqrt{n}}$ for $f \in C\left(Q_{4\sqrt{n}}\right)$. Then $\exists\ \epsilon_0, C > 0$ constants depending only on n, λ, Λ s.t. if*

$$u \geq 0\ in\ Q_{4\sqrt{n}}, \quad \inf_{Q_{1/4}} u \leq 1, \quad \|f\|_{L^n\left(Q_{4\sqrt{n}}\right)} \leq \epsilon_0$$

we have $\sup_{Q_{1/4}} u \leq C$.

Proof. (i) We prove that there exists constants $\theta > 1$ and $M_0 \gg 1$ depending only on n, λ, Λ *s.t.* if $u(x_0) = P > M_0$ for some $x_0 \in B_{1/4}$, then $\exists\ \{x_k\} \subset B_{1/2}$ *s.t.*

$$u(x_k) \geq \theta^k P \quad for\ k \in \mathbb{N}$$

which contradicts boundedness of u on $B_{1/4} \implies \sup_{B_{1/4}} u \leq M_0$.

(ii) Suppose $\exists\ x_0 \in B_{1/4}$ *s.t.* $u(x_0) = P > M_0$ with M_0, θ to be determined. Also take $Q_r(x_0)$ with side length r to be determined. The idea is to find $x_1 \in Q_{4\sqrt{n}r}(x_0)$ so that $u(x_1) \geq \theta P$. We start by choosing r so that $\{u > \frac{P}{2}\}$ covers less than half of $Q_r(x_0)$, using the power decay for distribution function of u. Note $\inf_{Q_3} u \leq \inf_{Q_{1/4}} u \leq 1$. Hence by Lemma 1.4, we know $\exists\ \epsilon > 0$ *s.t.* taking $t = \frac{P}{2}$

$$\left|\{u > \frac{P}{2}\} \cap Q_1\right| \leq C\left(\frac{P}{2}\right)^{-\epsilon}$$

Choose r *s.t.* $r \leq \frac{1}{4}$ and $\frac{r^n}{2} \geq C\left(\frac{P}{2}\right)^{-\epsilon}$. The former gives $Q_r(x_0) \subset Q_1$ while the latter gives

$$\frac{1}{|Q_r(x_0)|}\left|\{u > \frac{P}{2}\} \cap Q_r(x_0)\right| \leq \frac{1}{2}$$

(iii) Next we show for $\theta > 1$ with $\theta - 1$ small, $\exists\, x_1 \in Q_{4\sqrt{n}r}(x_0)$ *s.t.* $u(x_1) \geq \theta P$. Show by contradiction. Suppose $u < \theta P$ in $Q_{4\sqrt{n}r}(x_0)$. Consider transformation

$$x = x_0 + ry \quad for\ y \in Q_{4\sqrt{n}} \implies x \in Q_{4\sqrt{n}}(x_0)$$

and the function $\tilde{u}$ defined on $Q_{4\sqrt{n}} \supset B_{2\sqrt{n}}$

$$\tilde{u}(y) = \frac{\theta P - u(x)}{(\theta - 1)P} = \frac{\theta P - u(x_0 + ry)}{(\theta - 1)P}$$

We observe $u < \theta P$ in $Q_{4\sqrt{n}r}(x_0) \implies \tilde{u} \geq 0$ in $B_{2\sqrt{n}}$, and $\tilde{u}(0) = 1$, $Q_3 \subset B_{2\sqrt{n}} \implies \inf_{y \in Q_3} \tilde{u}(y) \leq 1$. Also, defining

$$\tilde{f}(y) := -\frac{r^2}{(\theta - 1)P} f(x) \quad for\ y \in B_{2\sqrt{n}}$$

We see, upon choosing P *s.t.* $r \leq (\theta - 1)P$

$$\left\|\tilde{f}\right\|_{L^n\left(B_{2\sqrt{n}}\right)} \leq \frac{r}{(\theta - 1)P}\|f\|_{L^n\left(B_{2\sqrt{n}}\right)} \leq \epsilon_0$$

Notice since $u \in \mathcal{S}(\lambda, \Lambda, f) = \mathcal{S}^+(\lambda, \Lambda, f) \cap \mathcal{S}^-(\lambda, \Lambda, f)$, we make use of $u \in \mathcal{S}^-(\lambda, \Lambda, f)$ to see that

$$\forall\, x_0 \in Q_{4\sqrt{n}}\ and\ \forall\, \varphi \in C^2\left(Q_{4\sqrt{n}}\right)\ s.t.\ u - \varphi\ has\ local\ maximum\ at\ x_0 \implies \Lambda\sum_{e_i > 0} e_i + \lambda\sum_{e_i < 0} e_i \geq f(x_0)$$

$$\iff -\Lambda\sum_{e_i > 0}(-e_i) - \lambda\sum_{e_i < 0}(-e_i) \geq f(x_0) \iff \lambda\sum_{e_i < 0}(-e_i) + \Lambda\sum_{e_i > 0}(-e_i) \leq -f(x_0)$$

rewriting "$u - \varphi$ has local maximum at x_0" as "$-(u - \varphi) = -u - (-\varphi)$ has local minimum at x_0", and observe now $-e_i$ are positive eigenvalues for $D^2(-\varphi)$ at x_0 and e_i negative eigenvalues. We see $-u \in \mathcal{S}^+(\lambda, \Lambda, -f)$. Hence obviously $\tilde{u} \in \mathcal{S}^+\left(\lambda, \Lambda, \tilde{f}\right)$ in $B_{2\sqrt{n}}$, we may apply Lemma 1.4 to $\tilde{u}$. Note $u(x) \leq \frac{P}{2} \iff \tilde{u}(y) \geq \frac{\theta - \frac{1}{2}}{\theta - 1}$, and is large for θ close to 1. Hence we've reached a contradiction

$$\frac{1}{|Q_r(x_0)|}\left|\{u \leq \frac{P}{2}\} \cap Q_r(x_0)\right| = \left|\{\tilde{u} \geq \frac{\theta - \frac{1}{2}}{\theta - 1}\} \cap Q_1\right| \leq C\left(\frac{\theta - \frac{1}{2}}{\theta - 1}\right)^{-\epsilon} < \frac{1}{2} \quad for\ \theta\ close\ to\ 1 \implies 1 < 1$$

(iv) Notice we've proved $\exists\, \theta = \theta(n, \lambda, \Lambda) > 1$ *s.t.* if $u(x_0) = P$ for some $x_0 \in B_{1/4}$, then

$$\exists\, x_1 \in Q_{4\sqrt{n}r}(x_0) \subset B_{2nr}(x_0)\ s.t.\ u(x_1) \geq \theta P$$

provided $0 < r \leq \frac{1}{4}$ and

$$C(n, \lambda, \Lambda)P^{-\frac{\epsilon}{n}} \leq r \leq (\theta - 1)P$$

Hence we need to choose P so that $P \geq \left(\frac{C}{\theta - 1}\right)^{\frac{n}{n+\epsilon}}$. We take $r = CP^{-\frac{\epsilon}{n}}$.

(v) Now we iterate above result. For k^{th} iteration, we treat P above as $\theta^{k-1}P$, so

$$r_k = C\left(\theta^{k-1}P\right)^{-\frac{\epsilon}{n}} = C\theta^{-(k-1)\frac{\epsilon}{n}}P^{-\frac{\epsilon}{n}}$$

and we've obtained a sequence $\{x_k\}$ $s.t.$ $\forall\ k \in \mathbb{Z}_+$

$$u\left(x_k\right) \geq \theta\left(\theta^{k-1} P\right) = \theta^k P \quad for\ some\ x_k \in B_{2nr_k}\left(x_{k-1}\right)$$

In order to ensure $\{x_k\} \subset B_{1/2}$, we let $\sum_{k\in\mathbb{Z}_+} 2nr_k < \frac{1}{4}$. So choose M_0 $s.t.$ $\forall\ P > M_0$ the sum holds, $i.e.$

$$\sum_{k\in\mathbb{Z}_+} 8nr_k < 1 \implies 8nC \sum_{k=1}^{\infty} \theta^{-(k-1)\frac{\epsilon}{n}} \leq M_0^{\frac{\epsilon}{n}} < P^{\frac{\epsilon}{n}} \implies choose\ M_0 \geq \left(\frac{C}{\theta - 1}\right)^{\frac{n}{n+\epsilon}}$$

□

Now Harnack's Inequality is a direct consequence for above lemma.

Theorem 1.2 (Harnark's Inequality - Viscosity Version)**.** *$u \in \mathcal{S}(\lambda, \Lambda, f)$ in B_1 for $f \in C(B_1)$ with $u \geq 0$ in B_1. Then for $C = C(n, \lambda, \Lambda) > 0$*

$$\sup_{x\in B_{1/2}} u \leq C\left\{\inf_{x\in B_{1/2}} u + \|f\|_{L^n(B_1)}\right\}$$

Proof. Take $u \in \mathcal{S}(\lambda, \Lambda, f)$ in $Q_{4\sqrt{n}}$ $s.t.$ $u \geq 0$ in $Q_{4\sqrt{n}}$. To apply Lemma 1.5 we need 2 more assumptions on the solution. Hence consider

$$u_\delta := \frac{u}{\inf\limits_{x\in Q_{1/4}} u + \delta + \frac{1}{\epsilon_0}\|f\|_{L^n\left(Q_{4\sqrt{n}}\right)}} \quad for\ \delta > 0 \quad \forall\ x \in Q_{4\sqrt{n}}$$

which indeed gives $\inf\limits_{x\in Q_{1/4}} u_\delta \leq 1$ and $\|f_\delta\|_{L^u\left(Q_{4\sqrt{n}}\right)} \leq \epsilon_0$ where

$$f_\delta(x) := \frac{f(x)}{\inf\limits_{x\in Q_{1/4}} u + \delta + \frac{1}{\epsilon_0}\|f\|_{L^n\left(Q_{4\sqrt{n}}\right)}} \quad for\ \delta > 0 \quad \forall\ x \in Q_{4\sqrt{n}}$$

Hence we apply Lemma 1.5 to u_δ and see, letting $\delta \to 0$

$$\sup_{x\in Q_{1/4}} u \leq C\left\{\inf_{x\in Q_{1/4}} u + \|f\|_{L^n\left(Q_{4\sqrt{n}}\right)}\right\}$$

We conclude by choosing a finite cover for $B_{1/2}$ with cubes $Q_{1/4}$. □

Corollary 1.2 (Interior Hölder Continuity - Viscosity Version)**.** *$u \in \mathcal{S}(\lambda, \Lambda, f)$ in B_1 for $f \in C(B_1)$. Then $u \in C^\alpha(B_1)$ for $\alpha = \alpha(n, \lambda, \Lambda) \in (0, 1)$. Moreover for $C = C(n, \lambda, \Lambda) > 0$*

$$|u(x) - u(y)| \leq C|x - y|^\alpha \left\{\sup_{x\in B_1} |u| + \|f\|_{L^n(B_1)}\right\} \quad for\ any\ x,\ y \in B_{1/2}$$

1.3 Schauder Estimates

We provide the problem setting for Schauder's Estimate for viscosity solutions.

(i) Uniformly Elliptic. Given $\lambda, \Lambda > 0$, let $a_{ij}(x) \in C(B_1)$ $s.t.$ $\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$ $\forall\ x \in B_1,\ \xi \in \mathbb{R}^n$.

(ii) $f \in C(B_1)$.

(iii) Recall Definition 1.1, $u \in C(B_1)$ is viscosity solution of $a_{ij}D_{ij}u = f$ in B_1 if

$$\forall\ x_0 \in B_1\ and\ \forall\ \varphi \in C^2(B_1)\ \ s.t.\ \begin{cases} if\ \ u - \varphi\ has\ a\ local\ minimum\ at\ x_0 \implies a_{ij}(x_0)\,D_{ij}\varphi(x_0) \leq f(x_0) \\ if\ \ u - \varphi\ has\ a\ local\ maximum\ at\ x_0 \implies a_{ij}(x_0)\,D_{ij}\varphi(x_0) \geq f(x_0) \end{cases}$$

Lemma 1.6 (Approximation)**.** *Let $u \in C(B_1)$ be viscosity solution of $a_{ij}D_{ij}u = f$ in B_1 with $|u| \le 1$ in B_1. Assume for some $0 < \epsilon < \frac{1}{16}$,*

$$\|a_{ij} - a_{ij}(0)\|_{L^n(B_{3/4})} \le \epsilon$$

Then $\exists\ h \in C\left(\overline{B}_{3/4}\right)$ with $a_{ij}(0)D_{ij}h = 0$ in $B_{3/4}$ and $|h| \le 1$ in $B_{3/4}$ s.t.

$$\|u - h\|_{L^\infty(B_{1/2})} \le C\left\{\epsilon^\gamma + \|f\|_{L^n(B_1)}\right\}$$

for $C = C(n, \lambda, \Lambda) > 0$ and $\gamma = \gamma(n, \lambda, \Lambda) \in (0, 1)$.

Proof. (i) By Poisson's Integral Formula, we solve for $h \in C\left(\overline{B}_{3/4}\right) \cap C^\infty\left(B_{3/4}\right)$ explicitly *s.t.*

$$\begin{aligned} a_{ij}(0)D_{ij}h &= 0 \quad \text{in } B_{3/4} \\ h &= u \quad \text{on } \partial B_{3/4} \end{aligned}$$

where $u \in C\left(\partial B_{3/4}\right) \subset C(B_1)$. Here the maximum principle for harmonic functions implies that $|h| \le \sup_{x\in\partial B_{3/4}} u(x) \le 1$. Note that $u \in \mathcal{S}(\lambda, \Lambda, f)$ in B_1, so by Interior Hölder's Continuity 1.2 for viscosity solutions $\implies$ we have $u \in C^\alpha\left(\overline{B}_{3/4}\right)$ with $\alpha = \alpha(n, \lambda, \Lambda) \in (0, 1)$ and estimate

$$\|u\|_{C^\alpha\left(\overline{B}_{3/4}\right)} \le C\left(1 + \|f\|_{L^n(B_1)}\right)$$

Then by Global Hölder Continuity for $h \in C\left(\overline{B}_{3/4}\right)$ harmonic function in $B_{3/4}$ with $h = u$ on $\partial B_{3/4}$, and that $u \in C^\alpha\left(\partial B_{3/4}\right) \implies$ we have $h \in C^{\alpha/2}\left(\overline{B}_{3/4}\right)$ with estimate

$$\|h\|_{C^{\alpha/2}\left(\overline{B}_{3/4}\right)} \le C\|u\|_{C^\alpha\left(\partial B_{3/4}\right)} \le C\|u\|_{C^\alpha\left(\overline{B}_{3/4}\right)} \le C\left(1 + \|f\|_{L^n(B_1)}\right)$$

(ii) We need 2 ingredients before estimating $u - h$.
First, since $u - h = 0$ on $\partial B_{3/4}$, *i.e.*, $\|u - h\|_{L^\infty\left(\partial B_{3/4}\right)} = 0$, consider $\forall\ 0 < \delta < \frac{1}{4}$

$$\frac{\|u-h\|_{L^\infty\left(\partial B_{3/4-\delta}\right)} - \|u-h\|_{L^\infty\left(\partial B_{3/4}\right)}}{\delta^{\alpha/2}} \le \|u-h\|_{C^{\alpha/2}\left(\overline{B}_{3/4}\right)} \le \|u\|_{C^{\alpha/2}\left(\overline{B}_{3/4}\right)} + \|h\|_{C^{\alpha/2}\left(\overline{B}_{3/4}\right)} \le C\left(1 + \|f\|_{L^n(B_1)}\right)$$

$$\implies \|u-h\|_{L^\infty\left(\partial B_{3/4-\delta}\right)} \le C\delta^{\alpha/2}\left(1 + \|f\|_{L^n(B_1)}\right) \tag{1.1}$$

Second, consider $\forall\ 0 < \delta < \frac{1}{4}$, $\forall\ x_0 \in B_{3/4-\delta}$, we take some $x_1 \in \partial B_\delta(x_0)$ and apply Interior C^2-estimate to $h - h(x_1)$ in $B_\delta(x_0) \subset B_{3/4}$, using $h \in C^{\alpha/2}\left(\overline{B}_{3/4}\right)$

$$\left|D^2h(x_0)\right| \le C\frac{1}{\delta^2}\sup_{x\in B_\delta(x_0)}|h(x) - h(x_1)| \le C\frac{1}{\delta^2}\delta^{\alpha/2}\|h\|_{C^{\alpha/2}\left(\overline{B}_{3/4}\right)} \le C\delta^{\alpha/2-2}\left(1 + \|f\|_{L^n(B_1)}\right)$$

$$\implies \left\|D^2h\right\|_{L^\infty\left(B_{3/4-\delta}\right)} \le C\delta^{\alpha/2-2}\left(1 + \|f\|_{L^n(B_1)}\right) \tag{1.2}$$

(iii) Note $u - h$ is viscosity solution to

$$a_{ij}D_{ij}(u-h) = f - (a_{ij} - a_{ij}(0))D_{ij}h \equiv F \quad \text{in } B_{3/4}$$

By ABP Method Theorem 1.1, the above estimates (1.1), (1.2) and assumption on a_{ij}

$$\begin{aligned}\|u-h\|_{L^\infty\left(B_{3/4-\delta}\right)} &\leq \|u-h\|_{L^\infty\left(\partial B_{3/4-\delta}\right)} + C\,\|F\|_{L^n\left(B_{3/4-\delta}\right)} \\ &\leq \|u-h\|_{L^\infty\left(\partial B_{3/4-\delta}\right)} + C\left\|D^2 h\right\|_{L^\infty\left(B_{3/4-\delta}\right)} \|a_{ij}-a_{ij}(0)\|_{L^n\left(B_{3/4}\right)} + C\,\|f\|_{L^n(B_1)} \\ &\leq C\left(\delta^{\alpha/2}+\delta^{\alpha/2-2}\epsilon\right)\left\{1+\|f\|_{L^n(B_1)}\right\} + C\,\|f\|_{L^n(B_1)}\end{aligned}$$

Hence take $\delta = \epsilon^{1/2} < \frac{1}{4}$, so $\delta^{\alpha/2}+\delta^{\alpha/2-2}\epsilon = 2\epsilon^{\alpha/4}$. Take $\gamma = \frac{\alpha}{4}$.

□

Now we're ready to state the Schauder estimate with definition for the weighted Hölder semi-norm below.

Definition 1.3 (Hölder Continuity in the L^n Sense)**.** *g is Hölder Continuous at 0 with exponent α in the L^n sense if*

$$[g]_{C^\alpha_{L^n}}(0) := \sup_{0<r<1} \frac{1}{r^\alpha}\left(\frac{1}{|B_r|}\int_{B_r} |g-g(0)|^n\right)^{\frac{1}{n}} < \infty$$

Theorem 1.3 (Schauder Estimate - Viscosity Version)**.** *$u \in C(B_1)$ be viscosity solution of $a_{ij}D_{ij}u = f$ in B_1. Assume both $\{a_{ij}\}$ and f are Hölder Continuous at 0 with exponent α in the L^n sense for some $\alpha \in (0,1)$. Then u is $C^{2,\alpha}$ at 0. Moreover, $\exists$ polynomial P of degree 2 s.t.*

$$|P(0)| + |DP(0)| + \left|D^2 P(0)\right| \leq C\left(\|u\|_{L^\infty(B_1)} + |f(0)| + [f]_{C^\alpha_{L^n}}(0)\right)$$

with estimate

$$\|u-P\|_{L^\infty(B_r(0))} \leq C r^{2+\alpha}\left(\|u\|_{L^\infty(B_1)} + |f(0)| + [f]_{C^\alpha_{L^n}}(0)\right) \quad \forall\ 0<r<1$$

where $C = C\left(n,\lambda,\Lambda,\alpha,[a_{ij}]_{C^\alpha_{L^n}}(0)\right) > 0$.

Proof. (i) We first restate our target problem. Assume $f(0) = 0$, since if we consider $v = u - b_{ij}x_i x_j \frac{f(0)}{2}$ for constant matrix $\{b_{ij}\}$ *s.t.* $a_{ij}(0)\,b_{ij} = 1$,

$$a_{ij}D_{ij}v = f - a_{ij}f(0)\,b_{ij} \implies a_{ij}(0)\,D_{ij}v(0) = f(0) - f(0) = 0$$

Also assume $[a_{ij}]_{C^\alpha_{L^n}}(0)$ is small by rescaling, and by considering for $\delta > 0$ the problem

$$\frac{u}{\|u\|_{L^\infty(B_1)} + \frac{1}{\delta}[f]_{C^\alpha_{L^n}}(0)}$$

we may assume $\|u\|_{L^\infty(B_1)} \leq 1$ and $[f]_{C^\alpha_{L^n}}(0) \leq \delta$.
Hence it suffices to prove that $\exists\ \delta > 0$ depending on n,λ,Λ,α *s.t.* if $u \in C(B_1)$ is viscosity solution of

$$a_{ij}D_{ij}u = f \quad \textit{in } B_1 \textit{ with}$$

$$\|u\|_{L^\infty(B_1)} \leq 1, \quad [a_{ij}]_{C^\alpha_{L^n}}(0) \leq \delta, \quad \left(\frac{1}{|B_r|}\int_{B_r}|f|^n\right)^{\frac{1}{n}} \leq \delta r^\alpha \quad \forall\ 0<r<1$$

Then $\exists$ polynomial P of degree 2 *s.t.*

$$|P(0)| + |DP(0)| + \left|D^2 P(0)\right| \leq C$$

with estimate

$$\|u-P\|_{L^\infty(B_r(0))} \leq C r^{2+\alpha} \quad \forall\ 0<r<1$$

for positive constant $C = C(n, \lambda, \Lambda, \alpha) > 0$.

(ii) **Claim** $\exists\ 0 < \mu < 1$ depending on $n, \lambda, \Lambda, \alpha$ and a sequence of polynomials of degree 2

$$P_k(x) := a_k + b_k \cdot x + \frac{1}{2} x^\top C_k x$$

s.t. $\forall\ k = 0, 1, \cdots$, and $P_0 = P_{-1} \equiv 0$

$$a_{ij}(0) D_{ij} P_k = 0, \quad \|u - P_k\|_{L^\infty(B_{\mu^k})} \le \mu^{k(2+\alpha)} \tag{1.3}$$

$$|a_k - a_{k-1}| + \mu^{k-1} |b_k - b_{k-1}| + \mu^{2(k-1)} |C_k - C_{k-1}| \le C\mu^{(k-1)(2+\alpha)} \quad for\ C = C(n, \lambda, \Lambda, \alpha) > 0 \tag{1.4}$$

We justify that our target theorem follows from the claim. Note by (1.4), a_k, b_k and C_k all converges and we define the limiting polynomial

$$p(x) = a_\infty + b_\infty \cdot x + \frac{1}{2} x^\top C_\infty x$$

Notice $\forall\ |x| \le \mu^k$

$$|P_k(x) - p(x)| \le C \left\{ \mu^{(\alpha+2)k} + |x| \mu^{(\alpha+1)k} + |x|^2 \mu^{\alpha k} \right\} \le C\mu^{(2+\alpha)k}$$

Hence $\forall\ |x| \le \mu^k$

$$|u(x) - p(x)| \le |u(x) - P_k(x)| + |P_k(x) - p(x)| \le C\mu^{(2+\alpha)k} \implies |u(x) - p(x)| \le C|x|^{2+\alpha} \quad \forall\ x \in B_1$$

(iii) We prove the claim by induction. Case $k = 0$ holds trivially. Assume for ℓ, and prove for $k = \ell + 1$. Define

$$\tilde{u}(y) := \frac{1}{\mu^{\ell(2+\alpha)}} (u - P_\ell)(\mu^\ell y) \quad for\ y \in B_1 \implies \|\tilde{u}\|_{L^\infty(B_1)} \le 1\ by\ assumption\ on\ \ell$$

Now $\tilde{u} \in C(B_1)$ is viscosity solution of $\tilde{a}_{ij} D_{ij} \tilde{u} = \tilde{f}$ *in* B_1 with

$$\tilde{a}_{ij}(y) = \frac{1}{\mu^{\ell\alpha}} a_{ij}(\mu^\ell y), \quad \tilde{f}(y) = \frac{1}{\mu^{\ell\alpha}} \left\{ f(\mu^\ell y) - a_{ij}(\mu^\ell y) D_{ij} P_\ell(\mu^\ell y) \right\}$$

To apply Lemma 1.6, we see, due to Hölder Continuity of $\{a_{ij}\}$, f at 0 in L^n sense

$$\|\tilde{a}_{ij} - \tilde{a}_{ij}(0)\|_{L^n(B_1)} \le \frac{1}{\mu^{\ell\alpha}} \|a_{ij} - a_{ij}(0)\|_{L^n(B_{\mu^\ell})} \le [a_{ij}]_{C^\alpha_{L^n}}(0) \le \delta$$

$$\begin{aligned} \left\|\tilde{f}\right\|_{L^n(B_1)} &\le \frac{1}{\mu^{\ell\alpha}} \|f\|_{L^n(B_{\mu^\ell})} + \frac{1}{\mu^{\ell\alpha}} \sup_{y \in B_{\mu^\ell}} \left|D^2 P_\ell\right| \|a_{ij} - a_{ij}(0)\|_{L^n(B_{\mu^\ell})} \\ &\le \delta + \left(\sum_{i=1}^{\ell} \sup_{y \in B_{\mu^\ell}} \left|D^2 P_i - D^2 P_{i-1}\right| \right) \delta \\ &\le \left(1 + \sum_{i=1}^{\ell} \mu^{(i-1)\alpha} \right) \delta \le (1 + C)\delta \quad for\ C = C(n, \lambda, \Lambda) > 0 \end{aligned}$$

Now take $\epsilon = C\delta$ and apply Lemma 1.6, $\exists\ h \in C\left(\overline{B}_{3/4}\right)$ with $\tilde{a}_{ij} D_{ij} h = 0$ in $B_{3/4}$ and $|h| \le 1$ in $B_{3/4}$ *s.t.*

$$\|\tilde{u} - h\|_{L^\infty(B_{1/2})} \le C\{\epsilon^\gamma + \epsilon\} \le 2C\epsilon^\gamma$$

We write $\tilde{P}(y) := h(0) + Dh(0) + y^\top D^2 h(0) y/2$. By Interior Estimate for h

$$\left\|\tilde{u} - \tilde{P}\right\|_{L^\infty(B_\mu)} \le \|\tilde{u} - h\|_{L^\infty(B_\mu)} + \left\|h - \tilde{P}\right\|_{L^\infty(B_\mu)} \le 2C\epsilon^\gamma + C\mu^3 \le \mu^{2+\alpha}$$

choosing μ small and then ϵ small. Rescaling back we see

$$\left\|u(x) - P_\ell(x) - \mu^{\ell(2+\alpha)}\tilde{P}\left(\mu^{-\ell}x\right)\right\| \le \mu^{(\ell+1)(2+\alpha)} \quad \forall\, x \in B_{\mu^{\ell+1}}$$

Hence for $k = \ell + 1$, we define

$$P_k(x) = P_{\ell+1}(x) := P_\ell(x) + \mu^{\ell(2+\alpha)}\tilde{P}\left(\mu^{-\ell}x\right)$$

□

Theorem 1.4 (Cordes-Nirenberg type Estimate). *$u \in C(B_1)$ be viscosity solution of $a_{ij}D_{ij}u = f$ in B_1. Then $\forall\, \alpha \in (0,1)$, $\exists\, \theta = \theta(n, \lambda, \Lambda, \alpha) > 0$ s.t. if*

$$\left(\frac{1}{|B_r|}\int_{B_r} |a_{ij} - a_{ij}(0)|^n\right)^{\frac{1}{n}} \le \theta \quad \forall\, 0 < r \le 1$$

then u is $C^{1,\alpha}$ at 0. Moreover, there $\exists$ affine function L s.t.

$$|L(0)| + |DL(0)| \le C\left(\|u\|_{L^\infty(B_1)} + \sup_{0<r<1} r^{1-\alpha}\left(\frac{1}{|B_r|}\int_{B_r}|f|^n\right)^{\frac{1}{n}}\right)$$

with estimate

$$\|u - L\|_{L^\infty(B_r(0))} \le Cr^{1+\alpha}\left(\|u\|_{L^\infty(B_1)} + \sup_{0<r<1} r^{1-\alpha}\left(\frac{1}{|B_r|}\int_{B_r}|f|^n\right)^{\frac{1}{n}}\right) \quad \forall\, 0 < r < 1$$